Beat The Market

Fan Wang

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Abstract

Speculation in asset market is modeled as a stochastic betting game of incomplete information played by finite players and repeated infinite times. With stochastic asset return and unkown quality of public signal, a generic adaptive learning rule and the corresponding evolutionary dynamics is analyzed. In the learning rule, the impact of historical events on players' belief decays over time. It is proved to be a robust approach to adapt to stochastic regime shifts in the market over time. The market dynamics has characteristics commonly observed in financial market, i.e. endogenous boom-bust cycle, positive correlation in return and volume series, and negative first order autocorrelation in return series, but inexplicable by conventional rational expectations theory.

Keywords: evolutionary games, adaptive learning, bounded rationality, behavioral finance.

1 Introduction

Financial market is at the center stage of profit-seeking and risk-taking in an ever changing environment. Economic condition is volatile and volume of information is enormous. Speculators come to the market with various presumptions and idiosyncratic characteristics. Only the fittest, with the acumen to discover changes in underlying rules and adapt, can endure the market. Unlike traditional approach, which focuses on analyzing strategic behaviors of market participants and finding equilibrium condition, this paper explores a new avenue and shows how adaptive learning and evolution principle explains market dynamics.

Everybody wants to make money when come to trade in a financial market. It is truely a zero-sum game: One's gain is another's loss. What makes a great investor, knowledge, experience, skill, rationality, or something else? The famed speculator, George Soros, wrote the following in *Alchemy of Finance*, a book summarizing essence of his life-long success in trading and investment management.

> *As a money manager I was emotionally engaged in managing my fund. I managed it as if my existence depended on it, as indeed it did. I relied on my instincts and intuition as well as my conceptual framework to guide me through uncertainty. ......I was not well positioned to perform better than others if I had tried to play the market by a particular set of rules: my competitive advantage lays in recognizing changes in the rules of the game.*

His fund delivered supernormal growth consistently, 3000 times the initial investment over a twenty-eight year horizon. It is not likely to be a rare chance outcome. Soros identified his investment decisions as emotional and instinctive. This account is rather to the contrary of most literatures on financial market.

Traditional rational expectations literature assumes that economic agents are fully rational and derives equilibrium conditions to describe the market. In equilibrium market price is efficient in the sense that it aggregate all public and private information. However, what drives market participants to seek costly information and how their speculation depletes arbitrage opportunities is left untouched. Market microstructure literature emerged to bridge this gap. It analyzes how agents with superior insider information will behave to profit from it at the cost of liquidity traders. It did not suppass rational expectations framework. The concept of "insiders" and "liquidity traders" are introduced. Insiders are assumed to have superior information source and trade for profit. Liquidity traders enter the market for liquidity reasons. But it is not plausible to rely on liquidity reasons as a sufficient driver for people to enter the market and perpetually take the wrong side of the trade.

After Internet bubble bursted, there is a growing volume of literature collecting evidence of market anomalies such as bubbles and crashes. Findings in double auction theory and the fact that more and more stock exchanges adopt electronic clearing system, it is clear that pricing anomalies do not root in the market clearing mechanism itself. Besides traditional reasons such as liquidity need, asset allocation, or trasaction cost, explanations of those "abnormal" phenomena are offered from new perspectives. It fuels a new wave of investigation to challenge and re-examine assumptions in rational expectation models. These research offers an alternative explanation of market behaviors.

Many scholars try to explore possible leaks in information structure and rationality assumptions. Bounded rationality approach relaxes the common knowledge assumption. When the chain of infinite depth of knowledge breaks in any link, there is possibility that the entire empire may collapse and follow an entirely new path. Abreu and Brunnermier (2003) shows that speculators cannot promptly coordinate private information and this delay in coordination leads to a market bubble. Heterogenous belief approach explores trading behavior induced from heterogenous prior belief. Harris and Arthur (1993) shows how trading volume and change in price can result from different opinion about economic fundamentals. David (2005) explains equity premium by heterogenous belief among traders.

Psychological findings, especially Prospect Theory, opens the burgeoning field of behavioral finance and offers an alternative explaination of market anormalies. Barberis and Thaler (2003) has a thorough review. Decision bias such as framing, mental accounting, risk-seeking in loss and risk-averse in gain are applied to analyze financial decision making. The conventional symmetric concave utility function applied to aggregate wealth is being challenged in this case.

This paper explores the territory of dynamic learning and adaption in financial market. The market is modeled under the dynamic learning and adjustment framework of Kandori, Mailath and Rob (1993) and Young (1993). It relates to both bounded rationality models and behavioral finance. We do not assume infinite memory and boundless knowledge so that strategic behaviors are present but limited. Similar to ficticious play, market participants apply a simple and intuitive strategy as a best response to the past when speculating but not necessarily exploit all possible arbitrage opportunities. Based on research findings in cognitive science, people overweight recent events and ignore distant events. A simple adaptive learning rule encompassing this feature is proposed for speculators. Market dynamics derived in this model resembles Best-Reponse Dynamics but is much less volatile. In a stochasitc environment, pure strategy Nash equilibria does not exist and market evolves with approximate probability distribution around intertemperal mixed strategy steady state.

