On the *Potential* of State Dependent Mutations as an Equilibrium Refinement Device*

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Abstract

This paper focuses on modelling the mutation process in evolutionary models. First, we develop a link between the nature of the mutation process, the detailed balance property and the nature of the game: we show that a game has a detailed balanced and utility monotonic perturbation if and only if it is an ordinal potential game. Then, we show that for ordinal potential games the utility monotonicity property is insufficient to generate robust equilibrium predictions. Therefore, we argue that the mutation induced solution concept only has limited potential as an equilibrium refinement device.

1 Introduction

To overcome path dependence of the dynamic process, Kandori et al. [1993] and Young [1993] introduced noise (mutations, perturbations) in evolutionary models: a state is called stochastic stable if it is, as the mutation rate converges to zero, a limiting state of the process with mutation/perturbation. Both Kandori et al. [1993] and Young [1993] use a uniform mutation rate.

Their (strong) predictions, however, are criticized by Bergin and Lipman [1996] who show that by allowing the mutation rate to vary across states, *any* stable state of the model without mutation can be obtained as a stochastic stable state of a process with mutation. In other words, the equilibrium correspondence which maps the process with mutations (perturbation) to the set of stable states is surjective. Therefore, Blume [2003] draws on state dependent mutation rates to get a more profound understanding of the mutation process: he assumes that when an individual i prefers x to y, the probability that i will

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move from y to x is larger than the probability that i will move from x to y. The mutation process is said to exhibit utility monotonicity.

A more methodological critique on Kandori et al. [1993] and Young [1993] concerns the graph-theoretic principles which underly the results. Although formally correct, the graph-methodology fails to add intuition and transparency to their analysis and, more importantly, it does not allow for an exact computation of the limiting distribution of the perturbed process. Therefore, Markov chains which satisfy the detailed balance conditions are very interesting: they don't require the Markov chain tree theorem to find the stochastic stable states and it is straightforward to find their limiting distribution.

Interestingly, in the literature, it appears that for potential games, the stochastic stable states of the detailed balanced and utility monotonic Markov chains turn out to be the states which maximize the potential function of the game (cf. Young [1998]; Blume [2003]; Baron et al. [2003]). However, we know of no result which formally links these three concepts (potential games, utility monotonicity and detailed balancedness) together.

The first part of this paper derives the following relationship: a game has a detailed balanced and utility monotonic perturbation if and only if it is an ordinal potential game.

The second part of this paper elaborates on the utility monotonicity property of the mutation process within the class of ordinal potential games. Although utility monotonicity it is an intuitive property, it is insufficient to generate robust predictions: we find that for all ordinal potential games and all stable states of a deterministic evolutionary process, we can find a utility monotonic and detailed balanced perturbation such that the stochastic stable states coincide with those stable states. It is, therefore, no solution to the Bergin and Lipman–critique.[1]

The paper is organized as follows: section 2 presents the necessary notation and section 3 derives the link between ordinal potential games, utility monotonicity and detailed balancedness of the mutation process. Section 4 elaborates on the potential of state dependent mutation as an equilibrium selection device in potential games. Finally, in section 5 we present the conclusions of this paper.

2 Notation

2.1 Markov Chains, stable sets and invariant distributions

Consider a finite set of states S. A *Markov chain* is a set of positive $\{p(x,y)\}_{x,y\in S}$ such that for all $x \in S$:

$$\sum_{y\in S} p(x,y) = 1.$$

The element $p(x,y)$ is the *transition probability* of going from state x to state y. With

[1] Note that, with a refinement of the allowed mutation processes, our result generalizes Bergin and Lipman [1996], while at the same time, by only considering ordinal potential games, our result is weaker than theirs. As such our result is neither stronger or weaker than theirs.

each Markov chain M, we can correspond a binary relation $R_M \subseteq S \times S$ such that

$$(x, y) \in R_M \text{ if and only if } p(y, x) > 0.$$

We denote by $T(R_M)$, the transitive closure of R_M[2]

A set $A \subseteq S$ is a *stable set* of the Markov process M if (i) for all $x, y \in A$, $(x, y) \in T(R)$ and (ii) for all $x \in A, y \in S - A$, $(y, x) \notin R_M$, i.e. for all $x, y \in A$ there is positive probability of going from x to y in finite time and the probability of going from a state in A to a state outside A is zero. Each Markov chain has at least one stable set and two different stable sets must have nonempty intersection. The elements in the stable sets of M are called the *stable states* of the Markov process.

