Welfare and Stability in Senior Matching Markets

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Abstract

We consider matching markets at senior level, where workers might be assigned to firms at an unstable matching- the status- quo- which might not be Pareto efficient. It might also be the case that none of the matchings Pareto superior to the status- quo is Core- stable. We propose two weakenings of Core- stability: status- quo stability and weakened stability, and the respective mechanisms which leads any status- quo to matchings meeting the stability requirements above mentioned. The fist one is inspired by the top trading cycle procedure, the other one belongs to the family of Branch and Bound algorithms. Last procedure find a core stable matching in many-to-one markets whenever it exists, dispensing on the assumption of substitutability.

1 Introduction

1.1 Motivation

Reports by Roth (1984) and Roth and Parenson (1999) lead to a non ambiguous conclusion: matching institutions should provide core stable outcomes. While in theoretical settings the normative appeal of the core yields from its

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characterization, the argument, here, is factual. Specifically, clearinghouses that produce core stable outcomes survive, others do not. In our view, the relevance of core stability for clearinghouse is tautological: a core stable outcome is robust to attempts of self- resignation by coalitions of agents. If it was not the case, groups of agents would have good reasons to oppose the outcome proposed by the central institution for their freedom to engage in economic activities. Thus, clearinghouses which design core stable outcomes makes them easier to enforce.

Nevertheless, inefficiencies might prevent decentralized labor markets from reaching core stability. Among others, the agenda of offers and acceptances may bias the assignment of agents; a worker might accept an offer by a firm and, once committed, receive the offer of a preferred firm she cannot accept anymore. One might also think about changes in the preferences of agents. The adoption of centralized mechanisms in matching markets at junior level allowed to tackle these inefficiencies.

Theses are not the only difficulties experienced by decentralized markets at senior level. The theoretical analysis is pioneered by Roth, Blum and Rothblum (1998) in the case of one-to-one markets. The authors define senior markets as those where some agents are matched to one another, and matchings are disrupted by changes in the population of agents. They show that a stable matching disrupted by the retirement of some workers or the creation of firms leads to a firm quasi-stable matching, namely it is such that only unmatched firms are involved in blocking pairs. Moreover, their upgraded version of the Deferred Acceptance (D.A.) Algorithm where firms make offers, originally introduced by Gale and Shapley (1962), always restabilizes such matchings. Cantala (2004) extends the result to many-to-one markets when firms have q- substitutable preferences and also consider the case where the disruption is due to the closure of positions and the entering of workers. There, the market reaches stability again if offers are emitted by workers. He observes two features that explain why instability might be persistent in the markets. In the case where workers do not have tenure, the market reaches stability again *only if* the disruption is the one studied in Blum et al. (1998) *and* firms make offers. Furthermore, the procedures above mentioned may not be successful anymore, in the respective cases of disruption, if the side of the market that makes the offers is reversed. Hence instability may last, as well as Pareto inefficiency.

The academic market in Mexico is an example of such markets. First, the universities are autonomous institutions, in particular their agenda of offers to senior professors is not coordinated. Thus matchings are likely to be unstable. Second, professors may hold a tenure. This protective status guarantees to senior workers a minimum level of welfare, not only by preventing them from unemployment, but also by guaranteeing that any switch of job will be for a preferred position. Third, even if universities are autonomous, it exists a council that cap them all, the Association National de Universidades i Instituciones de Educacion Superior (A.N.U.I.E.S.). The aim of the council is to reach an harmonious development of the institutions, homologizing of syllabus and academic grades ... . Hence, it exists an institution that might debate, adopt and implement the centralized procedures that we propose. We insist that an agreement has to be reached by universities. We take into account the status- quo matching previous to the negotiation by ensuring them a match at least as preferred as their present match. More generally, any situation where an administration wishes to reallocate a staff to departments at a Pareto superior assignation is an application we are dealing with.

Suppose that the set of matchings Pareto superior to the status- quo is non empty, is one of those matchings core stable? The answer is negative[1], reassigning all workers might not be compatible with fulfilling some blocking coalitions. Hence we are restricted to look for core consistent procedures, namely those which select a core stable matching whenever it exists.

1.2 On manipulability

Roth (1982) shows that there is no stable matching mechanism for which stating the true preferences is a dominant strategy for *all* agents. We believe, however, that clearing houses should not worry so much about the negative result. Dubins and Freedman (1981) and Roth (1982, 1984) consider markets where preferences are strict and shows that mechanisms which select the optimal stable matching for one side of the market is strategy-proof for this side of the market. Demange, Gales and Sotomayor (1986) establish a general result, when preferences might be not strict and, thus, the optimal stable matching above defined may not exist. Strategic questions for the other side of the market are analyzed in Roth (1982a, 1984b) and Gale and Sotomayor (1985). More recently and specifically about the D.A. algorithm, Ehlers (2004) considers that workers evaluate the probability to be matched to desirable firms. In this set- up, manipulating seems to be a

[1] See Example 2 in the Appendix.

very sophisticated behavior.

Our issue is also related to the literature on (one- sided) assignment when agents own property rights, which is comparable to our status- quo. While in these markets there is conflict between equal treatment of equals, Pareto optimality and strategy - proofness (Zhou 1990), it exists a large literature, following Shapley and Scarf (1974) and their “top trading cycle” procedure, that combines core stability and group- strategy proofness (Roth (1982), Ma (1994), Svenson (1999), Bird (1984), Moulin (1995), Abdulkadiroğly and Sönmez (1998, 1999) and Papaï (2000)[2]). Our first result, in contrast, is that no core- consistent procedure is strategy- proof, and manipulating might be straightforward.

1.3 Two core consistent solutions

We propose two weakenings of the core. Both intend to capture the idea developed earlier: the “less” agents oppose a matching, as formalized by blocking coalitions, the easier it is to enforce. First, status- quo stability. We guarantee to all agents an outcome at least as preferred as the status- quo. Thus, a blocking coalition that is not compatible with a re- assignation of all agents to matches at least as preferred as their status- quo is not a valid objection. Thus, a matching where all blocking coalitions are not valid, faces no legitimate opposition. In this sense it is stable as a status- quo, or status- quo stable. We define a procedure inspired by the family of “top trading cycle” mechanisms, which finds a status- quo stable matching. In particular, whenever a core stable matching exists, the procedure picks the core stable matching unanimously preferred by workers among all status- quo matchings Pareto superior to the status- quo. Notice that status- quo stability itself is not a Core consistent solution concept. Moreover, the procedure only applies to one-to-one markets. Our second approach, however, does not suffer such drawbacks.