In the next section, the model of a asset market with a single risky asset is presented. It is simple enough to be tractable and rich enough to show the impact of adaptive learning and the role of evolution on market dynamics. Section 3 defines and discusses adaptive learning rule. Section 4 introduces several evolutionary concepts and analyzes the consequent market dynamics. Section 5 concludes with future extentions.

2 The Model

There are N potential speculators participating in the market with a single risky asset. Trading of the risky asset occurs among speculators and a market maker at time $t = 1, 2, ...$ with equal time interval. In each round of trading, a public signal about the asset return is announced to all market participants. Speculators then submit bids of the asset. The orders are cleared by the market maker at a single clearing price. The realized risky asset value becomes known and gains and losses are settled between parties taking opposite side of a trade. The market advances to the next round of trading.

We assume that value of the risky asset changes at time t by,

$$D_t = \begin{cases} z, & \text{with probability } 0.5, \\ -z, & \text{with probability } 0.5, \end{cases}$$

where $z > 0$ and D_t is identical and independently distributed at different time period. The sequence $(D_1, D_2, ..., D_t, ...)$ is thus a simple random walk. This feature simplifies the determination of market clearing price and dramatically reduces the number of states in the market dynamics. The partial sum of the incremental change series is the value of the asset at time t,

$$V_t = V_0 + \sum_{t=1}^{t} D_t.$$

If asset value hits zero, the asset will not be traded in the future. We assume that the magnitude of increment z is very small compared with the initial asset value V_0 so that zero asset value is not likely to happen. Our analysis will focus on the change instead of the value series.

Before bidding, speculators all observe a public signal $\zeta_t \in \{-z, z\}$. It is common knowledge. Denote the probability that the public signal is correct by,

$$\phi_t = P(\zeta_t = R_t).$$

We can easily verify that the public signal is either high or low with equal probability,

$$\begin{aligned} P(\zeta_t = z) &= P(D_t = z)P(\zeta_t = D_t) + P(D_t = -z)P(\zeta_t \neq D_t) \\ &= 0.5\phi_t + 0.5(1 - \phi_t) = 0.5. \end{aligned}$$

The correlation between the true asset value and the public signal at time t is,

$$Corr(\zeta_t, D_t) = 2\phi_t - 1.$$

If $\phi_t > 1/2$, the public signal is of high quality and it is positively correlated with the real asset return. The expected return of the asset at time t is,

$$E(D_t|\zeta_t) = (2\phi_t - 1)\zeta_t.$$

The quality of public signal, ϕ_t, is neither publicly known or privately observed by any market participants. There is an underlying Possion process $N(t)$ with rate λ that reflects the time when a change in the quality of public signal occurs. If $N(t_n) > N(t_{n-1})$, a change happens between time t_{n-1} and t_n and a new ϕ_t is randomly drawn from interval $[0, 1]$. The uncertainty in quality of public signal represents changes in public consensus due to changes in fundamental economic structure. At the beginning of Internet era, new technology brought new channels to disseminate information and new opportunities and there was less consensus as to how much value the new technology would add to the economy.

Based on public signal, potential speculators decide whether to bid or not and if to bid whether to take a long or short position. There is no short-selling constraint so that speculators can take any position as they desire. For simplicity, it is assumed that each speculator is allowed to buy or sell only one unit of the asset. There assures that speculators have equal market influence. Buy or sell orders are all market orders, meaning that speculators are not allowed to condition their orders on a preset price threshold. After bidding, the market marker clears all orders at price P_t given that there are x_t buy orders and y_t sell orders,

$$P_t = P_{t-1} + z\frac{x_t - y_t}{x_t + y_t}.$$

Denote the change in price by R_t,

$$R_t = P_t - P_{t-1} = z\frac{x_t - y_t}{x_t + y_t}.$$

Change in asset price is proportional to the imbalance in buy and sell orders. When there are more buy orders, the price at time t will be higher than the previous period. When there are more sell orders, the price will be driven down. This pricing rule is similar in spirit to that in Kyle (1985) except that the underlying return series is assumed to follow a random walk. The market maker operates more like a automated clearing system than a profit maximizer. We assume that the market maker has no liquidity constraint. Since he does not need to be compensated for liquidity risk, and there is no bid and ask spread nor strategic price setting behavior. We further assume that positions are terminated at the end of each trading round and profit and loss from are accounted for among parties. If a speculator enters a long position at time t, her payoff at the end of this trading round will be $-P_t + \zeta_t$. If a speculator enters a short position at time t, her payoff at the end of the trading round will be $P_t - \zeta_t$.

There are two possible pure strategies that speculators can adopt, either to follow the public signal or bet against it. We call the first strategy "trend-following", denoted by s_1, and the latter "contrarian", s_2, and denote the strategy set by $S = \{s_1, s_2\}$. When there is a high signal, i.e. $\zeta_t = z$, trend-followers will take a long position and contrarian will take a short position in the risky asset. When there is a low public signal, i.e. $\zeta_t = -z$, trend-followers will take a short position and contrarian will take a long position in the risky asset.

Let m_t be the number of speculators adopting s_1 at time t, $m_t \in \{0, 1, 2, ..., N\}$. Given the realization of ζ_t, D_t, and m_t in each round of trading, the market price and realized payoff of trend-followers and contrarians are shown in Table 1.

