CONSISTENT COST SHARING AND RATIONING

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ABSTRACT. A new concept of *consistency* for cost sharing models is discussed, analyzed, and related to the homonymous property within the rationing context. Central in the discussion is the Moulin-Shenker (1994) characterization of cost sharing mechanisms in terms of rationing methods. It is used to characterize the class of consistent *incremental mechanisms*, which includes most of the prevalent solutions such as average, serial, and Shapley-Shubik cost sharing.

Keywords: consistency, cost sharing, mechanism design, rationing, additivity, incremental mechanism

JEL-Classification: C70, D63, D70

1. INTRODUCTION

This paper studies the connection between two models in the literature on distributive justice, i.e., that of rationing and cost sharing, respectively. This paper fits in the stream of axiomatic literature (see Thomson (2001)) which discusses structural and characterizing properties of solutions. The central property is *consistency*, pertaining to variations of the relevant set of agents. It envisions the idea of *fairness* of solutions at all levels of cooperation, for any subgroup of agents, according to which 'no subgroup should want to "re-contract"' (Young (1985, p19)). Davis and Maschler (1965) and Hart and Mas-Colell (1989) introduce the

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property of consistency to the field of cooperative games, ideas which Sudhölter (1998) applied to solutions for cost sharing games. Other examples include Young (1987, 1988) on taxation problems and Moulin (1987) on a model of surplus sharing. More recently, Friedman (1997) considers consistency properties in heterogeneous cost sharing problems. See Thompson (1996) for a general overview on the property of consistency in economic theory.

Formally, a *rationing problem* among agents in $N = \{1, 2, \ldots, n\}$ consists of a profile q of nonnegative individual demands, q_i being the demand of agent i in N, and an amount t to be allocated to the agents. What makes it a rationing problem is the assumption $t \leq \sum_{i \in N} q_i$, so that the aggregate demand (weakly) exceeds the available amount. Many applications fit in this simple framework, ranging from bankruptcy (O'Neill (1982), Aumann and Maschler (1985)) to taxation (Young (1988)). Solutions are the proportional, uniform gains, and uniform losses methods (see e.g. Moulin (2002) and Thomson (2003) for overviews).

Suppose that the agents in N jointly own a production facility for some good Y. A *cost sharing problem* among agents in N consists of a demand profile q, q_i being the demand of agent i for good Y, and the description of the technology in terms of a *cost function* c, relating each nonnegative level y of production to the minimum required level of inputs or cost $c(y)$. Instead of a fixed amount t, the agents now have to divide a variable cost $c(\sum_{i \in N} q_i)$. Many solutions have been proposed in the *mechanistic* cost sharing literature, including the average cost sharing mechanism, the Shapley-Shubik mechanism (Shubik (1962), Sudhölter (1998), Young (1985, 1994)), and the serial mechanism (Moulin and Shenker (1992,1994)).

Where a natural and very intuitive formulation of the consistency principle exists for the rationing model, in the cost sharing literature such unified approach is absent. This is best illustrated by a number of studies, e.g. by Sudhölter (1998), Moulin and Shenker (1994), and Tijs and Koster (1998), each of them proposing a different concept, based on distinct notions of a reduced cost sharing problem – the basic construct in the notion of consistency. The ambiguity in choosing the 'proper' reduction is expressed by the many ways that production levels may correspond to an agents' cost share, each choice leading to a different truncated and reduced cost function. As these truncations may alter the very nature of the cost sharing problem at hand, I suggest, to avoid too much arbitrariness in the choice of the reduction, to include all 'sensible' ones. Then under consistency the corresponding induced set of solutions includes those derived from the status quo solution. Where the foregoing notions of a reduction roughly concentrate only on an agent's cost share, here I will suggest a more subtle notion requiring a match with the size of an agents' demand as well.

Moulin and Shenker (1994) show a class of cost sharing solutions which naturally corresponds to the class of monotonic rationing methods, i.e., the class of all *additive* cost sharing mechanisms with the property *constant returns*. Additivity is a decomposition property and is usually advocated for the ease of accounting, whereas the constant returns property specifies the natural allocation for cost sharing problems without externalities. Each cost sharing mechanism with these two properties represents a functional which allows for a Stieltjes-integral representation with respect to the rationing method. For example, the average mechanism corresponds to the proportional, and the serial mechanism to the uniform gains method. Moulin (2000) leaves it as an open problem whether a notion of consistency for cost sharing problems exists which, under the above correspondence, transfers smoothly from the cost sharing model to the rationing model and *vice versa*. This paper provides a partial answer.

Overview of the paper and results Section 2 provides the basic setup for rationing and cost sharing problems, as well as the notion of solution in these contexts. Numerous examples of solutions and mechanisms are provided. Section 3 introduces the concept of *consistency* for cost sharing solutions. Section 4 focuses on the Moulin-Shenker (1994) characterization of cost sharing mechanisms in terms of rationing methods, and some refinements thereof. Each consistent mechanism in the corresponding class is represented by a family of *consistent rationing methods* (Young (1987), Moulin (2000)). In addition, the counterpart of Theorem 1 of Young (1987) is that each family of rationing methods defining a continuous, consistent and symmetric mechanism is *parametric*. Section 5 introduces the class of *incremental mechanisms*, each being characterized by a family of *piecewise linear* rationing methods. Herein the consistent mechanisms induce consistent rationing methods, and *vice versa*. Then the average, serial, and Shapley-Shubik mechanism are all consistent as members of this class. Each incremental mechanism for 2-agent cost sharing problems is uniquely extended to a consistent mechanism – which then is incremental as well. An incremental mechanism satisfies *interval consistency* iff it is the *composition* of average and marginal mechanisms. This leads us to conclude that the average sharing mechanism is the unique incremental and *strongly* consistent mechanism.

2. Rationing, cost sharing, and preliminaries

2.1. **Rationing.** In this paper the focus is on a given finite set of agents $N = \{1, 2, \ldots, n\}$. A *rationing problem* for $S \subseteq N$ consists of a pair $(q, t) \in \mathbb{R}^S_+ \times \mathbb{R}_+$ such that $q(S) := \sum_{i \in S} q_i \geq t$. A *rationing method* r associates to any rationing problem $(q, t) \in \mathbb{R}^S_+$ a vector $r(q, t) \in \mathbb{R}^S_+$ such that $r_i(q, t) \leq q_i$ for all $i \in S$ and $\sum_{i \in S} r_i(q, t) = t$. Then r is *monotonic* whenever $t \leq t'$ implies $r(q, t) \leq r(q, t')$ for all $t, t', q \in \mathbb{R}^S_+$. Then each such rationing method defines for all $q \in \mathbb{R}^S_+$ a monotonic (and continuous) path $t \mapsto r(q, t)$ from 0 to q. A rationing method is called *piecewise linear* if the path $t \mapsto r(q, t)$ is piecewise linear. The *parametric rationing methods* (Young (1987)) constitute a rich class of solutions. Let $f : D \rightarrow \mathbb{R}$ be a real-valued function where $D \subset \mathbb{R}^2$ is a set in $\mathbb{R}_+ \times [0, \Omega]$ for some $\Omega \in \mathbb{R}_+ \cup \{\infty\}$. It is assumed that for any $(z, \omega) \in D$ it holds that $f(z, 0) = 0, f(z, \Omega) = z$ and $\omega \mapsto f(z, \omega)$ is non-decreasing and continuous. Then for such an f there is a unique rationing method r such that $r_i(x, t) = f(x_i, \omega)$ where ω solves $\sum_{i \in S} f(x_i, \omega) = t$. This r is then called the *parametric rationing method* for f. Notice that the focus is on the continuous formulation of the model, where t may be arbitrarily divided. In this setting the proportional and uniform gains methods are considered as the prevalent symmetric solutions. Moulin (2000) focuses on discrete formulation of the problem and asymmetric priority rules (see Moulin (2002) and Thomson (2003) for overviews). The prevalent symmetric solutions include the *proportional* rationing method and *uniform gains* method. The proportional rationing method is defined by $r^{\mathrm{P}}(q, t) = q/q(N)t$, and the uniform gains method is defined by $r_i^{\mathrm{UG}}(q,t) = \min\{q_i, \omega\}$, where ω solves $\sum_{j\in N} \min\{q_j, \omega\} = t$. Its dual is the *uniform losses* method defined by $r_i^{\mathrm{UL}}(q,t) = q_i - \min\{q_i, \omega\}$, where ω solves $\sum_{j\in N}(q_i - \min\{q_i, \omega\}) = t$. For further reference see Moulin (2002).

2.2. **Cost sharing.** Consider a production facility for some perfectly divisible good Y, of which the technology is summarized by a *cost* function $c : \mathbb{R}_+ \to \mathbb{R}$; $c(y)$ denotes the minimal (monetary) input to generate y units of Y. First of all it is assumed that there are no fixed costs, or $c(0) = 0$. In addition we shall assume *absolutely continuous* cost functions[1]. This technical condition implies that a cost function is differentiable almost everywhere. With slight abuse of notation c' is the *marginal cost function*, i.e., it coincides with the derivative of c whenever the latter exists, and assumes the value 0 otherwise. In particular, c' is is Lebesgue-integrable and costs for output level y may be expressed as $c(y) = \int_0^y c'(t)dt$.[2] The set of all cost functions is denoted by $\mathcal{C}$.

