On the dynamics of stable matching markets

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Abstract: We study the dynamics of stable marriage and stable roommates markets. Our main tool is the incremental algorithm of Roth and Vande Vate and its generalization by Tan and Hsueh. Beyond proposing alternative proofs for known results, we also generalize some of them to the nonbipartite case. In particular, we show that the lastcomer gets his best stable partner in both of these incremental algorithms. Consequently, we confirm that it is better to arrive later than earlier to a stable roommates market. We also prove that when the equilibrium is restored after the arrival of a new agent, some agents will be better off under any stable solution for the new market than at any stable solution for the original market. We also propose a procedure to find these agents.

Keywords: stable marriage problem, stable roommates problem, matching mechanism

1 Introduction

The stable marriage problem was introduced and solved by Gale and Shapley [9]. In terms of graphs, this is the bipartite case of the stable matching problem, where the two sets of agents are

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that of men and women. The solution obtained by the authors' deferred-acceptance algorithm was proven to be optimal for men if men make proposals. This means that each man gets his best stable partner, so no man can have a better partner in some other stable matching.

The nonbipartite version, the stable roommates problem, is also defined in [9]. It is shown by an example that a stable matching does not always exist. Irving [12] constructed the first algorithm that finds a stable matching if one exists at all. Later, Tan [23] gave a compact characterisation by a half-integer solution.

For the bipartite case, Knuth [15] asked whether it is possible to obtain a stable matching by starting from an arbitrary matching and successively satisfying blocking edges. Roth and Vande Vate [19] gave a positive answer by a decentralized algorithm, in which pairs or single vertices enter the market in a random order, and stability is achieved by a natural proposal-rejection process. Knuth's question for the bipartite case was also answered by Abeledo and Rothblum [1] by a common generalization of the Roth-Vande Vate and the Gale-Shapley algorithms. Later, Diamantoudi *et al.* [8] solved the same problem for the roommates case. They proved that one can always reach a stable matching by successively satisfying blocking edges from an arbitrary matching if a stable matching exists.

Roth and Vande Vate modelled the dynamics of the two-sided matching market by considering the situation when a new agent enters the market and the stability is restored by that natural process. This mechanism also yields an algorithm to find a stable matching for a market by letting the agents enter the market in a random order. Independently, Tan and Hsueh [22] constructed an algorithm, that finds a stable half-matching for general graphs by using a similar incremental method. In the bipartite case, the Tan-Hsueh algorithm is equivalent to the Roth-Vande Vate algorithm. The difficulty of the Tan-Hsueh algorithm is that infinite repetitions can occur. These are handled by the introduction of cycles. These two algorithms are abbreviated hereafter as "incremental algorithms".

Blum, Roth and Rothblum [4] described the properties of a dynamic two-sided matching market. They showed that their proposed algorithm is similar to the McVitie-Wilson's version [16] of the original deferred-acceptance algorithm. So, the output of the process is predictable: if some men enter the market then each man either remains matched with the same partner (if it is possible) or gets a worse (but his best) stable partner for the new market. Blum and Rothblum [5] pointed out that these results imply that the lastcoming agent gets his best stable partner in the Roth-Vande Vate algorithm. Moreover, an agent can only benefit from entering the market later (we assume here that the others enter the market in the same order). Independently, Ma [14] observed on an example of Knuth, that if agents enter the market successively then the Roth-Vande Vate algorithm may not find all stable matchings in general. Cechlárová [7] sthrengtened Ma's result by justifying that in a stable matching output by the incremental algorithm for a bipartite graph some agent gets his best stable partner. Here we give direct proofs for the above results in the bipartite case, and we generalize most of them to general graphs with the help of our Key Lemma.

Gale and Sotomayor [10] showed that if some man expand his preference-list then no man is better off in the new men-optimal stable matching. This implies that the same statement is true if a number of men enter the market. Roth and Sotomayor [18] proved that if a man arrives and becomes matched, then certain women will be better off, and some man will be worse off under any stable matching for the new market than at any stable matching for the original market. We generalize this theorem by using an improved version of a result of Irving and Pittel [17] on the core configuration.

Our results have an economic interpretation. Matching markets are well-known applications of the stable matching problem. A detailed description of two-sided markets can be found in the book of Roth and Sotomayor [18]. An important example is job matching. Blum *et al.* [4] studied the dynamics of the two-sided matching market in this context by analysing the formation of the “vacancy chains”.

The dynamic formation of social and economic networks can be described by stable matching models as Jackson and Watts considered in [13]. In the nonbipartite case, the connections between individuals can model mutual “best friend” relationships. An other important new application of the stable roommates problem, the pairwise kidney exhange was discovered by Roth *et al.* [20] recently, however in this one-sided matching market the dynamic processes are not typical.

This paper is organized as follows. In section 2, we define stable matchings and half-matchings. In section 3, the Roth-Vande Vate and the Tan-Hsueh algorithm are described. We prove our main results in section 4.

2 Stable matchings and half-matchings

Let us model the *stable matching problem* with a graph G, where the agents are represented by vertices, and two vertices are linked with an edge if the agents are both acceptable to each other. For every vertex v, let $<_v$ be a linear order on the edges incident with v. That is, every agent has strict preferences on his possibles partnerships. We say that agent v prefers edge f to e (in other words f *dominates* e at v) if $e <_v f$ holds. A *matching* M is a set of edges with pairwise distinct vertices. If an edge $e = \{u, v\}$ belongs to M, then u and v are *matched*[2] in M, so u and v are partners in the market. An agent remains single if his vertex is uncovered in M.

A matching M is called *stable* if every nonmatching edge e is dominated by some edge f of M. Alternatively, a stable matching can be defined as a matching without a *blocking edge*: an edge preferred by both of its vertices to the eventual matching edges. For a matching market, this means that no pair of agents can benefit by leaving their actual partners and establishing a new mutual partnership.