The market price is $P_t = P_{t-1} + z(2\frac{m_t}{N} - 1)$ when $\zeta_t = z$ and $P_t = P_{t-1} - z(2\frac{m_t}{N} - 1)$ when $\zeta_t = -z$. It can be expressed in a concise way as

$$P_t = P_{t-1} + z(2\frac{m_t}{N} - 1)(2\chi_{\{\zeta_t = z\}} - 1). \tag{1}$$

		R_t	$\Pi_{s_1}(m_t)$	$\Pi_{s_2}(m_t)$
$\zeta_t = z,$	$D_t = z$	$z(2\frac{m_t}{N} - 1)$	$2z(1 - \frac{m_t}{N})$	$-2z(1 - \frac{m_t}{N})$
$\zeta_t = -z,$	$D_t = -z$	$-z(2\frac{m_t}{N} - 1)$	$2z(1 - \frac{m_t}{N})$	$-2z(1 - \frac{m_t}{N})$
$\zeta_t = z,$	$D_t = -z$	$z(2\frac{m_t}{N} - 1)$	$-2z\frac{m_t}{N}$	$2z\frac{m_t}{N}$
$\zeta_t = -z,$	$D_t = z$	$-z(2\frac{m_t}{N} - 1)$	$-2z\frac{m_t}{N}$	$2z\frac{m_t}{N}$

Table 1: Price and Payoff

Since outcome of D_t and public signal are both known after price is released, the proportion of the population adopting each strategy is fully revealed and the number of trend-followers m_t is thus public information.

Trend-followers make a profit if and only if public signal is correct. Contrarians are the opposite. The magnitude of profit or loss depends on the distribution in the current population adopting each stategies. In the extreme, if the public signal is correct and all speculators are trend-followers, they will earn zero profit. If the public signal is correct and there is only one trend-follower, his gain will be the largest ever, close to $2z$. The outsized gain is attributed to his correct judgement and the rest of the speculators' wrong judgement. Speculators face two uncertainties, the true quality of the public signal and distribution of the current population adopting each strategies. The first dictates the chance of winning and the second determines the size of the reward. The expected payoff of a trend-follower is,

$$\begin{aligned} E[\Pi_{s_1}(m_t)] &= 2z(1 - \frac{m_t}{N})P(\zeta_t = D_t) + (-2z\frac{m_t}{N})P(\zeta_t \neq D_t) \\ &= 2z(\phi_t - \frac{m_t}{N}) \end{aligned} \tag{2}$$

The expected payoff of a contrarian is,

$$\begin{aligned} E[\Pi_{s_2}(m_t)] &= 2z(\frac{m_t}{N} - \phi_t) \\ &= -E[\Pi_1(m_t)] \end{aligned} \tag{3}$$

When the portion of trend-followers in the population is higher than the probability that the public signal is correct, the trend-followers will take a loss and the contrarians will make a profit on average. It then follows.

Proposition 1 (Symmetric Baysian Nash Equilibrium): *The symmetric Baysian*

Nash equilibrium in the speculation game is when speculators play strategy s_1 with probability ϕ_t and strategy s_2 with probability $1 - \phi_t$.

The symmetric Baysian Nash equilibrium is satisfactory in projecting limiting behavior of speculators. However, it is unclear how such an equibrium can be acheived. When the quality of the public signal ϕ_t is unkown and possibly change over time, it is not clear how individual speculators can play exactly according to an unknown odd. The market will be more often in a disequilibrium than in an equilibrium. In order to gain full perspective of the market dynamics, it is therefore necessary to analyze how individual speculators learn of the quality of public signal and how the market behaves as a result of this learning process.

As shown in Equation (2) and (3), a speculator's expected payoff from taking each strategy dependes very much on the relationship between the true quality of public signal and the behavior of the population. Since the true quality ϕ_t is unknown to the speculator, she has to rely on her own belief to make a decision. Without further complication of strategic behavior, we assume that speculators treat m_t as the best prediction of number of trend-followers at time t+1. Let θ_t^i denote speculator i's belief of the public signal at time t. Speculator i's expected payoff to be a trend-follower at time t+1 is,

$$E[\Pi_{s_1}(m_{t+1})|(\theta_t^i, m_t)] = 2z(\theta_t^i - \frac{m_t}{N})$$

and expected payoff to be a contrarian is

$$E[\Pi_{s_2}(m_{t+1})|(\theta_t^i, m_t)] = -E[\Pi_{s_1}(m_{t+1})|(\theta_t^i, m_t)].$$

The decision rule is to optimize the expected payoffs conditional on speculator's individual information set at time t. Call $\frac{m_t}{N}$ the population belief of public signal quality at time t. Speculators always compare their own belief with the population belief to determine their strategy in the next period. If $\theta_t^i > m_t/N$, speculator i will be better off being a trend follower in the next round of trading. If $\theta_t^i < m_t/N$, she will be better off being a contrarian. If the equality holds, it is assumed that speculator chooses the two strategies with equal probability. Since belief space is a continuum, this assumption does not have much influence on the market dynamics.

Before we move on to the discussion of learning and evolutionary dynamics, we will prove efficiency of the pricing mechanism.

