Existence of equilibrium, core and fair allocation in a heterogeneous divisible commodity exchange economy

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We consider the problem of exchange and allocation of a heterogeneous divisible commodity. One notable example of such a commodity is land. This problem is coined in literature as the 'cake division' or 'land division' problem. A heterogeneous divisible commodity is modelled as a measurable space (X, Σ). In theoretical models of land economics X is assumed to be a Borel measurable subset of Euclidean space R^2 (or more generally R^k) and Σ to be the Borel $\sigma-$algebra $\mathcal{B}(X)$ of subsets of X. It is usual to consider this measurable space with the Lebesgue measure.

Berliant [1] is the first to study a competitive equilibrium in this context. He shows the existence of a competitive equilibrium in the case when preferences over land plots are represented by utility functions of the form $U(B) = \int_B u(x)dx$, so that U is a measure on $\mathcal{B}(X)$ absolutely continuous with respect to the Lebesgue measure.

Dunz [3] studies the existence of the core for substantially more general preferences. In [3] preferences are given by the utility functions that are compositions of quasi-concave functions with a finite number of characteristics of land parcels. Dunz proves that under these assumptions on preferences the weak core of a land trading economy is nonempty. These chracteristics are countably-additive over land parcels. Assigning a finite number of additive characteristics to land parcels is a common assumption made in empirical literature on land trading.

Our approach to the problem at hand will be abstract. That is, a heterogeneous divisible commodity will be modelled as an abstract measurable space with nonatomic characteristics measures. We will call a model of exchange of such commodity the land trading economy.

One of the goals of the present paper is to show that a competitive equilibrium exists in land trading economy with rather general unordered preferences. Then, using the standard scheme, we show that a competitive allocation is a weak core allocation. This core existence result generalizes Dunz's core existence theorem in two directions; first, it considers the land trading problem in the setting of abstract measurable space and does not assume the existence of a reference measure, and the second, preferences are not assumed to be ordered.

The third topic that is dealt with in this paper is the existence of a fair division. Weller [4] consideres a problem of fair division of a measurable space (X, Σ) with agents' preferences that are atomless measures. He shows the existence of an envy-free and efficient partition in this problem. In a somewhat different setting, namely when X is a measurable subset of the Euclidean space R^k and preference measures are nonatomic and absolutely continuous with respect to Lebesgue measure Berliant, Thomson and Dunz [2] show the existence of a group envy-free and efficient partition. Our approach to the fairness problem will be abstract and we will consider much more general preferences over measurable pieces. The result established here will imply both of the above discussed results.

Let (X, Σ) be a measurable space (the cake or land plot) and let $\bar{P} = \{A_1, A_2, \ldots, A_n\}$

be a measurable ordered partition of X. Let $\mu_1, \mu_2, \ldots, \mu_n$ be nonatomic finite vector measures on (X, Σ) of dimension $s_1, s_2, \ldots, s_n$, respectively. The interpretation is that there are n persons each contributing his piece A_i $(i \in N)$ of the land, X, and plots of the land are valued by individuals according to their measures $\mu_1, \mu_2, \ldots, \mu_n$, respectively. The components of vector-measure $\mu_i(B)$ are interpreted as measures of different attributes of a measurable piece B attached to this piece by individual i. We assume that individual i has a preference $\succ_i$ over his subjective attributes profiles $\mu_i(B)$, $B \in \Sigma$ and hence over measurable sets $B \in \Sigma$. Every ordered measurable partition $\{B_1, B_2, \ldots, B_n\}$ will be interpreted as a feasible allocation (of land X).

Definition 1. A pair $(P = \{B_1, B_2, \ldots, B_n\},\ \mu)$ consisting of a feasible partition P and a measure μ is said to be a *competitive equilibrium* if for each individual i subset B_i maximizes his preference $\succ_i$ in his *budget set* $\mathcal{B}_i(\mu) = \{B \in \Sigma \mid \mu(B) \leq \mu(A_i)\}$. In this case $P = \{B_1, B_2, \ldots, B_n\}$ is called an *equilibrium allocation* and measure μ is called an *equilibrium price.*

(Weak) Pareto efficient and weak (core) allocations are defined in the usual way.