Alternatively, stable matchings can be described with compact formulas. If M is a set of edges then let $x_M : E(G) \longrightarrow \{0,1\}^{E(G)}$ its *characteristic function* i.e.

$$x_M(e) = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$$

Subset M of $E(G)$ is a stable matching if the following conditions hold:

(M) Matching:
$\sum_{v \in e} x_M(e) \leq 1$ for every vertex $v \in V(G)$

(S) Stability:
for every edge $e \in E$ there exists a vertex $v \in e$ such that $\sum_{v \in f, f \geq_v e} x_M(f) = 1$

We consider a *stable marriage* problem if the graph is bipartite, and the *stable roommates* problem if the graph is general. Gale and Shapley [9] proved that stable matching always exists

[2]Equivalently, a matching can be described by an involution μ on the set of agents, where $\mu(u) = v$ implies $\mu(v) = u$, and means that u and v are matched. $\mu(w) = w$ corresponds to the case when w is unmatched. An advantage of the graph terminology is that it can handle parallel edges that corresponds to the case where two agents can make several types of partnership with each other.

for the marriage problem but may not exists for the roommates problem. They gave the following example to show the non-existence:

Example 1

Agents	*Preference-lists*
A :	$[B, C, D]$
B :	$[C, A, D]$
C :	$[A, B, D]$
D :	*arbitrary*

Let us imagine that these agents are tennis-players, each is looking for a partner to play with for one hour a week. For example Andy would like to play mostly with Bill, then with Cliff and finally he prefers Daniel the least. (In fact, everybody tries to avoid Daniel.) There is no stable solution. If a pair is formed from the first three players, say Andy plays with Bill, then the third, Cliff must be matched with Daniel, but in this case Bill and Cliff block this matching.

Tan [23] discovered, that if the agents can create half-time partnerships then a stable solution always exists in the sense that no pair of agents would like to increase the intensity of their partnership mutually.

Considering the above example, we suppose that Andy, Bill and Cliff agree to meet once a week and play half-time games in each formation. Thus, each of them play one hour in sum, only Daniel remains without any tennis-partner. The stability in this case is that no pair of tennis-players want to play more time mutually with each other. For example Andy plays with Daniel no time at all, because Andy fills his one-hour by playing two half-hour games with better partners. Andy and Bill will not play more than a half-hour, because Bill fills his rest of his time (a half-hour) by playing with a better partner, Cliff.

A *half-matching* hM consists of matching edges M and half-weighted edges H, so that each vertex is incident either with at most one matching edge or with at most two half-weighted edges. In a matching market an agent can have at most one partner or at most two half-partners. A half-matching is *stable* if for each edge not in hM there exists a vertex v, where e is dominated either by one matching edge or by two half-weighted edges, and for every half-weighted edge h there exists a vertex v, where h is dominated by an other half-weighted edge. So no pair of agents wants to improve their partnership mutually, because for each pair of agents who are not matched, one of them fills his capacities with better partnership(s).

If $x_{hM} : E(G) \longrightarrow \{0, \frac{1}{2}, 1\}^{E(G)}$ is a weight-function that describes the set of matching edges, M and the set of half-weighted edges, H so that $hM = H \cup M$ and

$$x_{hM}(e) = \begin{cases} 1 & e \in M \\ \frac{1}{2} & e \in H \\ 0 & e \notin hM \end{cases}$$

then the same (M) and (S) inequalities preserve the half-matching and the stability-property.

The fact, that every half-weighted edge must be dominated by an other half-weighted edge at one of its endvertices imply that the half-weighted edges form cycles, where preferences are cyclic. Tan [23] observed that an even-cycle can be separated into matched pairs, but if an odd-cycle C occurs then C must belong to the H-part of any stable half-matching so no stable matching exists. He characterized the stable half-matching[3] in the following way:

[3]Originally, Tan called it *stable partition* . We have several reasons to use this new notion. The expression

Theorem 2 (Tan) *For a stable roommates problem there always exists a stable half-matching*[4] *that consists of matched pairs and odd-cycles formed by half-weighted pairs. The set of agents can be partitioned into:*

a) unmatched (or single) agents,

b) cycle-agents and

c) matched agents.

Furthermore, for any instance the same agents remains unmatched and the same odd-cycles are formed in each stable half-matching.

If for a half-matching $hM = H \cup M$ an edge $e = \{u, v\}$ is in M, then we say that the agents u and v are *partners*. If two agents can be partners in a stable half-matching we call them *stable partners*. If an edge $e = \{u, v\} \in H$ is in an odd-cycle, then u and v are *half-partners*. If u prefers v to his other half-partner, then v is the *successor* of u and u is the *predecessor* of v.

To consider the stable half-matchings of a matching market can have many motivations. First of all, if the stable half-matching does not contain any odd-cycle, then we receive a stable matching, otherwise we know the reason of the non-existence. Secondly, we can obtain a matching, by leaving one agent from each odd-cycle and forming pairs from the rest of the cycles, that is stable for the remaining agents. In other words every blocking edge is incident with one of the removed agents, so by compensating them somehow we can reach a kind of stability for the market.[5] Thirdly, in some real applications (like in the case of the tennis-players) the half-solutions are feasible in practice.

3 The incremental algorithms

In an equilibrium state matching market with a stable matching a natural question is how the situation changes if a new player enters the game and the preferences over the former partnerships are unchanged. Let the newcomer make proposals according to his preference order. If no one accepts, then everybody has a better partner, so the former matching remains stable. If somebody accepts a proposal, then a new pair is formed. The left alone partner, has to leave the market and enter as a newcomer.