Proposition 2 (Efficiency):

(1) The market price is informationally efficient.

(2) Market maker adopting the above pricing mechanism expects zero profit.

Proof of Proposition 2: Given x_t buy orders and y_t sell orders, the population belief of the quality of the public signal is

$$\psi_t = \frac{x_t}{x_t + y_t} = \frac{x_t}{N}.$$

The market clearing price changes by,

$$P_t - P_{t-1} = z\frac{x_t - y_t}{x_t + y_t} = z(2\psi_t - 1),$$

which equals the expected asset value change given all public information,

$$E[D_t|\theta_t = \psi_t] = z(2\psi_t - 1).$$

Therefore, the market price is weak form informationally efficient. It can also be easily checked that the expected profit of the market maker is

$$E[\Pi_t|\theta_t = \psi_t] = (x_t - y_t)[\psi_t(-z + P_t) + (1 - \psi_t)(z + P_t)] = (x_t - y_t)((1 - 2\psi_t)z + P_t) = 0.$$

Q.E.D.

3 Adaptive Learning

Speculators' action depends very much on their belief. Initially, they have a rough opinion about the market. As they participate in trading, their belief are impacted by outcomes of trading activities. In this section, we will propose and discuss the adaptive learning rule based on findings in cognitive science. Compared with empirical mean widely adopted in economic literature, adaptive learning rule emphasizes recent outcomes and overlooks distant outcomes.

In general, learning can be represented by a belief updating process. speculator i starts with an initial belief of the public signal quality, θ_0^i, $i = 1, 2, ...N$. She updates her belief after observing the true value of the asset at the end of each round of trading according to,

$$\theta_t^i = a_0^i(t)\theta_0^i + (1 - a_0^i)\sum_{l=1}^{t} a_l^i(t)\chi_{\{V_l = \zeta_l\}}. \tag{4}$$

where $\chi_{\{V_t=\zeta_t\}}$ is an indicator function, $0 \leq a_l^i(t) \leq 1$ for $l = 0, 1, 2, ...,$, $t = 0, 1, 2, ...,$, and $\sum_{l=1}^{t} a_l^i(t) = 1$. This representation is generic. It does not make assumptions about the value of the coefficients except that current belief at time t is a weighted average of initial belief and past outcomes. The weights at different time on the same previous outcome or initial belief could be different, e.g $a_0^i(1) \neq a_0^i(2)$. It only requires that weights are nonnegative and sum to one at any time period. If the public signal is correct at time t, it will increase speculator's belief at time t and onwards. For simplilcity, denote $\chi_{\{V_t=\zeta_t\}}$ by χ_t.

If speculator i with initial belief θ_0^i updates belief according to empirical mean, then $a_t^i = \frac{1}{t}$,

$$\theta_t^i = \frac{1}{t+1}\theta_0^i + \frac{1}{t+1}\sum_{l=1}^{t}\chi_l. \tag{5}$$

It assumes that all outcomes have equal influence on belief no matter when they occur. This belief updating rule is used in most evolutionary game literature. Foster and Young (1990) adopted a variant of this rule by assuming that players randomly select m out of n previous outcomes and take the average as their belief. The randomness helps to prevent the game to be locked into certain evolutionary unstable equilibrium without mutation.

Despite its popularity in game theory liturature, there are two major reasons to reject the belief updating rule based on empirical mean. First, innovation and scientific discovery bring structural changes to the economy from time to time. It is not reasonable to assume that the underlying factors are homogenous across time and expect the future to follow the pattern of long-dated outcomes to the same extent as near-dated outcomes. Rail road used to be high growth sectors in 1900's. It fueled the economy with cheaper means of mass transportation and generated supernormal returns to equity investors. But as the expansion reached boundary and new ways of transportation emgerged, the high return can no longer be sustained. Automobile industry in the 1940's, telephone in the 1960's, and semiconductor in the 1980's all follow similar development pattern but their growth fell off the radar later on as the industries mature. Recent data is thus more relevant to predict the near future. Second, empirical studies in cognitive science have shown that first impression and recent experience has bigger mental impact and thus bigger influence on people's belief than things happen in the middle. This phenomenon was initially studied by Miller and Campbell in psychology literature and fully researched in Shiffrin (1973). The bigger impact of first impression is termed "primacy effect" and the discriminate emphasize on recent events is termed "recency effect".

In this paper, we will focus our attention on the "recency effect" and formulate a learning rule reflectng this property. The primacy effect can be readily included in the learning rule by adding another parameter on impact from initial periods. Since we are more interested in adaptive behavior in a changing environment, we will use a simpler learning rule without "primacy effect".

The model in Foster and Young (1990) has "recency effect" feature, too. At the end of each period, the oldest outcome is droped and the recent outcome is added to the state of the dynamics so that there are only a fixed number n of outcomes in the history of events. Then m out of the n outcomes in the history queue will be picked randomly for each agent. The length of the history queue is a crucial assumption, which determines the number of states of the game and the dynamics over time. However, both n and m have to be arbitrarily chosen in order for the game to have a clear definition of states. The following specification is not restrictive in that sense and allows individuality in cognitive process and assumes only the general functional form.

