Interaction Patterns with Hidden Complementarities*

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Abstract

We consider a finite population simultaneous move game with heterogeneous interaction modes across different pairs of players. We allow for general interaction patterns, but restrict our analysis to linear-quadratic payoffs so that we can formulate the Nash equilibrium problem as the solution to a linear complementarity problem. More generally, our results potentially hold in any set up where equilibrium conditions boil down to a set of piece-wise linear conditions.

We introduce the new class of games with hidden complementarities. Games with hidden complementarities are such that a suitable linear transformation of the interaction matrix produces an induced game with complementarities. We provide general conditions on the interaction matrix such that the equilibrium is unique and/or interior, in which case we characterize equilibrium actions by means of a closed-form expression that involves a generalized version of the Katz-Bonacich network measure of node centrality.

Keywords: Nash equilibrium, uniqueness, complementarity, interaction matrix, linear complementarity problem. **JEL Classification:** A14, C72, L14.

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1 Introduction

It is a feature of most economic groupings that the behavior of each member may affect the behavior and well-being of every other member. This cross influence can sometimes be exerted through some public good nature of the group interaction. In a given competitive industry, for instance, the common market price at which all firms sell their output, and that enters their individual profit calculations, results from the production decision of each such firm. It can also be directly embodied in the preferences of the agents. When cross influences operate, every single action taken by an individual in isolation affects the well-being of other individuals in the group. Cross influences thus naturally yield to interdependent decisions.

The aim of this paper is to analyze the equilibrium behavior for general interaction patterns, where cross influences are allowed to vary in sign and value across different pairs of players.

More precisely, we consider a finite population simultaneous move game with heterogeneous interaction modes across different pairs of players. When an interaction is positive (negative), the decisions of the linked agents are said to be strategic complements (substitutes). Beyond its differences in sign, interactions can also differ in intensity across different pairs of players. For a given population of players, we gather together the characteristics of each possible bilateral interaction (sign and intensity) in a matrix, the interaction matrix. We consider very general interaction matrices, that can reflect both strategic complementarity or strategic substitutability in actions of any intensity, for a same group of players and depending on the pair of players considered. Although we allow for general interaction patterns, we restrict our analysis to linear-quadratic payoffs. For these games, the interaction matrix coincides with second-order derivatives of individual payoffs. With such payoff structure, best-response functions are piece-wise linear.

In fact, beyond the assumption on quadratic payoffs, it is the piece-wise linearity of best-responses that proves crucial for our analysis. Our results thus potentially apply to any set up where equilibrium conditions boil down to a set of piece-wise linear conditions. Bayesian games or team problems with linear information structures, for instance, fall into this category.[1]

Borrowing from the extensive literature on *complementarity problems*, we find conditions on the interaction matrix such that our game with heterogeneous interaction modes has a unique Nash equilibrium in pure strategies. The uniqueness and existence of a solution is a desirable property in order to be able to make unambiguous predictions and to make consistent comparative statics. The most general property that guarantees uniqueness of Nash equilibrium the $P-$matrix property, that is, the fact that all the principal minors of the matrix are positive. Unfortunately, checking that a matrix fulfills this condition is impractical in most cases. We thus opt by singling out some subclasses of matrices for which uniqueness of the equilibrium outcome still holds and that are easier to identify. More importantly, these classes of matrices deserve an intuitive and straight economic interpretation.

More precisely, we introduce a new class of games that we call *games with hidden complementarities*. This class of games includes in particular games with complementarities, for which the cross-payoffs derivatives between every pair of players are non-negative. Games with complementarities have been extensively dealt with in the literature. Supermodular games, for instance, correspond to games with complementarities with a lattice strategy space.[2] In the class of games we consider, we deal with an unbounded strategy space, the non-negative real line. Without boundedness, we loose some properties characteristics of supermodular

[1] See, e.g., Radner (1962) and an application in Calvó-Armengol and De Martí (2006).

[2] In our case, this is equivalent to a bounded strategy space. See Topkis (1979), Vives (1990) and Milgrom and Roberts (1990).

games, such as the lattice structure of the Nash equilibrium set. But some other interesting properties, like existence and uniqueness of the equilibrium correspondence, are obtained from simple and intuitive conditions on the pattern of interactions.

However interesting, games with complementarities constitute only a subclass of the richer family of games with hidden complementarities. By definition, games with hidden complementarities are such that a linear transformation of the corresponding interaction matrix yields an induced game with complementarities. This broader and new class of games includes, for instance, the public good network game analyzed by Bramoullé and Kranton (2006), where players' actions are strategic substitutes across network-linked players. The games with local complementarities and global substitutabilities analyzed in Ballester *et al.* (2006) constitute another example of a game with hidden complementarities.

We provide a characterization of games with hidden complementarities possessing a unique Nash equilibrium, a result that follows from Pang (1979a). When the equilibrium is interior, there is a simple one-to-one correspondence between the Nash equilibrium of the original game with hidden complementarities and that of the induced game with complementarities. We are also able to provide a closed-form expression for this equilibrium that involves a generalized version of the Katz-Bonacich network measure of node centrality.

Although seemingly very demanding, the $P-$matrix property turns out to be a necessary condition for the existence of Nash equilibria in situations particularly compelling for the economist. For instance, when utilities are increasing functions in own actions at the origin, then the $P-$matrix property is necessary for equilibrium existence, and the game has either a unique equilibrium or no equilibrium at all, so that equilibrium multiplicity is also ruled out.

We also analyze a subclass of games with hidden complementarities with particular interest, games with shifted complementarities. Games with shifted complementarities are obtained with an upwards shift of a game with complementarities. In this case, the pattern of hidden complementarities emerges by wiping out the substitutabilities that are latent in the pattern of cross influences through a simple translation upwards of the matrix entries. In other words, games with shifted complementarities can be additively decomposed into local complementarities and local substitutabilities. For this class of games, the equilibrium isomorphism between the original game and the induced game is particularly simple: equilibrium actions are proportional to each other in the two games. The additive shift in the interaction matrix thus translates into a multiplicative shift in the equilibrium actions. Besides, this multiplicative isomorphism almost characterizes the whole class of games with shifted complementarities. Here, again, the Katz-Bonacich network measure characterizes equilibrium behavior. These results generalize previous findings in Ballester *et al.* (2006).

Although the linear form of the complementarity problem (or the linear-quadratic form of the utilities) is a considerably strong assumption, we want to stress that many results of existence and uniqueness on linear complementarity theory have been used to derive existence and uniqueness in the nonlinear case. For instance, in Kolstad and Mathiesen (1987) and Simsek *et al.* (2005) the uniqueness of a solution is essentially determined by the property of uniqueness in the linear complementarity problem induced by the Jacobian of the function at every point.[3] Further boundedness conditions allow these authors to prove the existence of an equilibrium. Thus, our analysis may constitute the starting point for a more general analysis of games with arbitrary (but smooth enough) utility functions.

For the class of shifted complementarity games, we illustrate the benefits provided by the analytical closed-form expression for the equilibrium actions by designing a targeted policy that is able to discriminate across

[3] See, also, Mas-Colell (1979). Notable exceptions to this approach are Bamon and Frayssé (1985) and Rosen (1965).

players depending on their their role in the interaction pattern.[4] More precisely, given a game with general interaction modes between pairs of players, we study the *key group problem* which amounts to choosing optimally the group whose removal from the game disrupts the most the aggregate activity. We provide an analysis of the complexity of the key group problem, and describe an approximation polynomial-time algorithm to the optimal solution for which we bound from above the error term.

Finally, we also discuss the stability properties of the equilibrium solution for the class of games with hidden complementarities.

The paper is organized as follows. Section 2 introduces the interaction patterns and its corresponding game. Section 3 deals with games with hidden complementarities and provides the main result for uniqueness and for the correspondence of the equilibrium to that of the hidden game. Section 4 is devoted to some special subclass of games with hidden complementarities: games with generalized diagonal dominance and games with shifted complementarities. In section 5, we use our results in the context of a game of public goods played in a network. Section 6 examines some extensions, focusing on the design of a network-based policy.

2 The model

We define a finite population simultaneous move game with heterogeneous interaction modes across different pairs of players.

The set of players is $N = \{1, ..., n\}$. Each player i chooses an action x_i in the positive half-space, $x_i \in \mathbb{R}_+$. Given an action profile $x = (x_i, x_{-i}) \in \mathbb{R}^n_+$, individual payoffs are:

$$u_i(x) = \alpha_i(x_{-i}) + \theta_i x_i - \frac{1}{2}\sigma_{ii}x_i^2 - \sum_{j \neq i} \sigma_{ij} x_i x_j, \text{ for all } i \in N. \quad (1)$$

We assume, without any loss of generality for the equilibrium analysis, that $\alpha_i(x_{-i}) = 0$, for all $x_{-i} \in \mathbb{R}^{n-1}_+$. We formulate two additional conditions on payoffs:

(C1) $\theta_i = \partial u_i(\mathbf{0})/\partial x_i \geq 0$.

(C2) $-\sigma_{ii} = \partial^2 u_i(\mathbf{0})/\partial x_i^2 < 0$, for all $i \in N$.

When condition (C1) holds, marginal utilities are non-decreasing at the origin. Condition (C2) corresponds to concavity in own action. When (C1) and (C2) hold, the individual optimization problem $\max\{u_i(x_i, 0) : x_i \in \mathbb{R}_+\}$ has a well-defined and unique solution equal to θ_i/σ_{ii}. If only (C2) holds but $\theta_i < 0$, the maximizer is 0.

Throughout the paper, we impose condition (C2). Sometimes, we also resort to condition (C1) to strengthen some of our results.

The payoff interdependence in (1) is captured by the $n(n-1)$ cross derivatives:

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j}(x) = -\sigma_{ij}, \text{ for all } i \neq j \text{ and } x \in \mathbb{R}^n_+.$$

[4] For instance, a specific model of delinquent behavior where this type of policy emerges is Calvó-Armengol and Zenou (2004). For the case of labor markets, Calvó-Armengol and Jackson (2004) explain how employment outcomes vary across otherwise identical agents with their informational location in the social setting, which again opens the door to pattern–taylored interventions.

These cross derivatives can take arbitrary value and sign across different pairs of players, $-\sigma_{ij} \in \mathbb{R}$. When $-\sigma_{ij} > 0$ (resp. $-\sigma_{ij} < 0$) we say that the actions of players i and j are strategic complements (resp. substitutes) within the pair (i, j) and from i's perspective. The intensity of this strategic linkage is captured by $|\sigma_{ij}|$.

Let $\mathbf{\Sigma} = [\sigma_{ij}] \in \mathbb{R}^{n \times n}$, so that $-\mathbf{\Sigma} = [\partial^2 u_i / \partial x_i \partial x_j]$ is the matrix of cross effects. The matrix $-\mathbf{\Sigma}$ is sometimes referred to in the literature as the Jacobian matrix of the implicit best-response function. Let $\theta = (\theta_1, ..., \theta_n)$ be the profile of first derivatives at the origin.

We denote by $\Gamma(\theta, \mathbf{\Sigma})$ the game of bilateral interactions given by $\langle N, \theta, \mathbf{\Sigma}, \mathbb{R}^n_+ \rangle$ and (1).

Square matrices with non-positive off-diagonal entries are called $Z-$matrices. We denote the class of $Z-$matrices of size n by $\mathbb{Z}_n$. By definition, every $Z-$matrix $\mathbf{M}$ can be written as $\mathbf{M} = s\mathbf{I} - \mathbf{G}$ where $\mathbf{G} \geq \mathbf{0}$, and $\mathbf{I}$ is the identity matrix. Our results focus on a particular class of games that we now define.

Definition 1 $\Gamma(\theta, \mathbf{\Sigma})$ *is a game with hidden complementarities (GHC) if there exist* $\mathbf{X}, \mathbf{\Psi} \in \mathbb{Z}_n$ *such that* $\mathbf{\Sigma X} = \mathbf{\Psi}$.

We call $\Gamma(\theta, \mathbf{\Psi})$ the game hidden behind $\Gamma(\theta, \mathbf{\Sigma})$. We also say that the game $\Gamma(\theta, \mathbf{\Psi})$ is obtained from the game $\Gamma(\theta, \mathbf{\Sigma})$ through the transformation matrix $\mathbf{X}$.

The hidden game $\Gamma(\theta, \mathbf{\Psi})$ has the particularity that $\mathbf{\Psi} = [\psi_{ij}] \in \mathbb{R}^{n \times n}$ belongs to $\mathbb{Z}_n$. Formally, $\psi_{ij} \leq 0$, for all $i \neq j$. We say that $\Gamma(\theta, \mathbf{\Psi})$ is a *game with complementarities* (GC) that is, individual actions are strategic complements for all pair of players. Games with complementarities are a special case of games with hidden complementarities for the transformation matrix $\mathbf{X} = \mathbf{I}$.

Consider a game $\Gamma(\theta, \mathbf{\Sigma})$ satisfying condition (C2). We analyze its set of pure strategy Nash equilibria. We focus more particularly on existence and uniqueness of this equilibrium set. One of the interesting properties that arises in some classes of GHCs is the correspondence between the (unique) Nash equilibrium of the original game and that of the associated hidden game.

As an example of a GHC, consider the classical Cournot competition in a symmetric oligopoly, with linear inverse demand $p(\mathbf{x}) = \alpha - \sum_i x_i$ and quadratic costs $C_i(\mathbf{x}) = \frac{1}{2} c x_i^2$, where x_i is the output of firm i. The profits of each firm are (up to the normalization scalar $r = (1 + c)^{-1}$):

$$\pi_i(\mathbf{x}) = r \left[p(\mathbf{x}) x_i - C_i(\mathbf{x}) \right].$$

Let $\mathbf{J}$ be the square matrix of all ones, and $\mathbf{e}$ the unit vector. Taking first and second derivatives of profits, it is readily checked that the linear oligopoly game is $\Gamma(r\alpha\mathbf{e}, \mathbf{\Sigma})$, where $\mathbf{\Sigma} = \mathbf{I} + r\mathbf{J}$. Although the quantities chosen by competing firms are strategic substitutes, we can find an associated hidden game with complementarities as follows. Let $\mathbf{X} = \mathbf{I} - r\mathbf{J}$ be a transformation matrix. Clearly, $\mathbf{X} \in \mathbb{Z}_n$. Let now $\mathbf{\Psi} = \mathbf{\Sigma X} = \mathbf{I} - r^2 n \mathbf{J}$. Again, we have $\mathbf{\Psi} \in \mathbb{Z}_n$. The game $\Gamma\left(r\alpha\mathbf{e}, \mathbf{I} - r^2 n \mathbf{J}\right)$ is the hidden game associated with the original Cournot game for the transformation matrix $\mathbf{I} - r\mathbf{J}$. The intuitive relationship between both games is the following. In the original Cournot game $\Gamma(r\alpha\mathbf{e}, \mathbf{\Sigma})$ firm i reacts to an increase in x_j by reducing its quantity x_i and all other firms do so. This, in turn, implies that firm i reacts to all those movements by increasing its quantity. This is precisely what is captured by the hidden game $\Gamma(r\alpha\mathbf{e}, \mathbf{\Psi})$, where only these "second-order" reactions of players are accounted for.