For a preference $\succ_i$ on $R_+^{s_i}$ we denote $P_i(x_i) = \{x_i' \in R_+^{s_i} \mid x_i' \succ_i x_i\}$. We assume that preferences $\succ_i$ or P_i $(i \in N)$ are continuos, that is graph of correspondence P_i is open relative to $R_+^{s_i} \times R_+^{s_i}$, and that they satisfy the following assumption.
Assumption (Weak Monotony). If for $x_i,\ x_i' \in R_+^{s_i}$ and $x_i' \geq x_i$, then $P_i(x_i') \subset P_i(x_i)$ for all $i \in N$.

Theorem 1. *If attribute measures μ_i, $i \in N$ are nonatomic, preferences $\succ_i$, $i \in N$ are irreflexive continuos and weakly monotone, then there exsists a competitive equlibrium $(P = \{B_1, B_2, \ldots, B_n\},\ \mu)$ in the land trading economy. Moreover, the equilibrium price measure μ is absolutely continuous with respect to the sum of all component measures of vector-measures measures μ_i, $i \in N$.*

Corollary 2. *Under the conditions of Theorem 1 the weak core of the land trading economy is nonempty.*

Proposition 3. *If preferences $\succ_i$ are the strict parts of rational continuous preferences $\succsim_i$, monotone (for $x_i, x_i' \in R_+^{s_i}$, $x_i' \geq x_i$ implies $x_i' \succ_i x_i$,) and if measures $\nu_i = \sum_{j=1}^{s_i} \mu_i^j$ $(i \in N)$ are absoluely continuous with respect to each other, then the weak core (the weak Pareto set) and the core (the Pareto set) coincide.*

Corollary 2 and Proposition 3 imply

Corollary 4. *If in addition to the assumptions of Propsition 3 preferences are convex, then the core in the land trading economy is nonempty.*

Definition 2. A division $P = \{A_1, A_2, \ldots, A_n\}$ of X is said to be *fair* if it is
(a) *Pareto optimal,* that is if there is no other division $Q = \{C_1, C_2, \ldots, C_n)\}$ such that

$\mu_i(C_i) \in P_i(\mu_i(A_i))$ for $i \in N$, and
(b) *envy-free,* that is if $\mu_i(A_j) \notin P_i(\mu_i(A_i))$ [otherwise, not $A_j \succ_i A_i$] for $i, j \in N$.

Definition 3. A division $\{A_1, A_2, \ldots, A_n\}$ is *weak group envy-free* if for every pair of coalitions N_1, N_2 with $|N_1| = |N_2|$ there is no division $\{C_i\}_{i \in N_1}$ of $\cup_{j \in N_2} A_j$ such that $C_i \in P_i(A_i)$ for all $i \in N_1$.

This definition is adapted from Berliant-Thomson-Dunz [2]. Obviously if an allocation is weak group envy-free then it is envy-free and weak Pareto efficient.

Definition 4. A division $\{A_1, A_2, \ldots, A_n\}$ is *group envy-free* if for every pair of coalitions N_1, N_2 with $|N_1| = |N_2|$ there is no division $\{C_i\}_{i \in N_1}$ of $\cup_{j \in N_2} A_j$ such that $A_i \notin P_i(C_i)$ for all $i \in N_1$ and $C_i \in P_i(A_i)$ for at least one $i \in N_1$.

When preferences P_i are rational preferences then the last part of Definition 4 will be read as "$C_i \succcurlyeq_i (A_i)$ for all $i \in N_1$ and $C_i \succ_i A_i$ for at least one $i \in N_1$. Under the assumptions of Proposition 3 every weak group envy-free division is group envy-free, that is two concepts coincide.

Theorem 5. *Under the assumptions of Theorem 1 there exists a group envy-free and Pareto efficient allocation.*

In the case when the cake X is a subset of Euclidean space R^k and preferences $\succ_i$ are given by scalar measures on the Borel $\sigma-$algebra of sets in X absolutely continuous with respect to the Lebesgue measure we Theorem 2 of Berliant-Dunz-Thomson [2]. Notice that in their approach there is a reference measure (the Lebesgue measure) while our approach does not involve any such measure.

Corollary 6. *Under the assumptions of Theorem 1 there exists a fair division of a measurable space* (X, Σ).

When preferences are given by scalar measures on we get Weller's fairness result [4].

References

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