A *cost sharing problem for* $S \subseteq N$ is an ordered pair $R = (q,c) \in \mathbb{R}_+^S \times \mathcal{C}$. The interpretation of R is that the agents in S jointly own the production facility, and $q = (q_i)_{i\in S}$ summarizes the individual demands of the agents for good Y; then $q(S)$ is produced and cost $c(q(S))$ has to be shared. The set of all cost sharing problems for S is denoted $\mathcal{R}^S$, and put $\mathcal{R} := \bigcup_{S\subseteq N} \mathcal{R}^S$. For $(q,c) \in \mathcal{R}^S, y \in \mathbb{R}_+^S$ is called *vector of cost shares* if $y(S) = c(q(S))$. The set of all cost shares for $R \in \mathcal{R}^S$ is denoted $\mathcal{A}(R)$. A *solution* is a mapping $\Psi : \mathcal{R} \to \cup_{S\subseteq N}\mathbb{R}_+^S$ such that $\Psi(R) \subseteq \mathcal{A}(R)$ for all $R \in \mathcal{R}$; Ψ is a *mechanism* if it is single-valued, i.e., if the set $\Psi(R)$ consists of precisely one element for all $R \in \mathcal{R}$. The class of all solutions and mechanisms are denoted $\mathfrak{S}$ and $\mathfrak{M}$, respectively. With slight abuse of notation I shall write $\Psi(R) = x$ whenever x is the unique element in $\Psi(R)$. The class of solutions and mechanisms, resp., with properties $P_1, P_2, \ldots, P_k$ is denoted $\mathfrak{S}(P_1, P_2, \ldots, P_k)$ and $\mathfrak{M}(P_1, P_2, \ldots, P_k)$. Now we will discuss the solutions

[1] For such functions it holds that for all intervals $[a,b] \subset \mathbb{R}_+$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for every finite collection of pairwise disjoint intervals $(a_k, b_k) \subset [a,b], k = 1,2,\ldots,n$ with $\sum_{k=1}^n (b_k - a_k) < \delta$, we have $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$.

[2] This follows by the Fundamental Theorem in Lebesgue (1904).

that will be most prominent in this paper. The focus will be on a fixed problem $R = (q, c) \in \mathcal{R}^S$.

Examples of multi-valued solutions

These are all motivated by concepts for the cooperative stand alone cost game as in Young (1985, 1994). The *imputation set* and *core*, respectively, are defined by

$$\begin{aligned}\mathfrak{I}(q,c) &= \{x \in \mathcal{A}(q,c) \mid x_i \leq c(q_i) \text{ for all } i \in S\}\\ \mathfrak{C}(q,c) &= \{x \in \mathcal{A}(q,c) \mid x(T) \leq c(q(T)), \text{ for all } T \subset S\}.\end{aligned}$$

As in Tijs and Koster (1998), each $R = (q, c) \in \mathcal{R}^S$ is associated with a *pessimistic cost sharing problem* $(q, c_R^{\mathrm{P}}) \in \mathcal{R}^S$ with $c_R^{\mathrm{P}} \in \mathcal{C}$ defined by

$$c_R^{\mathrm{P}}(y) = \begin{cases} \inf\left\{ \int_T c'(t)\,\mathrm{d}t \mid T \in \mathcal{B}([0, q(S)]); \lambda(T) = y \right\} & \text{if } y \in [0, q(S)], \\ c(y) & \text{if } y > q(S), \end{cases} \tag{1}$$

i.e. the *pessimistic cost function* for R. Here $\mathcal{B}([0, q(S)])$ stands for the Borel-σ-algebra on the interval $[0, q(S)]$ and λ is the Lebesgue measure. So c_R^{P} relates each level of aggregate demand $y \in [0, d(S)]$ to a specific upper bound on costs, given by the minimum of corresponding aggregate marginal cost on $[0, q(S)]$. [3] It is easy to show that each such pessimistic cost function c_R^{P} is concave on $[0, q(S)]$. The *pessimistic imputation set* is the set $\mathfrak{I}^{\mathrm{P}}(q,c) = \mathfrak{I}(q, c_R^{\mathrm{P}})$, the imputation set of the pessimistic cost sharing problem. Similarly, the *pessimistic core* is defined by $\mathfrak{C}^{\mathrm{P}}(q,c) = \mathfrak{C}(q, c_R^{\mathrm{P}})$.

Examples of mechanisms

The *average cost sharing mechanism* μ^{AV} determines the vector

$$\mu^{\mathrm{AV}}(q,c) = \begin{cases} 0 & \text{if } q(S) = 0, \\ \dfrac{c(q(S))}{q(S)} \cdot q & \text{otherwise.} \end{cases}$$

Let $\Pi(S)$ be the set of all mappings $S \to \{1, 2, \ldots, |S|\}$. For each $\sigma \in \Pi(S)$ and $q \in \mathbb{R}_+^S$ let $q_0^\sigma, q_1^\sigma, \ldots, q_{|S|}^\sigma$ be defined by $q_j^\sigma = \sum_{\ell \leq j} q_{\sigma(j)}$. Then for any $\sigma \in \Pi(S)$,

[3] A generalization of the pessimistic cost sharing problem to heterogeneous cost sharing problems is in Koster (2000).

the corresponding *marginal mechanism* $\mu^{\sigma} \in \mathcal{R}^S$ is given by

$$\mu_i^{\sigma}(q,c) := c\left(q^{\sigma}_{\sigma^{-1}(i)}\right) - c\left(q^{\sigma}_{\sigma^{-1}(i)-1}\right). \tag{2}$$

So for each cost sharing problem $\mu^{\sigma}(q,c)$ is the marginal vector with respect to the stand alone game (see Young (1985)). The *Shapley-Shubik mechanism* Φ (Shubik (1962)) averages all marginal or incremental cost shares, i.e., $\Phi(q,c) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)}\mu^{\sigma}(q,c)$. Define the *pessimistic marginal mechanism* with respect to $\sigma \in \Pi(N)$ by $\mu_p^{\sigma}(q,c) := \mu^{\sigma}(q,c_R^{\mathrm{P}})$, for all $R = (q,c) \in \mathcal{R}^S$. The *pessimistic Shapley-Shubik mechanism* Φ^{P} is defined by $\Phi^{\mathrm{P}}(q,c) = \Phi(q,c_R^{\mathrm{P}})$. Weber (1988) discusses the class of *random order values* consisting of all mechanisms that are a convex combination of marginal mechanisms.

The *serial mechanism* μ^{SR} (see, e.g., Moulin and Shenker (1992)) is defined as follows. For $q \in \mathbb{R}_+^S$ let $\sigma \in \Pi(S)$ be such that $q_{\sigma(i)} \le q_{\sigma(j)} \Leftrightarrow i \le j$. Define numbers $q_0^*, q_1^*, \ldots, q_{|S|}^*$ by $q_0^* = 0$ and $q_j^* = \sum_{\ell\le j-1} q_{\sigma(\ell)} + (|S|-j+1)q_{\sigma(j)}$. Then put

$$\mu_i^{\mathrm{SR}}(q,c) = \sum_{\ell\le\sigma(i)} \frac{c(q_\ell^*) - c(q_{\ell-1}^*)}{|S|-\ell+1} \text{ for all } c \in \mathcal{C}, i \in S. \tag{3}$$

3. Consistency in cost sharing

In the literature on distributive justice, the invariance property of solutions with respect to varying sets of agents is usually referred to as *consistency*. Within the rationing context the idea of consistency is transparent and intuitive: a rationing method r is called *consistent* if for all rationing problems (q,x) among agents in S, $r_{S\setminus\{j\}}(q,x) = r(q_{S\setminus\{j\}}, x - r_j(q,x))$ for all $j \in S$.[4] Hence, consistency states that with removing an agent from the cooperative S, and taking all the resources that are allocated to this agent, renewed allocation of the remaining pieces within the reduced society does not make a difference as long as r is used. As Moulin (2000) puts it, 'changing the status of an agent from *active participant* to *passive expense of resources* does not alter the overall distribution'. Consistency puts forward an idea of fairness on the level that 'no subgroup of agents should

[4]In fact the notion is usually defined in terms of general sets of agents leaving, but is derived from repeated application of this statement.

want to "re-contract"' (Young (1985, p19)).

Less trivial is the notion of consistency within the cost sharing context. Suppose that the group of agents $S \subseteq N$ face a cost sharing problem $(q, c) \in \mathcal{R}^S$ and that the mechanism μ is used to calculate the individual shares. One of the agents in S, say i, leaves the problem, takes his demand q_i and pays $\mu_i(q, c)$. The mechanism μ is *consistent* if it determines the same allocation for the agents $S\backslash\{i\}$ in the new situation. Then, just as in rationing problems, agents should not bother about renegotiating as it will not help them to improve upon the status quo $\mu(q, c)$. But the crucial point here is that, in order to be able to apply a solution like μ again, first there has to be a clear understanding of the new cost sharing problem. This amounts to a translation of the original problem into a *reduced cost sharing problem* $(\bar{q}, \bar{c}) \in \mathcal{R}^{S\backslash\{i\}}$ where the pre-paid amount $\mu_i(q, c)$ is taken into account. Although it seems fairly reasonable to use $\bar{q} = q_{S\backslash\{i\}}$ as the new demand profile, the choice of $\bar{c}$ seems at least debatable. This is illustrated by the literature where several reductions are proposed, see, e.g., Moulin and Shenker (1994), Kolpin (1994), and Sudhölter (1998).

Here I depart from this approach and propose, as long it is not clear which reduction fits the situation best, to include *all* problems that could possibly serve as a proper reduction. Then I shall call a solution *consistent* if any vector of cost shares in the original solution is still available for the remaining agents in the solution induced by *some* reduced cost sharing problem. A cost sharing problem is admitted as reduction with respect to agent i if it is derived from the original problem through truncation of the cost function over q_i production levels, matching the cost share $\mu_i(q, c)$. The following example explains the basic idea.