Other interesting features should be pointed out about GHCs. First, a game with hidden complementarities can have many associated hidden games, one for each admissible transformation matrix. Second, hidden games should be understood as "virtual" games played by the agents, the true game being $\Gamma(\theta, \mathbf{\Sigma})$.

3 Equilibrium in Games with Hidden Complementarities

In this section, we describe conditions for existence and uniqueness of a Nash equilibrium in GHCs. We also identify when this equilibrium is interior, and relate the Nash equilibrium of the true game with that of its associated hidden game.

3.1 The Katz-Bonacich Centrality Index

The spectral index of a matrix is the largest modulus of its eigenvalues. Given a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$, we denote by $\rho(\mathbf{M})$ its spectral radius.

The closed-form equilibrium expression that we obtain in the next section resorts to a network centrality index due to Bonacich (1987) and Katz (1953). The Katz-Bonacich node centrality in a network counts the number of all weighted and directed paths of any length in the network stemming from that node. We define an extended weighted version of the Katz-Bonacich centrality where different paths connecting different nodes receive different weights.

More precisely, consider some matrix $\mathbf{G} \geq \mathbf{0}$ such that $0 \leq g_{ij} \leq 1$, for all $i, j \in N$. We can interpret $\mathbf{G}$ as the adjacency matrix of a weighted and directed network on N, where the directed link ij receives the weight g_{ij}. If $g_{ij} = g_{ji}$, for all $i \neq j$, we say that the network is un-directed. If $g_{ij} \in \{0, 1\}$, for all $i, j \in N$, we say that the network is un-weighted. If $g_{ii} = 0$, the network has no self-loops. Notice that un-weighted, un-directed networks without self-loops are only one particular example of the more general (weighted, directed and with self-loops) class of networks that we work with.

Let $\mathbf{G}$ be the adjacency matrix of a network on N, $a \geq 0$ a small enough scalar, and $\mathbf{u} \in \mathbb{R}^n_+$ a vector on the nonnegative orthant of $\mathbb{R}^n$. The vector of $\mathbf{u}$-Bonacich node centralities of parameter $a \geq 0$ in the network $\mathbf{G}$ is:

$$\mathbf{b}_{\mathbf{u}}(a, \mathbf{G}) = [\mathbf{I} - a\mathbf{G}]^{-1} \mathbf{u} = \sum_{p=0}^{+\infty} a^p \mathbf{G}^p \mathbf{u}. \tag{2}$$

Notice that the matrix $\mathbf{G}^p = [g_{ij}^{[p]}]$ keeps track of all the paths of length k in the network with adjacency matrix $\mathbf{G}$ (possibly with an intensity reflecting the weight of the links on any such path). Then, when $\mathbf{u} = \mathbf{e}$, the corresponding Katz-Bonacich centrality index

$$b_{i,\mathbf{e}}(a, \mathbf{G}) = \sum_{j=1}^{n} \sum_{p=0}^{+\infty} a^p g_{ij}^{[p]}$$

counts all paths of any length stemming from i in $\mathbf{G}$ weighted by the geometrically decaying factor a. More generally,

$$b_{i,\mathbf{u}}(a, \mathbf{G}) = \sum_{j=1}^{n} u_j \sum_{p=0}^{+\infty} a^p g_{ij}^{[p]}$$

counts the very same paths except that paths yielding to an arbitrary node j are now pondered by u_j.

More generally, consider any matrix $\mathbf{G} \geq \mathbf{0}$. Let $\lambda_{\mathbf{G}} = \max\{g_{ij} : i, j = 1, ..., n\}$. We can then write $\mathbf{G} = \lambda_{\mathbf{G}} \mathbf{G}'$ with $0 \leq g'_{ij} \leq 1$, for all $i, j \in N$. $\mathbf{G}'$ is the adjacency matrix of a weighted and directed network on N. Abusing slightly we write $\mathbf{b}_{\mathbf{u}}(\mathbf{G}) = \mathbf{b}_{\mathbf{u}}(\lambda_{\mathbf{G}}, \mathbf{G}')$.

3.2 Main Result

We now establish necessary and sufficient conditions for the existence and uniqueness of the Nash equilibria in pure strategies of game with hidden complementarities. Under these conditions, we relate the Nash equilibrium of the original game to a Katz-Bonacich index computed on the induced hidden game.

Let $\Gamma(\theta, \mathbf{\Sigma})$ be a GHC, and $\mathbf{X}, \mathbf{\Psi} \in \mathbb{Z}_n$ such that $\mathbf{\Sigma X} = \mathbf{\Psi}$, Thus, we can always decompose the transformation matrix $\mathbf{X}$ and the interaction matrix of the induced game $\mathbf{\Psi}$ as follows:

$$\begin{aligned} \mathbf{\Psi} &= s_1 \mathbf{I} - \mathbf{G}_1 \\ \mathbf{X} &= s_2 \mathbf{I} - \mathbf{G}_2, \end{aligned}$$

where $\mathbf{G}_i \geq \mathbf{0}$ for $i = 1, 2$.

Mangasarian (1976) first introduced the following condition. Let $\Gamma(\theta, \mathbf{\Sigma})$ be a GHC.

(C3) $\mathbf{\Sigma X} = \mathbf{\Psi}$ for some $\mathbf{X}, \mathbf{\Psi} \in \mathbb{Z}_n$ such that $\mathbf{r}^t \mathbf{X} + \mathbf{s}^t \mathbf{\Psi} > \mathbf{0}$ for two non-negative vectors $\mathbf{r}, \mathbf{s} \geq \mathbf{0}$.

Examples of GHC satisfying (C3) are, e.g., games with complementarities (GCs), or games for which $\mathbf{G}_1/s_1$ or $\mathbf{G}_2/s_2$ are contractions, that is, $\rho(\mathbf{G}_2/s_2) < 1$ or $\rho(\mathbf{G}_1/s_1) < 1$.

The following result characterizes existence and uniqueness of equilibria. Its immediate corollary characterizes the set of the Nash equilibria that can be obtained from the Katz-Bonacich centrality measure through the transformation matrix $\mathbf{X}$.

We say that a game $\Gamma(\theta, \mathbf{\Sigma})$ is a GHC* when it is a GHC that admits a unique Nash equilibrium in pure strategies, for all θ.

THEOREM **1** *Consider a game with hidden complementarities $\Gamma(\theta, \mathbf{\Sigma})$ such that (C2) holds. Then,*

(*a*) *$\Gamma(\theta, \mathbf{\Sigma})$ is a GHC* and it satisfies (C3) if and only if there exists some transformation matrix $\mathbf{X} \in \mathbb{Z}_n$ such that $\mathbf{Xu} > \mathbf{0}$ and $\mathbf{\Psi u} = \mathbf{\Sigma X u} > \mathbf{0}$ for some $\mathbf{u} \in \mathbb{R}^n_+$.*

(*b*) *Moreover, when (C1) also holds and $\mathbf{X b}_{\boldsymbol{\theta}}(\mathbf{G}_1/s_1) \geq \mathbf{0}$, then the unique pure strategy Nash equilibrium of $\Gamma(\theta, \mathbf{\Sigma})$ is precisely*

$$\mathbf{x}^* = \frac{1}{s_1} \mathbf{X b}_{\boldsymbol{\theta}}(\mathbf{G}_1/s_1).$$

Note that $\mathbf{\Psi u}, \mathbf{Xu} > \mathbf{0}$ can be rewritten as $(s_i \mathbf{I} - \mathbf{G}_i)\mathbf{u} > \mathbf{0}$ or, equivalently, $(\mathbf{I} - \mathbf{G}_i/s_i)\mathbf{u} > \mathbf{0}$, for $i = 1, 2$.[5] Given that $s_i > 0$ and $\mathbf{G}_i \geq 0$, this implies that both $\mathbf{G}_1/s_1$ and $\mathbf{G}_2/s_2$ are contractions, that is, their respective spectral indexes are strictly smaller than 1: $s_i > \rho(\mathbf{G}_i) \geq 0$ for $i = 1, 2$. The reverse, though, does not hold. Indeed, Theorem 1 (*a*) requires that both $\mathbf{G}_1/s_1$ and $\mathbf{G}_2/s_2$ are contractions ($\mathbf{\Psi u}, \mathbf{Xu} > \mathbf{0}$) *and* that they are so in a "similar" way, by sharing the condition over a common vector $\mathbf{u} > \mathbf{0}$. The following example shows that equilibrium uniqueness requires this extra condition.

Consider the game $\Gamma(\theta, \mathbf{\Sigma})$, with

$$\begin{aligned} \mathbf{\Sigma} &= \mathbf{\Psi X}^{-1} \\ \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1}. \end{aligned}$$

[5] Given that $\mathbf{G}_i \geq \mathbf{0}$, necessarily $s_i > 0$.

Both $\mathbf{X}, \mathbf{\Psi} \in \mathbb{Z}_n$. Also, $\mathbf{\Psi} = \mathbf{I} - \mathbf{G}_1$ and $\mathbf{X} = \mathbf{I} - \mathbf{G}_2$, with

$$\mathbf{G}_1 = \mathbf{G}_2^t = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Clearly, $\rho(\mathbf{G}_1) = \rho(\mathbf{G}_2) = 0$, implying that $\mathbf{G}_1/s_1$ and $\mathbf{G}_2/s_2$ are both contractions. But it turns out that $\mathbf{X}$ and $\mathbf{\Psi}$ cannot share a common semipositivity vector. Indeed, for any $\mathbf{u} = (u_1, u_2) \geq \mathbf{0}$, the inequality $(\mathbf{Xu})_1 = u_1 - 2u_2 > 0$ implies $(\mathbf{\Psi u})_2 = u_2 - 2u_1 < 0$. In fact, one can readily check that the set of Nash equilibria of the game $\Gamma(\theta, \mathbf{\Sigma})$ is empty for all $\boldsymbol{\theta} > \mathbf{0}$.

It is also important to see that, when $\Gamma(\theta, \mathbf{\Sigma})$ is a GHC*, condition (a) need not hold for all possible transformation matrix and associated hidden game. So, rejecting the common semipositivity condition for some given transformation matrix $\mathbf{X}$ does not allow to conclude that the game has either no equilibrium at all, or a multiplicity of equilibria.

When Theorem 1(b) holds, the pure strategy Nash equilibrium of the original game is isomorphic to that of the induced hidden game. The one-to-one mapping is linear, and involves the very same transformation matrix $\mathbf{X}$ that maps the interaction matrices of the original and the induced game with each other. If Theorem 1(b) holds with strict inequality, the unique Nash equilibrium is interior.

Notice, that condition (a) does not always imply condition (b) (see Section 5.1. for an example). In other words, even when $\Gamma(\theta, \mathbf{\Sigma})$ has a unique Nash equilibrium, there need not be a simple linear correspondence between this equilibrium and that of the associated hidden game.

In some cases, instead, conditions (a) and (b) are interrelated. The equilibrium then exists, is unique, interior, and of the Katz-Bonacich type. For instance, when both $\boldsymbol{\theta} > \mathbf{0}$ and $\mathbf{Xb}_{\boldsymbol{\theta}}(\mathbf{G}_1/s_1) > \mathbf{0}$, the common semipositivity condition in (a) holds for the vector $\mathbf{b}_{\boldsymbol{\theta}}(\mathbf{G}_1/s_1)$.

COROLLARY 1 *Suppose that (C2) holds, and let* $\boldsymbol{\theta} > \mathbf{0}$. *Then* $\Gamma(\theta, \mathbf{\Sigma})$ *has a unique pure strategy Nash equilibrium, which is interior, and it satisfies (C3) if and only if there exists some transformation matrix* $\mathbf{X} \in \mathbb{Z}_n$ *for which* $\mathbf{\Sigma X} = s_1 \mathbf{I} - \mathbf{G}_1 \in \mathbb{Z}_n$ *is such that* $s_1 > \rho(\mathbf{G}_1)$ *and* $\mathbf{Xb}_{\boldsymbol{\theta}}(\mathbf{G}_1/s_1) > 0$. *Then, this equilibrium is:*

$$\mathbf{x}^* = \frac{1}{s_1} \mathbf{Xb}_{\boldsymbol{\theta}}(\mathbf{G}_1/s_1).$$

When $\Gamma(\theta, \mathbf{\Sigma})$ is a game with complementarities, that is, all payoff cross effects in (1) are non-negative, we have $\mathbf{\Sigma} = \mathbf{\Psi} = s\mathbf{I} - \mathbf{G}$. The transformation matrix is $\mathbf{X} = \mathbf{I}$, and Theorem 1 reduces to $\mathbf{G}/s$ being a contraction.

The intuition is the following. With non-negative cross effects, upward shifts in players' actions feed positively into each other. If these cross effects are moderate, these feedback loops dampen, and players' actions eventually reach some equilibrium point. But, if these cross effects are too big, the positive feed-back loops can trigger an unbounded escalation in individual actions, and equilibrium fails to exist.[6] When $\mathbf{G}/s$ is a contraction, that is, $s > \rho(\mathbf{G})$, these payoff complementarities are bounded from above. This bound accounts for both the size and the pattern of these complementarities, measured by the spectral index of (roughly) the off-diagonal entries in $\mathbf{\Sigma}$.[7]

[6] Unless the strategy space is arbitrarily bounded from above, of course, in which case we can borrow directly from the literature on supermodular games.

[7] The class of matrices $\mathbf{\Sigma} \in \mathbb{R}^{n \times n}$ satisfying $\mathbf{\Sigma} \in \mathbb{Z}_n$ (i.e., $\mathbf{\Sigma} = s\mathbf{I} - \mathbf{G}$, $\mathbf{G} \geq \mathbf{0}$) and $s > \rho(\mathbf{G})$ is often referred to in the literature as the class of $K-$matrices ($\mathbb{K}_n$), also called nonsingular $M-$matrices.

Suppose, for instance, that $\mathbf{G}$ is a $(0,1)$ symmetric adjacency matrix of an un-weighted and un-directed network without self-loops.

Consider first regular networks, where each player has the same number of connections, that is, $\sum_{j=1}^{n} g_{ij} = k \leq n-1$, for all i. The index of a regular network is equal to its connectivity, that is, $\rho\left(\mathbf{G}^{regular_k}\right) = k$. The eigenvalue condition then boils down to the (standard) inequality $s > k$.

