Mixed Quantal Response Equilibria for Normal Form Games

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Abstract

We introduce the mixed quantal response equilibrium as an alternative statistical approach to normal form games with random utility function and prove its existence. Then we extend the quantal response equilibrium to payoff functions with disturbances outside the family of admissible distributions. Finally, we define the mixed logit quantal response equilibrium, we draw the correspondence between it and the multinomial mixed logit model and prove that any random utility game has a quantal response equilibrium, which additionally is the limit of a parametric mixed logit quantal response equilibrium.

Keywords: normal form games, quantal response equilibrium, mixed multinomial logit, type profile, admissible disturbance.

JEL Classification:

1 Introduction

In the seminal paper written by McKelvey and Palfrey (henceforth McKP; 1995), normal form games with random utility functions are being examined. Given that each player cannot predetermine with certainty the exact payoff that a strategy profile will entail, the best response function becomes stochastic. Hence, tools outside Nash's classical game-theoretic framework have to be used.

The problem of random utility functions was introduced by Luce (1959) and was later extensively studied by McFadden (1973) in his breakthrough work. Each of the finitely many choices gives some utility level consisting of two components, a deterministic and a random one. Under this framework, individuals, instead of maximizing utilities, they randomize between choices in a way such that higher probabilities to be assigned to choices that are more likely to entail greater utility level. This kind of models are called quantal choice models. McFadden (1973) imposes the assumption of iid Gumbel (extreme value type I) distributed random components, creating thus the famous Multinomial Logit model which became cornerstone in modern econometrics, mainly due to its tractability (closed form choice probabilities).

McKP study normal form games from this perspective by assuming that for each strategy profile the expected payoff vector is not deterministic, but a disturbance vector is also added. They follow McFadden's line by assuming that each player would behave according to a quantal response function, rather than a best response correspondence, and given some mild assumptions about the

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joint error distribution, they prove existence of Quantal Response Equilibria (QRE). Similarly to McFadden, they also introduce the Logit QRE as a special case with iid Gumbel distributed errors and examine its properties.

In practice, even though the logit model has been widely used, different extensions and generalizations (McFadden, Train; 2000, Chesher, Santos Silva; 2002) have appeared, basing their argument on its weakness to capture random taste variation and non-proportional substitution patterns between choices. In order to cover these cases, the Mixed Multinomial Logit model was introduced, which can model preferences with these characteristics. In order to do this, it is assumed that some of the parameters of the utility function (without specifying if they belong to the deterministic or the stochastic part) come from a different distribution. Then the choice probabilities are transformed into expected choice probabilities of ordinary logit models, under the measure of this superposed distribution.

In this paper we apply this idea to McKP's normal form games, by assuming existence of a random vector of parameters in each player's utility function (still without specifying if they belong to the deterministic or the stochastic part), which represents various characteristics of the players or the game. More precisely, it would be very relevant to assume that rationality is described by an unknown parameter drawn randomly from some population. Following McKP, we could model such a situation with a mixed logit model, where the parameter λ denotes the degree of rationality, which we assume that comes from some other distribution, f. In this case, we have to estimate the parameters of f rather than λ itself.

Notice that utilities can be affected by both the own and the opponents' characteristics, without this being necessary. An interesting point is that this specification allows for non-proportional substitution patterns between different choices, which implies relaxation of the assumption of Independence of Irrelevant Alternatives (IIA), which is the one among the VNM axioms that has mostly been attacked.

We define the Mixed Quantal Response Equilibrium (MQRE) as a fixed point of weighted quantal response vector, under the joint type distribution and we prove its existence. Notice that by using the law of total probability, it is straightforward to show that in equilibrium the probability assigned by each player to each pure strategy is equal to the total probability this strategy to best response. Hence the MQRE is not conceptually different to the ordinary QRE, but it refers to models with different informational background. Similarly to the case of deterministic utility functions, the Bayesian approach lays between the two cases. McKelvey and Palfrey (1996) combine the classical Bayesian approach with quantal responses, introducing thus incomplete information to random utility games. Even though they do not explicitly call it that, in order to be consistent with the terminology used throughout this paper, we use the concept of Bayesian Quantal Response (BQR) to describe the behavioral structure in this class of games. In the following sections there is a discussion about the relation between the BQR and the MQR.