Example Consider the cost sharing problem (q, c) for $N = \{1, 2, 3\}$ defined by $c(y) = y^2$ and $q = (1, 2, 3)$. Assume average cost sharing, so that the cost shares are determined by $\mu^{\text{AV}}(q, c) = (6, 12, 18)$. Suppose agent 2 leaves the cost sharing, claiming the production levels $X = [2, 4]$. These levels reflect his cost share of $12 = c(4) - c(2)$, the total of marginal costs involved with the production of his demand 2. In the same way the levels $Y = [0, 1] \cup [5, 6]$ fit agent 2's cost share, as $(c(1) - c(0)) + (c(6) - c(5)) = 12$. One possible reduction with respect to agent

2 would be the truncation c_X of c over X,

$$c_X(y) = \begin{cases} c(y) & \text{if } y \leq 2, \\ c(y+2) - 12 & \text{if } y > 2. \end{cases}$$

Similarly, another reduction c_Y is defined by the truncation of costs over Y,

$$c_Y(y) = \begin{cases} c(y+1) - c(1) & \text{if } y \leq 4, \\ c(y+2) - 12 & \text{if } y > 4. \end{cases}$$

Similarly, in case of reduction by agent 1 the sets $U = [2\frac{1}{2}, 3\frac{1}{2}]$ and $V = [0, \frac{1}{2}] \cup [5\frac{1}{2}, 6]$ reflect both his demand *and* cost share 6. Then truncation of c over these production levels leads to the reduced cost functions c_U and c_V, respectively,

$$c_U(y) = \begin{cases} c(y) & \text{if } y \leq 2\frac{1}{2}, \\ c(y+1) - 6 & \text{if } y > 2\frac{1}{2}. \end{cases} \quad c_V(y) = \begin{cases} c(y+\frac{1}{2}) - c(\frac{1}{2}) & \text{if } y \leq 5, \\ c(y+1) - 6 & \text{if } y > 5. \end{cases}$$

Summarizing, a plausible reduction will address different levels of output to the leaving agent, such that the aggregate production equals his demand and corresponding marginal costst reflect his cost share. $\triangleleft$

The *indicator function* $\mathbb{I}_A : \mathbb{R} \to \{0, 1\}$ for $A \subseteq \mathbb{R}$ is defined by $\mathbb{I}_A(t) = 1 \Leftrightarrow t \in A$. For any bounded set $U \in \mathcal{B}([0, \infty))$ and $y \in \mathbb{R}_+$ define $U_y \subset [0, \infty)$ as the smallest interval containing $[0, y]$ such that $\lambda\left(U_y \setminus U\right) = y$. Then the *reduced cost function with respect to* U, c_U, is defined by

$$c_U(y) = \int_{U_y} c'(t)\ \mathbb{I}_{\mathbb{R}_+ \setminus U}(t)\ \mathrm{d}t \text{ for all } y \in \mathbb{R}_+.$$

So $c_U \in \mathcal{C}$ takes for each input level y the total of marginal costs of the first y levels *outside* U. For any $R = (q, c) \in \mathcal{R}^S$ and $i \in S$ define $\mathcal{Q}(\Psi, R, i)$ as the set of all $T \in \mathcal{B}([0, q(S)])$ such that

$$\begin{array}{ll} (i) & \lambda(T) = q_i, \\ (ii) & \int_T c'(t)\,\mathrm{d}t \in \Psi_i(q, c). \end{array}$$

Then $T \in \mathcal{Q}(\Psi, R, i)$ can be interpreted as a set of demand levels that simultaneously represent agent i's individual demand (condition (i)) and his share in total costs (condition (ii)). If $Q(\Psi, R, i) \neq \varnothing$ for all $i \in S$, then R is called *reducible* with respect to Ψ.

A solution Ψ is *consistent* if each restriction of a share vector in the original solution is available for some reduced cost sharing problem. Then if this holds for all reductions, Ψ is called *strongly consistent*. Formally:

Consistency $\Psi \in \mathfrak{S}(\mathrm{CO})$ if for all $R = (q, c) \in \mathcal{R}^S, i \in S$, and each $y \in \Psi(R)$ there exists $U \in Q(\Psi, R, i)$ such that $y_{S\setminus\{i\}} \in \Psi(q_{S\setminus\{i\}}, c_U)$.

Strong Consistency $\Psi \in \mathfrak{S}(\mathrm{SCO})$ if $\Psi \in \mathfrak{S}(\mathrm{CO})$ such that for all $R = (q, c) \in \mathcal{R}^S, i \in S$, and each $y \in \Psi(R), U \in Q(\Psi, R, i) \implies y_{S\setminus\{i\}} \in \Psi(q_{S\setminus\{i\}}, c_U)$.

Note that these notions require that a problem be reducible with respect to the solution. Lemma 6.4 in the Appendix shows that this is actually not too much to ask, as each problem R with solution $\Psi(R) \subseteq \mathfrak{C}^{\mathrm{P}}(R)$ is reducible. Below we focus on a version of consistency weaker than SCO but stronger than CO. It requires not only CO, but also invariance of the solution with respect to all suitable reductions by intervals:

Interval Consistency $\Psi \in \mathfrak{S}(\mathrm{ICO})$ if $\Psi \in \mathfrak{S}(\mathrm{CO})$ such that for all $R = (q, c) \in \mathcal{R}^S, i \in S$, it holds that $\Psi(q_{S\setminus\{i\}}, c_U) \subseteq \Psi_{S\setminus\{i\}}(R)$ for all $U = [t, t + q_i] \in Q(\Psi, R, i)$.

Note that $\mathfrak{S}(\mathrm{SCO}) \subset \mathfrak{S}(\mathrm{ICO}) \subset \mathfrak{S}(\mathrm{CO})$. Below I will show that most of the discussed solutions are in full accordance with CO. In particular, it is not hard to show that $\mathcal{A}$ is strongly consistent, and that $\mathfrak{C}^{\mathrm{P}}$ is consistent. In fact, $\mathfrak{C}^{\mathrm{P}}$ is the maximal consistent solution in $\mathfrak{I}^{\mathrm{P}}$.

Theorem 3.1 *If $\Psi \in \mathfrak{S}(\mathrm{CO})$ and $\Psi(R) \subseteq \mathfrak{I}^{\mathrm{P}}(R)$ for all $R \in \mathcal{R}$, then $\Psi(R) \subseteq \mathfrak{C}^{\mathrm{P}}(R)$ for all $R \in \mathcal{R}$.*

Example 3.2 The marginal mechanisms μ^σ are consistent but not strong consistent. Just consider $R = (q, c) \in \mathcal{R}^S$ and define the sets $T_i := \left[q^\sigma_{\sigma^{-1}(i)-1}, q^\sigma_{\sigma^{-1}(i)}\right]$ Then for all $i \in S$, $T_i \in \mathcal{Q}\left(\mu^\sigma, R, i\right)$ and thus $\left(q_{S\setminus\{i\}}, c_{T_i}\right)$ is a reduced cost sharing problem with respect to i. Moreover, according to μ^σ the same ordering of the remaining agents is used to calculate the individual shares in the reduction, which means that nothing changes for the agents k with $\sigma(k) < \sigma(i)$. Now consider an agent k with $\sigma(k) > \sigma(i)$. Then with $y_t := \sum_{j \in S\setminus\{i\}:\sigma(j)\leq t} q_j, S$ we get

$$\mu_k^\sigma\left(q_{S\setminus\{i\}}, c_{T_i}\right) = c_{T_i}\left(y_{\sigma(k)}\right) - c_{T_i}\left(y_{\sigma(k)-1}\right) = c\left(q_i + y_{\sigma(k)}\right) - c\left(q_i + y_{\sigma(k)-1}\right) =$$
$$= c\left(\sum_{j:\sigma(j)\leq\sigma(k)} q_j\right) - c\left(\sum_{j:\sigma(j)<\sigma(k)} q_j\right) = x_k^\sigma(q,c).$$

This proves that μ^σ is consistent. To see that strong consistency is violated, consider the problem $R = (q,c) \in \mathcal{R}^{\{1,2,3\}}$ with $c(y) = \int_0^y \mathbb{I}_{[0,1]\cup[2,3]}(t)\,\mathrm{d}t$ and $q = (1,1,1)$. If σ is the identity permutation, $\mu^\sigma(q,c) = (1,0,1)$. Then $[2,3] \in \mathcal{Q}(\mu^\sigma, R, 1)$, but $x_2^\sigma\left(q_{\{2,3\}}, c_{[2,3]}\right) = 1 \neq \mu_2^\sigma(q,c)$. In a similar way one may prove that the pessimistic marginal sharing mechanisms μ_p^σ are consistent but not strongly so. $\triangleleft$

4. Moulin & Shenker (1994) and consistency

The two main characterizing properties in the cost sharing literature are *additivity* and *constant returns*. Additivity is propagated as an accounting convention, allowing for the decomposition of a cost sharing problem in several cost components without altering the final cost allocations. Constant returns declares price of the good equal to marginal costs in case of linear cost functions. Formally,

Additivity $\mu \in \mathfrak{M}(\text{ADD})$ if $\mu(q, c_1 + c_2) = \mu(q, c_1) + \mu(q, c_2)$ for all demand profiles q and $c_1, c_2 \in \mathcal{C}$.
Constant Returns $\Psi \in \mathfrak{S}(\text{CR})$ if $\Psi(q, c_\vartheta) = \vartheta q$ for $c_\vartheta \in \mathcal{C}$ given by $c_\vartheta(y) = \vartheta y$.

All the previously discussed solutions satisfy CR, except for $\mathcal{A}$, whereas most of the prevalent mechanisms in the literature belong to $\mathfrak{M}(\mathrm{ADD}, \mathrm{CR})$. Examples are $\mu^{\mathrm{AV}}, \mu^{\sigma}, \mu^{\mathrm{SR}}$ and Φ.[5] The following result is due to Moulin and Shenker (1994) and shows the close relationship between $\mathfrak{M}(\mathrm{ADD}, \mathrm{CR})$ and the class of monotonic rationing methods.