Second, weakened stability. Consider again academic markets. Suppose that a centralizer has to choose between two matchings, both Pareto superior to the status- quo and none Pareto dominates the other. The first matching is such that a university with a micro position hires a micro specialists and blocks the matching with a micro professor. The second matching is such that a university with a macro position hires a micro specialists and

[2]Papaï (2000) does not assume property rights.

blocks the matching with a macro professor. We argue that the first blocking coalition is a weaker opposition than the second one. It is so because it is desirable, on an educational point of view, that a position be held by the adequate specialist. Thus, in some applications it makes sense to assume that blocking coalitions are comparable and that this comparison follows from a social objective: the more a blocking pair impacts the social welfare, the stronger objection it constitutes to a matching. In our example of academic market, we observe that only preferences of university should be taken into account. They are represented by cardinally measurable and comparable utility functions. Moreover we adopt an utilitarian approach[3]. Among all matchings Pareto superior to the status- quo, we choose the one with the weakest opposition. Specifically, for all such matchings, we sum all utility improvements for firms from all blocking coalitions, and pick the matching which entails the smallest such summation (See (1) 4.1). We believe that our formalization of the problem is consistent, and appealing in the example of academic market. We do not claim, however, that it is fully general.

How would perform a D.A. algorithms in our setting? First, the procedures, adapted in Roth, Blum and Rothblum (1998) and Cantala (2004) to senior markets, do not take into account welfare restrictions above mentioned, except individual rationality. Second it might cycle. One type of cycling is harmless: even if we consider a case where a status-quo matching exists, one can easily design an example where a D.A. algorithm would cycle. To solve the difficulty one might adopt the solution proposed by Roth and Van de Vate (1988), namely introduce loops detectors in the algorithm that detects them and launch a new sequence of offers until finding the one that leads to a stable matching. The solution has no bite whenever there is no such matching. Finally these procedures require firms to have substitutable preferences, which is not a weak restriction. We believe that keeping on sophisticating the D.A. procedures would make it loose their original appeal.

Instead, we make use of a much more versatile family of procedures : Branch and Bound Algorithms. Four of their properties motivate the choice: a) they do not require any restriction on the preferences of firms, b) by construction they do not cycle, c) They can compute all the possible solutions of the problem- which means, in the case of junior markets, that they might compute all the stable matchings, d) whenever the problem to solve has no solution, they specify it.

[3]It is an abuse of language.

We establish that the outcome matching of our Weakened Stability Algorithm is the solution to our problem and it is status- quo stable. Moreover, when the input matching is the empty one, it is core stable whenever a core stable matching exists, even if the preferences of firms are not substitutable.

2 Preliminaries

2.1 The market

A many-to-one matching market is a quadruple $(\mathcal{F}, \mathcal{W}, q, \succ)$ where $\mathcal{F}$ and $\mathcal{W}$ are two disjoint finite sets of agents. $\mathcal{F} = \{f_1, ..., f_m\}$ is the set of firms and $\mathcal{W} = \{w_1, ..., w_n\}$ is the set of workers; generic firms and workers will be denoted by f and w respectively. Subsets of $\mathcal{F}$ and $\mathcal{W}$ are denoted by F and W. The vector of quotas associated with each firm is $q = (q_f)_{f \in \mathcal{F}}$, where q_f is the maximum number of workers that can be assigned to firm f. Preference relations are not symmetrically defined between firms and workers since a firm can be assigned to many workers whereas a worker can be assigned to at most one firm. Each firm f has a strict, transitive and complete preference relation $\succ_f$ over the family of subsets of workers $2^{\mathcal{W}}$. We interpret the empty set as firm f not being assigned to any worker. When a firm ranks the empty set better than a subset, it means that it prefers remaining unmatched to being assigned to this subset. Each worker w has a strict, transitive and complete preference relation $\succ_w$ over the set $\mathcal{F} \cup \{\emptyset\}$. We interpret the empty set in $\succ_w$ as w being unemployed. Preference profiles are $(m+n)$-tuples of preference relations and they are represented by $\succ = (\succ_{f_1}, ..., \succ_{f_m}, \succ_{w_1}, ..., \succ_{w_n})$.

For any firm f we define the *acceptable set of f under q and $\succ$* to be the subsets of workers with cardinality smaller or equal to q_f, strictly preferred to the empty set; namely

$$A_f(q, \succ) \equiv \{S \subseteq \mathcal{W} \mid S \succ_f \emptyset \text{ and } |S| \leq q_f\}.$$

Subsets in $A_f(q, \succ)$ are called acceptable. Since only acceptable subsets will matter, we will represent the preferences of the firm as a list of acceptable subsets. Likewise, for any w we define the *acceptable set of w under $\succ$* to be the set of firms strictly preferred to $\emptyset$. We denote it by $A_w(\succ)$. Firms in $A_w(\succ)$ are called acceptable. We will represent the preferences of firms and workers by ordered lists of acceptable partners. A pair (w, f) is acceptable under q and $\succ$ if both agents are mutually acceptable. Let $A(\mathcal{F}, \mathcal{W}, q, \succ)$ be the set of workers-firm coalitions (W,f) such that $W \subseteq A_f(q,\succ)$ and for all $w \in W$, $f \in A_w(\succ)$.

Let $\succ$ be a preference profile. Given a set $W \subseteq \mathcal{W}$, let the *Choice* of firm f, denoted $Ch(W, q_f, \succ_f)$, be f 's *most preferred* subset of W with cardinality at most q_f according to its preference ordering $\succ_f$.

Definition 1 *A matching μ is a mapping from the set $\mathcal{F} \cup \mathcal{W}$ into the set of all subsets of $\mathcal{F} \cup \mathcal{W}$ such that for all $f \in \mathcal{F}$ and $w \in \mathcal{W}$:*

(1) $\mu(f) \in 2^{\mathcal{W}}$ *and* $|\mu(f)| \leq q_f$,

(2) *either* $|\mu(w)| = 1$ *and* $\mu(w) \in \mathcal{F}$, *or* $\mu(w) = \emptyset$,

(3) $\mu(w) = f$ *if and only if* $w \in \mu(f)$.

We denote $\mathcal{M}$ the space of all possible matchings.

2.2 Stability concepts

A matching μ is *blocked by a worker* w if she prefers remaining alone than being matched to $\mu(w)$; i.e., $\emptyset \succ_w \mu(w)$. Similarly, μ is *blocked by a firm* f if $\mu(f) \neq Ch(\mu(f), q_f, \succ_f)$. We say that a matching is *individually rational* if it is not blocked by any individual agent. A matching is *blocked by a worker-firm pair (w,f)* if worker w prefers being matched to f than to $\mu(w)$ and f would like to hire w; i.e., $f \succ_w \mu(w)$ and $w \in Ch(\mu(f) \cup \{w\}, q_f, \succ_f)$.