Definition: *Speculator i updates her belief according to an* ***adaptive learning*** *rule if the new belief is a weighted average of previous belief and current outcome with constant* α^i *as weight on previous belief, i.e.*

$$\theta_t^i = (1-\alpha^i)\chi_{t-1} + \alpha^i \theta_{t-1}^i \tag{6}$$

$$= (1-\alpha^i)\sum_{l=0}^{t-1}(\alpha^i)^l \chi_{t-l} + (\alpha^i)^t \theta_0^i. \tag{7}$$

The weight α^i in an adaptive learning rule represents speculator i's responsiveness to new information. Like gene of a species in ecological evolution, a speculator's responsiveness is an inherited characteristic. It is type of speculator i and does not vary over time or depends on outcomes of the game. If $\alpha^i = 1$, speculator i is absolutely strong in initial belief and totally ignores subsequent signal outcomes. If $\alpha^i = 0$, speculator i's belief depends solely on the previous round's signal accuracy. If $0 < \alpha^i < 1$, the influence of initial belief and outcomes in earlier rounds of trading will gradually phase out over time and recent outcomes have bigger impact on speculator i's current belief. It is the "recency effect" feature.

Proposition 3 (Biasedness and Nonconvergence in Static Environment): *If quality of public signal* ϕ_t *stays at a constant* ϕ *over time and given* $0 < \alpha^i$*, the belief series derived* *from the adaptive learning rule*

(1) is a biased estimate of the true quality of public signal and;

(2) will not converge to the true quality of public signal when t goes to infinity.

Proof of Proposition 3: Notice that the coefficient sequence in Equation (7) satisfies,

$$(1-\alpha^i)\sum_{l=0}^{t-1}(\alpha^i)^l = 1-(\alpha^i)^t \tag{8}$$

Taking expectation of Equation (7),

$$\begin{aligned} E(\theta_t^i) &= (1-\alpha^i)\sum_{l=0}^{t-1}(\alpha^i)^l E(\chi_{t-l}) + (\alpha^i)^t\theta_0^i \\ &= \phi(1-\alpha^i)\sum_{l=0}^{t-1}(\alpha^i)^l + (\alpha^i)^t\theta_0^i \\ &= (1-(\alpha^i)^t)\phi + (\alpha^i)^t\theta_0^i \end{aligned}$$

Speculator i's belief at time t, θ_t^i, is thus a biased estimate of ϕ. It is unbiased only if $\theta_0^i = \phi$. The estimation bias at time t is $(\alpha^i)^t(\theta_0^i - \phi)$. It decreases over time. In the limit as t goes to infinity, the impact of initial belief approaches zero and $E(\theta_t^i)$ converges to ϕ,

$$\lim_{t\to\infty} E(\theta_t^i) = \lim_{t\to\infty}(1-(\alpha^i)^t)\phi + \lim_{t\to\infty}(\alpha^i)^t\theta_0^i = \phi.$$

It can be shown that the variance of the adaptive learning belief at time t is

$$\begin{aligned} Var(\theta_t^i) &= Var((1-\alpha^i)\sum_{l=0}^{t-1}(\alpha^i)^l\chi_{t-l}) \\ &= (1-\alpha^i)^2\sum_{l=0}^{t-1}(\alpha^i)^{2l}\phi(1-\phi) \\ &= \phi(1-\phi)\frac{1-\alpha^i}{1+\alpha^i}(1-(\alpha^i)^{2t}). \end{aligned}$$

The variance of the adaptive learning belief series converges to a constant as time approaches infinity,

$$\lim_{t\to\infty} Var(\theta_t^i) = \phi(1-\phi)\frac{1-\alpha^i}{1+\alpha^i},$$

therefore the adaptive learning belief series does not converge in probability to ϕ.

Q.E.D.

The empirical mean is also biased. Its estimation bias at time t is $\frac{1}{1+t}(\theta_0^i - \phi)$. The bias reduces over time. Empirical mean does not converge to the true quality of public signal in the limit unless $a_i(0) = 0$. The variance of belief based on empirical mean at time t is

$$Var(\theta_t^i) \quad = \quad \phi(1-\phi)(1-\alpha_0^i)^2 \frac{1}{t}.$$

The empirical mean belief series converges to 0 as time approaches infinity. By Strong Law of Large Numbers, the empirical mean belief series converges to ϕ in probability.

As shown before, the belief series generated from this rule does not converge to the true quality of public signal and has a larger variance in the limit than the empirical mean. But It has an advantage when the game is set with structural shifts.