The Roth-Vande Vate algorithm for the stable marriage problem

Suppose, that a bipartite graph G is built up step by step in the algorithm by adding vertices to the graph in some order. In a *phase* of the algorithm we add a new agent and restore the

"stable partition" is also used as a core-solution of a coalition formation game, that can be confusing. If we consider more general models (where agents can have several partners, or multiple activities are possible) definition of stable half-matching can be easily extended. Finally, the half-solution may interpret real partnerships with half-intensities.

[4]Aharoni and Fleiner [3] showed, that the existence of the stable half-matching is the consequence of the famous theorem of Scarf [21].

[5]This idea can be used as a heuristic to find a matching that contains as few blocking edges as possible. It is reasonable to apply such a method, since even to approximate the minimal number of the blocking pairs for general graphs is theoretically hard (see [2]).

stability. To describe a phase, let us add a vertex v to $G-v$, where a stable matching M_v exists. Our task is to find a stable matching M for G.

If v is not incident to any blocking edge, then M_v remains stable for G, too. In this case the phase is called *inactive*.

A phase is *active* if the newcomer agent v is incident to some blocking pair, say $\{v,u\}$ is the blocking pair that v prefers the best. Let $v = a_0$ and $u = b_1$. If b_1 was unmatched in M_{a_0}, then $M_{a_0} \cup \{a_0, b_1\}$ is a stable matching for G. In the other case b_1 had a partner a_1 in M_{a_0}, whom he leaves after receiving a better proposal. In this case, the matching $M_{a_1} = M_{a_0} \setminus \{a_1, b_1\} \cup \{a_0, b_1\}$ is stable for $G - a_1$. So we have a similar situation as in the beginning: a_1 enters the market and makes proposals. Continuing the process, a *proposal-rejection sequence*, $S = (A|B) = a_0, b_1, a_1, \ldots$ is constructed with the following properties:

1. $M_{a_k} = M_{a_{k-1}} \setminus \{a_k, b_k\} \cup \{a_{k-1}, b_k\}$ is a stable matching for $G - a_k$.

2. a_{k-1} is a better partner for b_k than a_k and

3. b_{k+1} is a worse partner for a_k than b_k.

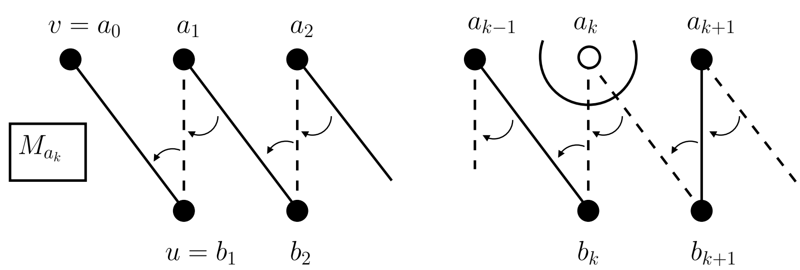

Figure 1: Proposal-rejection sequence in the Roth-Vande Vate algorithm

Observe that by this process, each $a_i \in A$ improves his situation and each $b_j \in B$ gets worse off. Consequensly, the same vertices cannot occur as new pairs. So a phase terminates in $O(m)$ time, when m denotes the number of the edges in the graph. It has two possible outcomes: either nobody accepts the proposals of some a_i (then the size of the matching remains the same) or the last b_j was unmatched, hence the size of the matching increases by one.

We illustrate with an example the mechanism of the incremental algorithm and we introduce briefly our results. The preferences of the agents on their possible partnerships in this two-sided market are the following:

Example 3

$$
\begin{array}{ll}
a_1 : e_1 > d_1 > f_1 & b_1 : f_3 > d_2 > n_1 > e_1 \\
a_2 : e_2 > d_2 > f_2 & b_2 : f_2 > d_1 > e_3 \\
a_3 : e_3 > d_3 > f_3 & b_3 : f_1 > d_3 > e_2 \\
a_4 : s > t & b_4 : s > n_2 \\
a_5 : m_1 > m_2 & b_5 : m_2 > n_3 \\
a_6 : n_1 > n_2 > n_3 > n_4 & b_6 : n_4 > m_1
\end{array}
$$

Let $e = \{e_1, e_2, e_3\}$, $d = \{d_1, d_2, d_3\}$ $f = \{f_1, f_2, f_3\}$. Suppose, that at the beginning a_6 is not present in the market. Partnerships $\{e, s, m_1\}$ form a stable matching in the market. (It is the best for every agent a_i.)

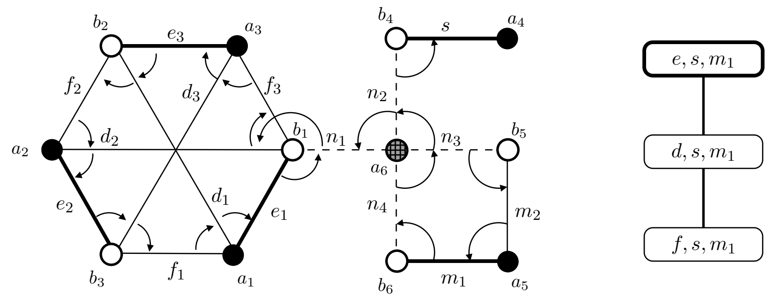

Figure 2: A stable matching and the lattice of the stable matchings before the arrival of a_6

When agent a_6 enters the market, four new possibles partnerships are created. The best one is n_1 for the newcomer, that is blocking the actual matching. Following the algorithm of Roth and Vande Vate let us satisfy this blocking edge: b_1 and a_6 form a new pair, and partnership e_1 terminates, so agent a_1 has to find a new partner as a newcomer. Continuing this process, the following edges will be satisfied and terminated in sequence: d_1, e_3; d_3, e_2; d_2, n_1. Here agent a_6 makes proposals again, that b_1 and b_4 refuse, because they have better partners than a_6 actually. We will prove, that if a new partnership is not blocking, then it cannot be present in any stable matching. In the last step of our example a single agent b_5 receives finally the proposal of a_1, and $\{d, s, m_1, n_3\}$ form a stable matching. This stable solution is the best possible for the newcomer a_6, since the better partnerships, that were refused by his possibles partners cannot appear in any stable matching. This argument proves also that every agent that receives partner by making proposal during the process gets his best stable partner.