Theorem 4.1 $\mu \in \mathfrak{M}(\mathrm{ADD}, \mathrm{CR})$ *if and only if there is a rationing method r such that for each $S \subseteq N, q \in \mathbb{R}_+^S$*

$$\mu(q, c) = \int_0^{q(S)} c'(t) \; \mathrm{d}r(q, t) \text{ for all } c \in \mathcal{C}. \tag{4}$$

So each mechanism $\mu \in \mathfrak{M}(\mathrm{ADD}, \mathrm{CR})$ is fully characterized through its rationing method r; I shall write $\mu = \mu^r$. The following result shows that, although the set of mechanisms in Theorem 4.1 can be considered as rather small, the set of cost shares generated by it is not.

Theorem 4.2 $\mathfrak{C}^{\mathrm{P}}(R) = \{\mu(R) \mid \mu \in \mathfrak{M}(\mathrm{ADD}, \mathrm{CR})\}$ *for all $R \in \mathcal{R}^S$.*

Note that, in our model, except for the labeling of the agents, it is only their individual demand that may influence a solution, *ceteris paribus*. Then, if two agents can not be distinguished for these characteristics it is reasonable that they be treated equally by the solution:

Equal Treatment $\Psi \in \mathfrak{S}(\mathrm{ET})$ if for all $x \in \Psi(q, c)$ it holds $q_i = q_j \implies x_i = x_j$.

Ranking (RANK) encompasses the idea that larger demanders have higher impact on total costs, and should therefore contribute more. ET is weaker than the *anonymity* property in Moulin and Shenker (1992) and implied by RANK. Formally:

[5] An excellent overview on additive cost sharing is Moulin (2002). In particular this work shows the power of the additivity as a mathematical tool. Non-additive mechanisms are proposed and analyzed in, e.g., Sprumont (1998), Tijs and Koster (1998), Koster (2001), and Hougaard and Petersen (2001).

Ranking $\Psi \in \mathfrak{S}(\text{RANK})$ if for all $x \in \Psi(q,c)$ it holds that $q_i \geq q_j \implies x_i \geq x_j$.

Refinements of Theorem 4.1 in terms of these properties are easily derived, as we have:

Proposition 4.3 $\mu \in \mathfrak{M}(\text{ADD}, \text{CR}, \text{RANK})$ *if and only if*

(i) *μ is generated by a rationing method r, i.e., $\mu = \mu^r$,*
(ii) *if for $q \in \mathbb{R}^S_+, q_i \geq q_j$ for some $i, j \in S$, then $r_i(q, \cdot) \geq r_j(q, \cdot)$.*

Corollary 4.4 $\mu \in \mathfrak{M}(\text{ADD}, \text{CR}, \text{ET})$ *if and only if*

(i) *μ is generated by a rationing method r, i.e., $\mu = \mu^r$,*
(ii) *if for $q \in \mathbb{R}^S_+, q_i = q_j$ for some $i, j \in S$, then $r_i(q, \cdot) = r_j(q, \cdot)$.*

By Theorem 4.1 it is possible to study for properties of cost sharing mechanisms by looking at corresponding rationing families, and *vice versa*. In this respect, the notion of consistency transfers smoothly from the cost sharing to the rationing model.

Theorem 4.5 *If $\mu = \mu^r$ is consistent, then r is consistent.*

Proof. For all $x \in [0, q(S)]$, $\mu(q, \Gamma_x) = \int_0^{q(S)} \Gamma'_x(t) \mathrm{d}r(q,t) = r(q,x)$. Moreover, for each $j \in S$ we have for $R = (q, \Gamma_x)$, $[x - \mu_j(q, \Gamma_x), x] = [x - r_j(q,x), x] \in Q(\mu, R, j)$. But then it holds by CO for all $x \in [0, q(S)]$ that

$$r_{S \setminus \{j\}}(q,x) = \mu_{S \setminus \{j\}}(q, \Gamma_x) = \mu(q_{S \setminus \{j\}}, \Gamma_{x - r_j(q,x)}) = r(q_{S \setminus \{j\}}, x - r_j(q,x)). \qquad \blacksquare$$

Young (1987) characterizes the class of parametric rationing methods by equal treatment, consistency and continuity, a result that is useful in the present framework of mechanisms as well. A rationing method r is called *continuous* if it is jointly continuous in both arguments, i.e., $(q,t) \mapsto r(q,t)$ is continuous for all rationing problems (q,t). Such r is then robust against small changes in the parameters defining the rationing problem. For cost sharing mechanisms the approach is similar. For $t \geq 0$, define the *base function* $\Gamma_t \in \mathcal{C}$ by $\Gamma_t(y) = \min\{y, t\}$ for all $y \in \mathbb{R}_+$. A mechanism will be called *continuous* if small changes in demands and Γ_t cause only small changes in cost shares. More specifically,

Continuity $\mu \in \mathfrak{M}(\text{CONT})$ if the mapping $(q,t) \mapsto \mu(q,\Gamma_t)$ is continuous on $\mathbb{R}_+^S \times \mathbb{R}_+$, for all $S \subseteq N$.[6]

Lemma 4.6 *If $\mu = \mu^r \in \mathfrak{M}(\text{ADD}, \text{CR}, \text{CONT})$ then r is continuous.*

Proof. Without loss of generality it may be assumed that $r(q,t) := \mu(q,\Gamma_t)$. Then clearly continuity of μ implies continuity of the method r. ■

Theorem 4.7 $\mu \in \mathfrak{M}(\text{ADD}, \text{CR}, \text{CO}, \text{ET}, \text{CONT})$ *if $\mu = \mu^r$ and r is parametric.*

Proof. By Theorem 4.1 there is a monotonic rationing method r such that $\mu = \mu^r$, and Corollary 4.4 shows that r satisfies equal treatment. Then the proof is completed by application of Theorem 1 in Young (1987) and Lemma 4.6. ■

Note that Theorem 4.5 and Theorem 4.7 do not fully characterize the set of all consistent cost sharing mechanisms within $\mathfrak{M}(\text{ADD}, \text{CR})$. In the next section I will show that consistency is transferred smoothly between piecewise linear rationing methods and *incremental mechanisms*.

5. Incremental cost sharing mechanisms

All of the earlier examples of additive mechanisms have in common that a finite number of intermediate levels of output determines the final solution, as the cost increments of two consecutive levels is split amongst the agents in a fixed ratio. It is similar to the calculation of the random order values of Weber (1988), by which cost shares are determined on the basis of the coalitional aggregate demands as intermediate output levels. Here I will discuss a more general class of mechanisms, each of one splits increments related to other intermediate levels of output as well.

A mechanism μ is called *incremental* if for each $q \in \mathbb{R}_+^S$ there is an integer $k \in \mathbb{N}$,

[6] In order to avoid the hybrid character of CONT one may consider the replacement by two requirements, continuity of the mappings $t \mapsto \mu(q,\Gamma_t)$ and $q \mapsto \mu(q,\Gamma_t)$.

vectors $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Delta(S) := \{y \in \mathbb{R}^S_+ \mid y(S) = 1\}$ and $x \in \mathbb{R}^{k+1}_+$ such that

$$\mu(q,c) = \sum_{\ell=1}^{k} \alpha_\ell \left(c(x_\ell) - c(x_{\ell-1})\right) \text{ for all } c \in \mathcal{C}, \tag{5}$$

and where $0 = x_0 \leq x_1 \leq \ldots \leq x_k = q(S)$. The set of all incremental mechanisms, denoted $\mathfrak{M}^I$, forms a generalization of the class of additive *incremental methods* and *random order mechanisms* as in Weber (1988). An incremental mechanism μ satisfies CR if only $\sum_{\ell=1}^{k} \alpha_\ell \left(x_\ell - x_{\ell-1}\right) = q(S)$. In the latter case the corresponding rationing method r is determined by

$$\tfrac{\partial}{\partial t} r(q,t) = \alpha_\ell \text{ for all } t \in (x_{\ell-1}, x_\ell) \text{ and } x_\ell \neq x_{\ell-1}. \tag{6}$$

5.1. **Examples of incremental mechanisms.** Obviously, Φ is incremental as random order value. Moreover, all previously discussed additive mechanisms belong to $\mathfrak{M}^I$:

- μ^{AV}: $k = 1, x_k = q(S), \alpha_k = \frac{q}{q(S)}$ whenever $q(S) > 0$.
- μ^{σ}: For each σ, take $x_\ell = \sum_{p \leq \sigma^{-1}(\ell)} q_{\sigma(p)}$ and $(\alpha_\ell)_i = 1 \Leftrightarrow \sigma(i) = \ell$.
- μ^{SR}: For $q \in \mathbb{R}^S_+$ there is an ordering permutation $\sigma \in \Pi(S)$ that such that $q_{\sigma(i)} \leq q_{\sigma(j)} \Leftrightarrow \sigma(i) \leq \sigma(j)$ for all $i, j \in S$. Then take $k = |S|$ and $x_\ell = \sum_{p \leq \sigma^{-1}(\ell)-1} q_{\sigma(p)} + (|S| - \ell + 1) q_{\sigma(\ell)}$. Define $(\alpha_\ell)_i = 1/(|S| - \ell + 1)$ if $\sigma(i) \geq \ell$ and 0 else.

5.2. **Consistent incremental mechanisms.** Theorem 4.5 does not represent a full characterization of the class of consistent mechanisms in $\mathfrak{M}(\mathrm{ADD}, \mathrm{CR})$. However, those mechanisms being incremental and consistent correspond one-to-one to consistent and piecewise linear rationing methods.