A Markov chain is said to be *irreducible* if and only if for all $x, y \in S$: $(x, y) \in T(R_M)$. An irreducible Markov chain has the property that the probability of going from any state in S to any other state in S in finite time is positive. The stable set of an irreducible Markov chain is unique and is equal to S.

A probability distribution $\{P(x)\}_{x \in S}$ over S is an invariant distribution of the Markov chain M if for all $x \in S$:

$$\sum_{x \in S} P(x)p(x, y) = P(y).$$

Lemma 1. *Every Markov chain has an invariant distribution*

Proof. Consider the S dimensional simplex Δ. Consider the function $f : \Delta \to \mathbb{R}^{|S|}$ such that for an element $P = \{P(x)\}_{x \in S}$ of Δ:

$$f(P)(y) = \sum_{x \in S} P(x) p(x, y).$$

As

$$\sum_{x \in S} f(P(x)) = \sum_{x \in S} \sum_{y \in S} P(y) p(y, x) = \sum_{y \in S} P(y) \sum_{x \in S} p(y, x) = \sum_{y \in S} P(y) = 1$$

We have that for all $P \in \Delta$, $f(P) \in \Delta$. The function f is a continuous function from Δ to Δ, hence by Brouwer's fixed point theorem, there must be an element $P \in \Delta$ such that $f(P) = P$. From the definition of f, this element must be an invariant distribution of the markov chain. □

We will mainly be interested in the elements of S for which an invariant distribution $\{P(x)\}_{x \in S}$ has positive support, i.e. $x \in S$ has positive support if $P(x) > 0$. We have the following results connecting the stable sets and the support of an invariant distribution.

Lemma 2. *If an invariant distribution $\{P(x)\}_{x \in S}$ of the Markov chain M has positive support on $x \in S$, then x is contained in a stable set of M.*

[2] $(x, y) \in T(R_M)$ if and only if there exists a sequence $x_1, \ldots, x_n$ of elements in S, such that $x_1 = x, x_n = y$ and $\forall t = 1, \ldots, n-1 : (x_t, x_{t+1}) \in R_M$.

Proof. We begin by stating following facts, which are straightforward derived from the definition of the invariant distribution:

- If x has positive support and $(y, x) \in R_M$, then y has positive support.
- If x has positive support, and $p(x, x) \neq 1$, then there is an element $y \neq x$ with positive support such that $(x, y) \in R_M$. We denote this by $(x, y) \in R^*$

We now continue with the rest of the proof. Assume on the contrary that x has positive support and that x is not in a stable set. Then from the definition of stables sets, there is an element y in a stable set of M such that $(y, x) \in T(R_M)$. As x has positive support, every element $z \in S$ for which $(z, x) \in R_M$ has also positive support (by the first fact above). Hence there must be an element $z \in S$ such that z has positive support, z is in no stable set of M and $(v, z) \in R_M$ for some element v in a stable set of M. As z has positive support (and $p(z, z) \neq 1$, which follows from the fact that $\{z\}$ is not a stable set), there must be at least one element $w \in S$ such that $(z, w) \in R^*$. Also from the definitions of stable sets, it follows that for all $w \in S$ for which $(z, w) \in R^*$, w is in no stable set. Consider the set:

$$V = \{w \in S \,|(z, w) \in T(R^*)\}.$$

This set is finite (as S is finite) and does not contain an element in a stable set of M. Further every element in V has positive support. Now we have that:

$$\sum_{w \in V} P(w) = \sum_{w \in V} \sum_{w' \in W} P(w') \, p(w', w) < \sum_{w' \in V} P(w') \sum_{w \in S} p(w', s) = \sum_{w' \in V} P(w')$$

A contradiction. The first equality follows from the definition of the invariant distribution. The inequality follows from the fact that $p(z, v) > 0$. □

From the first fact in above proof, if an element of a stable set has a positive support, then every other element of this stable set also has positive support. Therefore, an invariant distribution of an irreducible Markov chain has support on the whole of S.