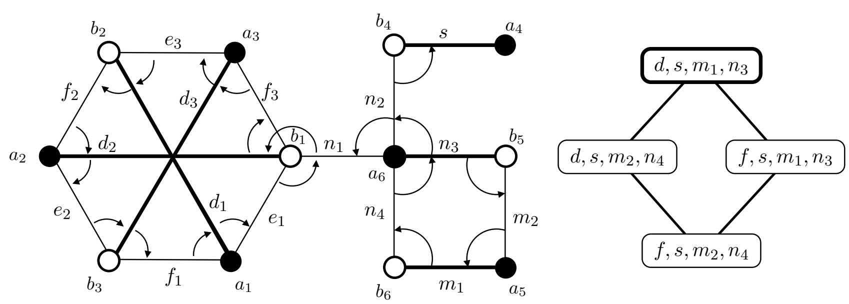

Figure 3: The obtained stable matching, and the lattice of the stable matchings

Note, that if we would start with the stable matching $\{f, s, m_1, n_3\}$, then the process would stop in one step, since b_5 accepts first the proposal of a_6. The obtained stable matching $\{f, s, m_1, n_3\}$ yields the best stable partner to the newcomer a_6 again, but the other agents a_i dont get necessarily their best stable partners.

The Tan-Hsueh algorithm for the stable roommates problem

Tan and Hsueh [22] proposed an incremental algorithm to find a stable half-matching. In this more general setting we use the terminology of the Roth-Vande Vate algorithm. The only difference is that G is not bipartite, so instead of a matching, we maintain a half-matching hM_v for $G - v$.

If nobody accepts the newcomer's proposal, then the phase is called *inactive* again and the stable half-matching is unchanged.

If some agent u accepts the proposal of v then three cases are possible:

a) If u is unmatched in hM_v, then $hM = hM_v \cup \{v, u\}$ is a stable half-matching for G.

b) If u is a cycle-vertex in hM_v, so $u = c_0$ for some cycle $C = (c_0, c_1, \dots, c_{2k-1}, c_{2k})$, then $hM = hM_v \setminus C \cup \{v, u\} \cup \{c_1, c_2\} \cup, \dots, \cup \{c_{2k-1}, c_{2k}\}$ is a stable half-matching for G.

c) If u is matched with x in hM_v, then $hM_x = hM_v \setminus \{u, x\} \cup \{v, u\}$ is a stable half-matching for $G - x$.

The actual phase end in cases a) and b). Here, unlike in the bipartite case, it can happen that an agent, that made a proposal earlier can receive a proposal later. So the proposal-rejection sequence might never end. One result of Tan and Hsueh [22] was that repetition always occurs along an odd-cycle.

Theorem 4 (Tan-Hsueh) *If $S = (A|B) = a_0, b_1, a_1, \dots$ is a proposal-rejection sequence and $a_i = b_k$ is the first return, then this proposal-rejection sequence can be extended so it will return to a_k at b_{k+m+1}, and the following properties are true: $\{a_k, b_{k+1}, \dots, b_{k+m}, a_{k+m}\}$ are distinct vertices, and in the same order they form an odd-cycle C, and $hM = hM_{a_k} \setminus \{a_{k+1}, b_{k+1}\} \setminus \cdots \setminus \{a_{k+m}, b_{k+m}\} \cup C$ is a stable half-matching.*

The following example illustrates the mechanisms of the Tan-Hsueh algorithm:

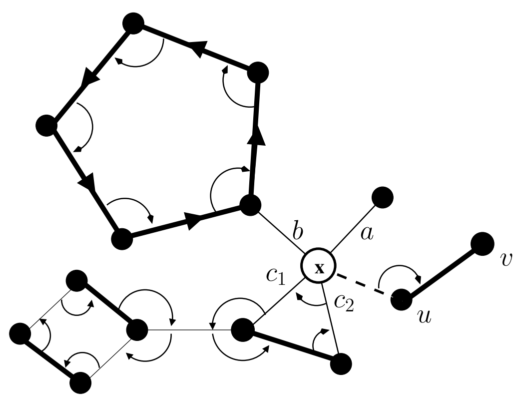

Figure 4: The Tan-Hsueh algorithm in an example

Here, vertex v enters. The first vertex accepting v's proposal is u, and u's previous partner x is left alone. In this figure there is a stable half-matching hM_x for $G - x$. In the next step x makes proposals. If nobody accepts it x remains uncovered and hM_x is stable for G, too. If somebody accepts x's proposal one of the following cases is true:

a) an uncovered vertex accepts x's proposal and they form a new pair.

b) a cycle-vertex accepts x's proposal and they form a new pair. The rest of the cycle breaks into stable pairs.

c) a matched vertex accepts x's proposal. The process continues and finally x receives a proposal, so the sequence returns in x. In this case the phase would never end, but by collecting the repeating vertices into an odd-cycle, the following stable half-matching is reached:

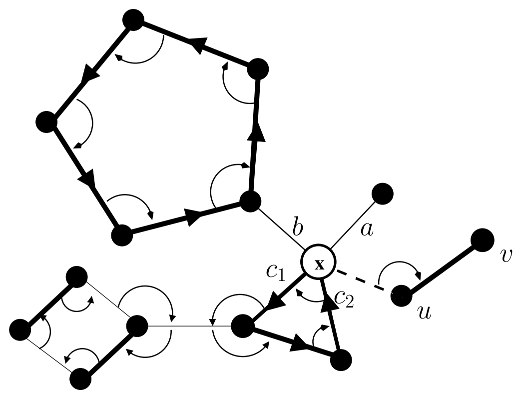

Figure 5: The obtained stable half-matching

4 Properties of the dynamic solutions

In this section we prove our results.