Theorem 5.1 *$\mu \in \mathfrak{M}^I(\mathrm{ADD}, \mathrm{CR}, \mathrm{CO})$ if and only if $\mu = \mu^r$ and r is consistent and piecewise linear.*

One may easily verify that the rationing methods corresponding to the earlier mentioned incremental mechanisms are all consistent. Then an implication of Theorem 5.1 is that the mechanisms $\mu^{\mathrm{AV}}, \mu^{\mathrm{SR}}, \mu^{\sigma}, \Phi$ are all consistent.

Theorem 5.2 *Let $\bar{\mu}$ be an incremental mechanism for 2-agent cost sharing problems with the property* CR. *Then $\bar{\mu}$ uniquely extends to $\mu \in \mathfrak{M}(\text{ADD}, \text{CR}, \text{CO})$. In particular μ must be incremental.*

For instance, this shows that there is only one consistent way of extending serial cost sharing for 2-agent cost sharing problems, and that is by application of the serial mechanism for all cost sharing problems.

5.3. **Reducible incremental mechanisms.** Moulin (2000) introduces the notion of a *reducible rationing method*. Basically, for such method there is an ordered partition of the set of agents and for each element in the partition there is give a (different) rationing method, such that the final allocation can be determined in two steps: (1) the available units are divided over the different elements in the partition, (2) the rationing method associated with a specific element in the partition determines the further allocation for the agents therein. This two step procedure will be used for the cost sharing model as well. A mechanism will be called reducible if there is a non-trivial ordered partition $\mathcal{N}$ of N, such that cost shares may be calculated by different mechanisms to certain cost sharing problems induced by the ordered partition. Formally this procedure is as follows. Consider an ordered partition $\mathcal{N} = (N(1), N(2), \ldots, N(\kappa))$ of N, where κ is an integer smaller than $|N|$. Given $\mathcal{N}, R = (q, c) \in \mathcal{R}^S$ define for $k = 1, \ldots, \kappa$ the cost function $c_R^k \in \mathcal{C}$ by

$$c_R^k(y) = c\left(\sum_{\ell \leq k-1} q(N(\ell) \cap S) + y\right) - c\left(\sum_{\ell \leq k-1} q(N(\ell) \cap S)\right)$$

for all $y \in \mathbb{R}_+$. A mechanism $\mu \in \mathfrak{M}$ is *reducible* if there is a non-trivial ordered partition $\mathcal{N} = (N(1), N(2), \ldots, N(\kappa))$ of N together with mechanisms $\mu^1, \ldots, \mu^\kappa \in \mathfrak{M}$ such that for each $(q, c) \in \mathcal{R}^S, i \in S \cap N(k)$ it holds $\mu_i(q, c) = \mu_i^k(q_{S \cap N(k)}, c_R^k)$. Then μ is considered as the *composition* of the mechanisms $\mu^1, \ldots, \mu^\kappa$. So, as an analogue to the rationing model, here the ordered partition is used to address the different cost levels to the elements in the induced partition. The proof of the following is in the Appendix.

Theorem 5.3 *$\mu \in \mathfrak{M}^I(\mathrm{ADD}, \mathrm{CR}, \mathrm{ICO})$ if and only if μ is composition of the average mechanism and marginal mechanisms.*

Corollary 5.4 *Suppose $Y \in \{\mathrm{ET}, \mathrm{RANK}\}$, then $\mathfrak{M}^I(\mathrm{ADD}, \mathrm{CR}, \mathrm{ICO}, Y) = \{\mu^{\mathrm{AV}}\}$.*

Proof. This follows from Theorem 5.5 in combination with Proposition 4.3, Corollary 4.4. ∎

Theorem 5.5 $\mathfrak{M}^I(\mathrm{ADD}, \mathrm{CR}, \mathrm{SCO}) = \{\mu^{\mathrm{AV}}\}$.

Proof. In order to show that μ^{AV} is strongly consistent, consider a reduction of the problem $R = (q, c) \in \mathcal{R}^S$ with respect to agent $i \in S$, say $(q_{S\setminus\{i\}}, c_T)$, where $T \in \mathcal{Q}\left(\mu^{\mathrm{AV}}, R, i\right)$. Then

$$\begin{aligned}\mu^{\mathrm{AV}}(q_{S\setminus\{i\}}, c_T) &= \frac{q_{S\setminus\{i\}}}{q(S\setminus\{i\})} c_T(q(S\setminus\{i\})) = \frac{q_{S\setminus\{i\}}}{q(S\setminus\{i\})}\left\{c(q(S)) - \frac{q_i}{q(S)} c(q(S))\right\} \\ &= \frac{q_{S\setminus\{i\}}}{q(S\setminus\{i\})}\frac{q(S\setminus\{i\})}{q(S)} c(q(S)) = \mu^{\mathrm{AV}}_{S\setminus\{i\}}(q, c).\end{aligned}$$

Finally, proceed along the lines of Example 3.2 in order to show that no other *reducible* mechanism is strongly consistent. ∎

Remark In fact one may prove that the unique strongly consistent mechanism in $\mathfrak{M}(\mathrm{ADD}, \mathrm{CR})$ is μ^{AV}. A proof is available upon request.

6. Appendix

Lemma 6.1 *Let $R = (q, c) \in \mathcal{R}^S$ and take $y \in \mathfrak{C}^{\mathrm{P}}(R)$. Then for each $i \in S$ there is $a \in [0, q(S \setminus \{i\})]$ such that $y_i = c^{\mathrm{P}}_R\left(a + q_i\right) - c^{\mathrm{P}}_R\left(a\right)$.*

Proof. Let $i \in S$ and define $g : [0, \infty) \to \mathbb{R}$ by $g\left(t\right) = c^{\mathrm{P}}_R\left(t + q_i\right) - c^{\mathrm{P}}_R\left(t\right)$. Then $g\left(0\right) = c^{\mathrm{P}}_R\left(q_i\right) \geq y_i$ and $g\left(q\left(S\setminus\{i\}\right)\right) = c^{\mathrm{P}}_R\left(q\left(S\right)\right) - c^{\mathrm{P}}_R\left(q\left(S\setminus\{i\}\right)\right) \leq y_i$. The latter inequality follows from the fact that $y \in \mathfrak{C}^{\mathrm{P}}\left(q, c\right)$ since $c^{\mathrm{P}}_R\left(q\left(S\right)\right) = y\left(S\right)$ and $c^{\mathrm{P}}_R\left(q\left(S\setminus\{i\}\right)\right) \geq y\left(S\setminus\{i\}\right)$. Recall that c^{P}_R is concave and thus continuous. Then by continuity of g there exists a such that $g\left(a\right) = y_i$. ∎

Lemma 6.2 *Let $R = (q, c) \in \mathcal{R}^S$. For each $x \in [0, q(S)]$ there is $T_x \in \mathcal{B}([0, q(S)])$ such that $c_R^{\mathrm{P}}(x) = \int_{T_x} c'(t)dt$ and $\lambda(T_x) = x$. The sets can be taken such that $x \leq y \implies T_x \subseteq T_y$.*

Proof. Take $x \in [0, q(S)]$. For $z \in \mathbb{R}_+$ we define $D_z := \{t \in [0, q(S)] \mid c'(t) \geq z\}$. Then let $z(x) := \sup \{z \in \mathbb{R}_+ \mid \lambda(D_z) \geq x\}$. We distinguish two cases, $\lambda(D_z) = x$ and $\lambda(D_{z(x)}) > x$. We will show that the choice of $T_x := D_{z(x)}$ serves our goal. To see this, just take an arbitrary $T \in \mathcal{B}([0, q(S)])$ with $\lambda(T) = x, T \neq T_x$. Then in particular for $t \in T \setminus T_x$ it holds that $c'(t) < z(x)$ and therefore

$$\int_{T \setminus T_x} c'(t)\mathrm{d}t \leq z(x) \cdot \lambda(T_x \setminus T) \leq \int_{T_x \setminus T} c'(t)\mathrm{d}t.$$

As a consequence $\int_{T_x \setminus T} c'(t)\mathrm{d}t = \int_{T \cap T_x} c'(t)\mathrm{d}t + \int_{T \setminus T_x} c'(t)\mathrm{d}t \leq \int_{T \cap T_x} c'(t)\mathrm{d}t + \int_{T_x \setminus T} c'(t)\mathrm{d}t = \int_{T_x} c'(t)\mathrm{d}t$. So $c_R^{\mathrm{P}}(x) = \sup \{\int_T c'(t)\mathrm{d}t \mid \lambda(T) = x\} = \int_{T_x} c'(t)\mathrm{d}t$. Now for the second case assume that $\lambda(D_{z(x)}) > x$. This means that

$$\lambda(\{t \in [0, q(S)] \mid c'(t) = z(x)\}) > \lambda(D_{z(x)}) - x.$$

Determine $t' \in [0, q(S)]$ with $\lambda([0, t'] \cap \{t \in [0, q(S)] \mid c'(t) = z(x)\}) = \lambda(D_{z(x)}) - x$. Now take $T_x := D_{z(x)} \setminus ([0, t'] \cap \{t \in [0, q(S)] \mid c'(t) = z(x)\})$. Then $\lambda(T_x) = x$ and the rest is proved analogously to the first case. Besides, it should be clear from the presented construction that $T_x \subseteq T_y$ whenever $x \leq y$. ∎

Lemma 6.3 *Let $c : [0, y] \to \mathbb{R}$ be increasing and concave such that $c(0) = 0$. Then for any $\alpha \in [0, 1]$ there is an interval $I = [t, t + \alpha y] \subseteq [0, y]$ such that $\alpha c(y) = c(t + \alpha y) - c(t)$.*

Proof. Define $g : [0, (1 - \alpha)y] \to \mathbb{R}_+$ by $g(t) = c(t + \alpha y) - c(t)$. Concavity of c implies $g(0) = c(\alpha y) \geq \alpha c(y) = c(y) - (1 - \alpha)c(y) \geq c(y) - c((1 - \alpha)y) = g((1 - \alpha)y)$. Hence by continuity of g there is $t \in [0, (1 - \alpha)y]$ such that $g(t) = \alpha c(y)$. ∎