Consider now minimally connected networks, also referred to as trees. The most irregular tree is the star. The most regular tree is the line. Both networks, though, have the same total number of links, $n-1$. In the star, the central node reaps complementarities from many different sources, while in the line the playing field is more even. We have $\rho\left(\mathbf{G}^{star}\right) = \sqrt{n-1} > 2\cos\frac{\pi}{n+1} = \rho\left(\mathbf{G}^{line}\right)$. Not surprisingly, the equilibrium existence and uniqueness condition, $s > \rho\left(\mathbf{G}\right)$, is more binding for the star than for the line

Corner solutions are also easily dealt with in games with complementarities. The next result summarizes all these findings.

Given $S \subset N$ and $\mathbf{M} \in \mathbb{R}^{n \times n}$, let $\mathbf{M}_S$ be the restriction of $\mathbf{M}$ to the rows and columns in S. For a vector $\mathbf{x} \in \mathbb{R}^n$, define $S^+\left(\mathbf{x}\right) = \{i \in N : x_i > 0\}$

Corollary **2** *Consider a game with complementarities $\Gamma\left(\theta, \mathbf{\Sigma}\right)$ such that (C2) holds. Then,*

1. *This game has a unique equilibrium for all θ if and only if $s > \rho\left(\mathbf{G}\right)$.*

2. *The Nash equilibrium x^* satisfies*

$$\begin{aligned} x_i^* &= \frac{1}{s} b_{i,\boldsymbol{\theta}}\left(\mathbf{G}_{S^+(\boldsymbol{\theta})}/s\right), \text{ for all } i \in S^+\left(\boldsymbol{\theta}\right) \\ x_i^* &= 0 \text{ otherwise.} \end{aligned}$$

4 The Linear Complementarity Problem

Let us describe now the tools that are the basis for the main results shown in the paper.

The linear complementarity problem is a very well studied problem in mathematics. We have borrowed from this literature to address several issues of interest for the economist, e.g. existence and uniqueness of Nash equilibrium. We have worked out the economics of some of these results and extended some others to characterize the Nash equilibria of the class of games with hidden complementarities.

Given a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{q} \in \mathbb{R}^n$, the linear complementarity problem $LCP(\mathbf{q}, \mathbf{M})$ consists on finding a vector $\mathbf{z} \in \mathbb{R}^n$ satisfying:

$$\begin{aligned} \mathbf{z} &\geq \mathbf{0} \\ \mathbf{q} + \mathbf{M}\mathbf{z} &\geq \mathbf{0} \\ \mathbf{z}^t\left(\mathbf{q} + \mathbf{M}\mathbf{z}\right) &= 0. \end{aligned} \tag{3}$$

A very well-known and central result in the literature (Samelson *et al.* 1958) on the linear complementarity problem is the following: the linear complementarity problem $LCP\left(\mathbf{q}, \mathbf{M}\right)$ admits a unique solution for all $\mathbf{q} \in \mathbb{R}^n$ if and only if all the principal minors of $\mathbf{M}$ are positive.[8] A matrix satisfying this property is called a $P-$matrix. We denote the class of $P-$matrices of size n by $\mathbb{P}_n$.

[8]The principal minor corresponging to $S \subset N$ is defined as $\det \mathbf{M}_S$.

For instance, positive definite matrices are in $\mathbb{P}_n$.[9] The class $\mathbb{P}_n$ also contains diagonally dominant matrices with positive diagonal, etc. Obviously, if $\mathbf{M}$ is a $P-$matrix, it is invertible because $\det \mathbf{M} > 0$. Cottle *et al.* (1992) is a standard reference in this literature to which we refer the reader for more details.

Most of our results use the fact that the Nash equilibria of a game $\Gamma(\theta, \mathbf{\Sigma})$ are precisely the solutions of a linear complementarity problem.

Lemma 1 *Suppose that (C2) holds. Then, the set of pure strategy Nash equilibria of* $\Gamma(\theta, \mathbf{\Sigma})$ *is given by the set of solutions to* $LCP(-\boldsymbol{\theta}, \mathbf{\Sigma})$.

The principal minors of dimension one of a matrix coincide with its diagonal terms, and thus $\mathbf{M} \in \mathbb{P}_n$ implies that all its diagonal terms are positive, $m_{ii} > 0$, for all $i = 1, ..., n$. Together with lemma 1, (C2) is thus a necessary condition for equilibrium uniqueness of $\Gamma(\theta, \mathbf{\Sigma})$ for arbitrary θ.

Example: Cournot oligopoly with differentiated products Consider a Cournot oligopoly with n firms, heterogeneous goods, linear inverse demand and quadratic costs. Firms decide on their output $x_i \geq 0$. Firm i's profits are:

$$\pi_i(\mathbf{x}) = p_i(\mathbf{x})x_i - \frac{1}{2}cx_i^2, \text{ where } p_i(\mathbf{x}) = \alpha + \sum_{i=1}^{n} \phi_{ij}x_j.$$

Here, $\partial p_i / \partial x_j = \phi_{ij}$ describes the complementarity between goods i and j. When $\phi_{ij} > 0$ (resp. < 0), we say that goods i and j are gross complements (resp. substitutes). When the demand for goods comes from the maximization of a concave utility function by a representative agent, the matrix $\mathbf{\Phi} = \left[\phi_{ij}\right]$ is symmetric and negative definite, implying that $\phi_{ii} < 0$ for all $i = 1, ..., n$ (Vives, 1999). In other words, $-\mathbf{\Phi} \in \mathbb{P}_n$.

In particular, let $\phi_{ij} \in \{0, 1\}$, for all $i \neq j$, and $\phi_{ii} = 0$, for all i. A discrete version of the canonical model of Salop corresponds with $\mathbf{\Phi}$ being the adjacency matrix of a wheel network. Instead, the Dixit-Stiglitz-Spence mode corresponds to $\mathbf{\Phi}$ being the adjacency matrix of the the complete networkl. Our more general formulation encompasses these two standard models, as well as the whole range of intermediate configurations. In particular, one can use Theorem 1 and its corollaries to work out the Cournot equilibrium for (general) asymmetric complementarity patterns and discuss, e.g., the introduction of new goods, etc.

Marginal and infra-marginal profits are:

$$\frac{\partial \pi_i}{\partial x_i}(\mathbf{0}) = \alpha, \text{ and } \frac{\partial^2 \pi_i}{\partial x_i \partial x_j}(\mathbf{x}) = \begin{cases} -c + 2\phi_{ii}, \text{ if } i = j \\ \phi_{ij}, \text{ if } i \neq j \end{cases}$$

Note that $-c + 2\phi_{ii} < 0$, and thus (C2) holds. By Lemma 1, the Nash equilibria of the Cournot oligopoly are given by the solutions to $LCP(-\alpha\mathbf{e}, \mathbf{D} - \mathbf{\Phi})$, where $\mathbf{D}$ is a diagonal matrix with $d_{ii} = c - 2\phi_{ii} > 0$. The fact that $-\mathbf{\Phi} \in \mathbb{P}_n$ implies that $\mathbf{D} - \mathbf{\Phi} \in \mathbb{P}_n$. As a consequence, this equilibrium exists and is unique for all $\alpha, c \geq 0$.

$\mathbb{P}_n$ is a very important class of matrices, as the result by Samelson *et al.* (1958) shows and the previous example illustrates. Unfortunately, detecting whether a matrix is in $\mathbb{P}_n$ is computationally very demanding.[10]. Games satisfying the conditions of Theorem 1, instead, are easier to detect. It amounts to first solving a linear program, and then checking the consistency of a linear inequality system (Tsatsomeros, 2002). In fact,

[9] $\mathbf{M} \in \mathbb{R}^{n \times n}$ is positive definite if $\mathbf{x}^t\mathbf{M}\mathbf{x} > 0$, for all $\mathbf{x} \neq \mathbf{0}$.

[10] Checking the $2^n - 1$ principal minors of an arbitrary matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ is an $O(n^3 2^n)$ task. See, e.g., Tsatsomeros (2002) for more details.

condition (a) is due to Pang (1979a) which characterizes a subset of $\mathbb{P}_n$ when (C3) holds. Very importantly, Theorem 1 encompasses a number of meaningful economic situations. For instance, dominance diagonal games with concave utility functions, a class of games familiar to the economist, belong to this class of games.

Notice that the $P-$matrix property guarantees that a solution exists and is unique for all $\boldsymbol{\theta} \in \mathbb{R}^n$. This can sometimes be a desirable property. For instance, in the linear oligopoly example, the $P-$matrix property guarantees existence and uniqueness of the Cournot equilibrium for all values of $\partial \pi_i / \partial x_i(\mathbf{0})$. One can then freely work out price subsidy/tax policies without worrying about equilibrium existence or multiplicity issues. In some other cases, though, requiring existence and uniqueness for all $\boldsymbol{\theta} \in \mathbb{R}^n$ can be superfluous, e.g., when only one particular parametric specification of the game $\Gamma(\theta, \mathbf{\Sigma})$ matters. In these cases, it is natural to ask whether requiring existence and uniqueness everywhere for θ does not impose too stringent conditions on the interaction matrix $\mathbf{\Sigma}$ that yield to the $P-$matrix property. It turns out that the $P-$matrix property is a necessary condition for the existence of Nash equilibria when $\boldsymbol{\theta} > \mathbf{0}$, which corresponds to situations particularly compelling for the economist. We deal with this issue in Section 6.1.

5 Classes of GHCs

Aside from the games with complementarities introduced in section 2, there are other classes of GHCs that deserve our attention.

5.1 Games with Generalized Diagonal Dominance (GGDD)

A game $\Gamma(\theta, \mathbf{\Sigma})$ is a GGDD if there is a vector $\mathbf{d} > \mathbf{0}$ such that $d_i \sigma_{ii} > \sum_{j \neq i} d_j |\sigma_{ij}|$ for $i = 1, ..., n$. Note that this condition implies that $\sigma_{ii} > 0$, i.e., individual payoffs are concave in own-action, which coincides with (C2).

GGDDs form a subclass of GHCs and they possess a unique Nash equilibrium for all θ (Berman and Plemmons, 1994). This class of games contains, in particular, the class of games with a dominant diagonal (when $d_i = 1$ for all $i \in N$).

An interesting feature of GGDD is the global stability properties of the unique equilibrium.

Given that $\sigma_{ii} > 0$ for all $i = 1, ...n$, a best response of player i to x_{-i} is

$$BR_i(x_{-i}) = \sigma_{ii}^{-1} \max\{\theta_i - \sum_{j \neq i} \sigma_{ij} x_j, 0\}. \tag{4}$$

Following Gabay and Moulin (1980), we consider the following iteration map where, at every stage, players combine previous actions with best replies to the actions taken in the previous round:

$$x_i^{k+1} = \left(1 - \xi_i^k\right) x^k + \xi_i^k BR_i(x_{-i}^k), \tag{5}$$

where $\xi_i^k \in (0, 1]$ for each player i and stage $k = 0, 1, ...$ The sequence is initiated at some arbitrary x^0.

Let $\mathbf{\Xi}^k$ denote the diagonal matrix with diagonal entries ξ_i^k, for all $i = 1, ..., n$. Let us assume that the sequence $\left(\mathbf{\Xi}^k\right)_{k=0}^{\infty}$ converges to $\mathbf{\Xi}$. Gabay and Moulin (1980) prove that the iteration (5) converges to the unique Nash equilibrium of the game when the matrix $\mathbf{\Sigma}$ has a dominant diagonal. In fact, their result holds for the slightly more general class of interaction matrices that we describe.

The comparison (or companion) matrix $\overline{\mathbf{M}}$ of a given matrix $\mathbf{M}$ is defined by $\overline{m}_{ii} = |m_{ii}|$ and $\overline{m}_{ij} = -|m_{ij}|$ for all $j \neq i$. By definition, $\overline{\mathbf{M}} \in \mathbb{Z}_n$, and so the comparison matrix displays complementarities. It is well-known that GGDDs are precisely those games for which their corresponding *comparison games* have also a unique equilibrium. Thus, GGDD's contain the class of games with diagonal dominance and, also, GCs with a unique equilibrium (GCs*).

PROPOSITION **1** *Let $\Gamma(\theta, \mathbf{\Sigma})$ be a GGDD. Then, for any weighting rule $\left(\mathbf{\Xi}^k\right)_{k=0}^{\infty}$ with a well-defined limit $\mathbf{\Xi}$ and for any x^0, the iteration procedure defined in (5) converges to the unique Nash equilibrium of this game.*

In particular, take a game with complementarities that has a unique equilibrium. Then, we can conclude that this equilibrium is globally stable for the tatônnement process (5). Notice that this tatônnement process uses best-responses, and that best-responses only use local information about the interaction matrix. Indeed, player i need only know $\sigma_{i1}, ..., \sigma_{in}$ and the actions taken by the other players to compute the best-response (4). Therefore, *only local information* is needed on the interaction matrix $\mathbf{\Sigma}$ for the players to be able to reach the unique Nash equilibrium using (5) and this local information, and irrespective of the initial action profile x^0. The Nash equilibrium that they eventually reach, characterized in Corollary 2, is of the Katz-Bonacich type, and each equilibrium action depends, in a well-specified way, on the whole interaction matrix $\mathbf{\Sigma}$.[11]

Although global stability of the equilibrium is granted for games with complementarities, we show in Lemma 2 that there is no guarantee that any arbitrary game with hidden complementarities, even with a unique equilibrium (GHC*), has this global stability property.

5.2 Games with Shifted Complementarities (GSC)

$\Gamma(\theta, \mathbf{\Sigma})$ is a game with shifted complementarities if there exists $\mathbf{\Psi} \in \mathbb{Z}_n$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_+$, $\mathbf{u}, \mathbf{v} > \mathbf{0}$, such that $\mathbf{\Sigma} = \mathbf{\Psi} + \mathbf{u}\mathbf{v}^t$. As will be shown below, games with shifted complementarities with a unique Nash equilibrium (GSC*) form a subclass of GHC*.

Games with shifted complementarities can be decomposed additively into a hidden game $\Gamma(\theta, \mathbf{\Psi})$ with complementarities *plus* a *substitutability* shift $\mathbf{u}\mathbf{v}^t$ that has rank one. In other words, a game with shifted complementarities is such that a suitable downwards translation of the matrix of cross effects $\mathbf{\Sigma}$ of the original game $\Gamma(\theta, \mathbf{\Sigma})$ induces a new game with complementarities.