Getting the best stable partner by making proposals

Key Lemma 5 *If hM_v is a stable half-matching for $G-v$, and edge $\{v,u\}$ is not blocking hM_v, then v and u cannot be matched in a stable half-matching for G.*

Proof: Let us suppose that $\{v,u\}$ is not blocking hM_v but there is a stable half-matching hM of G, where v and u are matched. Let $v = a_0$ and $u = b_1$. First we consider the case where none of hM and hM_v contains an odd-cycle. Then b_1 has a partner in hM (say a_1), who is better than a_0. So $\{a_0,b_1\} <_{b_1} \{a_1,b_1\}$, where $\{a_0,b_1\} \in hM \setminus hM_v$. Since hM cannot dominate $\{a_1,b_1\}$ at b_1, it must be dominated at a_1 by some edge $\{a_1,b_2\}$ of hM. As $\{a_1,b_2\}$ is not in hM_v, it must be dominated at b_2 by an edge $\{a_2,b_2\}$ of hM_v, and so on. The alternating sequence $(a_0,b_1,a_1,b_2,\dots)$ has the following property: $\{a_{i-1},b_i\} \in hM \setminus hM_v$ and $\{b_i,a_i\} \in hM_v \setminus hM$, furthermore the domination is also in sequence: $\{a_{i-1},b_i\} <_{b_i} \{a_i,b_i\}$ and $\{a_i,b_i\} <_{a_i} \{a_i,b_{i+1}\}$ for every i. We call this sequence *alternating preference sequence*. Because a_0 is not covered by the stable matching hM_v, the sequence can return neither to a_0, nor to any other vertex, a contradiction. (This part of the proof already confirms the bipartite case.)

The other case is, when hM_v or hM may contain odd-cycles. The properties of the alternating preference sequence remain the same, the diffefence is that the edges can be half-weighted edges as well. To avoid repetition, the idea is the following: when an edge $\{a_i, b_i\} \in hM_v$ is dominated at a_i in hM by two edges (so a_i is in a cycle in hM), then we chose for b_{i+1} that neighbour in the cycle which is less preferred by a_i. Edge $\{a_i, b_{i+1}\}$ is still not in hM_v, so it must be dominated at b_{i+1}. But then the edge(s) that dominate(s) $\{a_i, b_{i+1}\}$ is (are) better than either of the edges that cover b_{i+1} in hM, so they are not in hM. This is why every new edge in this sequence will be alternately in $hM \setminus hM_v$ and $hM_v \setminus hM$.

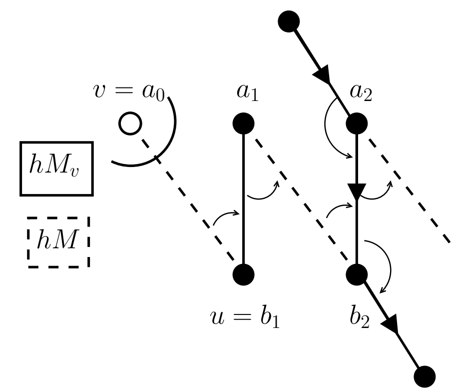

Figure 6: Alternating preference sequence with half-weighted edges

Let us suppose that $a_k = a_i$ for some $k \neq i$. This means that $\{b_k, a_i\}$ and $\{b_i, a_i\}$ are in the same odd-cycle in hM_v but the directions are opposite, because for b_i and also for b_k a_i is the less preferred neighbour in the cycle. In the other case, assume that $a_k = b_i$ for some $k \neq i$. This means that $\{b_k, b_i\}$ and $\{b_i, a_i\}$ are in the same odd-cycle in hM_v but the directions are opposite again. As a_i is less prefered for b_i in the cycle, and $\{b_k, b_i\} \in hM_v \setminus hM$ it must be dominated at b_i in hM, this means $\{b_k, b_i\} <_{b_i} \{a_{i-1}, b_i\} <_{b_i} \{a_i, b_i\}$, a contradiction.

By similar reasons, no repetition can occur at some b_k, so the sequence would never end, a contradiction. $\square$

The following Lemma is well-known.

Lemma 6 *If v is the best stable partner for u then u is the worst stable partner for v.*

PROOF: If indirectly, v and u are matched in a stable half-matching hM, but v has an even worse partner u' in a stable half-matching hM', then u would have some other partner v' worse than v, because v was u's best stable partner. So $\{u, v\}$ would be a blocking edge for hM', contradiction. $\square$

To generalize the results of Blum *et al.* [4] we prove that a newcomer gets his best stable partner in the output of the incremental algorithm in the graph case as well.

Theorem 7 *Suppose that an agent v enters the market and the stability is restored by a proposal-rejection process along the sequence $S = (A|B)$ then each agent $a \in A$, who became matched by making (accepting) proposal gets his best (worst) stable partner in the obtained stable half-matching.*

Proof: If an agent a is matched in the output, and receives a partner by making a proposal, then later he cannot accept any proposal because then he would be a cycle-agent. The last time when agent a makes a proposal during the process he does not prefer his last partner only to some agents that refused him. Because of the Key Lemma, no one of these agents can be a partner of a in a stable solution, so obviously agent a received his best stable partner. Similarly, each matched agent $b \in B$ gets his worst stable partner by Lemma 6. $\square$

Corollary 8 *If an agent enters the market last and becomes matched, then he gets his best stable partner.* $\square$

If a phase is inactive in an incremental algorithm, then each stable half-matching of the extended graph is also a stable half-matching in the original. That is, if hM is a stable half-matching for G not covering some vertex x, then hM is a stable half-matching for $G - x$, too. Because after deleting x from G no blocking edge can appear. So, by using the Key Lemma, we can confirm our main result:

Theorem 9 *Each matched agent, that gets a partner in the last active phase by making (accepting) a proposal, receives his best (worst) stable partner in the stable solution output by the incremental algorithm.* $\square$

Remark The vertices that remained uncovered in the last active phase or entered later in an inactive phase, will still be uncovered at the end of the algorithm, just like they are in every stable matching. The vertices that form an odd-cycle in the last active phase will form an odd-cycle at the end of the algorithm, just like they do in every stable half-matching. Hence these agents also get best stable partners in this sense.