Lemma 6.4 *Let $R \in \mathcal{R}^S$. If $\mu(R) \in \mathfrak{C}^{\mathrm{P}}(R)$ then $Q(\mu, R, i) \neq \varnothing$ for all $i \in S$.*

Proof. Take $R = (q, c) \in \mathcal{R}^S, i \in S$, and assume $\mu(R) \in \mathfrak{C}^{\mathrm{P}}(R)$. By Lemma 6.1 there is an interval $T = [a, a + q_i]$ such that $\int_T c'(t)\mathrm{d}t = c_R^{\mathrm{P}}(a + q_i) - c_R^{\mathrm{P}}(a)$. Moreover we may choose sets $U, V \in \mathcal{B}([0, q(S)]), U \subseteq V$ such that $\int_U c'(t)\mathrm{d}t = c_R^{\mathrm{P}}(a)$ and $\int_V c'(t)\mathrm{d}t = c_R^{\mathrm{P}}(a+q_i)$. Then $V \setminus U \in Q(\mu, R, i)$. ∎

Proof of Theorem 3.1

Part (i): Take $y \in \mathfrak{C}^{\mathrm{P}}(q,c)$ for $(q,c) \in \mathcal{R}^S, S \subseteq N$. Let $i \in S, Q \subset S \setminus \{i\}$. According to Lemma 6.1 there is an interval $[a, a+q_i] \subseteq [0, q(S)]$ such that $y_i = c_R^{\mathrm{P}}(a+q_i) - c_R^{\mathrm{P}}(a)$. Consider a family of measurable sets $\{T_z\}_{z \in [0,q(S)]}$ as in Lemma 6.2. Consider $T = T_{a+q_i} \setminus T_a$. Then it holds $\int_T c'(s)\, ds = y_i \in \mathfrak{C}^{\mathrm{P}}(q,c)_i$. Moreover $\lambda(T) = q_i$, so $T \in \mathcal{Q}(\mathfrak{C}^{\mathrm{P}}, S, i)$. Let $R_T := (q, c_T)$. By $y \in \mathfrak{C}^{\mathrm{P}}(q,c)$ it holds $y(Q \cup \{i\}) \geq c_R^{\mathrm{P}}(q(Q \cup \{i\}))$, so

$$\begin{aligned}
y(Q) &\geq c_R^{\mathrm{P}}(q(Q \cup \{i\})) - y_i = \\
&= \int_{T_{q(Q\cup\{i\})}} c'(s)\,\mathrm{d}s - \int_T c'(s)\,\mathrm{d}s = \int_{T_{q(Q\cup\{i\})}\setminus T} c'(s)\,\mathrm{d}s \\
&\geq \inf\left\{ \int_U c'(s)\,\mathrm{d}s \,\middle|\, U \in \mathcal{B}([0, q(S)] \setminus T), \lambda(U) = q(Q) \right\} \\
&= \inf\left\{ \int_U (c_T')(s)\,\mathrm{d}s \,\middle|\, U \in \mathcal{B}([0, q(S \setminus \{i\})]), \lambda(U) = q(Q) \right\} \\
&= (c_T)_{R_T}^{\mathrm{P}}(q(Q)).
\end{aligned}$$

By variation of Q and the fact that $y(S \setminus \{i\}) = c_T(q(S \setminus \{i\}))$ we conclude that $y_{S\setminus\{i\}} \in \mathfrak{C}^{\mathrm{P}}(R_T)$.

Part (ii): Let Ψ be a consistent solution. Then we need to show that $\Psi(R) \subseteq \mathfrak{C}^{\mathrm{P}}(R)$ for all problems $R = (q,c) \in \mathcal{R}$. We will start with a proof for $R \in \mathcal{R}^N$ and a similar reasoning applies for arbitrary $R \in \mathcal{R}^S$. So assume $\Psi \in \mathcal{S}(\mathrm{CO}), R \in \mathcal{R}^N$, and $x \in \Psi(R) \subseteq \mathfrak{I}^{\mathrm{P}}(R)$. Then it suffices to prove that for any $S \subseteq N$, $x(S) \leq c_R^{\mathrm{P}}(q(S))$. By consistency it holds that there is a $T_1 \in \mathcal{Q}(\Psi, R, \{1\})$ such that $x_{N\setminus\{i\}} \in \Psi(q_{N\setminus\{1\}}, c_{T_1})$. Put $R^1 = (q, c_{T_1})$, then by consistency there is a $T_2 \in \mathcal{Q}\left(\Psi, R^1, \{2\}\right)$ such that $x_{N\setminus\{1,2\}} \in \Psi(q_{N\setminus\{1,2\}}, (c_{T_1})_{T_2})$. Put $R^2 = (q_{N\setminus\{1,2\}}, (c_{T_1})_{T_2})$. This procedure may now be repeated for the agents $3, 4, ..., n$. In this way we obtain profit sharing problems $R^0, R^1, \ldots, R^{n-1}$, and R^n, such that $R^0 = R$, and for $i \in N, R^i = (q_{N\setminus\{1,2,\ldots,i\}}, c^i) \in \mathcal{R}^{N\setminus\{1,2,\ldots,i\}}$ is such that $c^i = c^{i-1}_{T_{i-1}}$ for some $T_{i-1} \in \mathcal{Q}(\Psi, R^{i-1}, \{i\})$ with the property that $x_{N\setminus\{1,2,\ldots,i\}} \in \Psi(R^i)$. In particular by definition of a reduction it holds for all $i \in N$

$$x_i = \int_{T_i} \left(c^i\right)' (s)\mathrm{d}s. \tag{7}$$

Define increasing bijections $g_i : [0, q(N\backslash\{1, 2, \ldots, i-1\}] \to [0, q(N\backslash\{1, 2, \ldots, i-1\}) - \lambda(T_i)]$ by $g_i(s) = \lambda([0, s]\backslash T_i)$ for all $s \in [0, q(N\backslash\{1, 2, \ldots, i-1\})]$. Note that

$$\lambda(g_i(U)) = \lambda(U) \text{ for all } i \in N. \tag{8}$$

Next, define $T_i^* := g_1^{-1} \circ g_2^{-1} \circ \ldots \circ g_{i-1}^{-1}(T_i)$ for $i \in N$. The collection $\{T_1^*, T_2^*, \ldots, T_n^*\}$ satisfies the following three properties:

(a) $\lambda(T_i^* \cap T_j^*) = 0$ for all $i \neq, j,$ since g_ℓ's are bijections,
(b) $\lambda(T_i^*) = q_i$ for all $i \in N,$ by (8), and
(c) $x_i = \int_{T_i^*} c'(s)\mathrm{d}s$ for all $i \in N,$ by (7).

Hence $x(S) = \sum\limits_{i \in S} \int_{T_i^*} c'(s)\mathrm{d}s \geq \inf\left\{ \int_T c'(s)\mathrm{d}s \mid T \in \mathcal{B}([0, q(N)]); \lambda(T) = q(S) \right\} = c_R^{\mathrm{P}}(q(S))$. ■

Proof of Theorem 4.2.

The proof of the inclusion "$\supseteq$" is easy and essentially done in Tijs and Koster (1998). We will now prove "$\subseteq$". Basically the argument is to choose a partition of the interval $[0, q(S)]$, such that each member in this partition is associated with a particular agent for which the corresponding marginal costs equal his cost share according to $y \in \mathfrak{C}^{\mathrm{P}}(q, c)$. Then, given q, we define solutions to the rationing problems (q, t) for $t \geq 0$ by $r(q, \cdot)$. By variation of q we obtain a rationing method r. At this point we may assume without loss of generality that $S = N$ and that we made a choice for $r(q', \cdot)$ for all $q' \neq q$. Our objective is to find $r(q, \cdot)$ such that $\mu^\pi(R) = y$. Let $T_1 = [a_1, a_1 + q_1]$ be an interval as in Lemma 6.1 for agent 1 and let $c_1 = c_R^{\mathrm{P}}$. Define a new cost sharing problem $R_2 = (q_2, c_2)$ by $q_2 := q_{N\backslash\{1\}}$ and

$$c_2(y) = \begin{cases} c_1(y) & \text{if } y \leq a_1, \\ c_1(a_1) + c_1(y + q_1) - c(a_1 + q_1) & \text{if } y > a_1. \end{cases}$$

Notice that c_2 is concave and that for this reason $c_{R_2}^{\mathrm{P}} = c_2$. Application of Lemma 6.1 to R_2 and agent 2 gives us a set $T_2 = [a_2, a_2 + q_2]$ such that $c_2(a_2 + q_2) -$

$c_2(a_2) = y_2$. Then we proceed as follows. Having defined $R_2, R_3, \ldots, R_i$ and intervals $T_1, T_2, \ldots, T_i, T_k = [a_k, a_k + q_k] \subseteq \left[0, \sum_{j \geq k} q_j\right]$, we define R_{i+1} by $\left(q_{N \setminus \{1,2,\ldots,i\}}, c_{i+1}\right)$ where $c_{i+1} \in \mathcal{C}$ is given by

$$c_{i+1}(y) = \begin{cases} c_i(y) & \text{if } 0 \leq y \leq a_i, \\ c_i(a_i) + c_i(y + q_i) - c_i(a_i + q_i) & \text{if } y > a_i. \end{cases}$$