Lemma 3. *If M is an irreducible Markov chain, it has a unique invariant distribution*

Proof. Assume that $\{P(x)\}_{x \in S}$ and $\{Q(x)\}_{x \in S}$ are two distinct invariant distributions. Let Δ be the $S - 1$ dimensional simplex. And consider the function $Z : \mathbb{R} \to \mathbb{R}^{|S|}$ with for $\beta \in \mathbb{R}$, $Z(\beta)$ given by:

$$\{\beta P(x) + (1 - \beta) Q(x)\}_{x \in S}.$$

The function Z defines a straight line in $\mathbb{R}^{|S|}$ and contains the elements P and Q in Δ. From the definition of the invariant distribution, it is easy to see that every element on this straight line in Δ is an invariant distributions of the Markov chain. Also, the function must intersect the boundary of Δ. This intersection defines a probability distribution on S, lets say $\{O(x)\}_{x \in S}$ for which there is an element $v \in S$ for which $O(v) = 0$. This contradicts with the fact that the support of an invariant distribution of an irreducible Markov chain has support on the whole of S. □

Consider an invariant distribution $\{P(x)\}_{x \in S}$ of a Markov chain M. We say that it satisfies the *detailed balance condition* if and only if, for all $x, y \in S$:

$$P(y)p(x,y) = P(x)p(y,x).$$

A distribution which satisfies the detailed balance condition is automatically an invariant distribution. A Markov chain which has an invariant distribution satisfying the detailed balance condition is said to be detailed balanced.

2.2 Markov chains and perturbations induced by a game

A game in strategic form, $G = (N, S, \{u_i\}_{i \in N})$, consists of a finite set of individuals $N = \{1, \ldots, n\}$; a set of strategy profiles (states) $S = \prod_{i \in N} S_i$, where S_i is the finite set of strategies of individual i; and a set of functions $u_i : S \to \mathbb{R}$, which present the payoffs to the individuals.

A strategy profile $s = (s_1, \ldots, s_n)$ can be written as (s_i, s_{-i}), where s_{-i} represents the strategies of the individuals in $N - \{i\}$. For each individual i we define a binary relation $\approx_i \subseteq S \times S$ such that: $x \approx_i y$ if and only if $x_{-i} = y_{-i}$. Since $\approx_i$ is an equivalence relation (transitive, reflexive and symmetric), it partitions S into equivalence classes: for an element $x \in S$ we denote the equivalence class of x with respect to $\approx_i$ as $[x]_i$.

Consider the following discrete–time model. At each time step one –random– individual $i \in N$ is given the opportunity to consider revising his strategy. He therefore maximizes his payoff, assuming that all the other individuals keep their strategy fixed. Formally, if at time t strategy s is played, and if at time $t+1$ individual i is selected, then at time $t+1$ strategy s' will be played, where $s' \in [s]_i$ and $s'_i \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$. We assume that when $\arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$ contains more than one element, each is chosen with equal probability.

This dynamic model defines a Markov chain $M_G = \{p(x,y)\}_{x,y \in S}$ with S as the set of states. If there is an $i \in N$ such that $y \in \arg\max_{z \in [x]_i} u_i(z)$ then $p(x,y) > 0$, else $p(x,y) = 0$.

Consider an element $\delta \in \mathbb{R}_0^+$. A *perturbation* of the game G is a set of Markov chains $\{\Sigma_G(\delta)\}_{\delta \in \mathbb{R}_0^+}$, such that: (i) $\Sigma_G(.)$ is continuous in δ; (ii) $lim_{\delta \to \infty} \Sigma_G(\delta) = M_G$; and, (iii) for all $\delta \in \mathbb{R}_0^+$, $\Sigma_G(\delta)$ is irreducible.