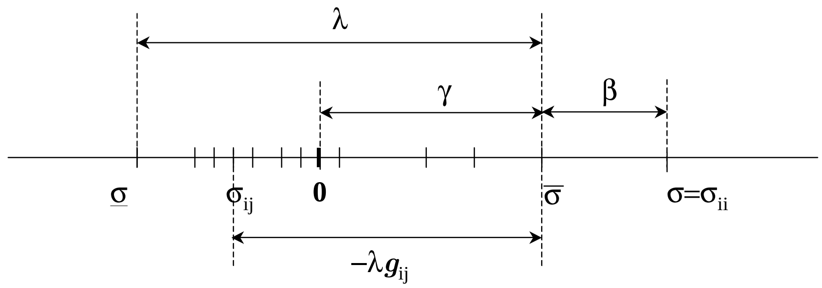
Example: homogeneous substitutability shift This model is due to Ballester *et al.* (2006).

Let $\mathbf{\Sigma} = [\sigma_{ij}] \in \mathbb{R}^{n \times n}$ such that $\sigma_{ii} > \max\{\overline{\sigma}, 0\}$, for all $i = 1, ..., n$, where $\overline{\sigma} = \max\{\sigma_{ij} \mid i \neq j\}$. For simplicity, we also assume that $\sigma_{ii} = \sigma$, for all $i \neq j$, although the model carries over to the more general case with heterogeneous diagonal values. We operate the following centralization, followed by a normalization, of the matrix entries.

Let $\underline{\sigma} = \min\{\sigma_{ij} \mid i \neq j\}$, $\lambda = \overline{\sigma} - \underline{\sigma} \geq 0$, and $\gamma = \max\{\overline{\sigma}, 0\} \geq 0$, and write $\sigma_{ij} = -\lambda g_{ij} + \gamma$, for $i \neq j$, and $g_{ii} = 0$. We assume that $\gamma > 0$, which is equivalent to $\overline{\sigma} > 0$.

By construction, $0 \leq g_{ij} \leq 1$, and we interpret $\mathbf{G} = [g_{ij}] \in [0, 1]^{n \times n}$ as the adjacency matrix of a network. Finally, let $\beta = \sigma - \gamma > 0$. See figure below.

[11] See Galeotti *et al.* (2006) for network games with incomplete information.



It is readily checked that $\mathbf{\Sigma} = \mathbf{\Psi} + \gamma\mathbf{J}$, where $\mathbf{\Psi} = \beta\mathbf{I} - \lambda\mathbf{G} \in \mathbb{Z}_n$ and $\mathbf{J} = \mathbf{e}\mathbf{e}^t$ is the matrix of all ones. The game $\Gamma(\theta, \mathbf{\Sigma})$ is thus a GSC.

We first notice that GSCs such that the induced game $\Gamma(\theta,\mathbf{\Psi})$ is a GC*, also have a unique equilibrium. We call this class of games GSC*.

Proposition 2 *Consider a GSC* $\Gamma(\theta, \mathbf{\Sigma})$ *such that* $\mathbf{\Sigma} = \mathbf{\Psi} + \mathbf{u}\mathbf{v}^t$, *for some* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_+$, $\mathbf{u}, \mathbf{v} > \mathbf{0}$ *and such that* $\Gamma(\theta, \mathbf{\Psi})$ *is a GC**. *Then,* $\Gamma(\theta, \mathbf{\Sigma})$ *is a GSC**.

Suppose that $\mathbf{\Sigma} = \mathbf{\Psi} + \mathbf{u}\mathbf{v}^t$ for some $\mathbf{\Psi} \in \mathbb{Z}_n$ and $\mathbf{u}, \mathbf{v} > \mathbf{0}$. The fact that $\Gamma(\theta, \mathbf{\Psi})$ is a GC* implies that $\mathbf{\Psi}$ is invertible, (because $\mathbf{\Psi} \in \mathbb{P}_n$). We can then write $\mathbf{\Sigma}\mathbf{X} = \mathbf{\Psi}$, where $\mathbf{X} = (\mathbf{I} + \mathbf{\Psi}^{-1}\mathbf{u}\mathbf{v}^t)^{-1}$ and, consequently, $\mathbf{\Sigma}$ is also invertible.[12] This linear transformation of $\mathbf{\Sigma}$ into $\mathbf{\Psi}$ is similar to the one used in Theorem 1. Condition (a) of Theorem 1 guarantees equilibrium existence and uniqueness, but an additional condition (b) is explicitly introduced to obtain equilibrium interiority. Therefore, Proposition 2 alone does not guarantee that the equilibrium is interior.

The next result derives uniqueness *and* interiority for games with shifted complementarities from a unique condition. As a matter of fact, this condition almost completely characterizes all games with shifted complementarities that have a unique and interior equilibrium.

The idea behind the result consists on operating the "righ" rank one shift $\mathbf{u}\mathbf{v}^t$, which is constructed the following way. Recall that we need both $\mathbf{\Sigma}-\mathbf{u}\mathbf{v}^t \in \mathbb{Z}_n$ *and* $\mathbf{\Sigma}-\mathbf{u}\mathbf{v}^t \in \mathbb{P}_n$. The first condition states that the shifted game has complementarities. The second condition guarantees equilibrium existence and uniqueness. The first condition amounts to having $\sigma_{ij} \leq u_i v_j$ for all $i \neq j$, and $\sigma_{ii} > u_i^2$ for at least one i, which asks for (roughly) vectors $\mathbf{u}, \mathbf{v}$ with high enough coordinates. The second condition requires that the complementarities in $\mathbf{\Sigma}-\mathbf{u}\mathbf{v}^t$ (roughly the off-diagonal terms) be bounded from above by the diagonal terms. This, instead, calls for as small off-diagonal terms as possible, and thus low coordinate vectors $\mathbf{u}, \mathbf{v}$. Given a vector $\mathbf{u}$, we thus chose the vector $\mathbf{v}$ given by $v_j = \max\{\sigma_{ij}/u_i : i \neq j\}$, for all j, so that the complementarity condition $\sigma_{ij} \leq u_i v_j$ for all $i \neq j$ is binding.

The following notations handle these expressions in full generality.

[12] Indeed, $\mathbf{\Sigma} = \mathbf{\Psi}(\mathbf{I} + \mathbf{\Psi}^{-1}\mathbf{u}\mathbf{v}^t)$ is the product of two invertible matrices. The inverse of $\mathbf{\Sigma}$ is given by the following expression:

$$\mathbf{\Sigma}^{-1} = \mathbf{\Psi}^{-1} - \frac{1}{1 + \mathbf{v}^t\mathbf{\Psi}^{-1}\mathbf{u}}\mathbf{\Psi}^{-1}\mathbf{u}\mathbf{v}^t\mathbf{\Psi}^{-1}.$$

Fix some $\varepsilon > 0$. For all vector $\mathbf{u} \in \mathbb{R}^n_+$, $\mathbf{u} > \mathbf{0}$ and for all matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$, define the vector $\mathbf{u}_{\mathbf{M},\varepsilon}$ as follows:

$$u_{i,\mathbf{M},\varepsilon} = \max\{\varepsilon, \max\{\frac{m_{ji}}{u_j} : j \neq i\}\}.$$

By construction, $\mathbf{M} - \mathbf{u}\mathbf{u}^t_{\mathbf{M},\varepsilon} \in \mathbb{Z}_n$. Also, if $\mathbf{v} \leq \mathbf{u}_{\mathbf{M},\varepsilon}$, $\mathbf{v} \neq \mathbf{u}_{\mathbf{M},\varepsilon}$, then $\mathbf{M} - \mathbf{u}\mathbf{v}^t \notin \mathbb{Z}_n$.

As a matter of fact, the "right" rank one shift uses $\mathbf{u} = \boldsymbol{\theta}$ and $\mathbf{v} = \boldsymbol{\theta}_{\boldsymbol{\Sigma},\varepsilon}$, so that the complementarity condition holds minimally.

Proposition **3** *Consider a game $\Gamma_1 = \Gamma(\theta, \boldsymbol{\Sigma})$ such that (C2) hold. Let $\boldsymbol{\theta} > \mathbf{0}$.*

(*a*) *If $\Gamma_2 = \Gamma\left(\theta, \boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\theta}^t_{\boldsymbol{\Sigma},\varepsilon}\right)$ is a GC* for some $\varepsilon > 0$, then Γ_1 is a GSC*. Moreover, if $\mathbf{y}^*$ is the (Katz-Bonacich) Nash equilibrium of Γ_2, then*

$$\mathbf{x}^* = \frac{1}{1 + \boldsymbol{\theta}^t_{\boldsymbol{\Sigma},\varepsilon}\mathbf{y}^*}\mathbf{y}^*$$

is the Nash equilibrium of Γ_1. Both equilibria are interior.

(*b*) *Reciprocally, suppose that Γ_1 has a unique interior equilibrium $\mathbf{x}^* > \mathbf{0}$, and that $\Gamma_2 = \Gamma\left(\theta, \boldsymbol{\Sigma} - \boldsymbol{\theta}\mathbf{v}^t\right)$ is a GC for some $\mathbf{v} > \mathbf{0}$ such that $\mathbf{v}^t\mathbf{x}^* < 1$. Then, Γ_2 is a GC* and its equilibrium $\mathbf{y}^*$ satisfies:*

$$\mathbf{x}^* = \frac{1}{1 + \mathbf{v}^t\mathbf{y}^*}\mathbf{y}^*.$$

Games with shifted complementarities with a unique Nash equilibrium form a subclass of games with hidden complementarities.

For games with shifted complementarities, the equilibrium isomorphism between the original game and the associated shifted (hidden) games of Theorem 1(b) takes a particularly simple form. The equilibrium in the shifted game is $\mathbf{y} = \mathbf{b}_{\boldsymbol{\theta}}(\mathbf{G}/s)/s$, as derives from Corollary 2. The equilibrium action in the original game is then proportional to that of the induced game, where the proportionality factor is identical for all players and equal $1/(1 + \mathbf{v}^t\mathbf{y})$. In other words, *the additive shift in the interaction matrix leads to a multiplicative shift in the equilibrium actions*. In particular, the *relative* equilibrium actions across players is the same for the original and for the associated shifted game.

The reason is the following. First, the equilibrium rank-one shift wipes out a common substitutability term from the original game. The resulting induced game with complementarities has an interaction matrix $s\mathbf{I} - \mathbf{G}$ that determines the values of the relative equilibrium actions. The impact of the rank-one global susbtitutability term has then a level effect on these actions, common to all players, and similar to that resulting from oligopoly competition.

This is well-illustrated in the example below.

Example Consider the following interaction matrix:

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & 1 \\ 1 & 1 & 3/2 \end{bmatrix}.$$

Note that this matrix does not have a dominant diagonal. We operate an $\mathbf{ee}^t$ rank-one shift that yields the following interaction matrix corresponding to a game with complementarities:

$$\mathbf{\Sigma} - \mathbf{ee}^t = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \in \mathbb{Z}_n$$

The equilibrium y^* for the induced game $\Gamma(\theta, \mathbf{\Sigma} - \mathbf{ee}^t)$ is $y^* = (2,2,1)/2$. The equilibrium x^* for the original game $\Gamma(\theta, \mathbf{\Sigma})$ is proportional to y^*, where the proportionality factor is $1/(1 + \mathbf{y}^{*t}\mathbf{e}) = 2/7$, that is, $x^* = 2y^*/7 = (2,2,1)/7$.

We now illustrate Proposition 3(b), and particularly the condition $\mathbf{v}^t\mathbf{x}^* < 1$.

Example Consider the following matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

It is readily checked that $\mathbf{\Sigma} \in \mathbb{P}_3$, and thus $\Gamma(\theta, \mathbf{\Sigma})$ has a unique equilibrium for all θ. In particular, when $\boldsymbol{\theta} > \mathbf{0}$, this unique equilibrium is interior and given by $x^* = (\theta_1 + \theta_3, \theta_1 + \theta_2, \theta_2 + \theta_3)/2$. Note also that $\boldsymbol{\theta}^t_{\mathbf{\Sigma},\varepsilon} = (1/\theta_3, 1/\theta_1, 1/\theta_2)$, for all $\varepsilon < \min\{1/\theta_i : i = 1,2,3\}$, and thus $\boldsymbol{\theta}^t_{\mathbf{\Sigma},\varepsilon}\mathbf{x}^* > 3/2 > 1$. Given that $\boldsymbol{\theta}^t_{\mathbf{\Sigma},\varepsilon}$ is the best candidate vector $\mathbf{v} > \mathbf{0}$ such that $\mathbf{\Sigma} - \boldsymbol{\theta}\mathbf{v}^t \in \mathbb{Z}_n \cap \mathbb{P}_3$,[13] we conclude that $\mathbf{\Sigma} - \boldsymbol{\theta}\mathbf{v}^t \notin \mathbb{P}_3$, for all $\mathbf{v} > \mathbf{0}$. Therefore, the unique Nash equilibrium of $\Gamma(\theta, \mathbf{\Sigma})$ is *not* proportional to a Bonacich centrality measure. As a matter of fact, notice that a sequence of rank-one shifts leaves at best with the following matrix:

$$\mathbf{\Psi} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \mathbf{I} - \mathbf{G},$$

which belongs to $\mathbb{Z}_3$ but not to $\mathbb{P}_3$ because $\det \mathbf{\Psi} = 0$.

Proposition 1 establishes a global stability property of the unique Nash equilibrium of games with complementarities. With shifted complementarities global stability is not guaranteed anymore, as the following example shows.

Let $\mathbf{\Sigma} = \beta\mathbf{I} + \mathbf{ee}^t$ and $\boldsymbol{\theta} = \alpha\mathbf{e}$, with $\alpha, \beta > 0$. This is a game with shifted complementarities with a unique equilibrium $x_i^* = \alpha/(n + \beta) > 0$, for all $i = 1, ..., n$.[14] The iteration process can be written as:

$$x_i^{k+1} = \frac{1}{\beta + 1} \max\{\alpha - \sum_{j \neq i} x_j^k, 0\}. \tag{6}$$

Lemma **2** *For any α, β, $n \geq 2$ and $\mathbf{x}^0 \neq \mathbf{x}^*$, the sequence (6) diverges away from the unique Nash equilibrium if $\beta < n - 2$.*

[13] The class of matrices $\mathbb{Z}_n \cap \mathbb{P}_n$ is referred to in the linear complementarity literature as $\mathbb{K}_n$.

[14] This corresponds to the classical Cournot competition in a symmetric oligopoly, with linear inverse demand $p(\mathbf{x}) = \alpha - \sum_i x_i$ and quadratic costs $c_i(\mathbf{x}) = \frac{1}{2}cx_i^2$, which results in $\beta = 1 + c$.