Notice that c_{i+1} is concave on $\left[0, \sum_{j \geq i+1} q_j\right]$ such that $c^{\mathrm{P}}_{R_{i+1}} = c_{i+1}$. Then by application of Lemma 6.1 there is a $a_{i+1} \in \left[0, \sum_{j \geq i+2} q_j\right]$ with the property that $c_{i+1}(a_{i+1} + q_{i+1}) - c_{i+1}(a_{i+1}) = y_{i+1}$. Then define $T_{i+1} = [a_{i+1}, a_{i+1} + q_{i+1}]$. Define for each $i \in N$ a function $g_i : [0, \infty) \to \mathbb{R}_+$ by

$$g_i(y) = \begin{cases} y & \text{if } y \leq a_i, \\ y + q_i & \text{if } y > a_i. \end{cases}$$

In addition define $U_1 := T_1$ and for $i \in N \setminus \{1\}$ let $U_i \subseteq [0, q(N)]$ be defined by $U_i := (g_2 \circ g_3 \circ \ldots \circ g_i)(T_i)$. Then these sets $U_1, U_2, \ldots, U_n$ have the following properties $\lambda(U_i \cap U_j) = 0$ if $i \neq j$, and $\lambda(U_i) = \lambda(T_i) = q_i$ for all $i \in N$. Assume w.l.o.g. that $\{U_i\}_{i \in N}$ constitutes a partition of $[0, q(S)]$, and define the rationing method $r_k(q, \cdot)$ by $r_k(q,t) = \int_0^t \mathbb{I}_{U_k}(s)\mathrm{d}s$ for all $t \in [0, q(S)], k \in N$. Then

$$\begin{aligned} y_k &= \int_{T_k} c_k'(t)\,\mathrm{d}t = \int_{g_k(T_k)} c_{k-1}'(t)\,\mathrm{d}t = \ldots = \int_{(g_2 \circ g_3 \circ \ldots \circ g_k)(T_k)} c_1'(t)\,\mathrm{d}t \\ &= \int_{U_k} \left(c^{\mathrm{P}}_R\right)'(t)\,\mathrm{d}t = \int_0^{q(S)} \mathbb{I}_{U_k}(t) \left(c^{\mathrm{P}}_R\right)'(t)\,\mathrm{d}t = \int_0^{q(S)} \left(c^{\mathrm{P}}_R\right)'(t)\,\mathrm{d}r_k(q,t). \end{aligned}$$

■

Proof of Proposition 4.3 The combination of properties ADD, CR implies the functional representation as in Theorem 4.1. Now suppose there is $q \in \mathbb{R}^S_+$ and $i, j \in S$ with $q_i \geq q_j$ but not $r_i(q, \cdot) \geq r_j(q, \cdot)$. Then by continuity there is an interval $U \subset [0, q(S)]$ such that $r_i(q, \cdot) < r_j(q, \cdot)$ on U. Consider $c \in \mathcal{C}$ defined by $c(y) = \int_0^y \mathbb{I}_U(t)\,\mathrm{d}t$ for all $y \in \mathbb{R}_+$. Then $\mu_i(q, c) = \int_0^{q(S)} \mathbb{I}_U(t)\,\mathrm{d}r_i(q,t) < \int_0^{q(S)} \mathbb{I}_U(t)\,\mathrm{d}r_j(q,t) = \mu_j(q, c)$, which gives the desired contradiction. ■

Proof of Theorem 5.1
'$\Rightarrow$': If $\mu \in \mathfrak{M}^I(\text{ADD}, \text{CR}, \text{CO})$ then by Theorem 4.5 $\mu = \mu^r$ with consistent rationing method r. Since μ is incremental, r must be piecewise linear.

'$\Leftarrow$': Suppose r is a consistent and piecewise linear rationing method. Fix $q \in \mathbb{R}^S_+$ and take $x_0, x_1, \ldots, x_k$ with $0 = x_0 < \ldots < x_k = q(S)$ and $\alpha_1, \ldots, \alpha_k \in \Delta(S)$ such that for any $\ell \in \{1, 2, \ldots, k\}$ and $t \in [x_{\ell-1}, x_\ell]$, $r(q,t) = \sum_{h=1}^{\ell-1} \alpha_h(x_h - x_{h-1}) + \alpha_\ell(t - x_{\ell-1})$. In particular $\mu = \mu^r$ implies $\mu(q,c) = \sum_{\ell=1}^k \alpha_\ell(c(x_\ell) - c(x_\ell))$ for all $c \in \mathcal{C}$. First notice that by piecewise linearity the mappings $t \mapsto r_i(q,t)$ and $f : y \mapsto r_i(q_{S\setminus\{j\}}, y)$ are both differentiable almost everywhere, for each $i \in S\setminus\{j\}$. Then consistency implies that for almost all $t \in [x_{\ell-1}, x_\ell]$

$$\begin{aligned} (\alpha_\ell)_i &= \frac{d}{dt} r_i(q,t) = f'(t - r_j(q,t)) \cdot \frac{d}{dt}(t - r_j(q,t)) \\ &= f'(t - r_j(q,t)) \cdot (1 - (\alpha_\ell)_j) = f'(t - r_j(q,t)) \cdot \alpha_\ell(S \setminus \{j\}). \end{aligned}$$

Then f' is constant on $(x_{\ell-1} - r_j(q, x_{\ell-1}), x_\ell - r_j(q, x_\ell))$, and equals $(\alpha_\ell)_i / \alpha_\ell(S\setminus\{j\})$ whenever $\alpha_\ell(S\setminus\{j\}) > 0$. Define $\tilde{x}_\ell, \tilde{\alpha}_\ell$ by

$$\tilde{x}_\ell = x_\ell - r_j(q, x_\ell) = \sum_{1 \le p \le \ell} (1 - (\alpha_p)_j)\left(x_p - x_{p-1}\right), \tag{9}$$

$$\tilde{\alpha}_\ell = \begin{cases} \dfrac{\alpha_\ell}{\alpha_\ell(S \setminus \{j\})} & \text{if } \alpha_\ell(S \setminus \{j\}) > 0, \\ 0 & \text{else.} \end{cases} \tag{10}$$

Then for all $t \in [\tilde{x}_{\ell-1}, \tilde{x}_\ell]$

$$r(q_{S\setminus\{j\}}, t) = \sum_{h=1}^{\ell-1} \tilde{\alpha}_h(\tilde{x}_h - \tilde{x}_{h-1}) + \tilde{\alpha}_\ell(t - \tilde{x}_{\ell-1}). \tag{11}$$

As a result $\mu(q_{S\setminus\{j\}}, c) = \sum_{\ell=1}^k \tilde{\alpha}_\ell(c(\tilde{x}_\ell) - c(\tilde{x}_{\ell-1}))$ for all $c \in \mathcal{C}$. Claim: there is a set $U \in \mathcal{Q}(\mu, R, j)$ such that $\mu_{S\setminus\{j\}}(q,c) = \mu(q_{S\setminus\{j\}}, c_U)$. Define for each $\ell \in \{1, 2, \ldots, k\}$ the convex function $c_\ell : [x_{\ell-1}, x_\ell] \to \mathbb{R}$ by

$$c_\ell(y) = \inf\left\{ \int_T c'(t)\mathrm{d}t \;\middle|\; T \in \mathcal{B}([x_{\ell-1}, x_\ell]), \lambda(T) = y \right\}.$$

According to Lemma 6.3 (in the Appendix) there is for each ℓ an interval $I_\ell = [t_\ell, t_\ell + (\alpha_\ell)_j\,(x_\ell - x_{\ell-1})] \subset [x_{\ell-1}, x_\ell]$ such that $c_\ell(t_\ell + (\alpha_\ell)_j\,(x_\ell - x_{\ell-1})) - c_\ell(t_\ell) = (\alpha_\ell)_j c_\ell\,(x_\ell - x_{\ell-1})$. As in Lemma 6.2 each interval I_ℓ corresponds to a set $U_\ell \in \mathcal{B}([x_{\ell-1}, x_\ell])$ such that $\int_{U_\ell} c'(t)\mathrm{d}t = (\alpha_\ell)_j c_\ell((x_\ell - x_{\ell-1}) = (\alpha_\ell)_j\,(c(x_\ell) - c(x_{\ell-1}))$, and in particular $U = \cup_\ell U_\ell \in \mathcal{Q}(\mu, R, j)$. By construction we have

$$c_U(\tilde{x}_\ell) - c_U(\tilde{x}_{\ell-1}) = \left(1 - (\alpha_\ell)_j\right)(c(x_\ell) - c(x_{\ell-1})).$$

Hence for all $i \in S \setminus \{j\}$,

$$\begin{aligned}
\mu_i(q_{S\setminus\{j\}}, c_U) &= \sum_{\ell=1}^{k} (\tilde{\alpha}_\ell)_i\,(c_U(\tilde{x}_\ell) - c_U(\tilde{x}_{\ell-1})) \\
&= \sum_{\ell;(\alpha_\ell)_j\neq 1} \frac{(\alpha_\ell)_i}{1 - (\alpha_\ell)_j} \cdot \left(1 - (\alpha_\ell)_j\right)(c(x_\ell) - c(x_{\ell-1})) \\
&= \sum_{\ell;(\alpha_\ell)_j\neq 1} (\alpha_\ell)_i\,(c(x_\ell) - c(x_{\ell-1})) \\
&= \sum_{\ell=1}^{k} (\alpha_\ell)_i\,(c(x_\ell) - c(x_{\ell-1})) \;=\; \mu_i(q, c).
\end{aligned}$$

Notice that the fourth equality is due to the fact that $(\alpha_\ell)_j = 1$ implies $(\alpha_\ell)_i = 0$. This proves the claim, and the theorem. ∎