Let $\Sigma_G(\delta)$ be a perturbation of G. As $\Sigma_G(\delta)$ is irreducible, it has a unique invariant distribution, say P_δ. This distribution has support on the whole of S and as $\Sigma_G(.)$ is continuous in δ, P_δ is also continuous in δ. A state $x \in S$ is said to be stochastic stable if $lim_{\delta \to \infty} P_\delta(x) > 0$. The distribution $P_\infty = \lim_{\delta \to \infty} P_\delta$ is the stochastic stable distribution of the perturbation Σ_G. As P_δ is continuous, P_∞ is also an invariant distribution of M_G. Therefore, every stochastic stable state is also a stable state: the set of stochastic stable states is a refinement of the set of stable states. Now remark that if a perturbation $\Sigma_G(\delta)$ of M_G satisfies the detailed balance condition, P_∞ necessarily satisfies the detailed balance condition for M_G.

3 Potential games, detailed balance conditions and 'utility monotonic' perturbations

A game G is an *exact* potential game (henceforth, potential game) if there exists a function $V : S \to \mathbb{R}$, such that, for all $i \in N, x \in S$ and $y \in [x]_i$:

$$u_i(x) - u_i(y) = V(x) - V(y).$$

Note that a potential function V of a potential game G is unique, up to a constant term (Monderer and Shapley [1996]).

A game G is a *weighted* potential game if there exists a function $V : S \to \mathbb{R}$, and for all $i \in N$ there are elements $w_i \in \mathbb{R}_0^+$, such that, for all $i \in N$, $x \in S$ an $y \in [x]_i$:

$$u_i(x) - u_i(y) = w_i(V(x) - V(y)).$$

Finally, a game G is an *ordinal* potential game[3] if there exists a function $V : S \to \mathbb{R}$, such that, for all $i \in N$, $x \in S$ and $y \in [x]_i$:

$$u_i(x) - u_i(y) \geq 0 \leftrightarrow V(x) - V(y) \geq 0.$$

Consider a perturbation $\Sigma_G(\delta) = \{p_\delta(x,y)\}_{x,y \in S}$ of the game G. We say that $\Sigma_G(\delta)$ is utility monotonic if there exists a function $h : S \times S \to \mathbb{R}$ such that such that, for all $i \in N$, $x \in S$ and $y \in [x]_i$:

$$\frac{p_\delta(y,x)}{p_\delta(x,y)} = e^{\delta h(x,y)}.$$

and $h(x,y) \geq 0$ if and only if $u_i(x) - u_i(y) \geq 0$. We call h a utility monotonic function of the perturbation Σ_G.

Lemma 4. *If $\Sigma_G(\delta)$ is detailed balanced and utility monotonic, with utility monotonic function h, there exists a function $V : S \to \mathbb{R}$, such that:*

$$h(x,y) = V(x) - V(y).$$

Proof. Let $x \in S$ and $y, z \in [x]_i$. Let $\{P_\delta(x)\}_{x \in S}$ be the limiting distribution of $\Sigma_G(\delta) = \{p_\delta(x,y)\}_{x,y \in S}$ and consider the identity:

$$\frac{P_\delta(x)}{P_\delta(y)} \frac{P_\delta(y)}{P_\delta(z)} = \frac{P_\delta(x)}{P_\delta(z)}.$$

As $\Sigma_G(\delta)$ is utility monotonic and detailed balanced, we know that, for all $i \in N$, $x \in S$ and $y, z \in [x]_i$:

$$h(x,z) = h(x,y) + h(y,z).$$

[3]For a characterization on ordinal potential games, see Voorneveld and Norde [1997].