6 Beyond linear quadratic payoffs

So far, we have only dealt with linear-quadratic utilities, which lead to linear (up to non-negativity) best-responses. Albeit somehow restrictive, we argue here that this assumption on payoffs sheds some light into the more general problem. On the one hand, linearity can be interpreted as a first-order approximation to more general best-replies. On the other hand, there are situations where a non-linear-quadratic problem can be re-stated as a linear-quadratic one, by a simple transformation, as illustrated below.

Consider a model with general utility funcions $u_i(x)$ for all $i = 1, \ldots, n$ santisfying:

$$\frac{\partial u_i}{\partial x_i}(x) = h_i(\eta_{ii} x_i + \sum_{j \neq i} \eta_{ij} x_j), \tag{7}$$

where $h_i : \mathbb{R} \to \mathbb{R}$ is differentiable and $\mathbf{H} = [\eta_{ij}] \in \mathbb{R}^{n \times n}$. Assume further that $h_i(r_i) = 0$ for some $r_i \in \mathbb{R}$ and that $\eta_{ii} h_i'(r) < 0$, for all $r \in \mathbb{R}$, and for all i.

Let $\widehat{\mathbf{r}} = (\widehat{r}_i, \ldots, \widehat{r}_n)$ where $\widehat{r}_i = r_i \cdot \operatorname{sgn}(\eta_{ii})$, for all i. Let us also define $\widehat{\mathbf{H}} = [\widehat{\eta}_{ij}]$, where $\widehat{\eta}_{ij} = \eta_{ij} \cdot \operatorname{sgn}(\eta_{ii})$ for all i and j. Note that $-\widehat{\eta}_{ii} < 0$ for all i, and that the sign of $-\partial^2 u_i / \partial x_i \partial x_j(r) = -h_i'(r)\eta_{ij}$ is equal to $\operatorname{sgn}(\widehat{\eta}_{ij})$, for all $i \neq j$.

Lemma **3** *The set of Nash equilibria of the game with payoffs satisfying (7) is the same than that of* $\Gamma(\widehat{\mathbf{r}}, \widehat{\mathbf{H}})$.

We now analyze an example.

6.1 A network public good game

This model is originally due to Bramoullé and Kranton (2006). Here, we analyze it as a game with hidden complementarities, and provide some results that complement their analysis.

There is a set of players N and a network with adjacency matrix $\mathbf{G}$ that connects them. The network can have general (weighted and directed) links, $0 \leq g_{ij} \leq 1$, for all $i, j \in N$, but we do not allow for self-loops, $g_{ii} = 0$, for all $i \in N$. Each player i exerts an effort level $x_i \geq 0$ with constant marginal cost $c > 0$.

Players receive benefits from own and neighbors' efforts in the network according to a (twice-differentiable) strictly concave benefit function $v : \mathbb{R}_+ \to \mathbb{R}$, with $v(0) = 0$, $v' > 0$ and $v'' < 0$. Player i's payoffs are thus:

$$u_i(x; \mathbf{G}) = v(x_i + \sum_{j=1}^{n} g_{ij} x_j) - c x_i.$$

Following Bramoullé and Kranton (2006), we assume that there exists some $\widetilde{x} > 0$ such that $v'(\widetilde{x}) = c$. A straight application of Lemma 3 leads to the following observation.

Corollary **3** *The set of Nash equilibria of the network public good game is the same than that of* $\Gamma(\widetilde{x}\mathbf{e}, \mathbf{I} + \mathbf{G})$.

From now on, we identify the network public good game with $\Gamma(\widetilde{x}\mathbf{e}, \mathbf{I} + \mathbf{G})$. We build on this linear-quadratic re-formulation and on our results on games with hidden complementarities to analyze the equilibrium effort levels.

The *original game* $\Gamma(\widetilde{x}\mathbf{e}, \mathbf{I} + \mathbf{G})$ is a game with *substitutabilities*: a higher contribution to the public good by one's neighbors reduces the incentives for contribution. Still, this is a game with hidden complementarities. Indeed, let $\mathbf{X} = \mathbf{I} - \mathbf{G} \in \mathbb{Z}_n$. Then, $\mathbf{\Sigma X} = \left(\mathbf{I} - \mathbf{G}^2\right) \in \mathbb{Z}_n$. The hidden game associated to the network public good game for the transformation matrix $\mathbf{X} = \mathbf{I} - \mathbf{G}$ is $\Gamma\left(\widetilde{x}\mathbf{e}, \mathbf{I} - \mathbf{G}^2\right)$. This hidden game is a game with complementarities for players who are two-link-away from each other in the network.

We illustrate Theorem 1 with this game.

First, invoking Theorem 1 (a), we conclude that the network public good game has a unique equilibrium for any $\widetilde{x}$ if $\rho(\mathbf{G}) < 1$.[15]

Second, when Theorem 1 (b) holds, this unique equilibrium is interior and given by $\widetilde{x}(\mathbf{I} - \mathbf{G})\,\mathbf{b_e}\left(\mathbf{G}^2\right)$. We show with two examples that condition (b) is sometimes redundant given condition (a), but not always.

Example Let $g_{ij} = \alpha h_{ij}$, with $h_{ij} \in \{0, 1\}$, for all $i \neq j$, that is, $\mathbf{H}$ is the $(0, 1)$ −adjacency matrix of an unweighted network. Then, $\rho(\mathbf{G}) = \alpha\rho(\mathbf{H})$. Suppose further that $\mathbf{H}$ is a regular network, that is, $\sum_{j=1}^{n} h_{ij} = h$, for all $i = 1, ..., n$. Then, $\rho(\mathbf{H}) = h$, and the existence and uniqueness condition $\rho(\mathbf{G}) < 1$ becomes $\alpha h < 1$. Straight algebra gives $\mathbf{b_e}\left(\mathbf{G}^2\right) = \mathbf{e}/(1-\alpha^2 h^2)$, so that condition (b) becomes $1/(1+\alpha h) > 0$, which is always true. Therefore, the public good game on a regular network has a unique equilibrium if and only if $ah < 1$, in which case the equilibrium is interior and given by $x_i^* = \widetilde{x}/(1 + \alpha h)$, for all $i = 1, ..., n$. Condition (b) for interiority is thus redundant given condition (a) for existence and uniqueness.

Example Let $\mathbf{G} = \alpha\mathbf{H}$ where $\mathbf{H}$ is now the star centered on player 1, with $n \geq 2$.

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \text{ and } \mathbf{H}^2 = \begin{bmatrix} n-1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \end{bmatrix}.$$

In this case, the existence and uniqueness condition $\rho(\mathbf{G}) < 1$ becomes $\alpha\sqrt{n-1} < 1$. Straight algebra gives $\mathbf{b_e}\left(\mathbf{G}^2\right) = \mathbf{e}/\left(1 - \alpha^2(n-1)\right)$, so that condition (b) becomes $\alpha(n-1) < 1$. Therefore, in this case, condition (b) is more demanding than condition (a): even when $\alpha\sqrt{n-1} < 1$, we may have $\alpha(n-1) > 1$. We can conclude the following. The public good game on a star has a unique equilibrium if and only if $\alpha < 1/\sqrt{n-1}$:

- if $1/(n-1) \leq \alpha < 1/\sqrt{n-1}$, this equilibrium is partially corner and given by $x_1^* = 0$ while $x_i^* = \widetilde{x}$, for all $i = 2, ..., n$;[16]
- if $\alpha < 1/(n-1)$, this equilibrium is interior and given by $x_1^* = \widetilde{x}(1 - \alpha(n-1))/\left(1 - \alpha^2(n-1)\right)$ while $x_i^* = \widetilde{x}(1-\alpha)/\left(1 - \alpha^2(n-1)\right)$, for all $i = 2, ..., n$.

We can also write the network public good game as a game with shifted complementarities, as follows. Without loss of generality, we can write $\mathbf{G} = \alpha\mathbf{H}$ where $\mathbf{H}$ is a matrix with entries in $[0, 1]$ and $\alpha = \max\{g_{ij} : i, j \in N\} \leq 1$. The matrix $\mathbf{H}$ corresponds to a weighted graph without loops ($h_{ii} = 0$ for all $i \in n$).

[15] Indeed, $\rho(\mathbf{G}) < 1$ is equivalent to $\mathbf{G}$ being a contraction, that is, there exists a $\mathbf{u} > \mathbf{0}$ such that $(\mathbf{I} - \mathbf{G})\,\mathbf{u} = \mathbf{w} > \mathbf{0}$. Note also that

$$\left(\mathbf{I} - \mathbf{G}^2\right)\mathbf{u} = (\mathbf{I} + \mathbf{G})(\mathbf{I} - \mathbf{G})\,\mathbf{u} = (\mathbf{I} + \mathbf{G})\,\mathbf{w} > \mathbf{0}.$$

Thus, $\mathbf{u}$ constitutes a common semipositivity vector for both $\mathbf{I} - \mathbf{G}$ and $\mathbf{I} - \mathbf{G}^2$.

[16] See, also, Proposition 7 in Bramoullé and Kranton for this equilibrium.

Given that $\boldsymbol{\theta} = \widetilde{x}\mathbf{e}$, a particular matrix that we can choose to obtain a suitable shift is $\mathbf{u}\mathbf{v}^t = [\widetilde{x}\mathbf{e}] \left[(\alpha/\widetilde{x})\,\mathbf{e}\right]^t = \alpha\mathbf{e}\mathbf{e}^t$. Then:

$$\boldsymbol{\Sigma} - \alpha\mathbf{e}\mathbf{e}^t = (1-\alpha)\,\mathbf{I} - \alpha\mathbf{H}^C,$$

where $\mathbf{H}^c$ is the complementary graph of $\mathbf{H}$:

$$h_{ij}^c = \begin{cases} 1 - h_{ij} & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j. \end{cases}$$

This matrix shift leads to a (shifted) hidden game $\Gamma\left(\widetilde{x}\mathbf{e}, (1-\alpha)\,\mathbf{I} - \alpha\mathbf{H}^C\right)$ played on the network $\mathbf{H}^c$.

A straight application of Proposition 3 leads to the following conclusion: when $\rho\left(\mathbf{H}^c\right) < -1 + 1/\alpha$, the network public good game has a unique and interior equilibrium given by

$$\mathbf{x}^* = \frac{\widetilde{x}}{\widetilde{x} + \alpha\mathbf{e}^t\mathbf{y}}\mathbf{y}^*,$$

where

$$\mathbf{y}^* = \frac{\widetilde{x}}{1-\alpha}\mathbf{b}_{\mathbf{e}}\left(\alpha\mathbf{H}^C/\left(1-\alpha\right)\right)$$

is the unique Nash equilibrium of the game with complementarities $\Gamma\left(\widetilde{x}\mathbf{e}, (1-\alpha)\,\mathbf{I} - \alpha\mathbf{H}^C\right)$. Players that are not neighbors in the original game decide on actions that are, in the hidden game, strategic complements. This is a consequence of taking the complementary graph $\mathbf{H}^c$ as the underlying network of the hidden game.

The network public good game is thus a hidden game that also belongs to the subclass of shifted hidden games when its Nash equilibrium is unique. Indeed, two different possible operations on the original game $\Gamma\left(\widetilde{x}\mathbf{e}, \mathbf{I} + \mathbf{G}\right)$ yield either to an associated hidden game $\Gamma\left(\widetilde{x}\mathbf{e}, \mathbf{I} - \alpha^2\mathbf{H}^2\right)$ (for the transformation matrix $\mathbf{X} = \mathbf{I} - \alpha\mathbf{H}$), or to an associated shifted game $\Gamma\left(\widetilde{x}\mathbf{e}, (1-\alpha)\,\mathbf{I} - \alpha\mathbf{H}^C\right)$ (for the rank one shift $\mathbf{u}\mathbf{v}^t = \alpha\mathbf{e}\mathbf{e}^t$). In the next example we show that the relative performance of these two linear transformations to establish equilibrium uniqueness depends on the characteristics of the original interaction matrix.

Example Consider a regular un-weighted network $\mathbf{H}$ with common degree $k = \sum_j h_{ij} = k$, for all $i \in N$. Then $\mathbf{H}^c$ is a regular network with degree $n-k-1$. The associated hidden game $\Gamma\left(\widetilde{x}\mathbf{e}, \mathbf{I} - \alpha^2\mathbf{H}^2\right)$ has a unique Nash equilibrium when $k < 1/\alpha$. Instead, the associated shifted game $\Gamma\left(\widetilde{x}\mathbf{e}, (1-\alpha)\,\mathbf{I} - \alpha\mathbf{H}^C\right)$ has a unique Nash equilibrium when $n-k < 1/\alpha$. Therefore, the hidden (resp. shifted) game is the "best" candidate when the original network is not too dense (resp. too sparse). In particular, consider the extreme case where $\mathbf{H}$ is a complete network and α is close to 1 but below it, so that $\mathbf{G}$ is arbitrarily close to the complete network. Then, $\rho\left(\mathbf{G}^2\right) = \alpha^2\left(n-1\right)^2 \nless 1$ and the associated hidden game $\Gamma\left(\widetilde{x}\mathbf{e}, \mathbf{I} - \mathbf{G}^2\right)$ has no Nash equilibrium. Nevertheless, $(1-\alpha)\,\mathbf{I} - \alpha\mathbf{H}^C = (1-\alpha)\,\mathbf{I}$, and the associated (hidden) shifted game has a unique equilibrium (because $\rho\left(\alpha\mathbf{H}^C\right) = 0 < (1-\alpha)$), from which we deduce that the network public good game has a unique (and symmetric) equilibrium. This equilibrium can then be written as a function of the equilibrium of the shifted hidden game on $\mathbf{H}^C$ ($y_i^* = \widetilde{x}/\left(1-\alpha\right)$, for all $i = 1, ..., n$):

$$x_i^* = \frac{1}{1 + \mathbf{v}^t\mathbf{y}^*}y_i^* = \frac{\widetilde{x}}{1 + (n-1)\,\alpha}, \text{ for all } i = 1, ..., n.$$

Uniqueness of the original game is suddenly disrupted when α takes the value of 1. In this case, there are $2^n - 1$ equilibria corresponding to the non-empty subsets of N.

As an application of Proposition 1, we can also establish properties regarding the stability of the solution in the public good game. We know that this game with hidden complementarities has a unique equilibrium when $\rho(\mathbf{G}) < 1$, where $\mathbf{G}$ is the adjacency network of substitutability links. The interaction matrix of this game is $\mathbf{\Sigma} = \mathbf{I} + \mathbf{G}$, and its corresponding comparison matrix is $\overline{\mathbf{\Sigma}} = \mathbf{I} - \mathbf{G}$. The fact that $\rho(\mathbf{G}) < 1$ implies that $\overline{\mathbf{\Sigma}}$ fulfills the conditions of Proposition 1, and global stability of the unique equilibrium is then guaranteed.