Then, we define the concept of Mixed Logit Quantal Response Equilibrium and using the techniques applied by McFadden and Train (2000) to prove that every random utility model can be approximated as closely as desired by a mixed logit model, we show that every normal form game with random utility functions has a QRE. In addition this QRE is the limit of a parametric MLQRE. This last result generalizes McKP's theorem, since it holds for almost every distribution function for the error components. That is, we extend their result to a class of disturbances outside the admissible ones that are originally assumed. No matter how complex the joint error distribution is, the QR function can be found by just calculating the limit of the parametric MLQR function. Finally, we discuss two specific mixing distribution families, Gamma and Triangular. We close this discussion by showing that the limiting behavior of the single parameter of the standard Gamma in a mixed logit QRE is almost surely identical to the limiting behavior of the single parameter of an ordinary logit QRE.

The paper has the following structure: in section 2 we present a quick description of normal form games with random utility functions, as they were introduced by McKP. We also present the notation style used throughout the whole paper. In section 3 we define the mixed quantal response equilibria, prove their existence and connect them to the the ordinary QRE and the BQRE. We also extend the equilibrium concept to utility functions with non-admissible error vectors. In section 4 the mixed logit quantal response equilibrium is introduced as a special case of the MQRE and we prove that every random utility normal form game possesses a QRE, which additionally is the limit of a parametric MLQRE. Section 5 concludes.

2 Normal Form Games with Random Payoff Functions

We consider a normal form game, consisting of a set of n players $N = \{1, ..., n\}$. Each one of them has a set of pure strategies $S_i = \{s_i^1, ..., s_i^{J_i}\}$ and a payoff function defined on the product of the pure strategy spaces, $u_i : S = S_1 \times ... \times S_n \to \mathbb{R}$. We denote this game by $\Gamma = (N, S, u)$. Let $\Delta(S_i) = \{(m_i^1, ..., m_i^{J_i}) \in [0,1]^{J_i} : \sum_{j=1}^{J_i} m_i^j = 1\}$ denote the set of player i's mixed strategies and $\Delta(S) = \Delta(S_1) \times ... \times \Delta(S_n)$ the set of mixed strategy profiles.

We assume, similarly to McKP, that each player's payoff function has two components: a deterministic and a random one. Therefore we can rewrite it as follows

$$u_i(s_i^j, s_{-i}) = \hat{u}_i(s_i^j, s_{-i}) + \epsilon_i^j \tag{2.1}$$

where $\epsilon_i = (\epsilon_i^1, ..., \epsilon_i^{J_i})$ follows some joint distribution, with density function $f_i(\epsilon_i)$, with expected value $E[\epsilon_i] = 0$ and such that all marginal distributions to exist. Every vector $\epsilon = (\epsilon_1, ..., \epsilon_n)$ that satisfies all these properties for every player is called admissible. The behavioral assumption that McKP are introducing, is that players substitute the Best Response (BR) correspondence with a Quantal Response (QR) function, according to which for every strategy profile played by the opponents, player i responds with a completely mixed strategy, which assigns to each pure strategy the probability to be the Best Response. Hence, player i's QR function, $\sigma_i = (\sigma_i^1, ..., \sigma_i^{J_i}) : \Delta(S_{-i}) \to \Delta(S_i)$ is defined as

$$\begin{aligned} \sigma_i^j(m_{-i}) &= \mathbb{P}\left[\ u_i(s_i^j, m_{-i}) \geq u_i(s_i^k, m_{-i}), \forall k = 1, ..., J_i\ \right] \\ &= \mathbb{P}\left[\ \epsilon_i^k - \epsilon_i^j \leq \hat{u}_i(s_i^j, m_{-i}) - \hat{u}_i(s_i^k, m_{-i}), \forall k = 1, ..., J_i\ \right] \end{aligned} \tag{2.2}$$

A Quantal Response Equilibrium (QRE) is a fixed point of $\sigma = (\sigma_1, ..., \sigma_n)$. More precisely, it is a mixed strategy profile $m = (m_1, ..., m_n) \in \Delta(S)$, such that $\sigma_i(m_{-i}) = m_i$, for every i. McKP prove the existence of QRE for every game with the structure described above.