Definition 2 *A matching μ is pair-wise stable if it is not blocked by any individual agent or any worker-firm pair.*

We denote by $PS(\mathcal{F}, \mathcal{W}, q, \succ)$ the set of pair-wise stable matchings of market $(\mathcal{F}, \mathcal{W}, q, \succ)$.

Let W be a subset of $\mathcal{W}$. A matching μ is *blocked by a workers-firm coalition (W,f)* if all workers w in W prefer being matched to f than to $\mu(w)$ and f would like to hire W; formally if for all $w \in W$, $f \succ_w \mu(w)$ and $W \subseteq Ch(\mu(f) \cup W, q_f, \succ_f)$. We say that (W,f) forms a blocking coalition of μ. Let $\mathcal{W}_{f,\mu}$ be the set of workers who prefer f to their match under μ and, thus, they are potential members of blocking coalitions of μ. Formally, $\mathcal{W}_{f,\mu} = \{w \in \mathcal{W} \mid f \succ_w \mu(w)\}$.

Definition 3 *A matching μ is group-stable if it is not blocked by any individual agent or by any workers-firm coalition.*

We denote by $GS(\mathcal{F}, \mathcal{W}, q, \succ)$ the set of group-stable matchings of market $(\mathcal{F}, \mathcal{W}, q, \succ)$. Obviously, if a group- stable matching is also pair-wise stable, moreover core stability defined by weak dominance and group stability coincide in such markets.[4]

3 Strategy proofness

We aim to design a core consistent procedure which assigns to all agents in the market a match at least as preferred as their status- quo, and Pareto undominated. Unfortunately, none of them is strategy- proof.

Definition 4 *A mechanism is strategy proof if it is a dominant strategy, for all agents, to report their true preferences.*

We now state the negative result.

Theorem 1 *In senior matching markets, there is no core- consistent and strategy- proof mechanism that chooses a matching Pareto undominated and which guarantees to all agents a match at least as preferred as the status-quo.*

Example 1 shows that any core- consistent procedure is manipulable.

Example 1 Consider the market $(\mathcal{F},\ \mathcal{W},\ q,\ P)$ where $\mathcal{F} = \{f_1,\ f_2,\ f_3\}$, $q_{f_1} = q_{f_2} = q_{f_3} = 1$, $\mathcal{W} = \{w_1,\ w_2,\ w_3\}$ and true preferences are

$\succ_{f_1}$	$\succ_{f_2}$	$\succ_{f_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_3	w_3	w_1	f_3	f_1	f_2
w_2	w_2	w_3	f_1	f_2	f_1
w_1					f_3

Suppose that the status- quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

There are two matchings Pareto- superior to the status- quo:

$$\mu^1 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_2 & w_3 & w_1 \end{pmatrix} \text{ and } \mu^2 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3 & w_2 & w_1 \end{pmatrix}.$$

[4]See Roth (1984).

Notice that μ^1 is stable while μ^2 is blocked by (f_2,w_3), thus a core- consistent procedure should pick μ^1. Nevertheless, if f_1 reports $\succ'_{f_1}$ where it prefers w_3 to w_1 and w_2 is not acceptable, the only matching Pareto superior to the status- quo is μ^2, which has to be selected, even if it is not core stable. Thus, in this market, firm 1 would gain by misrepresenting its preferences through $\succ'_{f_1}$ since, manipulating, it is matched to its favorite worker.

4 Status- quo stability

We guaranty the status- quo for all agents. Thus, to be considered as a valid objection to a matching, blocking coalitions have to be compatible with a reassignment that make all agents at least as well off as at the status- quo. In this sense, in Example 1, the blocking pair (f_2,w_3) is not valid when firm 1 reports $\succ'_{f_1}$ since, if f_2 and w_3 are matched, w_2 cannot be reassigned to a firm preferred to her status- quo, f_2.

Definition 5 *Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$, a matching μ is status-quo stable if for all blocking coalitions $(f, W) \subseteq \mathcal{F} \times 2^{\mathcal{W}}$ to μ, no matching where f and W are assigned to each other, possibly with other workers, is Pareto superior to μ.*

In this definition of stability, there is no conflict between blocking coalitions and Pareto optimality. Hence, given a status- quo μ_o, looking for matchings status- quo stable and Pareto superior to μ_o is equivalent to look for the set of matchings Pareto superior μ_o which is not Pareto dominated by another matching. Denote the set $SQS(\mu_o)$, by transitivity of preferences it is not empty whenever the there is at least one matching Pareto superior to the status- quo. Example 2 shows that picking any matching in $SQS(\mu_o)$ is not a core consistent procedure.

4.1 Core consistency

Next example shows that in $SQS(\mu_o)$, some matchings can be core stable and others not.

Example 2 Consider the market ($\mathcal{F}$, $\mathcal{W}$, q, $\succ$) where $\mathcal{F} = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9\}$, $q_{f_1} = q_{f_2} = q_{f_3} = 1$, $\mathcal{W} = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9\}$

and $\succ$ is given by the following profile

$$\begin{array}{ccccccccc} \succ_{f_1} & \succ_{f_2} & \succ_{f_3} & \succ_{f_4} & \succ_{f_5} & \succ_{f_6} & \succ_{f_7} & \succ_{f_8} & \succ_{f_9} \\ \hline w_1 & w_9 & w_4 & w_3 & w_5 & w_6 & w_7 & w_8 & w_2 \\ w_3 & w_2 & w_1 & & w_1 & w_5 & w_8 & w_9 & \\ w_2 & w_3 & & & w_6 & & & w_7 & \end{array}$$

$$\begin{array}{ccccccccc} \succ_{w_1} & \succ_{w_2} & \succ_{w_3} & \succ_{w_4} & \succ_{w_5} & \succ_{w_6} & \succ_{w_7} & \succ_{w_8} & \succ_{w_9} \\ \hline f_5 & f_2 & f_1 & f_3 & f_5 & f_6 & f_7 & f_8 & f_8 \\ f_1 & f_9 & f_4 & & f_6 & f_5 & f_8 & f_7 & f_2 \\ f_3 & f_1 & f_2 & & & & & & \end{array}$$

Suppose that the status- quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & \emptyset & \emptyset \\ w_2 & w_3 & w_1 & \emptyset & w_6 & w_5 & w_8 & w_7 & \emptyset & w_4 & w_9 \end{pmatrix}.$$

The two following matchings belong to $SQS(\mu_o)$:

$$\mu^1 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & \emptyset & \emptyset \\ w_3 & w_2 & w_1 & \emptyset & w_5 & w_6 & w_7 & w_8 & \emptyset & w_4 & w_9 \end{pmatrix},$$

$$\mu^2 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 \\ w_1 & w_9 & w_4 & w_3 & w_5 & w_6 & w_7 & w_8 & w_2 \end{pmatrix},$$

where μ^1 is blocked by (f_1, w_1) and μ^2 is core stable.