Corollary 10 *A stable matching, where no matched agent gets his best stable partner, cannot be output by the incremental algorithm.* $\square$

Let us remark that we did not prove that any stable matching where somebody gets his best stable partner or contains odd-cycle whether can be obtained with an incremental algorithm. Our result gives only a necessary condition not a sufficient one.

Blum *et al.* [4] proved, that if a man m enters the market and another man m' was matched with w' in M_m, then they remain matched in the obtained stable matching M for the new market if and only if they are stable partners for the new market. Otherwise m' and w' gets that agents whom they are matched in the men-optimal stable matching of the new market. (So m' receives his best stable partner, and w' receives her worst stable partner in this case.) Below, we generalize this statement for the nonbipartite case.

Theorem 11 *Suppose that w and u are matched in a stable half-matching hM_v for $G-v$. They remain matched in the stable half-matching hM, obtained by the proposal-rejection process if and only if they are stable partners for G as well. Otherwise, if they are not involved in a cycle, then one of them gets a better partner but receives his worst stable partner, the other one becomes single or gets a worse partner but receives his best stable partner in hM .*

Proof: If w and u are not involved in the proposal-rejection process, then obviously they remain matched. Otherwise, if $S = (A|B)$ is the proposal-rejection sequence, then some of them, w is in

A and the other one must be in B. As they are not involved in a cycle, u improves his situation and w gets worse off during the process, and finally (by Theorem 9) u gets his worst stable partner (better than w) and w gets his best stable partner (worse than u), so u and w cannot be stable partners in the output. □

Blum and Rothblum [5] realized that an agent can only benefit by arriving later to the market in the Roth-Vande Vate algorithm. By a similar argument, we can generalize this result for the nonbipartite case.

Lemma 12 *Assume agent u prefers stable half-matching hM_v to stable half-matching hM'_v, v arrives. The outputs received by the proposal-rejection process are hM and hM' respectively. Then u cannot prefers hM' to hM.*

Proof: Indirectly, u should get better off in hM' than in hM'_v or get worse off in hM than in hM_v. By Theorem 11, u would receive his worst stable partner in hM' or receive his best stable in hM respectively. A contradiction. □

Theorem 13 *In the incremental algorithm if two arrival orders of the agents differs only for one particular agent v, then v gets at least as good partner in the first output hM, where he arrives later as in the second output hM', where he arrives earlier.*

Proof: Assume that v arrives in the first proposal-rejection sequence. At that moment the same agents are present in the market, and from the Theorem 9 v cannot be better off in the second sequence. Afterwards the same agents arrive in each phase, so by the above Lemma v cannot be better off any more during the incremental algorithm. □

Improving the situation by accepting proposals

Our next goal is to generalize the following result of Roth and Sotomayor [18] (Thm. 2.26.).

Theorem 14 (Roth-Sotomayor) *Suppose a woman w is added to the market and let M^W be the woman-optimal stable matching for G and M^M_w is the man-optimal stable matching for $G-w$. If w is not single in M^W, then there exists a nonempty subset of men, S, such that each men in S are better off, and each women in S' are worse off under any stable matching for the new market than at any stable matching for the original market, when S' denotes the partners of men in S under matching M^M_w.*

Proof: After adding w to the market during the proposal-rejection process each man that gets a partner by accepting a proposal gets his worst possible partner at the end of the process. But this partner is strictly better than the best stable one for $G-w$. Similarly, each woman that gets a new partner during the process by making proposal gets her best stable partner for G , that is strictly worse than the worst stable partner was for $G-w$. □

Pittel and Irving [17] considered the following situation. A new agent v enters the market, and a perfect stable matching (i.e. a stable matching where no agent is single) is achieved in such a way that the proposal-rejection sequence is as short as possible. They called this special half-matching with the associated alternating sequence a *core configuration relative to v*. Irving and Pittel [17] proved the following interesting property.

Theorem 15 (Irving-Pittel) *If hM_v is a core configuration relative to v, then the associated proposal-rejection sequence $v = a_0, b_1, a_1, \ldots, a_{k-1}, b_k$ consists of $2k$ distinct person, it is uniquely defined, and for every $i = 1 \ldots k-1$*

1. *b_i is the worst stable partner of a_i for $G - v$;*
2. *a_i is the best stable partner of b_i for $G - v$.*

We generalize Theorem 15 by extending the notion of core configuration. A stable half-matching hM_v is a *core configuration* relative to v if after adding v to the graph the associated proposal-rejection sequence, $S(hM_v)$ is as short as possible, by assuming that in case of cycling the sequence is restricted till b_k, where $a_i = b_k$ is the first return.

Theorem 16 *If hM_v is a core configuration relative to v, and hM is the output by the proposal-rejection process, then the associated proposal-rejection sequence $S(hM_v) = (A|B) = a_0(= v), b_1, a_1, \ldots, a_{k-1}, b_k(, a_k)$ consists of $2k$ (or $2k+1$) distinct person. It is uniquely defined, and for every matched agent in the sequence the following properties are true:*

a) b_i is the worst stable partner of a_i for $G - v$ and b_{i+1} is the best stable partner of a_i for G;

b) a_i is the best stable partner of b_i for $G - v$ and a_{i-1} is the worst stable partner of b_i for G.

Theorem 16 implies the following generalization of Theorem 14.