3 Mixed Quantal Response Equilibrium

Up to this point we assume that every player has perfect knowledge about everybody's payoff function. That is they have complete information about the deterministic part, $\hat{u}_j$, for every j and the value of the parameters of the error distributions. Additionally, even though players do not know the realization of the payoff vector for every strategy profile, they do know the decision mechanisms that both their opponents and themselves use, in order to determine their responses to any given strategy played by the others. Namely they are aware of the fact that everybody is playing according to their QR function, which is commonly known.

Consider now a parameter vector, t_i for each player which denotes personal characteristics that enter the utility function. From this point and on we will refer to t_i as player i's *type*. The elements of t_i are included in (both or one of) the deterministic and the random component of the utility function, which takes the following general form

$$u_i(s) = \beta_i(t)\hat{u}_i(s) + \epsilon_i^j(\lambda_i(t)) \tag{3.1}$$

where $\hat{u}_i$ is a deterministic function and λ_i is the parameter of the error terms, which are assumed to be admissible for every given $t = (t_i, t_{-i})$. If both β_i and λ_i are constant for every t, then we are back to equation (2.1) which can be considered as a special case. The implication behind equation (3.1) is that utility can still be decomposed into an observed (deterministic) part and an unobserved (stochastic) one. It is straightforward that $\beta_i(t)\hat{u}_i(s) = \tilde{u}_i(s) + \tilde{\beta}_i(s,t)$. Hence, we can rewrite equation (3.1) as

$$u_i(s) = \tilde{u}_i(s) + \eta_i(s,t) \tag{3.2}$$

where $\tilde{u}_i$ is a deterministic function and $\eta_i(s,t) = \epsilon_i^j(t) + \tilde{\beta}_i(s,t)$ are the correlated and dependent on every player's strategy error terms. If $\beta_i(t)\hat{u}_i(s)$ are additively separable, implying that $\tilde{\beta}_i$ depends only on the type profile, then η_i is not a function of the opponents' strategy.

Assume now that the decision makers ignore the realizations of both their and their opponents' types. This could be justified in situations, where for instance the player is not a single individual, but represents a group of people. In such a case only one would make the choice about the strategy they would follow, say the leader, without having complete information about the "player's own type", since not all the members of the group are identical. However, she knows the type distribution of both her own and the opponent's group. From this point and on we will refer to the team as a single player. All players know the joint density function, $f_T(t_1, ..., t_n)$, of the private characteristics. Notice that the model allows for any kind of distribution (discrete or continuous). If all the probability mass is accumulated at one single type profile, then again we are back to McKP's model. That is, if everybody could observe the realization of every single t_i almost surely, then they would be answering to the opponent's actions according to the QR function.

Let us describe the structure of the random sources of the game with more details. Nature picks a type profile which remains unrevealed. The only available information is the joint type distribution. One could ask here why it is important to separate the two sources of uncertainty, error term and type profile, instead of taking into account a single joint distribution. The answer is quite straightforward and lays on the dependence among the error terms. Namely, as we are going to see later, it is quite often assumed that the initial error terms (ϵ_i^j) are independent. On the other hand, if we merge the two random components (ϵ_i^j and t) into one joint random variable, η_i^js will not be independent any more, which will not allow us to calculate the choice probabilities using the same techniques. Additionally, and more importantly, η may not admissible for every joint type distribution (f_T), which is a necessary condition for the existence of QRE. We denote this game by $\Gamma_M = (N, S, u, f_T)$ and we call it *Γ's corresponding mixed normal form game.* The normal form game that arises if we mix the disturbance and the type distribution into one joint random term, we call it *McKP representation of* Γ_M and technically it is equivalent to Γ_M.