If we put $x = y = z$, we have: $h(x, x) = 0$; if we put $x = z$, we have:

$$h(x, y) = -h(y, x).$$

Consider an element $z \in S$ and let $z_x^i = (z_i, x_{-i})$. Define the function $g(i, .)$ as:

$$g(i, x) = h(x, z_x^i).$$

We have that:

$$\begin{aligned} h(x, y) &= h(x, z_x^i)) + h(z_x^i, y). \\ &= h(x, z_x^i) - h(y, z_x^i). \\ &= g(i, x) - g(i, y). \end{aligned}$$

Now consider $i, j \in N$ and $x, y, z, v \in S$, such that $y \in [x]_i$, $z \in [y]_j$, $v \in [z]_i$ and $x \in [v]_j$. From the identity:

$$\frac{P_\delta(x)}{P_\delta(y)} \frac{P_\delta(y)}{P_\delta(z)} \frac{P_\delta(z)}{P_\delta(v)} \frac{P_\delta(v)}{P_\delta(x)} = 1,$$

we derive that:

$$(g(i, x) - g(i, y)) + (g(j, y) - g(j, z)) + (g(i, z) - g(i, v)) + (g(j, v) - g(j, x)) = 0.$$

This implies that the game $(N, S, \{g(i, .)\}_{i \in N})$ is a potential game (see Monderer and Shapley [1996], corollary 2.9). Therefore, there exists a function $V : S \to \mathbb{R}$ such that, for all $i \in N$, $x \in S$ and $y \in [x]_i$:

$$g(i, x) - g(i, y) = V(x) - V(y).$$ □

Corollary 1. *A game G has a detailed balanced and utility monotonic perturbation with utility monotonic function h, given by:*

$$\begin{array}{lll} h(x, y) = u_i(x) - u_i(y) & \leftrightarrow & G \text{ is a potential game} \\ h(x, y) = w_i(u_i(x) - u_i(y)) & \leftrightarrow & G \text{ is a weighted potential game} \\ u_i(x) - u_i(y) \geq 0 \leftrightarrow h(x, y) \geq 0 & \leftrightarrow & G \text{ is an ordinal potential game} \end{array}$$

Proof. Follows directly from Lemma 4. □

Corollary 2. *If G is an ordinal potential game, then, for an ordinal potential function V of G, any detailed balanced and utility monotonic perturbation $\Sigma_G(\delta) = \{p_\delta(x, y)\}_{x, y \in S}$ satisfies:*

$$\frac{p_\delta(y, x)}{p_\delta(x, y)} = e^{\delta(V(x) - V(y))}.$$

Furthermore, the stochastic stable states are those states which maximize the ordinal potential function V.

Proof. Follows from the proof of Lemma 4 and the definition of stochastic stability. □

This last result shows that for ordinal potential games, there is a one to one correspondence between the set of potential functions and the set of detailed balanced and utility monotonic perturbations.

If for all $i \in N$, $x \in S$ and $y \in [x]_i$: $h(x, y) = u_i(x) - u_i(y)$, the perturbation with utility monotonicity function h is called the log linear perturbation. This perturbation is detailed balanced if and only if the game is a potential game. This result, stated in Corollary 1, has also been noticed by various authors Baron et al. [2003], Blume [2003]) and has fruitfully been applied to many models, especially two by two symmetric population games. These games are always potential games. The state which maximizes the potential function, and hence is stochastic stable, is the risk dominant strategy. This result is analogous to the result that has been obtained by using the uniform mutation model (Kandori et al. [1993] and Young [1993]). This has given the widespread view that the risk dominant strategies quite robust against changes in the form of mutation rates. The next section however shows that this result is not as robust as may seem: there are utility monotonic and detailed balanced perturbations which does not select the risk dominant strategy.