Thus, our aim is not only to reach a matching in $SQS(\mu_o)$ but, whenever it exists, to select a core stable one. The Status- Quo stability procedure performs the task for one-to-one markets.

4.2 The Status- Quo Stability (SQS) procedure

The SQS- procedure begins by a graph representation of our problem.

1- Each node represents a match as defined by the status- quo μ_o; if $\mu_o(f) = w$, (f, w) is assigned a node, if $\mu_o(f) = \emptyset$, f is assigned a node and if $\mu_o(w) = \emptyset$, w is assigned a node.

2- From each node with a worker w, draw all arrows towards[5] firms f such that both w and f prefer each other to their respective status- quo.

[5]thus, it is a directed graph.

3- Identify all *cycles* and *paths* defined as follows.

A *cycle* is an ordered set S of pairs (f, w) which appear only once in S, where, in the graph constructed as mentioned in 1 and 2:
a. from each node (f, w) in S an arrow points another node in S,
b. (f, w) is pointed by an arrow from another node in S, moreover
c. (f', w') follows (f, w) in S only if w points f' in the graph, finally the first pair in S is said to follow the last one.

A *path* is an ordered set S with one and only one single worker w, one and only one firm f and possibly pairs (f', w'), they all appear only once in S and, in the graph constructed as mentioned in 1 and 2,
a. the node with the single worker w points another node in S and is the first element in the set,
b. for each node (f', w') in S there is one arrow that points another node in S and (f', w') is pointed by an arrow from another node in S,
c. the node with the single firm f is pointed by another node in S and is the last element in the set,
d. $[(f', w')$ or $f']$ follows $[(f, w)$ or $w]$ in S only of $[(f, w)$ or $w]$ points $[(f', w')$ or $f']$ in the graph.

Let $\mathcal{P}$ be the set of all pathes and cycles and denote p an element in $\mathcal{P}$. We are now ready to construct all possible Pareto improvements that may lead the market to status- quo stability, and select one of them.

4- A *composition* c $\mathcal{P}$ is a subset of $\mathcal{P}$ such that:
a. for all p, $p' \in c$, $p \cap p' = \emptyset$ and
b. for all $p" \in \mathcal{P}$ which does not belong to c, there is at least one $p \in c$ and $p" \cap p \neq \emptyset$.

Let $\mathcal{C}$ be the set of all compositions. We say that a worker w prefers composition c to composition c' if she prefers the firms which follows her in c to the one in c'.

5- Given a status- quo μ_o and a composition c in $\mathcal{C}$, the induced matching $\mu(\mu_o, c)$ is such that:
a. if a firm f' is involved in the composition c, it is assigned the worker w of the previous element in c;
b. else it is assigned the same match as at μ_o.
Let $I(\mu_o, \mathcal{C})$ the set of induced matching by all compositions in $\mathcal{C}$.

5.1- If $I(\mu_o, \mathcal{C}) = \{\emptyset\}$ then $SQ - S(\mu_o) := \mu_o$,
Else let $i := 1$,
5.2- If $I(\mu_o, \mathcal{C}) = \{\emptyset\}$, $SQ - S(\mu_o) := \mu^1$ as defined below.
Else pick a worker and let her choose within $I(\mu_o, \mathcal{C})$ her favorite matching in $I(\mu_o, \mathcal{C})$; if she is indifferent between different matchings, pick a second worker to break ties and so on and so forth until a single matching μ^i is selected.
5.3- Let all firms f make offers to workers preferred to their match $\mu^i(f)$.
5.4- If no offer is accepted, $SQ - S(\mu_o) := \mu^i$,
Else $I(\mu_o, \mathcal{C}) := I(\mu_o, \mathcal{C}) \diagdown \mu^i$, $i := i + 1$; go to 5.2.

Proposition 1 states that our SQ- S procedure finds a status- stable matching and it is a core consistent procedure.

Theorem 2 *Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$, $q_f = 1$ for all $f \in \mathcal{F}$ and a status quo μ_o then*

1- *$SQ - S(\mu_o)$ is status-quo stable and Pareto superior to μ_o,*
2- *whenever the set of core-stable matchings Pareto superior to μ_o is non-empty $SQ - S(\mu_o)$ is the core stable matching unanimously preferred by workers (and worst for firms).*

Proof of Theorem 2.
We observe that only arrows representing blocking pairs are drawn on the graph (step 2) since others cannot lead to a Pareto improvement. So as such a blocking pair to be completed and the market reach a Pareto improvement, dropped mates (if any) will also have to be assigned a blocking mate (by definition preferred to the status- quo). Thus, one needs to identify all the ordered sets of blocking pairs, with the interpretation that $[(f', w')$ or f' follows $[(f, w)$ or $w]$ if (w, f') is the blocking pair involving w and f' to be completed[6], such that:
a- if completed simultaneously, the market experiences a Pareto improvement and
b- if one or some of them is withdrawn from the set, there is no such Pareto improvement.
Obviously Cycles and Pathes are such sets; we show that they are the only ones. It is also clear that no blocking pair can appear twice in the sets since one cannot complete two blocking pairs simultaneously. We adopt the

[6] Both w and f' might be involved in blocking pairs with other agents in the set.

convention that a set begins by a node with a worker (and possibly a firm) pointing toward another node (if there is no "pointing" in the set, neither there are blocking pairs). Since blocking is simultaneous, the order only matters to keep track of who blocks with who. Thus, if there is an unmatched worker in the set, there is no loss of generality in shifting all elements, ranking this unmatched worker first and following the original ordering; that is why if there is an unmatched worker in the set, we put it first in the set.

Case 1 The set starts with an unmatched worker w.

If this worker blocks with an unmatched firm f, $\{w, f\}$ is the Pareto improving set as defined above, it is a path.

If this worker blocks with a matched firm f, the mate of f, w', will have to be assigned a firm f' in the set preferred to the status- quo. If this firm is unmatched, the set is $\{w, (f, w'), f'\}$, it is a path. Else a pair (f', w'') has to follow (f, w') so as to assign w' a firm preferred to her status quo. One can reiterate the argument, until an unmatched firm appear in the sequence. If such unmatched firms did not exist, the blocking pairs specified by the ordered set would not be Pareto improving for the worker of the last pair, who would remain unmatched. Thus, the set is a path in any case. If there is more than one unmatched worker in the set, by previous argument they would generate independent pathes since no pair can appear twice. Hence one of the pathes might be withdrawn from the set without altering the Pareto improvement of agents in the other set.