Then we define the Mixed Quantal Response (MQR) function, which is a mixture of Quantal Responses. Obviously, the MQR function assigns a QR to every type profile.

$$\sigma_i^j(m_{-i}) = \int_T \sigma_i^j(m_{-i}|t) f_T(t)dt \tag{3.3}$$

where T is the type profile support and $\sigma_i^j(m_{-i}|t)$ is the QR given some type profile, i.e. the QR in the degenerated case of error terms coming from a known distribution, as in equation (2.2).

Definition 3.1 (Mixed Quantal Response Equilibrium). *Consider a mixed normal form game* $\Gamma_M = (N, S, u, f_T)$. *We define MQRE to be a fixed point of the function* $\sigma = (\sigma_1, ..., \sigma_n)$, *where* σ_i *is given by equation (3.3).*

Theorem 3.1. *Every mixed normal form game* Γ_M *has a MQRE.*

Proof. See in the appendix. □

Corollary 3.1. *Every MQRE of a mixed normal form game* Γ_M *is a QRE of its McKP representation.*

Proof. See in the appendix. □

We could have assumed that each player's own type is private information possessed only by themselves and have consequently formed the corresponding Bayesian game, similarly to McKelvey and Palfrey (1996). In such a game players, with private information about their types, would be still responding playing mixtures of their quantal responses (Bayesian Quantal Response; BQR). Using the same underlying idea as Harsanyi (1973), one could easily see the correspondence between the sequence Nash Equilibrium- Bayesian Nash Equilibrium- Quantal Reponse Equilibrium and the sequence Quantal Response Equilibrium- Bayesian Quantal Response Equilibrium- Mixed Quantal Equilibrium. The role of joint error term in the first sequence is played by the joint type distribution in the second. Hence, while we move along the elements of the two sequences if we place them one after the other, we only decrease the ammount of information possessed by the players as follows:

- complete information (NE)
- incomplete with private information about error terms and complete about types (BNE)
- incomplete with no private information about error terms and complete about types (QRE)
- incomplete with no private information about error terms and incomplete with private information about types (BQRE)
- incomplete with no private information about error terms and incomplete with no private information about types (MQRE)

The turning point of this sequence of equilibrium concepts, is the substitution of best with quantal responses as a behavioral pattern applied by individuals. However, the existence of QRE has been proven only for admissible joint error terms that do not depend on the strategies played by the others, which although cover the huge majority of continuous density functions, they do not exhaust them. The following lemma goes one step further by relaxing these two restrictions.

Lemma 3.1. *Assume a normal form game with random utility functions, where the error term can be written as* $\eta_i(s,t) = \epsilon_i^j(t) + \tilde{\beta}_i(s,t)$, *where* t *is an arbitrary random variable,* ϵ *is admissible for every value of* t *and* $\tilde{\beta}_i$ *is a deterministic function. Then this game has a QRE.*

Proof. See in the appendix. □

The implication of the previous lemma is that every normal form game with random utility functions that can be written as McKP representation of some mixed normal form game Γ_M has a QRE. The utility functions with admissible disturbances form a subclass of these functions, since they coincide when $\tilde{\beta}(s,t) = \tilde{\beta}(s)$, for every t.

4 Mixed Logit Quantal Response Equilibrium

In the Classical Logit QRE, studied by McKP, the Quantal Response function is

$$\sigma_i^j(m_{-i}) = \frac{e^{\lambda \hat{u}_i(s_i^j, m_{-i})}}{\sum_{k=1}^{J_i} e^{\lambda \hat{u}_i(s_i^k, m_{-i})}}$$

This model is based on utility functions of the form of equation (2.1) with independent, extreme-value distributed error terms $\epsilon_i^j \sim F(x) = \exp\{-e^{-\lambda x}\}$. The parameter λ usually denotes the degree of rationality. This is a quite common assumption, since for values close to infinity the QR function converges to the Best Reply, while for $\lambda = 0$ players randomize blindly, using the uniform distribution as a mixed strategy, regardless of the opponent's move and the deterministic part of the payoff function.