Proof of Theorem 5.2 Let $\mu = \mu^r$ and let $\bar{r}$ be the rationing method for $\bar{\mu}$. Take $q \in \mathbb{R}^S_+, j \in S$, and assume without loss of generality that $q(S) > 0$. For every $i \in S\setminus\{j\}$ there are $h(i) \in \mathbb{N}, \bar{\alpha}^i \in \Delta(\{i,j\})^{h(i)}, \bar{x}^i \in \mathbb{R}_+^{h(i)}$ determining $\bar{\mu}(q_{\{i,j\}}, \cdot)$ as an incremental mechanism. Then define for $\ell = 1, \ldots, h(i), I^i_\ell := \{t \in [0, q(S)] \,|\, r_i(q,t) + r_j(q,t) \in (\bar{x}_{\ell-1}, \bar{x}_\ell)\}$, each being a closed interval by monotonicity of r. Let $\{b_0, b_1, \ldots, b_p\}$ be the set of endpoints of all the intervals I^i_ℓ, such that $0 = b_0 < b_1 < \ldots < b_p = q(S)$. Then for each $i \in S\setminus\{j\}, \ell \in \{1, 2, \ldots, p\}$ denote $\ell(i) \in \{1, 2, \ldots, h(i)\}$ for the index with $B_\ell := [b_{\ell-1}, b_\ell] \subseteq I^i_{\ell(i)}$. For any $t \in [0, q(S)]$, by repeated application of CO,

$$\begin{aligned}
r_i(q,t) &= \mu_i(q, \Gamma_t) = \mu_i(q_{\{i,j\}}, \Gamma_{r_i(q,t)+r_j(q,t)}) \\
&= \bar{\mu}(q_{\{i,j\}}, \Gamma_{r_i(q,t)+r_j(q,t)}) = \bar{r}_i(q_{\{i,j\}}, r_i(q,t) + r_j(q,t)).
\end{aligned}$$

The mappings $t \mapsto r_i(q,t)$ are monotonic and therefore differentiable almost everywhere. Hence, for almost all $t \in B_\ell, \ell \in \{1,2,\ldots,p\}$, by application of the chain rule, it holds that

$$\frac{d}{dt}r_i(q,t) = \frac{d}{dt}\bar{r}_i(q_{\{i,j\}}, r_i(q,t)+r_j(q,t)) = (\bar{\alpha}^i_{\ell(i)})_i\left(\frac{\partial r_i}{\partial t}(q,t)+\frac{\partial r_j}{\partial t}(q,t)\right).$$

By rearranging terms, we obtain

$$(\bar{\alpha}^i_{\ell(i)})_j\frac{\partial r_i}{\partial t}(q,t) = (\bar{\alpha}^i_{\ell(i)})_i\frac{\partial r_j}{\partial t}(q,t) \text{ for all } i \in S\backslash\{j\}. \tag{12}$$

We distinguish between two cases. Case (a): $(\bar{\alpha}^i_{\ell(i)})_j = 0$. Then (12) implies $\partial r_j/\partial t(q,t) = 0$ for $t \in B_\ell$, which means that $r_j(q,\cdot)$ is constant on B_ℓ. Case (b): $(\bar{\alpha}^i_{\ell(i)})_j > 0$. Then by (12), for all $i \neq j$,

$$\frac{\partial r_i}{\partial t}(q,t) = \frac{(\bar{\alpha}^i_{\ell(i)})_i}{(\bar{\alpha}^i_{\ell(i)})_j}\frac{\partial r_j}{\partial t}(q,t). \tag{13}$$

Since $\sum_{k\in S} r_k(q,t) = t$ it must hold for almost all $t \in [0,q(S)]$ that $\sum_{k\in S}\partial r_k/\partial t = 1$. Hence for almost all $t \in B_\ell$, (13) yields

$$\frac{\partial r_j}{\partial t}(q,t) = \left(\sum_{i\neq j}\frac{(\bar{\alpha}^i_{\ell(i)})_i}{(\bar{\alpha}^i_{\ell(i)})_j}+1\right)^{-1}.$$

This means that $r_j(q,\cdot)$ is linear on B_ℓ. Then as a result from (a) and (b) $r_j(q,\cdot)$ is piecewise linear. By varying j over S, $r(q,\cdot)$ is fully determined and, in particular, piecewise linear. Then μ is uniquely determined, and a member of $\mathfrak{M}^I$. ■

Proof of Theorem 5.3

Suppose $\mu \in \mathfrak{M}^I(\text{ICO})$, hence $\mu \in \mathfrak{M}^I(\text{CO})$. Consider $q \in \mathcal{R}^S_+$. Then there is $k \in \mathbb{N}$ and for each $\ell \in \{1,2,\ldots,k\}$ a real number $x_\ell \in [0,q(S)]$ and vector $\alpha_\ell \in \Delta(S)$ such that $\mu(q,c) = \sum_{\ell=1}^k \alpha_\ell\{c(x_\ell)-c(x_{\ell-1})\}$, for all $c \in \mathcal{C}$. Consider functions $c_p : \mathbb{R}_+ \to \mathbb{R}$ by $c_p(x) = e^{px}-1$ for all $x \geq 0, p \in \mathbb{N}$ and define $R_p = (q,c_p)$. Then c_p is convex so that for each $j \in N$ there is a unique interval $I(p,j) = [t^j_p, t^j_p+q_j] \in \mathcal{Q}(\mu,R_p,j)$ for each $p \in \mathbb{N}$. Define $c^j_p = (c_p)_{I(p,j)}$ for each $p \in \mathbb{N}$. Distinguish the following cases: (i) there is only one agent j with positive $(\alpha_k)_j$, or $(\alpha_k)_j = 1$, and (ii) there are at least two agents i and j with $(\alpha_k)_i, (\alpha_k)_j > 0$.

Case (i): Suppose that $x_k - x_{k-1} < q_j$. Then $t_p^j < x_{k-1}$ for all $p \in \mathbb{N}$ and for large p it holds $\mu_j(q, c_p) \geq (\alpha_k)_j \left\{c_p(x_k) - c_p(x_{k-1})\right\} = c_p(x_k) - c_p(x_{k-1}) > c_p(t_p^j + q_j) - c_p(t_p^j)$. But this means that $I(p, j) \notin \mathcal{Q}(\mu, R_p, j)$ for large p, contradiction. Hence it must be that $x_k - x_{k-1} = q_j$. Then $(\alpha_\ell)_j = 0$ for all $\ell \neq k$.

Case (ii): Suppose that there is a subsequence $\{t_{h(p)}^j\}_{p \in \mathbb{N}}$ of $\{t_p^j\}_{p \in \mathbb{N}}$ such that $t_{h(p)}^j \leq \tilde{x}_{k-1}$ for all $p \in \mathbb{N}$. Since μ is interval consistent we have for each $p \in \mathbb{N}$ that $\mu_{S \setminus \{j\}}(q, c_p) = \mu(q_{S \setminus \{j\}}, c_p^j)$. Following the proof of Theorem 5.1 it must hold for all $p \in \mathbb{N}, i \in S \setminus \{j\}$,

$$\sum_{\ell=1}^{k} \alpha_{\ell i} \left\{c_p(x_\ell) - c_p(x_{\ell-1})\right\} - \tilde{\alpha}_{\ell i} \left\{c_p^j(\tilde{x}_\ell) - c_p^j(\tilde{x}_{\ell-1})\right\} = 0.$$

By distinguishing the powers in this sum we must have

$$\lim_{n \to \infty} (\alpha_k)_i \left\{c_p(x_k) - c_p(x_{k-1})\right\} - (\tilde{\alpha}_k)_i \left\{c_p^j(\tilde{x}_k) - c_p^j(\tilde{x}_{k-1})\right\} = 0. \tag{14}$$

For all p we have

$$c_{h(p)}^j(\tilde{x}_k) - c_{h(p)}^j(\tilde{x}_{k-1}) = c_{h(p)}(\tilde{x}_k + q_j) - c_{h(p)}(\tilde{x}_{k-1}) - \mu_j(q, c_{h(p)}).$$

Then use the expression for $\mu_j(q, c_{h(p)})$ to see that (14) is equivalent with

$$\begin{aligned}&\lim_{p \to \infty} (\alpha_k)_i \left\{c_{h(p)}(x_k) - c_{h(p)}(x_{k-1})\right\} + \\ &-(\tilde{\alpha}_k)_i \left\{c_{h(p)}(x_k) - c_{h(p)}(x_{k-1} - q_j + (\alpha_k)_j(x_k - x_{k-1})) - (\alpha_k)_j(c_{h(p)}(x_k) - c_{h(p)}(x_{k-1}))\right\} = 0.\end{aligned}$$

Since $(\alpha_k)_i - (\tilde{\alpha}_k)_i(1 - (\alpha_j)_\ell) = 0$, the terms with the highest argument x_k vanish, so we get

$$\lim_{p \to \infty} -(\alpha_k)_i c_{h(p)}(x_{k-1}) - (\tilde{\alpha}_k)_i \left\{-c_{h(p)}(x_{k-1} - q_j + (\alpha_k)_j(x_k - x_{k-1})) + (\alpha_k)_j c_{h(p)}(x_{k-1})\right\} = 0.$$

Since $(\alpha_k)_i > 0$ it must hold that $(\alpha_k)_j(x_k - x_{k-1}) = q_j$. By interchanging the role of i and j we see that also $(\alpha_k)_i(x_k - x_{k-1}) = q_i$. In particular for any agent t with $(\alpha_k)_t > 0$ it holds that $(\alpha_k)_t(x_k - x_{k-1}) = q_t$ and therefore

$$x_k - x_{k-1} = \sum_{j \in S; (\alpha_k)_j > 0} (\alpha_k)_j(x_k - x_{k-1}) = \sum_{j \in S; (\alpha_k)_j > 0} q_j,$$

hence for each i with $(\alpha_k)_i > 0$ it holds that

$$(\alpha_k)_i = \frac{q_i}{\sum_{j\in S;(\alpha_k)_j>0} q_j} \text{ and } (\alpha_\ell)_i = 0 \text{ for all } \ell \neq k.$$

■

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