Case 2 The set starts with a pair (f, w).

By our convention, there is no unmatched worker in the set. So as to compensate f from the fact that w blocks with another firm f', the last element of the set in the sequence has to be a couples (f^n, w^n) where w^n blocks with f. We observe that no unmatched firm can be included in the set, since the firm will not point any other agent, in particular couples, as required. Thus, the set is a cycle.

Hence, $\mathcal{P}$ contains all sets of blocking pairs such that, if they are completed simultaneously, all agents involved in the set will improve with respect to the status- quo. Of course it might be that unmatched firms or workers, or matched worker- firm pairs are involved in many pathes and cycles and, nevertheless one cannot complete simultaneously many blocking pairs. A composition of $\mathcal{P}$ (Step 4) is a set of compatible cycles and pathes such that no other element in $\mathcal{P}$ is compatible with them.

We argue now that there is no matching Pareto superior to the one generated from a composition since the algorithm stops:

· either at step 5.4 when a matching is stable (in which case there is no matching Pareto superior to it, else some agents would block);
· or at 5.1, when μ_1 is selected. Consider step (5.2) that lead to the selection of μ_1. If only one worker is necessary to select μ_1, it means that this worker strictly prefers μ_1 to any other matching in $I(\mu_o, \mathcal{C})$. If many workers are necessary to pick μ_1, notice that each time a matching in $I(\mu_o, \mathcal{C})$ is discarded by a worker, the discarded matching is strictly worst than μ_1 for this worker. Thus, μ_1 is not Pareto dominated by any matching in $I(\mu_o, \mathcal{C})$.
We prove now that the procedure picks the workers optimal stable matching whenever it exists. We know from the lattice Lemma (Knuth 1976) that in one-to-one markets, if two stable matchings are not comparable for workers, by letting them choose their best mate between both matchings, not only the picking function leads to a matching but a stable one. Of course, if the two matchings are Pareto superior to the status- quo, so is the new matching. Thus, if there exist stable matchings Pareto superior to the status- quo, one of them is unanimously preferred by workers. That is why we let workers choose their favorite matchings in $I(\mu_o, \mathcal{C})$ and check if the chosen one is stable, i.e., if no offer emitted by firms is accepted by any worker, this is the outcome matching. Else another matching is chosen by new workers until a stable matching is found. If the all set of status- quo matching has been scrutinized and none of the matching is stable, the outcome matching is the first tentative matching.■

4.3 Comments

Note that, if none of the status- quo stable matchings is core stable, workers might not agreement on a ranking of matching in $SQS(\mu_o)$, thus the order in which they are picked in the procedure might affect the output matching.

Suppose now that the market is disrupted by changes in the population of agents. Then, a preferred status- quo does not insure a preferred outcome of the $SQ-S$ procedure. Indeed, if one is the best alternative for her/its match, she/it will not let one switch to another position. That is, the status- quo gives power to both matched agents. thus, the advantage of being guaranteed a minimum welfare might be balanced by the fact that switching to a better position is conditioned by the simultaneous improvement of the match.

This simultaneous improvement requires a central intervention since, unlike in "top trading cycle procedures", agents might belong to two different Pathes or Cycles, hence compatible reassignments will not occur without coordination. Moreover stage 5 is necessary for the SQ-S procedure to be core consistent. This suggests that dealing with the problem requires a central institution. Indeed weakened stability and the related procedure are centralized in nature.

Finally, the status- quo stability procedure suffers two main drawbacks: first, like any "top trading cycle" procedures, it is not operative in large markets, second, it is not adaptable to many- to- one markets when firms have preferences which are not responsive. We will argue that the Weakened Stability algorithm does not suffer such inefficiencies.

5 Branch and Bound Algorithms and weakened stability

5.1 The optimization problem

We assume that preferences of firms are represented by cardinally measurable and comparable utility functions, generically denoted u_f for firm f. Moreover we choose the reversed order representation: the lower the utility, the better; and the best subset of worker is assigned utility 0.

The more a blocking coalition improves the welfare of a firm, the stronger objection it constitutes to a matching. We follow an "utilitarian" approach and, for all matchings, we sum the utility improvement for firms from all blocking coalitions.

Definition 6 *Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$; for a matching μ, let*

$$i \equiv \sum_{\text{All blocking coalition } (S,f) \text{ of } \mu.} u_f(\mu(f)) - u_f(S),$$

then μ is said to be <u>*weakened stable of order i*</u>*.*[7]

Notice that a matching weakened stable of order 0 is core stable. Denote by $WS_i(\mu_o)$[8] the set of matchings that are weakened stable of order i for matching μ_o. We now define the utilitarian social welfare function $W(\mu) =$

[7] The order of stability depends on the utility representation chosen, which is no problem for our purpose.

[8] We let the reference to the market implicit so as to save notations.

$\sum_{f\in\mathcal{F}} u_f(\mu(f))$ that we aim to minimize, choosing a matching within the set of weakened stable matchings of the lowest order.

Formally, given a status- quo μ_o, our problem is

$$\begin{array}{ll} \min_{\mu \text{ is Pareto superior to } \mu_0} W(\mu) & \\ \text{s.t.} & \\ & \mu \in WS_i(\mu_o) \text{ and} \\ & WS_j(\mu_o) = \emptyset \text{ if } j < i. \end{array} \tag{1}$$

Hence, a matching μ is selected instead of another matching μ' if its order of weakened stability is lower or, in case of a tie, $W(\mu) < W(\mu')$. In other words, if one considers two matchings μ and μ', μ is preferred to μ' in the following cases: a- whenever the order of stability of μ is lower than the one of μ', b- whenever the order of stability of μ or μ' are the same but $W(\mu) < W(\mu')$; otherwise μ and μ' are indifferent. Notice that the status-quo is the solution to the program when it is not Pareto dominated. The following algorithm find this (these) optimal matching(s).

5.2 The Weakened Stability (W.S.) algorithm

Denote $WSP(\mu_o)$ the set of matchings produced by the algorithm when the input matching is μ_o. For all firm $f \in \mathcal{F}$, let $B_f(\mu_o) = \{W \subseteq 2^{\mathcal{W}} | W \succeq_f \mu_o(f)\}$ be the set of subsets of workers f prefers to its status- quo, and for all worker $w \in \mathcal{W}$, let $B_w(\mu_o) = \{f \in \mathcal{W} | f \succeq_w \mu_o(w)\}$ be the set of firms w prefers to her status- quo. Let $A = \times_{f\in\mathcal{F}}(B_f(\mu_o) \cup \{\emptyset\})$, where for all elements in A, the subset of worker in the f^{th} entry is interpreted as being assigned to firm f. Notice that A contains all matchings Pareto superior to μ_o, that is why we will restrict our attention to assignations in A. We also observe that some of the matchings in A may not be Pareto superior to μ_o since preferences of workers are not taken into account in A. Finally, some assignations in A may even not be matchings since, for instance, a worker might be assigned to many firms.