If we assume utility function of the form of equation (3.1) instead, then we can obtain the parametric form of the choice probabilities by conditioning with the type profile, t.

$$\sigma_i^j(m_{-i}|t) = \frac{\exp\{\lambda_i(t)\beta_i(t)\hat{u}_i(s_i^j, m_{-i})\}}{\sum_{k=1}^{J_i} \exp\{\lambda_i(t)\beta_i(t)\hat{u}_i(s_i^k, m_{-i})\}} \tag{4.1}$$

Taking expectations with respect to the type profile in the previous equation, we can define the Mixed Logit Quantal Response (MLQR) function.

$$\sigma_i^j(m_{-i}) = \int_T \frac{\exp\{\lambda_i(t)\beta_i(t)\hat{u}_i(s_i^j, m_{-i})\}}{\sum_{k=1}^{J_i} \exp\{\lambda_i(t)\beta_i(t)\hat{u}_i(s_i^k, m_{-i})\}} f_T(t)dt \tag{4.2}$$

This model is based on the Mixed Multinomial Logit model (McFadden, Train; 2000), which is defined as a mixture of ordinary logit models. The main advantage of this approach is that, unlike the ordinary logit model, since the ratio of two choice probabilities is not fixed, it does not exhibit Independence of Irrelevant Alternatives (IIA). That is, we do not restrict our model to fixed substitution patterns. In other words, changing the probability assigned by a mixed strategy to some choice does not necessarily leave unaffected the ratio of probabilities assigned to two other alternatives. Additionally, we allow for taste variation, implying that not only can changes in the deterministic part of the utility function be observed, but random taste variation is possible too.

An extremely interesting property possessed by Mixed Logit Models is that they can approximate as closely as desired any Random Utility Model (McFadden, Train; 2000). Using this idea we prove the following theorem.

Theorem 4.1. *Every n-player normal form game with random utility functions has a QRE almost surely, which is always the limit of a parametric MLQRE.*

Proof. See in the appendix. □

This theorem allows us to calculate the QRE in cases where the only information that is available is the joint probability of the error term, regardless of independency or admissibility.

After having described the structure and the properties of a MLQRE, it would be useful to compare this specification with an ordinary QRE. That is, we would like to check whether the introduction of a mixing distribution that determines the types of the players is necessary. McFadden and Train (2000) introduce the Lagrange multiplier test that aims is testing the hypothesis of fixed parameters. If this hypothesis is rejected, then a mixed logit specification would be more appropriate. Brownstone (2001) and Hensher and Greene (2003) also discuss this issue. It is quite easy to see that this test applied to discrete choice models can be extended to normal form games with random utility functions, since the likelihood function is constructed identically to the case with only one decision maker.

The most convenient way to apply a mixed logit specification in a game with random utilities would be to assume that the rationality level λ is coming from another distribution. There is a number of mixing distributions that could be used in this case, as long as their support contains only non-negative numbers. A very suitable distribution would be a standard Gamma, with density function

$$f_T(t) = \frac{t^{\gamma-1}e^{-t}}{\int_0^\infty u^{\gamma-1}e^{-u}du} \tag{4.3}$$

where $t \geq 0$ and $\gamma > 0$. In this case the only parameter that determines the type of the players is the shape parameter γ. The larger it is the greater mass probability is accumulated in the tail of the distribution, which results to higher probability of observing best replies. On the other hand, while γ becomes lower and gets closer to 0, there is high probability of observing blind randomization among the different choices. This implies that the limiting behavior of this parameter is similar to the one possessed by the parameter λ used in the case of the ordinary logit QRE, with the difference that in the case of the mixed logit model, perfect rationality ($\gamma \to \infty$) and blind randomization ($\gamma \to 0$) occur almost surely rather than surely, since γ is the parameter of the density function of the random rationality level.

Even though Gamma seems a reasonable choice (perhaps optimal among the commonly used distributions), its formula could cause various tractability problems. In this case we could consciously skip some theoretical controversies and use the triangular distribution which is much easier to manipulate (Hensher, Greene; 2003). The main drawback of such a mixing distribution would be its upper bound, which is finite, implying that we exclude the case of perfect rationality. However data from empirical studies (Lieberman; 1960), show that people systematically diverge from their best replies.