Theorem 17 *Suppose that a new agent is added to the market and a new stable solution is reached by the proposal-rejection process. There exists some agents that are better off, and some other agents that are worse off under any stable half-matching for the new market than at any stable half-matching for the original market.* □

Remark We can find agents in Theorem 17 algorithmically, as in the proof of Theorem 16.

The increasing side gets worse off

Finally, we give alternative proofs for some special results that occurs only in the two-sided matching markets.

Lemma 18 *If a man enters the game then no man can have better partner in the new men-optimal stable matching than in the former men-optimal stable matching.*

Lemma 18 is a straightforward consequence of Theorem 2 in [10] by Gale and Sotomayor. Here, we give an alternative proof.

Proof: Let m be the man that enters the game last. We shall prove that if a man m' in the same side gets w' in the men-optimal stable matching M^M, then m' cannot have a worse partner in the former men-optimal stable matching M^M_m for $G - m$. If m is unmatched in M^M, then M^M is also stable for $G - m$. If $\{m, w\} \in M^M$, then $M^M \setminus \{m, w\}$ is stable for $G - \{m, w\}$. After w reenters the game, during the proposal-rejection process m' either remains matched with w' or receives a proposal from a better woman for him. □

Theorem 19 *If some men enter the game one after another then at the end of the proposal-rejection processes they all get their best stable partners in the resulted stable matching.*

PROOF: Suppose that a man m' is matched with his best stable partner w' before a new man, m enters. If m' remains matched with w' in the new obtained matching, then by the Lemma 18 w' is still his best stable partner. If m' gets a new partner during the phase, then he must receive her by making a proposal, so the Theorem 9 proves that m' gets his best stable partner again. $\square$

The following Theorem of Blum *et al.* [4] can be proved in the same way by using Theorem 11.

Theorem 20 *If some men enter the market then any other man m either remain matched with his original partner w if w is still stable partner for m or m receives his best stable partner in the output.* $\square$

If the arrival order is such that women enter the game first and men follow after that, then the output will be the same as the output of the deferred-acceptance algorithm by Gale and Shapley [9]. Theorem 19 shows alternatively, that the received stable matching is optimal for the men.

Conclusion and further questions

We have studied a matching market, where agents enter and leave one after the other, and they are able to terminate and build new partnerships without restrictions. By this assumption, a new stable state is created for the market by a natural decentralized process if such an equilibrium exists. For a two-sided market a new stable matching, for a general market a new stable half-matching can always be obtained this way.

The main lesson of our study is that an agent can benefit if he enters the market as late as possible. This fact may encourage an agent to leave the market and enter again with the hope of getting a better partner. We can avoid this kind of instability if and only if the stable solution is unique.

Accepting a proposal always means an improvement for the agent. Moreover, among the agents that accept proposals during the process, some are strictly better off under any stable solution for the new market than at any stable solution for the former one. Finally, if in a two-sided market the number of men increases then the best stable partner for each man gets worse.

To generalize these results further, it is reasonable to consider the cases, where an agent can be matched with more partners. Cantala [6] studied the many-to-one matching markets under *q-substitutable* preferences and Ünver [24] considered the many-to-many matching markets under *categorywise-responsive* preferences. Cantala used the idea of Blum *et al.* [4] to analyse the restabiling mechanism of that market, Ünver proved that a pairwise stable matching can be obtained by successively satisfying blocking edges by an algorithm similar to the Roth and Vande Vate's [19]. A natural question is the study of nonbipartite versions of these dynamic matching markets.

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Appendix

Proof:[Theorem 16] In our proof we construct a core configuration. Suppose that hM^0 is an arbitrary stable half-matching for G. Let a new agent u enter the market such a way that u is acceptable only for v and u is the most preferred partner for v. Let us denote the proposal-rejection sequence by $S(hM^0)$ and the output stable half-matching for $G{+}u$ by hM^0_{+u}. Obviously, u and v are partners in any stable half-matching hM'_{+u} for $G+u$, moreover hM'_{+u} is a stable half-matching for $G{+}u$ if and only if $hM'_v = hM'_{+u} \setminus \{u,v\}$ is a stable half-matching for $G{-}v$. So, by deleting $\{u,v\}$ from hM^0_{+u} we get a stable half-matching, say hM_v for $G-v$. We prove that hM_v is a core configuration relative to v. (We denote the associated proposal-rejection sequence by $S(hM_v)$ and the output stable half-matching for G by hM.)

To prove that $S(hM_v)$ is as short as possible we show that each agent that are involved in $S(hM_v)$ must be involved in any other proposal-rejection sequence as well, and each agent occurs exactly once in $S(hM_v)$ (unless a new odd-cycle is created, when $a_i = b_k$ occurs twice.)

First, we prove that if $x \in S(hM_v)$ then $x \in S(hM'_v)$ for any stable half-matching hM'_v for $G-v$. We consider the cases according the status of x (unmatched, cycle-agent or matched) in the stable half-matchings for $G-v$ and G.

1-2. No agent can be unmatched for $G-v$ and cycle-agent for G, similarly no agent can be cycle-agent for $G-v$ and unmatched for G.

3-4. If an agent is unmatched/cycle-agent for $G-V$ and remains unmatched/cycle-agent for $G-v$ then he cannot be involved in any proposal-rejection sequence.

5. If x is matched for $G-v$ and become unmatched for G then $x = a_k$, so x is the last agent in $S(hM_v)$ (nobody accepts his proposal) and obviously x must be the last agent in any other $S(hM'_v)$ as well.

6. If x is unmatched for $G-v$ and become matched for G then $x = b_k$, so x is the last agent in $S(hM_v)$ (he accepts the last proposal) and obviously x must be the last agent in any other $S(hM'_v)$ as well.