7 Extensions

7.1 On the $P-$matrix condition

In this paper, we repeatedly invoke the P property on matrices in order to get equilibrium uniqueness in the game $\Gamma(\theta, \mathbf{\Sigma})$. In fact, the $P-$matrix property is equivalent to uniqueness of equilibria for arbitrary $\boldsymbol{\theta} \in \mathbb{R}^n$. One possible avenue to relax the $P-$matrix property would be to find out conditions for uniqueness on both $\mathbf{\Sigma}$ and $\boldsymbol{\theta}$. In what follows, we establish a partial impossibility result that blocks the chance of dispensing from $P-$matrices. Essentially, the $P-$matrix property turns out to be *necessary* in many cases for the existence of Nash equilibria in games with hidden complementarities.

PROPOSITION **4** *Let $\Gamma(\theta, \mathbf{\Sigma})$ be a game with hidden complementarities satisfying (C2) and (C3). Then, either $\Gamma(\theta, \mathbf{\Sigma})$ has a unique Nash equilibrium for all $\boldsymbol{\theta} \in \mathbb{R}^n$, or the set of Nash equilibria of $\Gamma(\theta, \mathbf{\Sigma})$ is empty for all $\boldsymbol{\theta} > \mathbf{0}$.*

Thus, under the assumption that $\boldsymbol{\theta} > \mathbf{0}$, the $P-$matrix property is a minimum requirement for the existence of a Nash equilibrium in games with hidden complementarities, which then also turn out to be unique. A corollary to the previous result is that multiplicity of equilibria is not possible in games $\Gamma(\theta, \mathbf{\Sigma})$ with hidden complementarities whenever $\boldsymbol{\theta} > \mathbf{0}$. Either there is one single equilibrium, or none.

7.2 On equilibrium uniqueness

So far, we have defined games with hidden complementarities, and we have established conditions such that these games possess a unique pure strategy Nash equilibrium. Sometimes, the uniqueness of pure strategy equilibria translates into uniqueness of any other sort of equilibrium, even if we allow for both mixed or correlated strategies.

PROPOSITION **5** *Suppose that (C1) and (C2) hold. If $\mathbf{\Sigma}$ is symmetric and positive definite, then the unique pure strategy Nash equilibrium of $\Gamma(\theta, \mathbf{\Sigma})$ is also its unique correlated equilibrium.*

Any matrix $\mathbf{\Sigma}$ which is both symmetric and a $P-$matrix is positive definite. Therefore, the uniqueness conditions in Proposition 2 for games with complementarities, in Proposition 1 for games with hidden complementarities, and in Propositions 2, 3 extend to any equilibrium beyond pure strategy equilibria (and even allowing for correlated strategies) whenever $\mathbf{\Sigma}$ is symmetric.

In another vein, Lemma 1 establishes equivalence between the set of pure strategy Nash equilibria of $\Gamma(\theta, \mathbf{\Sigma})$ and the solutions to the linear complementarity problem $LCP(-\boldsymbol{\theta}, \mathbf{\Sigma})$ under certain conditions. Suppose further that $\boldsymbol{\theta} > \mathbf{0}$ and define $\mathbf{D} = diag(1/\theta_1, ..., 1/\theta_n)$. Inspection of (3) then clearly shows that the solutions to $LCP(-\boldsymbol{\theta}, \mathbf{\Sigma})$ coincide exactly with the solutions to $LCP(-\mathbf{e}, \mathbf{D\Sigma})$. Noticing that the set of matrices $\mathbb{Z}_n \cap \mathbb{P}_n$ is closed for the left-multiplication by diagonal matrices with a positive diagonal (Pang, 1979a), we can rewrite Proposition 3(a) as follows.

Corollary 4 *Consider a game* $\Gamma(\theta, \mathbf{\Sigma})$ *such that (C2) hold, and* $\boldsymbol{\theta} > \mathbf{0}$. *If* $\mathbf{D\Sigma} - \mathbf{e}\mathbf{e}^t_{\varepsilon,\mathbf{D\Sigma}} \in \mathbb{Z}_n \cap \mathbb{P}_n$ *for some* $\varepsilon > 0$ *(so that* $\mathbf{D\Sigma} - \mathbf{e}\mathbf{e}^t_{\varepsilon,\mathbf{D\Sigma}} = s_\varepsilon \mathbf{I} - \mathbf{G}_\varepsilon$ *with* $s_\varepsilon > \rho(\mathbf{G}_\varepsilon)$*) then* $\Gamma(\theta, \mathbf{\Sigma})$ *is a game with shifted complementarities that possesses a unique Nash equilibrium in pure strategies, which is interior and given by:*

$$\mathbf{x}^* = \frac{1}{s_\varepsilon + \mathbf{e}^t_{\varepsilon,\mathbf{D\Sigma}} \mathbf{b}_\mathbf{e}(\mathbf{G}_\varepsilon / s_\varepsilon)} \mathbf{b}_\mathbf{e}(\mathbf{G}_\varepsilon / s_\varepsilon).$$

7.3 The dual game and efficiency

Let us assume that, in expression 1, we have $\alpha_i(x_{-i}) = \alpha_i$ for all $i \in N$. Given a game $\Gamma(\theta, \mathbf{\Sigma})$, its *dual* game is defined as $\Gamma(\theta, \mathbf{\Sigma}^t)$.

We say that x^* is an efficient action profile of $\Gamma(\theta, \mathbf{\Sigma})$ whenever it maximizes the total welfare:

$$x^* \in \arg\max\{\sum_{i=1}^{n} u_i(x) : \mathbf{x} \in \mathbb{R}^n_+\}.$$

Let $\mathbf{D}_\mathbf{\Sigma}$ be the diagonal matrix of the diagonal entries of $\mathbf{\Sigma}$.

Proposition 6 *Suppose (C2) holds. Then,* x^* *is an efficient action profile of the game* $\Gamma(\theta, \mathbf{\Sigma})$ *if and only if* x^* *is a Nash equilibrium of* $\Gamma\left(\theta, \mathbf{\Sigma} + \mathbf{\Sigma}^t - \mathbf{D}_\mathbf{\Sigma}\right)$.

The game $\Gamma\left(\theta, \mathbf{\Sigma} + \mathbf{\Sigma}^t - \mathbf{D}_\mathbf{\Sigma}\right)$ defines a situation where the externalities present in $\Gamma(\theta, \mathbf{\Sigma})$ are internalized by all players.

7.4 On network-based policies

The previous results establish conditions such that games with hidden complementarities have a unique and interior equilibrium, which is then proportional to the Katz-Bonacich centrality for some network adjacency matrix.

In this section, we identify the optimal target set in the population when the planner wishes to reduce (or to increase) optimally some function $F(\mathbf{\Sigma})$ of the equilibrium action profile, the key group problem. This function, of course, depends on the interaction matrix $\mathbf{\Sigma}$ for the underlying game.

More precisely, the key group problem consists on eliminating a targeted group of size s players from the current population. If we remove a set S of players, the interaction matrix becomes $\mathbf{\Sigma}_{-S}$. The problem is thus to minimize $F(\mathbf{\Sigma}_{-S})$ by picking the adequate set S.[17]. Formally, the $s-$key group problem is:

$$\min_{|S| \leq s} F(\mathbf{\Sigma}_{-S}) \tag{8}$$

This is a finite optimization problem, that admits at least one solution. Let S^* be a solution to (8). We call the set S^* a *key group* of the game $\mathbf{\Sigma}$. Removing S^* from the game has the highest overall impact on the value of F.

We analyze the key group problem for globally shifted games, a subclass of games with shifted complementarities. Also, invoking Corollary 4, we restrict our analysis to $\boldsymbol{\theta} = \mathbf{e}$.

Definition 2 $\Gamma(e, \mathbf{\Sigma})$ *is a game with globally shifted complementarities if there exists* $\mathbf{\Psi} \in \mathbb{Z}_n$ *and* $\gamma > 0$ *such that* $\mathbf{\Sigma} = \mathbf{\Psi} + \gamma \mathbf{e}\mathbf{e}^t$.

[17] The case in which the planner maximizes $F(\mathbf{\Sigma}_{-S})$ is analogous.

From Proposition 2(a), we deduce that a game with globally shifted complementarities has an interior and unique solution when $\mathbf{\Sigma} - \gamma \mathbf{e}\mathbf{e}^t \in \mathbb{Z}_n \cap \mathbb{P}_n$ for some $\gamma > 0$, which is equivalent to $\mathbf{\Sigma} = \beta\mathbf{I} - \lambda\mathbf{G} + \gamma\mathbf{e}\mathbf{e}^t$ for some $\beta > 0$, $\lambda \geq 0$, and $\mathbf{G} \in [0,1]^{n\times n}$ such that $\beta > \lambda\rho(\mathbf{G})$.

Note that if $\beta > \lambda\rho(\mathbf{G})$ then, $\beta > \lambda\rho(\mathbf{G}_{-S})$ for all $S \subseteq N$ (Debreu and Herstein, 1953).

Consider the problem of optimizing the aggregate activity at equilibrium[18]. Let $\lambda^* = \lambda/\beta$. From Proposition 2$(a)$ we obtain:

$$F(\mathbf{\Sigma}) = \frac{1}{\beta + \gamma \mathbf{e}^t \mathbf{b}_{\mathbf{e}}(\lambda^* \mathbf{G})} \mathbf{e}^t \mathbf{b}_{\mathbf{e}}(\lambda^* \mathbf{G}).$$

The key group problem (8) then becomes:

$$\min_{S\subseteq N, |S|=s} \mathbf{e}^t \mathbf{b}_{\mathbf{e}}(\lambda^* \mathbf{G}_{-S}), \tag{9}$$

where $|S| = s$. This is simply because $\mathbf{e}^t \mathbf{b}_{\mathbf{e}}(\lambda^* \mathbf{G}_{-S})$ is decreasing in S for the inclusion ordering (Ballester *et al.* 2006).

For simplicity, we omit the subscript $\mathbf{e}$. We also use the notation $Y = \mathbf{e}^t \mathbf{y}$.

Definition **3** *The group inter-centrality of $S \subseteq N$ in $\mathbf{H}$ is*

$$d_S(\mathbf{H}) \equiv \sum_{j\in N} b_j(\mathbf{H}) - b_j(\mathbf{H}_{-S}) = B(\mathbf{H}) - B(\mathbf{H}_{-S}).$$

Note that $b_j(\mathbf{H}_{-S}) = 0$ for all $j \in S$. Then, (9) reduces to choosing the set with highest group inter-centrality:

$$\max_{S\subseteq N, |S|=s} d_S(\lambda^* \mathbf{G}), \tag{10}$$

that is, the solution of (9) is $S^* \subseteq N$ such that $d_{S^*}(\lambda^* \mathbf{G}) \geq d_S(\lambda^* \mathbf{G})$, for all $S \subseteq N$ with $|S| = s$.

The version of the problem with $s = 1$ (the *key player* problem) is analyzed in Ballester *et al.* (2006), who provide a simple geometric criterion to single out the optimal target. Indeed, the key player i^* is the one that maximizes the following inter-centrality network measure:

$$i^* \in \arg\max\{\frac{b_i(\lambda^* \mathbf{G})^2}{b_{ii}(\lambda^* \mathbf{G})} : i \in N\}, \tag{11}$$

where $b_{ii}(\lambda^* \mathbf{G})$ is the ith diagonal entry of the matrix $[\mathbf{I} - \lambda^* \mathbf{G}]^{-1}$.

When $s \geq 2$, the optimal choice of the group of players requires, at least potentially, the study of all possible combinations of subsets of N of size s. We prove that the key group problem has an inherent complexity that calls for approximation algorithms.

Lemma **4** *The problem of finding a key group is NP-hard.*

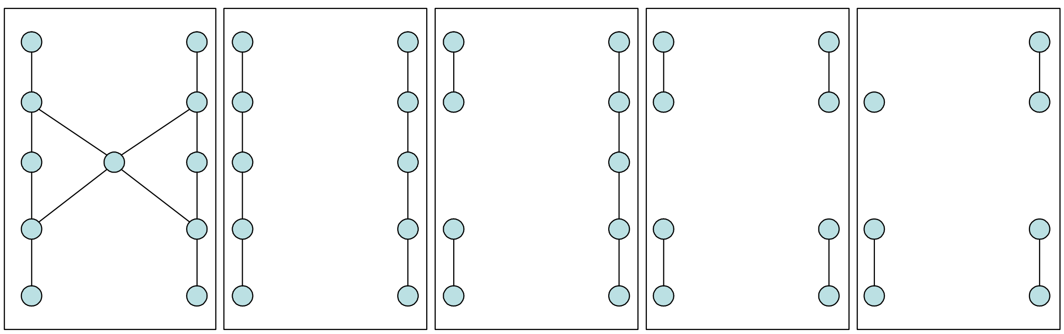
The proof relates the key group problem to that of finding a maximum vertex cover of a network. A *vertex cover* of a network $\mathbf{G}$ is a subset of vertices $S \subseteq N$ such that every link in the network is incident with some vertex in S. A maximum vertex cover is a vertex cover of maximum size. The problem of finding a maximum vertex cover in a network is known to be NP-hard (Karp, 1972).

Basically, the problem of finding the group with highest contribution to the game is essentially combinatorial. Th following example illustrates this idea.

[18] This problem may arise in decisions related to crime reduction, where the objective is to choose the set of criminals whose removal decreases crime activity in society. Calvó-Armengol and Zenou (2004) provide an economic model of crime decisions where this type of optimal choices could be applied.

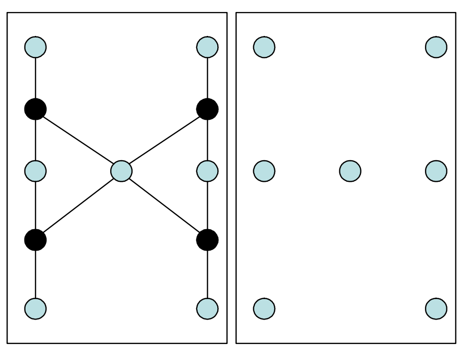
Example[19] There are $n = 11$ players, and utilities are given by (1), with $-\sigma_{ij} \in \{0, \delta\}$ for some $\delta \geq 0$. Complementarities, when present, are homogeneous across agents. The underlying complementarity network appears in the first panel of the figure below. A link between any two players i and j corresponds to the case when $-\sigma_{ij} = \delta$. For small enough δ, the condition of Proposition 2 holds and the game $\Gamma(\theta, \mathbf{\Sigma})$ has a unique equilibrium. Moreover, the condition also holds for any subgame $\Gamma(\theta, \mathbf{\Sigma}_S)$, because the spectral index is decreasing with respect to network inclusion. Thus, we can construct a path of unique equilibria that results from a succession of removals of individual players.