5 Concluding Discussion

In this paper we present an alternative theoretical model for normal form games with random utility functions. Namely we substitute random payoff functions with mixtures of random utilities. In this case we allow taste variation modelling and the participation of players who do not exhibit independence of irrelevant alternatives. This idea that has already been widely discussed in the framework of discrete choice models in statistics (McFadden, Train; 2000) and slightly mentioned in the literature related to normal form games with random utility functions (McKelvey, Palfrey; 1996) is being further developed. The first part incorporates the ideas of incomplete information to normal form games with random utilities and creates the framework that is used throughout the whole paper. The setup does not object the classical incomplete information models. Using this framework, combined with the concept of mixed models, we define the the Mixed Normal Form Games and the Mixed Quantal Response Equilibrium as an alternative specification, which allows for dependence of irrelevant alternatives and taste variation, as aimed. We also prove its existence. Then we establish relationships QRE and MQRE.

In the second part of the paper, keeping track with the previous work that have been carried out both in the field of statistics (McFadden; 1973, McFadden, Train; 2000) and of game theory (McKelvey, Palfrey; 1995) we specify the Mixed Logit QRE as a special case of our model. The basic result of the whole paper lays on a result that is strongly related to this special case. Namely, every normal form game has a QRE almost surely, which is always the limit of a parametric MLQRE. This result (i) generalizes the class of utility functions that possess QRE and (ii) reveals that QRE can also be seen as limits of sequences of equilibria in games with incomplete information, where players may not satisfy the requirements of IIA and taste invariance.

Appendix

Proof of Theorem 3.1. Since $\sigma_i^j(\cdot|t)$ is continuous (McKP), $\sigma_i^j(\cdot)$, which is a convex combination, will be continuous too. Therefore we apply Brouwer's fixed point theorem on the function

$$\sigma = (\sigma_1, ..., \sigma_n) : [0,1]^{\sum_{i=1}^n (J_i - 1)} \to [0,1]^{\sum_{i=1}^n (J_i - 1)}$$

where $\sigma_i = (\sigma_i^1, ..., \sigma_i^{J_i})$, which proves the theorem. The reason for defining player i's mixed strategy space to be $[0,1]^{J_i - 1}$ is that the probability assigned to the J_ith choice is linearly dependent to the other ones. □

Proof of Corollary 3.1. If we substitute the QR with the probability the s_i^j to be best response, given the type profile in equation (3.3) and we apply the law of total probability, we obtain that the MQR is equal to the probability $u_i(s_i^j, m_{-i})$ to be maximal, which implies QR for the McKP representation of the game. □

Proof of Lemma 3.1. From theorem 3.1 we obtain that Γ_M has a MQRE. Then what we would like to show follows directly from the equivalence between the MQRE in Γ_M and QRE in the game with error terms η_i^j. □

Proof of Theorem 4.1. Assume utilities given by

$$u_i(s_i^j, s_{-i}) = \hat{u}_i(s_i^j, s_{-i}) + \eta_i^j \tag{5.1}$$

Apparently, since we do not have any information about the error term, the existence of QRE is not ensured. However we know that $\eta = (\eta_1, ..., \eta_n)$ follows some joint distribution, g. If we knew the realized value of this random variable, then the QR function would take values in the set $\{0, 1\}$, according to the 0-1 indicator function. Therefore, using the law of total probability we see that the player i's QR becomes

$$\begin{aligned} \sigma_i^j(m_{-i}) &= \int_H \sigma_i^j(m_{-i}|\eta) dG(\eta) \\ &= \int_H I_{\{u_i(s_i^j, m_{-i}) \geq u_i(s_i^k, m_{-i}), \forall k\}} dG(\eta) \end{aligned} \tag{5.2}$$

where H is η's range.