7. If x is cycle-agent for $G-v$ and become matched for G then $x = b_k$, so x is the last agent in $S(hM_v)$ (he accepts the last proposal). We prove that for any stable half-matching hM'_v x is the last agent in $S(hM'_v)$ as well. Let $C = (c_0, c_1, \dots c_{2k})$ be the cycle that eliminates when v

enters the market. We suppose indirectly that two different cycle-agents $x = c_0$ and c_i accept the last proposals, maked by y and y' in $S(hM_v)$ and $S(hM'_v)$ respectively. Obviously, the agent who make the final proposal is better than the predecessor of that cycle-agent who accepts it, (so $y >_{c_0} c_{2k}$ and $y' >_{c_i} c_{i-1}$). From Theorem 9, we also know that c_0 and c_i get their worst stable partners in hM and hM' respectively. This is a contradiction, because if i is even then c_i would be matched with c_{i-1} in hM and if i is odd then c_0 would be matched with c_{2k} in hM'.
8. If x is matched for $G - v$ and became a cycle-agent for G then x must occur in any proposal-rejection sequence until the first return, since Tan and Hsueh [22] proved that no new agent occurs in the sequence after the first return.
9. Finally we consider the case where x is matched for $G - v$ and for G as well. Let us denote x's partners by y^0, y_v and y in hM^0, hM_v and hM respectively.

a) If $y <_x y_v$, then x must receive y during $S(hM_v)$ by making a proposal, so from Theorem 9 y is the best stable partner of x for G. Thus, $y^0 \leq y$ implies $y^0 <_x y_v$, it means that x must receive y_v during $S(hM^0)$ by accepting a proposal, so y_v is the worst stable partner of x for $G-v$. It is obvious now that x gets worse partner under any stable half-matching for G than at any stable half-matching for $G - v$, so x must be involved in any proposal-rejection sequence.

b) Similarly, if $y >_x y_v$, then x must receive y during $S(hM_v)$ by accepting a proposal, so from Theorem 9 y is the worst stable partner of x for G. Thus, $y^0 \geq y$ implies $y^0 >_x y_v$, it means that x must receive y_v during $S(hM^0)$ by making a proposal, so y_v is the best stable partner of x for G. It is obvious now that x gets better partner under any stable half-matching for G than at any stable half-matching for $G - v$, so x must be involved in any proposal-rejection sequence.

c) If $y = y_v$, then x cannot be involved in $S(hM_v)$

Now, we prove that each agent occurs exactly once in $S(hM_v)$. Let consider the above sequence with an extra stopping rule: if a_j looks for a new partner let choose the best one among those that either form a blocking pair with a_j or a b_i for $i < j$ such that b_i prefers a_j to a_i (and not to his actual partner a_{i-1}). Assume that the first repetition (according to the extra stopping rule) would occur at b_{j+1}. .

Case 1. If $b_i = b_{j+1}$ for some $i < j$ then let hM_{a_j} be the actual stable half-matching for $G - a_j$. We construct a new stable partition for $G - v$: $hM'_v = hM_{a_j} \cup \{a_j, b_i\} \setminus \{\{a_{p-1}, b_p\}, 1 \leq p \leq i\} \cup \{\{a_p, b_p\}, 1 \leq p \leq i-1\}$. It is stable, because by compairing with hM_v only the agent $\{a_q, i \leq q \leq j\}$ get worse partners, but the extra stopping rule preserves that no edge $\{\{a_q, b_p\}, 1 \leq p < i \leq q \leq j\}$ can block hM'_v (and obviously no other edge).

Since in hM'_v every agent $\{b_q, i \leq q \leq j\}$ gets better partner than in hM_v, and every agent $\{a_q, i \leq q \leq j\}$ gets worse partner than in hM_v, if some of these agents is matched for $G-v$ and G as well, then it is a contradiction, because in hM_v they are matched with their best/worst stable partners respectively.

The last case that we have to consider, that all of these agents are matched for $G - v$ and become cycle-agent for G. These agents are obviously in the same cycle (let say $(c_0, c_1, \ldots, c_{2k})$) in hM^0 as well. So, when $S(hM^0)$ ends at c_0 by eliminating this cycle, each of these agents become matched in hM_v to either with his successor or with his predecessor (so $\{c_{2i-1}, c_{2i}\} \in hM_v$ for all $1 \leq i \leq k$). We show that a_{i-1} must also be a cycle-agent for G. Otherwise a_{i-1} must receive a worse partner than b_i in hM, and for b_i his predecessor is also worse than a_{i-1} (that is why b_i accepted the proposal of a_{i-1}), so a_{i-1} and b_i would block hM. By continuing this argument, for some $p < i$, a_p must be c_0, (the cycle-agent in hM_v that accepted the last proposal in $S(hM^0)$). But then b_{p+1} must be the predecessor of c_0: c_{2k}. Otherwise, if for some $1 \leq r < 2k$, $c_r = b_{p+1}$ then $c_{2k} <_{c_0} c_r$ (since c_{2k} is matched with c_{2k-1} in hM_v, so he would accept the proposal of c_0) and $c_{r-1} <_{c_r} c_0$ (since c_r accepted the proposal of c_0), so c_0 and c_r would form a blocking pair in hM. Similarly, we can prove that the sequence goes along this odd-cycle, so for each d $(0 < d < j-p)$ $a_{p+d} = c_{2(k-d)+1}$ and $b_{p+d} = c_{2(k-d)}$. Finally, $b_i = b_{j+1}$ cannot be the predecessor of a_j in hM, a contradiction.

Case 2. If the first repetition is such that $a_i = b_{j+1}$, then the extra stopping rule was not used. This proves that a new odd-cycle can be created, so $hM = hM_{a_j} \setminus \{\{a_q, b_{q+1}\}, i \leq q \leq j\} \cup (a_i, a_j, b_j, a_{j-1}, \ldots, a_{i+1}, b_{i+1})$ is the output stable half-matching for G.

□