The W.S. algorithm belongs to the family of Branch and Bound (B.B.) algorithms. This technique is one of the most commonly used in optimization problems[9] when all or some of the decision variables are discrete (integer or mixed programing) and no characterization of optima exists; namely unlike

[9] Branch and Bound algorithms are used to solve, for instence, the classical assignment problem in operation research.

first and second order conditions in differential calculus environments. As a consequence, the all set of decision variables, A in our case, has to be scrutinized.

In our problem, there are as many decision variables as firms in the market, hence, the number of solutions can be very large: we call solution any matching, a matching that solves (1) is an *optimal* solution. The efficiency of B.B. algorithms relies on the fact that, instead of analyzing a particular solution at a time, they discard sets of solutions. We denote $R \equiv (W_1, \dots, W_n, \emptyset, \dots, \emptyset)$, $R \subseteq A$, the set of solutions where the subset of workers W_f is assigned firm f for $f = 1, \dots, n$, and there is not specific subset assigned to firms $f = n+1, \dots, F$.

The stack, S, is the set of solutions that the algorithm still has to scrutinize. At each iteration, the algorithm picks a set of solution $R \equiv (W_1, \dots, W_n, \emptyset, \dots, \emptyset)$ in S, deletes it from the stack ($S := S \setminus R$), and perform the following tests:

a) When the tentative optimal solution[10] is core stable, is the objective function of the tentative solution smaller than the upper bound of R?

b) Are all unassigned firms worst than the status- quo for some of the unassigned workers?

c) Can one assign to each of the unassigned firms in R a group of workers preferred to the status- quo?

If the answer to at least one question is positive, the optimal solution cannot belong to R, another set of solution in the stack is considered. Else, one cannot discard solutions in $R = (W_1, ..., W_n, \emptyset, ..., \emptyset)$, we break off R in subfamilies of the form $R' = (W_1, \dots, W_n, W_{n+1}, \emptyset, \dots, \emptyset)$. There are as many subfamilies as subsets of workers unassigned in R preferred to the status- quo by firm $n+1$. Thus, for each W_{n+1} in B_{n+1} and $\mathcal{W} \setminus \cup_{f=1}^{n} W_f$ Pareto superior to the status- quo, a subfamily of solutions $R' = (W_1, \dots, W_{n+1}, \emptyset, \dots, \emptyset)$ has to be inspected. These solutions are included in the stack, i.e., $S := S \cup \{R'\}$ for all such R'.

To formalize the algorithm, for all $R = (W_1, \dots, W_n, \emptyset, \dots, \emptyset)$ we define $Z_L(R)$, the upper bound of the objective function of problem (1)[11] reached by solutions in R; formally

[10] The tentative optimal solution is the solution which is optimal within the set of solutions already scrutinized.

[11] $Z_L(R) \geq \min_{\mu \in R} W(\mu)$

$$Z_L(R) = \sum_{f=1}^{n} u_f(W_f) + \sum_{f=n+1}^{F} \min\{u_f(W_f) | W_f \in B_f(\mu_o), W_f \subseteq \mathcal{W} \backslash \cup_{f=1}^{n} W_f\}.$$

Thus, $Z_L(R)$ is the minimal value reached by the objective function when all firms $n+1, \ldots, F$ are assigned their favorite subset of workers among those not assigned at R. We call $\overline{R}$ the assignment in R for which the value of the objective function is $Z_L(R)$. It might be that $\overline{R}$ is neither a matching nor stable, in any case if this lower bound does not improve upon the tentative optimal solution when the last one is stable of order 0, no matching in R will be optimal, therefore solutions in R are discarded[12].

We keep the record of the following information: in $WSP(\mu_t)$ the best current solution in the process, in i_t its order of weakened stability and in Z_U the value of its objective function. Notice that $WSP(\mu_t)$, i_t and Z_U are ordered sets where the first entry corresponds to the first tentative solution and the last entry corresponds to the tentative current solution.

We now describe the algorithm in detail, given a market $(\mathcal{F}, \mathcal{W}, q, \succ)$ and a status- quo μ_o.

1. Initial Round

   - For all $f \in \mathcal{F}$ define the function $B_f : \mathcal{M} \to 2^{2^{\mathcal{W}}}$ such that $B_f(\mu) = \{S \subseteq 2^{\mathcal{W}} | \ \#S \leq q_f \text{ and } S \succsim_f \mu(f)\}$. [Define the subsets of workers preferred by firms to a matching μ.]
   - For all $w \in \mathcal{W}$ define the function $B_w : \mathcal{M} \to 2^{\mathcal{F} \cup \{\emptyset\}}$ such that $B_w(\mu) = \{m \in \mathcal{F} \cup \{\emptyset\} | m \succsim_w \mu(w)\}$. [Define the set of firms preferred by workers to a matching μ.]
   - For all $f \in \mathcal{F}$ define the function $W_f : 2^{2^{\mathcal{W}}} \times 2^{\mathcal{F} \cup \{\emptyset\}} \to 2^{2^{\mathcal{W}}}$ such that
     $W_f(\mu) = \{W \in \mathcal{W} \mid W \subseteq B_f(\mu) \text{ and for all } w \in W,\ f \in B_w(\mu)\}$. [Define the set of subsets of workers who block μ with f.]
   - Define the function $i_0 : (2^{2^{\mathcal{W}}})^{\#\mathcal{F}} \to \Re$ such that
     $i(\mu) = \sum_{f \in F} \sum_{W \in W_f(\mu)} u_f(\mu(f)) - u_f(W)$ [$i(\mu)$ is the order of stability of matching μ.]

[12] The use of the tentative optimal objective values motivates the term Bound in Branch and Bound Algorithm.

- For all $R \subset A$, define the function $Z_L : A \to \Re$ such that $Z_L(R) = \sum_{f=1}^{n} u_f(W_f) + \sum_{f=n+1}^{F} \min\{u_f(W_f) | W_f \in B_f(\mu_o), W_f \subseteq \mathcal{W} \backslash \cup_{f=1}^{n} W_f\}$.
- $WSP(\mu_t) = \mu_o$. [The initial tentative optimal solution is the status- quo.]
- $Z_U = Z_L(\mu_o)$. [The objective value of the initial tentative solution is the one of the status- quo.]
- $i_0 := i(\mu_o)$ [i_o is the order of stability of the status- quo.]
- $S = \{(\emptyset, ..., \emptyset)\}$. [At the beginning, we have to review all possible solutions.]
- $t \equiv 1$.