We consider here the key group problem for $s = 4$ players. Let us proceed by removing the most inter-central player at each stage. Intuitively, inter-centrality of a player accounts for paths in the network that flow through that player, that is, the betweenness of that player in network. After the iterative removal of the four players, we are left with three interacting pairs and one isolated agent.



At each step, the player with the highest centrality is removed from the game.

Nevertheless, this greedy procedure does not remove the key group. In fact, figure 7.4 shows that a different choice of the group leads to a higher reduction in overall activity.



Removing the four black nodes yields
to seven isolated players.

Although the key group problem is combinatorially demanding, and that greedy algorithms may fail to find the exact solution, it is natural to wonder whether they still provide an acceptable approximation to the

[19] We thank Joan de Martí and Sergio Vicente for this example.

true solution. We now show that the simple greedy procedure, where a group is constructed by iteratively choosing an optimal player (key player) from the remaining population, provides a fairly good approximation for the key group problem. These approximation results rely on submodular functions, that we first define.

DEFINITION 4 *The set function $z : 2^N \to \mathbb{R}$ is submodular if for all $S, T \subseteq N$, $z(S) + z(T) \geq z(S \cup T) + z(S \cap T)$.*

Without loss of generality we can normalize z such that $z(\varnothing) = 0$. We only consider nondecreasing functions, $z(S) \leq z(T)$ for all $S \subseteq T \subseteq N$. Let us denote individual contributions by $\zeta_i(S) = z(S \cup \{i\}) - z(S)$. A function is submodular when individual contributions are increasing with respect to the set inclusion, that is, $\zeta_i(S) \geq \zeta_i(T)$, for all $i \in N \backslash T$ and $S \subseteq T \subseteq N$.

The problem of maximizing a submodular function, or equivalently, minimizing a supermodular function, is NP-hard, in general. Nemhauser *et al.* (1978) propose a polynomial-time greedy heuristic for approximating this kind of problems. At each step, the algorithm augments the solution set with the agent with highest contribution.

ALGORITHM 1 *Let $S_0 = \varnothing$. At step t set $S_t = S_{t-1} \cup i_t$, where $i_t \in \arg\max_{i \in N \backslash S_{t-1}} \zeta_i(S_{t-1})$. Stop whenever $\zeta_{i_t}(S_{t-1}) \leq 0$ or $|S_t| = s$.*

For the key group problem, this algorithm is adapted by taking $z(S) = d_S(\lambda^* \mathbf{G})$ and $\zeta_i(S_{t-1}) = d_{S_{t-1} \cup \{i\}}(\lambda^* \mathbf{G}) - d_{S_{t-1}}(\lambda^* \mathbf{G}) \geq 0$. The inequality holds because i adds new paths to the paths contributed by the set S_{t-1}. The algorithm should stop if and only if $|S_t| = s$.

PROPOSITION 7 *The key group problem (10) can be approximated in polynomial-time by the use of algorithm 1, where, at each step, the agent i^* who will become a member of the approximated key group is obtained from (11). Let d_{S^*} be the optimal value of (10) and d_{SG} be the value obtained by applying the greedy algorithm. Then, the approximation error is bounded like:*

$$\frac{d_{S^*} - d_{SG}}{d_{S^*}} \leq \left(\frac{s-1}{s}\right)^s < \frac{1}{e} \approx 36.79\%.$$

8 Discussion

Our main result, Theorem 1, establishes necessary and sufficient conditions for Nash equilibrium existence for general hidden games, and provides with a closed-form expression for equilibrium actions. In a linear-quadratic set-up, a closed-form expression for equilibrium payoffs is then readily obtained. Indeed, let $x^* = (x_1^*, ..., x_n^*)$ be an interior Nash equilibrium for the game with payoffs (1). The corresponding equilibrium payoffs are:

$$u_i(x^*) = \alpha_i\left(x_{-i}^*\right) + \frac{1}{2}\sigma_{ii} x_i^{*2}, \text{ for all } i \in N, \tag{12}$$

linear with the square of own equilibrium action.

For a rich class of games, the Katz-Bonacich closed-form expression for equilibrium actions, and the simple quadratic formula for equilibrium payoffs in (12), allow for clear-cut comparative statics results where monotonicity of actions and payoffs is tied down to the pattern of complementarities across players.

More precisely, consider two games with complementaries with a unique equilibrium. Formally, let $\Gamma_1 = \Gamma(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ and $\Gamma_2 = \Gamma(\boldsymbol{\theta}, \boldsymbol{\Sigma} - \mathbf{B})$ in GC^*, with $\mathbf{B} \geq \mathbf{0}$ and Nash equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$, respectively.

Complementarities are stronger in Γ_1 than in Γ_2 across every pair of players. Corollary 2, together with the fact that the Katz-Bonacich network centrality trivially increases with the network density, imply that $\mathbf{x}_1^* \geq \mathbf{x}_2^*$. In words, individual equilibrium actions increase when pair-wise complementarities are strengthened. Assuming, e.g., that α_i is constant, expression (12) then implies that individual equilibrium payoffs are also monotone in the interaction matrix for games in GC^*.

Of course, the monotonicity of individual equilibrium actions with the interaction matrix for GC^* implies that the aggregate equilibrium action is also monotone with $\mathbf{\Sigma}$ for such games. Beyond GC^*, the monotonicity of aggregate equilibrium action also holds for GSC^* with a homogenous substitutability shift $\gamma\mathbf{J}$, as already established in Ballester *et al.* (2006).

Again for games in GC^*, one can easily check that equilibrium actions are monotone in $\boldsymbol{\theta}$, the vector of marginal payoffs at $(0, ..., 0)$. Formally, let $\Gamma_1 = \Gamma(\boldsymbol{\theta}, \mathbf{\Sigma})$ and $\Gamma_2 = \Gamma(\boldsymbol{\theta}', \mathbf{\Sigma})$ in GC^*, with $\boldsymbol{\theta} \geq \boldsymbol{\theta}'$ and Nash equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$, respectively. Then, $\mathbf{x}_1^* \geq \mathbf{x}_2^*$. Monotonicity of equilibrium payoffs follows from (12).

Both our emphasis on equilibrium existence, and the comparative statics for games with complementarities, are reminiscent of the literature on supermodular games (see Topkis (1979) and Vives (2005) for up-to-date results). The main differences between our model and this literature are the following.

First, the strategy space of a supermodular game is a lattice, and thus is bounded. Here, instead, we deal with an unbounded strategy space. Unlike with supermodular games, equilibrium existence is then an open question that calls for specific conditions on the size and pattern of the prevailing hidden complementarities. We also show that equilibrium existence and equilibrium uniqueness are two sides of same token. Beyond gaining insights into the exact working of positive feed-back loops in a population with general interaction modes, we believe that our results call for a word of caution, namely, imposing an arbitrary bound on a strategy space need not be an innocuous modelling choice. Indeed, while this arbitrary bound solves equilibrium existence concerns, when the resulting lattice of equilibria does not boil down to a single outcome, this equilibrium lattice turns out to depend critically on the arbitrary choice of this upper bound.

Second, while we are able to characterize fully the (Katz-Bonacich) unique Nash equilibrium in games where complementarities need not be apparent, but are only hidden, our analysis is restricted to games with piece-wise linear best-responses and an unidimensional strategy space. The literature on supermodular games, instead, has a much wider scope. Future research should relax the linearity and unidimensionality assumptions and explore the connections between general (non-linear) games with hidden complementarities and the nonlinear complementarity problem.

Our paper also belongs to the nascent literature on games on networks. We have already discussed some connections with the network public good game in Bramoullé and Kranton (2006), and games with homogeneous shifted complementarities in Ballester *et al.* (2006). Broadly stated, these papers explore the role of network substitutabilities and complementarities in a complete information set up.

A recent work by Galeotti *et al.* (2006) relaxes the assumption of complete information and analyses network substitutabilities and complementarities under incomplete information (when players do not know the whole pattern of interaction modes). While we assume complete information throughout, Proposition 1 establishes the global stability of the unique Nash equilibrium for the class of games with generalized dominant diagonal that include, in particular, games with complementarities or with a dominant (positive) diagonal. Given that the tatônnement process for which stability is granted uses only local information about the interaction matrix (any player need only know the interaction modes in which he or she is involved, and the actions of his or her partners), one can invoke this global stability result to relax the requirement of complete information, at least for this subclass of games.[20]

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Proof of Theorem 1. As a matter of fact, this problem has a unique solution for all possible parameters $\boldsymbol{\theta}$ if and only if $\boldsymbol{\Sigma}$ is a $P-$matrix that is, all its principal minors are positive (see Lemma 1, below).

(a) This result is due to Pang (1979) and we reproduce his proof. A matrix $\boldsymbol{\Sigma}$ is said to be semipositive if there is a vector $\mathbf{r} > \mathbf{0}$ such that $\boldsymbol{\Sigma}\mathbf{r} > \mathbf{0}$. If $\boldsymbol{\Sigma}$ is a P-matrix then it is semipositive. In the special case where $\boldsymbol{\Sigma}$ is a $Z-$matrix, it is the case that both classes are equivalent (Debreu and Herstein, 1953). Pang proves the equivalence for the more general situation where $\boldsymbol{\Sigma}$ satisfies the two following conditions:

1. Hidden complementarities: there are $Z-$matrices $\mathbf{X}$ and $\boldsymbol{\Psi}$ such that $\boldsymbol{\Sigma}\mathbf{X} = \boldsymbol{\Psi}$.
2. Condition (C3): there are vectors $\mathbf{r}, \mathbf{s} \geq \mathbf{0}$ such that $\mathbf{r}^t\mathbf{X} + \mathbf{s}^t\boldsymbol{\Psi} > \mathbf{0}$.
   Using this result, we now formulate the proof of the theorem.

(If part) Suppose that $\mathbf{X}\mathbf{u} = \mathbf{w} > \mathbf{0}$ and $\boldsymbol{\Psi}\mathbf{u} > \mathbf{0}$. This implies that $\mathbf{X}$ is a $P-$matrix, so that it is nonsingular. Then $\boldsymbol{\Sigma}\mathbf{w} = \boldsymbol{\Sigma}\mathbf{X}\left(\mathbf{X}^{-1}\mathbf{w}\right) = \mathbf{Y}\mathbf{u} > \mathbf{0}$, implying that $\boldsymbol{\Sigma}$ is semipositive. Because $\mathbf{X}$ is a $P-$matrix, its transpose $\mathbf{X}^t$ is also P: there exists $\mathbf{r} > \mathbf{0}$ such that $\mathbf{X}^t\mathbf{r} > \mathbf{0}$ and condition (C3) is clearly satisfied. Because $\boldsymbol{\Sigma}$ has hidden complementarities, we conclude that $\boldsymbol{\Sigma}$ is a $P-$matrix.

(Only if part) Since Σ is a $P-$matrix, it is is semipositive, which implies that there exists a vector $\mathbf{u}$ such that $\mathbf{X}\mathbf{u}, \mathbf{Y}\mathbf{u} > \mathbf{0}$. As Pang shows, condition (C3) guarantees that, $\mathbf{u} > \mathbf{0}$.

(b) The condition $\mathbf{X}\mathbf{b}_{\boldsymbol{\theta}}\left(\mathbf{G}_1/s_1\right) > 0$ is equivalent to $\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta} = \mathbf{X}\boldsymbol{\Psi}^{-1}\boldsymbol{\theta} \geq \mathbf{0}$. Then it is straightforward to see that $\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}$ is the solution to $LCP\left(-\boldsymbol{\theta}, \boldsymbol{\Sigma}\right)$. Because $\mathbf{b}_{\boldsymbol{\theta}}\left(\mathbf{G}_1/s_1\right)/s_1 = \boldsymbol{\Psi}^{-1}\boldsymbol{\theta}$, we conclude that $\mathbf{X}\mathbf{b}_{\boldsymbol{\theta}}\left(\mathbf{G}_1/s_1\right)/s_1$ is the Nash equilibrium of $\Gamma\left(\theta, \boldsymbol{\Sigma}\right)$.

Proof of Corollary 1. The if part is immediate. For the only if part, given that $\mathbf{G}_1/s_1$ is a contraction and $\boldsymbol{\theta} > \mathbf{0}$, we have that $\boldsymbol{\Psi}$ is a $P-$matrix and $\mathbf{b}_{\boldsymbol{\theta}}\left(\mathbf{G}_1/s_1\right) = \left(\mathbf{I} - \mathbf{G}_1/s_1\right)^{-1}\boldsymbol{\theta} > \mathbf{0}$ is well-defined. Thus, $\boldsymbol{\Psi}\mathbf{b}_{\boldsymbol{\theta}}\left(\mathbf{G}_1/s_1\right) = s_1\boldsymbol{\theta} > \mathbf{0}$. On the other hand, $\mathbf{X}\mathbf{b}_{\boldsymbol{\theta}}\left(\mathbf{G}_1/s_1\right) > \mathbf{0}$ and uniqueness follows from Theorem 1(a).

Proof of Corollary 2. The result is a direct consequence of Theorem 1, when $\mathbf{X} = \mathbf{I}$. Moreover, here we deal easily with corner solutions. Without loss of generality, let $S^+\left(\boldsymbol{\theta}\right) = \{1, \ldots, s\}$. Consider the $s \times s$ matrix $\boldsymbol{\Sigma}_{S^+(\boldsymbol{\theta})}$, which is $P-$matrix because it a principal submatrix of $\boldsymbol{\Sigma}$. Thus, let $\mathbf{x}' \in \mathbb{R}^s_+$ be the unique Nash equilibrium of $\Gamma\left(\boldsymbol{\theta}_{S^+(\boldsymbol{\theta})}, \boldsymbol{\Sigma}_{S^+(\boldsymbol{\theta})}\right)$. Let $\mathbf{x}^* = (\mathbf{x}', \mathbf{0}) \in \mathbb{R}^n_+$. It is obvious that all players in $S^+\left(\boldsymbol{\theta}\right)$ playing the action profile $\mathbf{x}'$ best-respond to the actions of all the other players under the whole profile $\mathbf{x}^*$. For any player i her best reply x_i to the profile x^*_{-i} satisfies:

$$\sigma_{ii}x_i + \sum_{j \neq i}\sigma_{ij}x^*_j \geq \theta_i.$$

Because $\sigma_{ii} > 0$ for all i and $\sigma_{ij} \leq 0$ for all $j \neq i$, we conclude that when $\theta_i \leq 0$, we have that $x_i = 0$ is a best response to x^*_{-i}. Uniqueness of the equilibrium implies that this is, in fact, the only possibility.