Proposition 4: *Assume that the change in quality of public signal ϕ_t occurs at time $(T_1, T_2, ..., T_n, ...)$ and the number of changes at time t, $N(t)$, is a Poisson process with parameter λ. Given $0 < \alpha^i$, there exists time T, such that for all $t > T$, the expected belief series derived from the adaptive learning rule is closer to ϕ_t than that of the empirical mean belief series, conditional on, $\mathscr{F}_{T_n}$, the information set at T_n, where T_n is the last time before T when a change happens and $\mathscr{F}_{T_n} \equiv (\phi_{T_1}, \phi_{T_2}, ..., \phi_{T_m}) \bigcup (\chi_l)_{l=1}^{T_n}$.*

$$E[(1-\alpha^i)\sum_{l=0}^{t-1}(\alpha^i)^l \chi_{t-l} + (\alpha^i)^t \theta_0^i - \phi_t | \mathscr{F}_{T_n}]$$
$$< E[\frac{1}{t+1}\theta_0^i + \frac{1}{t+1}\sum_{l=1}^{t}\chi_l - \phi_t|\mathscr{F}_{T_n}] \tag{9}$$

Proof of Proposition 4: It is a property of Poisson process that the average time lapse from a change in the quality of public signal to the next is $1/\lambda$,

$$E[T_{n+1} - T_n | T_n] = 1/\lambda.$$

Denote $k \equiv \lfloor 1/\lambda \rfloor + 1$, the smallest integer greater than $1/\lambda$. Let

$$T = \left\lfloor \frac{k+1}{1-\alpha^{k+1}} \right\rfloor + 1,$$

then for any $t > T$,

$$1 - \alpha^{k+1} > \frac{k+1}{t+1}. \tag{10}$$

Notice that the left hand side of Condition (10) is sum of the k most recent belief weights in adaptive learning rule and the right hand side is that in empirical mean. For any $t > T$, in the adaptive learning rule total belief weight on outcomes happened after the recent change in quality of public signal is thus larger than that in empirical mean. Since $E[\chi_l] = \phi_l = \phi_{T_n}$ for any $l > T_n$,

$$E[(1-\alpha^i)\sum_{l=0}^{t-1}(\alpha^i)^l\chi_{t-l} + (\alpha^i)^t\theta_0^i - \phi_t|\mathscr{F}_{T_n}] = (1-\alpha^i)\sum_{l=t-T_n}^{t-1}(\alpha^i)^l(\chi_{t-l} - \phi_t) + (\alpha^i)^t(\theta_0^i - \phi_t)$$

and,

$$E[\frac{1}{t+1}\theta_0^i + \frac{1}{t+1}\sum_{l=1}^{t}\chi_l - \phi_t|\mathscr{F}_{T_n}] = \frac{1}{t+1}(\theta_0^i - \phi_t) + \frac{1}{t+1}\sum_{l=1}^{T_n}(\chi_l - \phi_t)$$

Condition (9) holds.

Q.E.D.

4 Evolutionary Dynamics

The model with adaptive learning feature distinguishes itself from Stochasitic Fictitious Play and the Adaptive Play model of Young (1993). In this section, we will focus on implication of adaptive learning on market dynamics.

Let θ_t denote the vector of speculators' belief, $(\theta_t^1, \theta_t^2, ..., \theta_t^N)$. As explained before, the clearing price P_t and realized outcome of the public signal and asset value fully reveal the proportion of population adopting each strategies at time t. In other words, m_t is public information before bidding begins at time $t+1$. The state of the game at time t is a tuple $< \theta_t, m_t >$. For speculator i, she knows only her own belief θ_t^i but not the others'. Also speculators do not have perfect recall of past outcomes. The past impacts speculators' future actions only through belief. Therefore, speculator i's information set at time t is (θ_t^i, m_t).

Definition: *Let $S = \{s_1, s_2, ..., s_K\}$ denote the set of pure strategies. Let δ_t be the population distribution adopting each strategies at time t, $\delta_t = (\delta_t^{(k)})_{k=1}^K$, where $\delta_t^{(k)} \in \{0, 1, ..., N\}$ and $\sum_{k=1}^K \delta_t^{(k)} = N$. Let Δ be the space of δ_t. Let Θ be the space of players' belief $\theta_t = (\theta_t^1, \theta_t^2, ..., \theta_t^N)$. The mapping $b : (\Delta, \Theta) \to (\Delta, \Theta)$ represents an **evolutionary dynamic**.*

An evolutionary dynamic is composed of a belief updating rule and choice decision of pure strategies based on belief at each point in time. The system evolves from one state to another according to the mapping $b(.)$. If the transition probability is determined, the evolutionary dynamics can be represented by a Markov chain. We have defined the belief updating rule and strategy choice rule for this model, the dynamics are hence fully determined given an initial state.

To show characteristics of the evolutionary dynamics in this model, we introduce the concept of payoff monotonic evolutionary dynamic.

Definition: *An evolutionary dynamic is* ***payoff monotonic*** *if in the next period the proportion of population adopting a higher payoff strategy is at least as large as that adopting a lower payoff strategy in the current period. Mathematically,*

$$\pi_k(\delta_t) > \pi_l(\delta_t) \Rightarrow \delta_{t+1}^{(k)} - \delta_t^{(k)} \geq \delta_{t+1}^{(l)} - \delta_t^{(l)}, \quad \forall \quad k, l \in \{1, 2, ..., K\}$$

where $\delta_{t+1} = b(\delta_t)$.

In the setting of this model, payoff monotone means that if strategy j generated higher payoff in the previous round of trading then the proportion of speculators adopting strategy j in the next round is non-decreasing.