4 Stochastic stable states and ordinal potential games

Previous section shows that ordinal potential games are a very interesting class of games when considering utility monotonic perturbations. Namely, they are the only class for which some of these perturbations also satisfy the detailed balance conditions. Furthermore, the value of the potential function determines which states are stochastic stable and which are not. However, a potential of an ordinal potential game is not unique. It is therefore of interest to investigate which stochastic stable states are robust against variation of the potential function, or equivalent, variation of the perturbation. This section establishes a rather disappointing result (which proof can be found in the appendix):

Theorem 1. *If G is an ordinal potential game, and y is a stable state of M_G, then there exists a detailed balanced and utility monotonic perturbation of G such that y is a stochastic stable state of this perturbation.*

Theorem 2. *If G is an ordinal potential game, and $\{P(x)\}_{x \in S}$ is an invariant distribution of M_G, then there exist a detailed balanced and utility monotonic perturbation of G such that the stochastic stable invariant distribution of this perturbation is equal to $\{P(x)\}_{x \in S}$.*

5 Conclusions

The introduction of mutation yielded evolutionary models considerable predictive power. Bergin and Lipman [1996], however showed that this predictive power crucially depends on the nature of the mutation process. Blume [2003], therefore, draws on state dependent mutation rates and assumes the mutation process to exhibit utility monotonicity.

For a special class of games, i.c. ordinal potential games, we check whether the utility monotonicity property indeed allows us to refine the equilibrium selection. First, we derive a link between the nature of the mutation process, the detailed balance property and the nature of the game: we show that a game has a detailed balanced and utility monotonic perturbation if and only if it is an ordinal potential game. Then, we show that for all ordinal potential games and all stable states of a deterministic evolutionary process, we can find a utility monotonic and detailed balanced perturbation such that the stochastic stable states coincide with those stable states. Therefore, for ordinal potential games, utility monotonicity of the mutation process is insufficient to overcome the Bergin and Lipman–critique.

We therefore argue that where the mutation induced solution concept proves very useful in solving the path dependence of dynamical processes, its effectiveness as an equilibrium refinement device is limited.

Appendix: Proof of theorem 1

The proof draws heavily on the theory of binary extensions. It is therefore of interest to give a short overview of this research field.

Consider a binary relation $R \subseteq S \times S$. The asymmetric part of R, denoted by $P(R)$ is defined by $(x, y) \in P(R)$ iff $(x, y) \in R$ and $(y, x) \notin R$. The symmetric part of R, denoted by $I(R)$ is given by $(x, y) \in I(R)$ iff $(x, y) \in R$ and $(y, x) \in R$. The transitive closure of R is denoted as in section 2 by $T(R)$.

A binary relation R is reflexive if for all $x \in S$: $(x, x) \in R$, it is transitive if for all $x, y, z \in S$: $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ and R is complete if for all $x, y \in R$: $(x, y) \in R$ or $(y, x) \in R$. A reflexive and transitive relation is called a quasi-ordering and a complete quasi-ordering is called an ordering.

For a binary relation R, the set of maximal elements of R, denoted by $M(R)$ is given by:

$$M(R) = \{x \in S \,|\text{there is no} y \in S : (y, x) \in P(R)\}.$$

The set of greatest elements of a binary relation R is denoted by $G(R)$ and is defined as:

$$G(R) = \{x \in S \,|\text{ for all } y \in S : (x, y) \in R\}.$$

A binary relation R' is said to be an extension of R if $R \subseteq R'$ and $P(R) \subseteq P(R')$. The following result is from Suzumura Suzumura [1976]:

Lemma 5. *A binary relation R has an ordering extension if and only if for all $x, y \in S$: if $(x, y) \in T(R)$ implies $(y, x) \notin P(R)$.*

Another result reads:

Lemma 6. *If R satisfies the conditions of Lemma 5, then every ordering extension of R is also an ordering extension of $T(R)$.*

Proof. Let R' be an extension of R. If $(x, y) \in T(R)$, then obviously from the requirement that R' is transitive: $(x, y) \in R'$. Now assume that on the contrary that $(x, y) \in P(T(R))$ and $(x, y) \notin P(R')$. From completeness of R', we have that $(y, x) \in R$. From $(x, y) \in T(R)$, we have that there is a sequence $x - 1, ..., x_n$ such that $x_1 = x$, $x_n = y$ and for all $t = 1, .., n - 1$: $(x_t, x_{t+1}) \in R$. There must also be a t such that $(x_t, x_{t+1}) \in P(R)$, otherwise we would have that $(y, x) \in T(R)$, in contradiction with $(x, y) \in P(T(R))$. From transitivity of R', we have that $(x_{t+1}, x_t) \in R'$ in contradiction with $(x_t, x_{t+1}) \in P(R)$ and the requirement that R' is an extension of R. □