Proof of lemma 1. Given the condition (C2), an action profile $x^* \in \mathbb{R}^n_+$ is a pure strategy Nash equilibrium of $\Gamma\left(\theta, \boldsymbol{\Sigma}\right)$ if and only if the following holds:

$$\begin{aligned}\frac{\partial u_i}{\partial x_i}(x^*) &= 0, \text{ for all } i \in N \text{ such that } x^*_i > 0 \\ \frac{\partial u_i}{\partial x_i}(x^*) &\leq 0, \text{ for all } i \in N \text{ such that } x^*_i = 0\end{aligned}$$

In matrix notation, these Nash equilibrium necessary and sufficient conditions become:

$$\begin{aligned} \mathbf{x}^* &\geq \mathbf{0} \\ -\boldsymbol{\theta} + \boldsymbol{\Sigma}\mathbf{x}^* &\geq \mathbf{0} \\ \mathbf{x}^{t*}\left(-\boldsymbol{\theta} + \boldsymbol{\Sigma}\mathbf{x}^*\right) &= 0 \end{aligned}$$

This is a linear complementarity problem $LCP(-\boldsymbol{\theta}, \boldsymbol{\Sigma})$. It is a well-known fact that this problem has a unique solution for all possible parameters $\boldsymbol{\theta}$ if and only if $\boldsymbol{\Sigma}$ is a $P-$matrix, that is, all its principal minors are positive.

Proof of Proposition 1. We can discard the weights by setting $\boldsymbol{\Xi} = \mathbf{I}$, and concentrate on the best responses $\mathbf{BR}(\cdot)$. The proof then appears in Ahn (1981), Theorem 4.1 and corollary 4.2. Let $\mathbf{D}$ be diagonal matrix with entries σ_{ii}, for all i. Ahn (1981) shows that the sequence

$$\mathbf{x}^{k+1} = \max\{\mathbf{x}^k - \omega\mathbf{E}\left(-\boldsymbol{\theta} + \boldsymbol{\Sigma}\mathbf{x}^k + \mathbf{K}\left(\mathbf{x}^{k+1} - \mathbf{x}^k\right)\right), 0\},$$

converges, where $\omega > 0$, $\mathbf{E}$ is a positive diagonal matrix and $\mathbf{K}$ is strictly lower or upper triangular matrix. By setting $\mathbf{K} = \mathbf{0}$ and $\omega\mathbf{E} = \mathbf{D}^{-1}$, we have

$$\left|\mathbf{x}^{k+1} - \mathbf{x}^k\right| < \left|\mathbf{I} - \mathbf{D}^{-1}\boldsymbol{\Sigma}\right|\left|\mathbf{x}^k - \mathbf{x}^{k-1}\right|$$

and a spectral radius condition for convergence is:

$$\rho\left(\left|\mathbf{I} - \mathbf{D}^{-1}\boldsymbol{\Sigma}\right|\right) < 1.$$

Under this condition the sequence converges to the unique Nash equilibrium of the game. The comparison matrix of $\mathbf{D}^{-1}\boldsymbol{\Sigma}$ is $\mathbf{I} - \left|\mathbf{I} - \mathbf{D}^{-1}\boldsymbol{\Sigma}\right|$. This is a $P-$matrix (or equivalently, $\boldsymbol{\Sigma}$ is generalized diagonally dominant) if and only if the same spectral radius condition holds.

Proof of Proposition 2. The fact that $\Gamma(\theta, \boldsymbol{\Psi})$ is a GC* is equivalent to $\boldsymbol{\Psi}^{-1}\mathbf{u} > 0$. This implies that, by choosing,

$$\mathbf{X} = \left(\mathbf{I} + \boldsymbol{\Psi}^{-1}\mathbf{u}\mathbf{v}^t\right)^{-1} = \mathbf{I} - \frac{1}{1 + \mathbf{v}^t\boldsymbol{\Psi}^{-1}\mathbf{u}}\boldsymbol{\Psi}^{-1}\mathbf{u}\mathbf{v}^t,$$

we have that $\mathbf{X}$ is a $Z-$matrix. Noting that $\boldsymbol{\Sigma}\mathbf{X} = \mathbf{Y}$ and that condition (C3) is met because $\boldsymbol{\Psi}$ is a $P-$matrix, we will conclude, by Theorem 1, that there is a unique Nash equilibrium in $\Gamma(\theta, \boldsymbol{\Sigma})$: it is readily checked that

$$\begin{aligned} \mathbf{X}\left(\boldsymbol{\Psi}^{-1}\mathbf{u}\right) &= \frac{1}{1 + \mathbf{v}^t\boldsymbol{\Psi}^{-1}\mathbf{u}}\boldsymbol{\Psi}^{-1}\mathbf{u} > \mathbf{0} \\ \boldsymbol{\Psi}\left(\boldsymbol{\Psi}^{-1}\mathbf{u}\right) &= \mathbf{u} > \mathbf{0}. \end{aligned}$$

Proof of Proposition 3

First, it is clear that if $\Gamma\left(\theta, \boldsymbol{\Sigma} - \boldsymbol{\theta}\mathbf{v}^t\right)$ is a GC* for some $\mathbf{v} > \mathbf{0}$, then necessarily Γ_2 is a GC*. This is so because of the monotonicity of the spectral index of nonnegative matrices with respect the usual ordering in $\mathbb{R}^{n\times n}$ (Debreu and Herstein, 1953). Suppose now that the condition in (a) is satisfied. Then, by Proposition 2, the game has a unique equilibrium $\mathbf{x}^*$. Consider the system $\boldsymbol{\Sigma}\mathbf{x} = \boldsymbol{\theta}$, where $\boldsymbol{\Sigma} = s_\varepsilon\mathbf{I} - \mathbf{G}_\varepsilon + \boldsymbol{\theta}\boldsymbol{\theta}^t_{\varepsilon,\boldsymbol{\Sigma}}$, that is:

$$\left[s_\varepsilon\mathbf{I} - \mathbf{G}_\varepsilon\right]\mathbf{x} + \boldsymbol{\theta}\boldsymbol{\theta}^{\mathbf{t}}_{\boldsymbol{\varepsilon},\boldsymbol{\Sigma}}\mathbf{x} = \boldsymbol{\theta}.$$

Noting that the Nash equilibrium of Γ_2 is $\mathbf{y}^* = [s_\varepsilon \mathbf{I} - \mathbf{G}_\varepsilon]^{-1} \boldsymbol{\theta} = \mathbf{b}_{\boldsymbol{\theta}} (\mathbf{G}_\varepsilon / s_\varepsilon) / s_\varepsilon$, this equation is equivalent to:

$$\mathbf{x} = (1 - \boldsymbol{\theta}^t_{\varepsilon,\boldsymbol{\Sigma}} \mathbf{x}) \mathbf{y}^*, \tag{13}$$

which implies that

$$\boldsymbol{\theta}^t_{\varepsilon,\boldsymbol{\Sigma}} \mathbf{x} = \frac{\boldsymbol{\theta}^t_{\varepsilon,\boldsymbol{\Sigma}} \mathbf{y}^*}{1 + \boldsymbol{\theta}^t_{\varepsilon,\boldsymbol{\Sigma}} \mathbf{y}^*}.$$

Plugging back into (13), we finally obtain:

$$\mathbf{x} = \frac{1}{1 + \boldsymbol{\theta}^t_{\varepsilon,\boldsymbol{\Sigma}} \mathbf{y}} \mathbf{y},$$

which is clearly positive and we conclude that the unique solution is interior: $\mathbf{x}^* = \mathbf{x}$. We now prove (b). The unique $\mathbf{x}^*$ solution to $\boldsymbol{\Sigma}\mathbf{x} = \boldsymbol{\theta}$ is such that $\mathbf{x}^* = (1 - \mathbf{v}^t \mathbf{x}^*) \left[\boldsymbol{\Sigma} - \boldsymbol{\theta} \mathbf{v}^t\right]^{-1} \boldsymbol{\theta}$. Given that $\mathbf{x}^* > \mathbf{0}$ and $\mathbf{v}^t \mathbf{x}^* < 1$, we have $\left[\boldsymbol{\Sigma} - \boldsymbol{\theta} \mathbf{v}^t\right]^{-1} \boldsymbol{\theta} > \mathbf{0}$. But for any $Z-$matrix $\mathbf{M}$ we have that semipositivity implies the $P-$property. Therefore, Γ_2 is a GC* and the result follows.

Proof of Lemma 2. Let $\mathbf{H}$ be a (nonsingular) matrix where $h_{ii} = 0$ for all i, and $h_{ij} = 1$ for $i \neq j$. Note that the best reply can be written as:

$$x_i^k = \frac{1}{\beta + 1} \max \left\{ \left(\alpha \mathbf{e} - \mathbf{H} \mathbf{x}^{k-1} \right)_i, 0 \right\},$$

and $\rho(\mathbf{H}) = n - 1$. Suppose that we achieve convergence. If it is achieved in $K < \infty$ steps, the fact that $\mathbf{H}$ is nonsingular, implies that $\mathbf{x}^0 = \ldots = \mathbf{x}^{K-1} = \mathbf{x}^*$, a contradiction. On the other hand, consider convergence in the long-run (thus having $\mathbf{x}^0 \neq \mathbf{x}^*$). Then, there is a stage K from which all best responses are positive, $\left(\mathbf{x}^k\right)_{k \geq K} > \mathbf{0}$

$$\mathbf{x}^k = \frac{1}{\beta + 1} \left(\alpha \mathbf{e} - \mathbf{H} \mathbf{x}^{k-1} \right), \text{ for } k \geq K.$$

When $\beta < n - 2$ this subsequence diverges because $\rho\left((\beta + 1)^{-1} \mathbf{H} \right) < 1$, a contradiction to the convergence of the sequence $\left(\mathbf{x}^k\right)$.

Proof of Lemma 3. We first transform the game into another game with linear best response functions. Since utilities are strictly concave ($\eta_{ii} h_i' < 0$), $\mathbf{x}$ is a Nash equilibrium of the game if and only if it satisfies:

$$\begin{aligned} h_i(\eta_{ii} x_i + \sum_{j \neq i} \eta_{ij} x_j) &\leq 0 \\ h_i(\eta_{ii} x_i + \sum_{j \neq i} \eta_{ij} x_j) x_i &= 0, \end{aligned}$$

for all i. When $\eta_{ii} > 0$ ($\eta_{ii} < 0$), we must have, by assumption that $h_i' < 0$ and the conditions are equivalent to:

$$\begin{aligned} \eta_{ii} x_i + \sum_{j \neq i} \eta_{ij} x_j &\geq (\leq)\, r_i \\ (\eta_{ii} x_i + \sum_{j \neq i} \eta_{ij} x_j - r_i) x_i &= 0. \end{aligned}$$

Then, the result follows from lemma 1.

Proof of Corollary 3 The result follows from lemma 3 by taking $\eta_{ii} = 1$ for all i, $\eta_{ij} = g_{ij}$ for all i and $j \neq i$, $h_i(\cdot) = v'(\cdot) - c$. Note then that $\widehat{\mathbf{r}} = \widetilde{x}\mathbf{e}$ and $\widehat{\mathbf{H}} = \mathbf{I} + \mathbf{G}$.

Proof of Proposition 4. This is a direct result of Pang (1979) that shows that the $P-$matrix property of a matrix $\mathbf{\Sigma}$ is equivalent to its semipositivity under the assumption that $\mathbf{\Sigma}$ has hidden complementarities and (C3) holds. The result then follows because semipositivity is easily checked to be equivalent to the existence of a Nash equilibrium for some $\boldsymbol{\theta} > \mathbf{0}$.

Proof of Proposition 5. By the symmetry of $\mathbf{\Sigma}$, $\Gamma(\theta, \mathbf{\Sigma})$ is a potential game (Ui, 2004). This, together with the positive definiteness of $\mathbf{\Sigma}$, implies uniqueness of the correlated equilibrium (Neyman, 1997). But, a positive definite matrix belongs to $\mathbf{\Sigma} \in \mathbb{P}_n$.Then $\Gamma(\theta, \mathbf{\Sigma})$ has a unique pure strategy Nash equilibrium, and the result follows.

Proof of Proposition 6. Consider the following maximization problem:

$$\max_{\mathbf{x} \geq \mathbf{0}} W(\mathbf{x}) = \sum_{i=1}^{n} u_i(\mathbf{x}).$$

Because $\partial^2 W / \partial x_i^2 = -\sigma_{ii} < 0$, this problem is equivalent, to

$$\begin{aligned} \sigma_{ii} x_i + \sum_{j \neq i} (\sigma_{ij} + \sigma_{ji}) x_j &\geq \theta_i \\ (\sigma_{ii} x_i + \sum_{j \neq i} (\sigma_{ij} + \sigma_{ji}) x_j - \theta_i) x_i &= 0, \end{aligned}$$

for each $i = 1, \ldots n$. This is simply $LCP(-\boldsymbol{\theta}, \mathbf{\Sigma} + \mathbf{\Sigma}^t - \mathbf{D}_{\mathbf{\Sigma}})$. By lemma 1, we get the desired result.

Proof of lemma 4. Given the maximum vertex cover problem, we can solve it by means of a polynomial-time parsimonious reduction to our key group problem. Given a maximum vertex cover S^* in $\mathbf{G}$ of size s^*, it is obvious that S^* is a key group of size s^*, because the removal of players in S^* results in a network of completely isolated vertices (empty network). Starting with $s = 1$ we can solve the key group problem by increasing s iteratively. When the removal of the key group S results in an empty network, we can stop iterating and conclude that S is a vertex cover of $\mathbf{G}$.

Proof of Proposition 7. The proof amounts to establishing that the function $d_S(\lambda^* \mathbf{G})$ is submodular in S. The result then follows from Nemhauser *et al.* (1978). Take $S \subseteq T \subseteq N$. Let $b_{ji}^{[k]}(\mathbf{G})$ be the $(j,i)-$entry of the matrix $\mathbf{G}^k$. Then, for all $i \in N \backslash T$:

$$\begin{aligned} \zeta_i(S) &= d_{S \cup \{i\}}(\lambda^* \mathbf{G}) - d_S(\lambda^* \mathbf{G}) = \left(B(\lambda^* \mathbf{G}) - B(\lambda^* \mathbf{G}_{-(S \cup \{i\})})\right) - (B(\lambda^* \mathbf{G}) - B(\lambda^* \mathbf{G}_{-S})) \\ &= B(\lambda^* \mathbf{G}_{-S}) - B(\lambda^* \mathbf{G}_{-(S \cup \{i\})}) = d_i(\lambda^* \mathbf{G}_{-S}) \geq d_i(\lambda^* \mathbf{G}_{-T}) = \zeta(T). \end{aligned}$$