Iteration

2. Selection within the stack S, of a solution.

   If $S = \emptyset$ then stop. [If the stack is empty, there are no more subsets to analyze and the tentative optimal solution is the solution to (1).] Otherwise, let R be such that $R = \arg\min_{R' \in S} Z_L(R')$, $S := S \backslash \{R\}$. [We select the family of solutions with minimal lower bound.][13]

3. Fathoms. One discards R or checks whether the optimal solution may belong to R.

   3.1 If $i_{t-1} = 0$ and $Z_U < Z_L(R)$ then go to 2. [If the tentative optimal solution is core stable and its objective function is smaller than the lower bound of R, solutions in R are discarded.]

   3.2 If $\{f_{n+1}, ..., f_F, \{\emptyset\}\} \cap B_w(\mu_o) = \emptyset$ for (at least) one $w \in \mathcal{W} \backslash \cup_{f=1}^{n} W_f$, then go to 2. [If all unassigned firms are worst than the status-quo, the solution cannot belong to R for (at least) one unassigned worker, R is discarded.]

[13] So as the algorithm to be more efficient, one would idealy choose the family of solution with lower bound of stability. Nevertheless this lower bound is not computable, that is why we use as lower bound the value of the objective function.

3.3 If for some firm $f \in \{f_{n+1}, ..., f_F\}$ no $W_f \in B_f(\mu_o)$ is such that $W_f \subseteq \{\mathcal{W} \cup \{\emptyset\}\} \setminus \cup_{f=1}^{n} W_f$, then go to 2. [If one cannot assign a group of workers preferred to the status- quo to each of the unassigned firms, R is discarded.]

3.4 If $n + 1 < F$ go to 4 [If more than one firm is not assigned any subset of workers, the solution is portioned in subsets of solutions ...].

Else for $f = F$ define $W_F = \{W \subseteq \mathcal{W} \setminus \cup_{f=1}^{F-1} W_f$ such that [... else subsets in W_F are the only ones which complete R to form a matching Pareto superior to the status- quo ...]

a- $W \in B_F(\mu_o)$,

b-$F \in B_w(\mu_o)$ for all $w \in W$

c- if $\mu_o(w) \neq \emptyset$ for $w \in \mathcal{W} \setminus \cup_{f=1}^{F-1} W_f$, then $w \in W_F\}$. [... in particular matched workers at the status quo have to be included.]

3.4.1 If $W_F = \emptyset$, go to 2.

Else, let $N \equiv \# W_F$ and $l \equiv 1$.

3.4.2.1 If $l \leq N$, select one $W \in W_F$, delete it from W_F and construct $R' = (W_1, \ldots , W)$.

Else $t = t + 1$, go to 2. [One completes R assigning F to an acceptable subset of workers, including a fortiori those who are matched at the status- quo.]

3.4.2.2 If $i(R') < i_{t-1}$ or ($i(R') = i_{t-1}$ and $Z_L(R') \leq Z_U$) then $i_t = i(R')$, $WSP(\mu_t) = R'$, $Z_U = Z_L(R')$. [A new tentative solution has been detected.]

In any case $l = l + 1$, go to 3.4.2.1..

4. Branching: in case we cannot discard R, we break it off in smaller subsets. Notice that only Pareto superior matchings are included in the stack.

   $S := S \cup \{(W_1, \ldots , W_{n+1}, \emptyset, \ldots , \emptyset) \subseteq A$ such that

   a- $(W_1, \ldots , W_n, \emptyset, \ldots , \emptyset) = R$, [New solutions in S are subfamilies of R ...]

b- $W_{n+1} \subseteq \mathcal{W} \backslash \cup_{f=1}^{n} W_f$, [... obtained by complementing R with subsets of available workers ...]

c- $W_{n+1} \in B_{f_{n+1}}(\mu_o)$, $f_{n+1} \in B_w(\mu_o)$ for all $w \in W_{n+1}\}$. [... compatible with the Pareto criterion.]

Then go to 2.

We are not ready to state our main result.

Theorem 3 *Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$ and a status quo μ_o then $WSP(\mu_t)$ is a solution to (1).*

Proof of Theorem 3.

We observe that the algorithm is well- behaved in the sense that it always ends. To see this, notice first that, when an iteration ends up by a branching, one does not add new solutions to the stack but keep the subset of solutions selected within a partition of the solution consider during the iteration (only the solutions that might be Pareto superior to the status- quo). Since the number of firms is finite, so is the number of iterations which end up by a branching. Furthermore, because at iterations which do not end up by a branching, a solution is deleted from the stack and the algorithm does not cycle by construction, the stack will end up empty.

So as to prove that the algorithm gives the optimal solution to problem (1), we argue that none of the three following errors occurs.

Error 1: A solution has not been scrutinized when it should have been.
At the initial Round, all possible solutions preferred to the status- quo by firms are included in the stack. Solutions are eliminated from the stack when it is analyzed. Then, either it is discarded, selected as a new tentative solution or one proceeds to branching. In this case only solutions which are Pareto superior to the status- quo are introduced in the stack (other solutions cannot be optimal for (1)) and, thus, will be analyzed later on.

Error 2: A solution has been discarded which should not have been discarded. In a given iteration, assume that the tentative optimal solution, μ_t, registered in $WSP(\mu_t)$, i_t and the corresponding lower bound Z_U, is correct, i.e., its is optimal within the set of solutions already scrutinized. The solution R is discarded at the following steps:

3.1. When the tentative matching is core stable and the lower bound of R is greater than the objective value of the tentative solution, no solution in R can be optimal.
3.2. If for (at least) one worker unassigned at R none of the firms unassigned at R is at least as good as the status- quo, no solution in R can be incentive compatible with μ_o for this worker.
3.3 If for (at least) one firm unassigned at R none of the subsets of workers unassigned at R is at least as good as the status- quo, no solution in R can be incentive compatible with μ_o for this firm.
3.4. solutions in R which are not Pareto Superior to the status- quo are discarded, they cannot be optimal solutions to (1).
3.4.2 and 3.4.3 All matchings in R Pareto superior to the status- quo are compared to the tentative solution and discarded if their order of stability is higher than the one of status- quo or. in case of a tie, when their objective value is higher.
Hence, if the tentative solution is correct, so is the fact to discard families of solutions at 3.1, 3.2, 3.3, 3.4, 3.4.2 and 3.4.3.