Every quasi-ordering satisfies the requirement of Lemma 5, and has therefore an ordering extension. For a quasi-ordering R, let $\mathcal{E}(R)$ be the nonempty set of ordering extensions of R. The following result is due to Banerjee and Pattanaik Banerjee and Pattanaik [1996]:

Lemma 7. *The set of maximal elements of a quasiordering is equal to the union of the sets of greatest elements of its ordering extensions. Or formally; for a quasi-ordering R:*

$$M(R) = \bigcup_{R' \in \mathcal{E}(R)} G(R').$$

Proof. Now we can begin with the proof of Theorem 1: Consider an ordinal potential game $G = \left(N, S, \{u_i\}_{i \in N}\right)$. The better than relation B is defined by:

$$(x, y) \in B \text{ if and only if } \exists i \in N : y \in [x]_i \text{ and } u_i(x) \geq u_i(y).$$

The following result if from Voorneveld and Norde Voorneveld and Norde [1997]

Lemma 8. *A game, with finite strategy set, is an ordinal potential game if and only if for all $x, y \in S$: $(x, y) \in T(B)$ implies $(y, x) \notin P(B)$.*

From Lemma 5, we have that a game is an ordinal potential game if and only if the better than relation B has an ordering extension. Consider such a relation R and define a real valued function $V : S \to \mathbb{R}$ such that $(x, y) \in B$ if and only if $V(x) \geq V(y)$. Such a function can be found there the set of strategy profiles, S is finite. It is easy to see that the function V is a potential function of the game G. Furthermore, every potential function of the game G defines in a similar way an ordering extension of the relation B. The set of elements of S which maximizes the potentials of G is then equal to the set:

$$\bigcup_{R' \in \mathcal{E}(B)} G(R') = \bigcup_{R' \in \mathcal{E}(T(B))} G(R') = M(T(B)).$$

The first equality sign comes from Lemma 6 and the second equality sign comes from Lemma 7 and the fact that B is reflexive. The set of elements which maximizes a potential of G is thus equal to the maximal elements of the relation $T(B)$. To complete the proof, we only have to show that $M(T(B))$ is equal to the union of the stable sets of the Markov chain M_G.

Consider the relation R_{M_G}. From its definition we can derive that:

$$R_{M_G} \subseteq B; P(R_{M_G}) \subseteq P(B); T(R_{M_G}) \subseteq T(B).$$

Further if there exist an element $y \in S$ such that $(y, x) \in B$ then there is an element $z \in S$ such that $(z, x) \in R_{M_G}$. If in addition for no $z \in S$: $(z, x) \in P(R_{M_G})$, then $(y, x) \in I(B)$ and $(y, x) \in I(R_M)$.

It is easy to see that the union of the stable sets of G is equal to the set $M(T(R_{M_G}))$. If $x \in M(B)$ and there is an $y \in S$ such that $(y, x) \in P(T(R_{M_G}))$, we have that $(y, x) \in P(T(B))$, a contradiction with maximality of x. Now assume that $x \in M(T(R_{M_G}))$ and there is an $y \in S$ such that $(y, x) \in P(T(B))$. This implies that there is a sequence $x_1, ..., x_n$ such that $x_1 = y, x_n = x$ and for all $t = 1, ..., n-1$: $(x_t, x_{t+1}) \in B$ and for at least one t: $(x_t, x_{t+1}) \in P(R)$. Consider the element x_{n-1}. From maximality of x and $(x_{n-1}, x) \in R$, we must have that $(x_{n-1}, x) \in I(R_{M_G})$, hence x_{n-1} is also maximal. By iteration, we have that y is maximal. From this $(x, y) \in I(T(R_{M_G})) \subseteq I(T(B))$. This contradicts with $(y, x) \in P(T(B))$. $\square$

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