Theorem 1 (Payoff Monotonicity): *If* $\exists i \in \{1, 2, ..., N\}$, *s.t.* $\alpha^i > 0$, *the decision rule in the speculating game is payoff monotone.*

Proof of Theorem 1: We claim that if trend-following was the higher payoff strategy at time t, then there will be at least as many trend-followers at time t+1 as before. To see this, first note that for any speculator i with $\alpha^i > 0$,

$$\theta_{t+1}^i = \alpha^i \chi_{\{V_t = \zeta_t\}} + (1 - \alpha^i)\theta_t^i > \theta_t^i,$$

since $\chi_{\{V_t=\zeta_t\}} = 1$. If trend-following generated higher payoff at time t, any speculator i with $\alpha^i > 0$ have belief $\theta_t^i > \frac{m_t}{N}$. They will either continue to be a trend-follower at time t+1 or switch from a contrarian to a trend-follower. For any speculator j with $\alpha_j = 0$, his belief $\theta_{t+1}^j = \theta_t^j$. His strategy will not change. Therefore our claim is valid. It can be easily proved that if contrarian is the higher payoff strategy at time t, there will be at least as many contrarians at time t+1 as before. This completes the proof that the decision rule is payoff monotone.

Q.E.D.

When $\alpha^i = 1$ for all $i = 1, 2, ...N$, the above belief updating rule will generate the Best-Response dynamics. If a speculator previously followed the public signal and it turned out to be correct, she will be a trend-follower in the next round. Otherwise, they will be contrarians. Similarly, contrarians will bid against the public signal in the next round if they won and become trend-followers if they lost in the previous round.

Note that payoff monotone does not necessarily imply that the strategy distribution with that property will converge to the symmetric Baysien Nash equilibrium. In this game, due to speculators' myopic emphasis of the most recent outcomes, their choice of strategies will be myopic.

Definition: *Given that the state of the game is (θ_t, m_t),*
a ***marginal contrarian*** *is a speculator with belief, $\theta_t^i < \frac{m_t}{N}$, whose belief updating weight α^i is such that*

$$(1-\alpha^i)\theta_i^t + \alpha^i \geq \frac{m_t}{N} \quad \textit{or equivalently,} \quad \alpha^i \geq \frac{m_t - N\theta_t^i}{N(1-\theta_t^i)};$$

a ***marginal trend-follower*** *is a speculator with belief, $\theta_t^i > \frac{m_t}{N}$, whose belief updating weight α^i is such that*

$$(1-\alpha^i)\theta_i^t + \alpha^i \leq \frac{m_t}{N} \quad \textit{or equivalently,} \quad \alpha^i \geq \frac{N\theta_t^i - m_t}{N(1+\theta_t^i)}.$$

A marginal contrarian will become a trend-follower at time t+1 if the public signal is correct at time t. Any contrarian at time t with belief θ_t^i and a belief updating weight less than the marginal contrarian with the same belief will still be a contrarian at time t+1 even if the signal is correct at time t. Similarly, a marginal trend-follower will become a contrarian at time t+1 if the public signal is incorrect at time t. Any trend-follower at time t with belief θ_t^i and a belief updating weight less than the marginal trend-follower with the same belief will still be a trend-follower at time t+1 even if the signal is incorrect at time t.

For each value of the individual belief θ^*, there is a cutoff value of belief updating weight, $\alpha_C(\theta^*)$, that solely determines whether a contrarian with this belief will possibly change strategy in the next period,

$$\alpha_C(\theta^*) = \frac{m_t - N\theta^*}{N(1-\theta^*)}.$$

The cutoff value $\alpha_C(\theta^*)$ is a convex and increasing function of θ^*. It is also increasing in m_t. Similarly, there is a cutoff value of belief updating weight, $\alpha_T(\theta^*)$, that solely determines whether a trend-follower with this belief will possibly change strategy in the next period,

$$\alpha_T(\theta^*) = \frac{N\theta^* - m_t}{N(1+\theta^*)}.$$

The cutoff value $\alpha_T(\theta^*)$ is a convex and decreasing function of θ^*. It is a decreasing function in m_t. Figure (**??**)shows an example of $\alpha_C(\theta^*)$ and $\alpha_T(\theta^*)$.

Given state of the game $< \theta_t, m_t >$, denote the number of marginal contrarians at time t by MC_t and the number of marginal trend-followers by MT_t. Since the game has finite number of players, it is possible that either MC_t or MT_t is 0 from time to time. If at time t, the public signal is correct, $\zeta_t = D_t$, then speculator i's belief will be updated, $\theta^i_{t+1} = \alpha^i + (1-\alpha^i)\theta^i_t$. It also drives marginal contrarians to switch strategy, $m_{t+1} = m_t + MC_t$. If at time t, the public signal is incorrect, i.e. $\zeta_t \neq D_t$, then speculator i's belief θ_{t+1} will be decreased to $\theta^i_{t+1} = -\alpha^i + (1-\alpha^i)\theta^i_t$. It also drives marginal trend-followers to switch strategy, $m_{t+1} = m_t - MT_t$. The state of the game changes to $< \theta_{t+1}, m_{t+1} >$, where $\theta_{t+1} = (\theta^1_{t+1}, \theta^2_{t+1}, ..., \theta^N_{t+1})$,

$$\theta^i_{t+1} = \alpha^i(2\chi_{\{\zeta_t=z\}} - 1) + (1-\alpha^i)\theta^i_t \quad \text{for i=1,2,...N} \tag{11}$$

$$m_{t+1} = m_t + \chi_{\{\zeta_t=z\}} MC_t + (\chi_{\{\zeta_t=z\}} - 1) MT_t. \tag{12}$$

Define $\bar{\alpha} = \sum_{i=1}^N \alpha^i$ to be the average belief updating weight. Since $\alpha^i \geq 0$, for all i, $\bar{\alpha} \geq 0$. Whenever the public signal is correct, speculators' belief of the quality of the public signal increases by a fixed amount both at individual and population level. Likewise, whenever the public signal is incorrect, speculators' belief of the quality of the public signal decreases by a fixed amount both at individual and population level.