Error 3: a solution has been selected as tentative optimal solution which should not have been selected.
In a given iteration, assume that the tentative optimal solution, μ_t, registered in $WSP(\mu_t)$, i_t and the corresponding lower bound Z_U, is correct, i.e., its is optimal within the set of already scrutinized solutions. The solution $\overline{R}$ is selected at the following steps:
3.4.3 All solutions in R which are Pareto superior to the status- quo are compared to the tentative solution and selected as the new tentative solutions if their indicia of stability and their value are lower than those of the tentative solution.
Hence, if the tentative solution is correct, so is the fact to select a new tentative solution at 3.4.3.■

In particular, if the status quo is the empty matching, our algorithm finds a core stable matching whenever such matching exists, dispensing of the condition of q-substitutability.

Corollary 1 *Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$ and let the status-quo μ_o be the empty matching. Then, when a core stable matching exists, the output of the WS algorithm is core stable.*

Moreover a solution to (1) cannot be Pareto dominated since, by transitivity of preferences, all blocking coalitions to a matchings are also blocking coalitions to a Pareto inferior matching.

Corollary 2 *$WSP(\mu_t)$ is status- quo stable.*

6 Concluding remarks

The WS algorithm selects the best core stable for firms in particular settings where it always exists. There, results by Dubins and Freedman (1981), Roth (1982-1984) and Demange, Gale and Sotomayor (1986) commented in the introduction apply. Nevertheless, for simple, Examples 3 in the Appendix and Proposition 1 suggest that the lack of existence of a core stable solution is no pathological case in such markets. Unfortunately, neither seems manipulability of core consistent procedures (Example 1) to be a sophisticated behavior. We believe that the Weakened Stable procedure is a convincing approach to deal with the problem for the following reasons. First, it is a core consistent procedure and core stability has shown to be a remarkable property of enforceability. Second there is no conflict between Weakened stability and Pareto efficiency: if a matching dominates another in Pareto terms, its order of stability is lower. Third, comparability of workers' career is a usual practice. In many countries civil servants are associated an index taking into account their seniority, professional performance or family situations that make them comparable. Thus, building up a social welfare function does seem reasonable. In our example of academic market, the social welfare function is indeed an objective function for universities. Moreover, these functions depends on observable variables, coping partially with the problem of manipulability. Finally Branch and Bound algorithm are so versatile tools that a large scope of variations from problem (1) is certainly solvable by these procedures.

7 References

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8 Appendix

Example 3 Senior market where no matching Pareto superior to the status-quo is stable.

Consider the market $(\mathcal{F}, \mathcal{W}, q, \succ)$ where $\mathcal{F} = \{f_1, f_2, f_3\}$, $q_{f_1} = q_{f_2} = q_{f_3} = 1$, $\mathcal{W} = \{w_1, w_2, w_3\}$ and $\succ$ is given by the following profile

$$\begin{array}{ccc} \succ_{f_1} & \succ_{f_2} & \succ_{f_3} \\ \hline w_3 & w_1 & w_1 \\ w_1 & w_2 & w_2 \\ & & w_3 \end{array} \quad \text{and} \quad \begin{array}{ccc} \succ_{w_1} & \succ_{w_2} & \succ_{w_3} \\ \hline f_3 & f_3 & f_1 \\ f_2 & f_2 & f_3 \\ f_1 & & \end{array}$$

Suppose that the status- quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 & w_2 & w_3 \end{pmatrix},$$

The only matchings Pareto superior to belong to μ^0 is:

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3 & w_1 & w_2 \end{pmatrix},$$

which is blocked by (f_3, w_1).

We investigate now the sufficient conditions which guarantee the existence of a group stable matching Pareto superior to a status- quo. We recall the following definitions.

Definition 7 *A matching μ is* <u>*worker quasi-stable*</u> *if it is individually rational and for any blocking coalition (S, f), $\mu(w) = \emptyset$, for all $w \in S$.*

Definition 8 *A matching μ is* <u>*firm quasi-stable*</u> *if it is individually rational and for any firm f, worker $w \in \mu(f)$ and subset of workers $S \subseteq \mathcal{W}_{f,\mu}$, $w \in Ch(\mu(f) \cup S, q_f, \succ_f)$.*

Definition 9 *A matching μ is* <u>*quasi-stable*</u> *if it is individually rational and for all blocking coalition $(S,\ f)$, for all $w \in \mu(f)$, $w \in Ch(\mu(f) \cup S, q_f, \succ_f)$ and $\mu(w) = \emptyset$, for all $w \in S$.*

Proposition 1 *Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$ and a matching μ_0. Suppose that the set of matchings Pareto superior to μ_o is non- empty, we know that one of them is core-stable when firms have q-substitutable preferences and the input matching is quasi-stable.*

Proof. The argument is constructive: if the matching of departure is quasi-stable, in particular it is firm quasi- stable. Since firms have q-substitutable preferences, Proposition 1 in Cantala (2004) shows that applying his modified version of the D.A. algorithm leads to a core stable matching and that all along the sequence of tentative matchings, workers are never dismissed and all assignations are firm quasi-stable. Since original blocking pairs only involve unmatched workers by quasi-stability, resolving them makes no firm worst off and no new blocking coalition appear along the process. Thus, all agents get better assignment, no new blocking pair appears, all tentative matchings are quasi- stable and the resulting matching, say μ, is Pareto superior to the status-quo matching. Finally since μ is stable, it is Pareto efficient. ■

One cannot dispense of q-substitutability since, then, it might be that no stable matching exists. Next example shows that quasi-stability is also necessary for Proposition 1 to hold.

Example 4 Consider the market $(\mathcal{F},\ \mathcal{W},\ q,\ P)$ where $\mathcal{F} = \{f_1, f_2, f_3\}$, $q_{f_1} = q_{f_2} = q_{f_3} = 1$, $\mathcal{W} = \{w_1,\ w_2,\ w_3\}$ and $\succ$ is given by the following profile

$$\begin{array}{cccccc} \succ_{f_1} & \succ_{f_2} & \succ_{f_3} & \succ_{w_1} & \succ_{w_2} & \succ_{w_3} \\ \hline \{w_1\} & \{w_3\} & \{w_2\} & \{f_1\} & \{f_1\} & \{f_3\} \\ \{w_2\} & \{w_1\} & \{w_3\} & \{f_2\} & \{f_3\} & \{f_2\} \end{array}$$

Suppose that the worker quasi stable status- quo is

$$\mu^0 = \left(\begin{array}{cccc} f_1 & f_2 & f_3 & \emptyset \\ w_2 & \emptyset & w_3 & w_1 \end{array} \right),$$

which is worker quasi- stable. The only matching Pareto superior to μ^0 is

$$\mu^1 = \left(\begin{array}{ccc} f_1 & f_2 & f_3 \\ w_2 & w_1 & w_3 \end{array} \right)$$

which is blocked by (f_1, w_1).