Consider now the following utility functions

$$\tilde{u}_i(s_i^j, s_{-i}; \mu) = u_i(s_i^j, s_{-i}) + \epsilon_i^j(\mu) \tag{5.3}$$

where ϵ_i^j iid Gumbel random variables with parameter $\mu > 0$. Notice that when $\mu \to \infty$, we obtain $\epsilon_i^j = 0$ almost surely and as a matter of fact we are back to the original game. If we rewrite it as follows

$$\tilde{u}_i(s_i^j, s_{-i}; \mu) = \hat{u}_i(s_i^j, s_{-i}) + \eta_i^j + \epsilon_i^j(\mu) \tag{5.4}$$

and we apply lemma 3.1, we obtain that the McKP representation of this normal form game has a QRE, which coincides with the MLQRE of the original mixed logit game. For some given $\mu > 0$, the choice probabilities in equilibrium are the ones that solve the system $\sigma_i(m_{-i};\mu) = m_i$, for every i, where

$$\sigma_i^j(m_{-i};\mu) = \int_H \frac{\exp\{\mu(\hat{u}_i(s_i^j, m_{-i}) + \eta_i^j)\}}{\sum_k \exp\{\mu(\hat{u}_i(s_i^k, m_{-i}) + \eta_i^k)\}} dG(\eta) \tag{5.5}$$

If we take the limit $\mu \to \infty$, we obtain that $\sigma_i^j(m_{-i};\mu) \to \sigma_i^j(m_{-i})$ for every $i = 1, ..., n$ and every $j = 1, ..., J_i$.

We know that $\sigma_i(m_{-i};\mu) = m_i(\mu)$ is an identity for every $\mu > 0$. Therefore, $\lim_{\mu\to\infty}[\sigma_i(m_{-i};\mu) - m_i(\mu)] = 0$. We have already proven that the limit of $\sigma_i(m_{-i};\mu)$ exists. Therefore, the limit of $m_i(\mu)$ will also exist. Hence,

$$\sigma_i^j(m_{-i}) = \lim_{\mu\to\infty} \sigma_i(m_{-i};\mu) = \lim_{\mu\to\infty} m_i(\mu)$$

which implies that $\lim_{\mu\to\infty} m(\mu)$ is a QRE of the original game almost surely. □

References

[1] Brownstone D. (2001). "Discrete Choice Modelling for Transportation", *Travel Behaviour Research: The Leading Edge* 97-124, Oxford: Pergamon Press.

[2] Chesher A., Santos Silva J.M.C. (2002). "Taste Variation in Discrete Choice Models", *Review of Economic Studies* 69, 147-168.

[3] Harsanyi J.C. (1967). "Games with Incomplete Information Played by "Bayesian" Players. Part I. The Basic Model", *Management Science* 14(3), 159-182.

[4] Harsanyi J.C. (1968). "Games with Incomplete Information Played by "Bayesian" Players. Part II. Bayesian Equilibrium Points", *Management Science* 14(5), 320-334.

[5] Harsanyi J.C. (1973). "Games with Randomly Disturbed Payoffs: A New Rationale for Mixed-Strategy Equilibrium Points", *International Journal of Game Theory* 2, 1-23.

[6] Hensher D.A., Greene W.H. (2003). "The Mixed Logit Model: The State of Practice", *Transportation* 30, 133-176.

[7] Lieberman B. (1960). "Human Behavior in a Strictly 3×3 Matrix Game", *Behavioral Sciences* 5, 317-322.

[8] Luce D. (1959). "Individual Choice Behavior", Wiley and Sons, New York.

[9] McFadden D. (1973). "Conditional Logit Analysis on Qualitative Choice Behavior", *Frontiers of Econometrics*, Zarembka, New York.

[10] McFadden D., Train K. (2000). "Mixed MNL Models for Discrete Response", *Journal of Applied Econometrics* 15, 447-470.

[11] McKelvey R.D., Palfrey T.R. (1995). "Quantal Response Equilibria for Normal Form Games", *Games and Economic Behavior* 10, 6-38.

[12] McKelvey R.D., Palfrey T.R. (1996). "A Statistical Theory of Equilibrium in Games", *Japanese Economic Review* 47, 186-209.

[13] McKelvey R.D., Palfrey T.R. (1998). "Quantal Response Equilibria for Extensive Form Games", *Experimental Economics* 1, 9-41.

[14] Train K. (2003). "Discrete Choice Methods with Simulation", Cambridge University Press.

[15] Weizsäcker G. (2003). "Ignoring the rationality of others: evidence from experimental normal form games", *Games and Economic Behavior* 44, 145-171.