When state of the game moves to $< \theta_{t+1}, m_{t+1} >$, price at time t+1 will be affected by change in belief as well. As shown in Equation (1), state of the dynamics affects price P_{t+1} through m_{t+1}. The absolute belief updating impact on price is $2z\frac{MC_t}{N}$ when the public signal is correct and $2z\frac{MT_t}{N}$ when the public signal is incorrect at time t. Because MC_t and MT_t are both state dependent, the impact on price is state dependent.

Theorem 2 (Extreme Aversion): *The evolutionary dynamics stays at extreme states (with one strategy dominating the population) less often than moderate states.*

Proof of Theorem 2: The cutoff value of marginal contrarian's belief updating weight, $\alpha_C(\theta^i)$, is increasing in the difference between her own belief and the population belief, $|\theta^i - \frac{m_t}{N}|$. The cutoff value of marginal trend-follower's belief updating weight, $\alpha_T(\theta^j)$, is also increasing in $|\theta^j - \frac{m_t}{N}|$. When the state is at m_t, assume there is a marginal contrarian with belief θ^i_C and belief updating weight $\alpha_C(\theta^i_C)$. Denote the distance between this marginal contrarian's belief and the population belief by $d = |\theta^i_C - \frac{m_t}{N}|$. A marginal trend-follower j whose belief is as close to the population belief as that of the marginal contrarian will have a belief updating weight $\alpha_T(\theta^j_T)$, s.t. $\theta^j_T = \frac{m_t}{N} + d$.

When there are more trend-followers in the market, i.e. a higher $\overline{m_t} > m_t$, belief of the marginal contrarian with $\overline{\theta^i_C} = \overline{m_t} + d$ and belief of the marginal trend-follower with $\overline{\theta^j_T} = \overline{m_t} - d$ with symmetric distance to the population belief are both higher. Notice that $\alpha_T(\theta^j_T) = \frac{d}{1+\theta^j_T}$ is increasing in θ^j_T and $\alpha_C(\theta^i_C) = \frac{d}{1-\theta^i_C}$ is decreasing in θ^i_C. Holding d constant, at $\overline{m_t} > m_t$,

$$\alpha_T(\overline{\theta^j_T}) > \alpha_T(\theta^j_T) \quad \text{and} \quad \alpha_C(\overline{\theta^i_C}) < \alpha_C(\theta^i_C).$$

It implies that when there are more existing trend-followers in the market, contrarians will be less inclined to switch strategy than trend-followers. The reverse holds when there are more contrarians. The evolutionary dynamic is therefore less likely to be at extreme states when m_t is either close to 0 or N.

Q.E.D.

Theorem 3 (Boom-Bust Cycle): *The price series in this market exhibits boom-bust cycle.*

Proof of Theorem 3: As shown in the dynamics of Equation (eq:eq2) and (eq:eq3), consecutive correct public signals increase price well as belief. The oppotimism as reflected in belief elevates price increment further. When a incorrect public signal occurs afterwards, the price down turn is bigger than if there are no previous favorable outcomes. The reverse is true when there are consecutive incorrect public signals. Consequently, belief is endogeneous force that generates boom-bust cycle in price.

Q.E.D.

5 Conclusion and Further Extension

In this paper, we introduced a model of financial market with dynamic adaptive learning. Speculators are inherited with idiosyncratic characters and learn of the stochastic quality of public signal. They are passive in the sense that their reaction to market events are determined by heredity instead of personal will. They are active in the sense that they try to beat the market and make a profit. It is a reflection of emotional behavior human beings instead of fully rational optimizers. This is the first attempt to model these aspects of financial market.

From the modelling point of view, the simple market order trading mechanism circumvents the usual cumbersome assumptions. Speculation is driven by heterogenous belief and dynamic learning. Nobody expects themselves to lose on average and still trade. At the same time, speculators adopt all pure strategies and market maker sets market clearing price without information scooping. The market clearing price is informationally efficient and does not involve complication of strategic interactions. As we have seen that as a result of adaptive learning price fluctuates irrespective economic fundamentals and boom-bust cycles are endogenous.

There are several directions worth further exploration. In the current model, the only information source is an exogenous public signal. It will be interesting to see how inclusion of speculators with private information on asset value will impact the market dynamic. Also current model has not touched elimination rule that represents the priciple of "survival of the fittest." To introduce wealth constraints to eliminate inept speculators and show the evolutionary impact on public learning and market dynamics will be the next step. In that case, public signal could be the public belief revealed by market clearing price and endogenized.

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