AN ELEMENTARY NON-CONSTRUCTIVE PROOF OF THE NON-EMPTINESS OF THE CORE OF THE HOUSING MARKET OF SHAPLEY AND SCARF.

by

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ABSTRACT

Shapley and Scarf, by using the theory of balanced games, prove, in a well-known paper of 1974 (*Journal of Mathematical Economics*, 1, 23-28), the non-emptiness of the core of the Housing Market. This paper provides a non-constructive, simple and short proof that gives some intuition about how blocking can be done by players who have not traded.

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INTRODUCTION

The Housing Market was introduced by Shapley and Scarf in their well-known paper of 1974. Each agent initially owns one indivisible good (say a house), and has complete and transitive preferences on the set of all houses in the economy. An allocation is a permutation of the houses among the agents[2].

Important results for this model have been obtained by several authors. Among them Roth and Postlewaite (1977), Wako (1984-1991), Roth (1982), Ma (1994), Quint (1997), Sonmez (1996-1999) and Konishi, Quint and Wako (1997).

These studies have caused this model to move from simply being an interesting mathematical model to become an important part of the emerging field of Market Design[3].

Two existence proofs of the non-emptiness of the core are presented in Shapley and Scarf (1974): One proof is non-constructive, while the other one is obtained through a simple algorithm called Top Trading Cycles, due to David Gale.

The non-constructive existence proof employs mathematical arguments much more complex than those used in the other proof. Indeed, the sophistication of the balanced games theory, used by Shapley and Scarf to prove the existence of a core outcome, vehemently contrasts with the simplicity of the combinatorial arguments in Gale's proof. This led us to seek an elementary nonconstructive proof of the non-emptiness of the core of the Housing Market.

The non-constructive existence proof presented here is short and only uses simple combinatorial arguments. It provides economic intuition about how blocking can be done by players who have not traded. The starting point is the identification of a convenient non-empty restriction of the set of feasible allocations, called here the *set of simple allocations.*

The idea is that every transaction in a simple allocation is stable in a very precise sense: None of the agents involved in exchange is part of a blocking coalition. We will see that stable trades need not be undone in case agents reach the core. Hence, if a simple

[2] Moulin (1995) has a nice discussion on this economy.

[3] Recently, Roth, Sonmez and Unver (2004) have designed a market for transplants of kidneys, by applying theoretical results that have been proved for the model of Shapley and Scarf.

allocation is not in the core then the blocking coalitions are formed by agents who have not traded, i.e. by agents who still own their initial endowment.

The crucial argument of the existence proof is that a simple allocation, which is not in the core of the Housing Market, has a very special property. It is dominated by another simple allocation that keeps every original stable trade, via a coalition formed with non-trading agents. Therefore, the new transaction in which these non-trading agents are involved is also stable. Thus, the number of stable trades in a simple allocation, out of the core, can always be augmented until the core is reached. Consequently, a Pareto optimal simple outcome must be in the core. Such an outcome always exists because of the transitivity of the preferences and the fact that the set of simple allocations is non-empty and has a finite number of elements.

The non-constructiveness of the proof presented here is reinforced by an example in which the set of Pareto optimal simple allocations does not contain any of the core allocations yielded by the Top Trading Cycles algorithm. (See the discussion in section 3).

In the next section we describe the model and prove the existence theorem. Final remarks are given in section 3.

2. THE HOUSING MARKET AND THE EXISTENCE THEOREM

There is a set of agents, $N=\{1,2,...,n\}$. Each agent i owns one house w_i and has complete and transitive preferences on the set of houses $\{w_1,...,w_n\}$. These preferences may be strict or non-strict. Hence, they can be given by an ordered list of preferences. An allocation x is a permutation of the houses of the players. It is **feasible** if no agent gets a house less preferred than his/her own house. If player i gets house w_j under x, we write $x(i)=w_j$. If $x(i)=w_i$ we say that i is **non-trading at *x*.** Player i prefers x to y if he/she prefers $x(i)$ to $y(i)$. Player i weakly prefers x to y if he/she likes $x(i)$ at least as well as $y(i)$. The feasible allocation x is **dominated** by the feasible allocation y via coalition $S \subseteq N$ if, for every $i \in S$, i prefers $y(i)$ to $x(i)$ and $y(i)$ is the house of some player in S. Coalition S is said to **block** the allocation x. The **core** is the set of all feasible allocations that are not dominated by any feasible allocation via some coalition.

Definition 1. The allocation x is **simple** if, in case a blocking coalition S exists, x(i)=w_i for all $i \in S$.

In other words, x is simple if either $x(i)=w_i$ for all $i \in N$ or, in case $x(i)=w_j$, for some $i \neq j$, then i does not belong to any blocking coalition of x. Since the allocation according to the initial endowments is simple, **the set of simple allocations is non-empty.**

Lemma 1 is the key result. It implies that we can always extend a simple allocation x, which is not in the core, to another allocation y, by keeping the trades done under x and adding a new trade. The crucial point is that the coalition of non-trading agents involved in the new transaction can be chosen so that none of the agents is part of a blocking coalition of y. This implies that y is simple.

Lemma 1. *Suppose x is a simple allocation which is not in the core. Then, there is a simple allocation y, which dominates x via some coalition S, such that $y(i)=x(i)$ for all $i \notin S$.*

Proof. Since x is not in the core, then it is dominated by some feasible allocation via some coalition and, since x is simple, $x(i) = w_i$ for every i in the blocking coalition. Therefore, the set of all non-trading agents at x who can be part of some blocking coalition is non-empty. Call this set of agents A. Any agent in A can be made better off by exchanging his initial endowment with some other player in the blocking coalition he is part of. This implies that, for every agent $i \in A$, there is some agent $j \in A$, $j \neq i$, such that i prefers w_j to w_i and weakly prefers w_j to w_h for all $h \in A$.

Now, choose any $i \in A$. Then, there is some other agent in A, say j, such that w_j is a favorite house for i among all houses of the players in A; since j is in A, there is some agent in A, other than j, say k, whose house is a favorite house for j among all houses of the players in A, and so on. Since A is finite, this sequence will cycle. This cycle is a blocking coalition of x. Call it S. Now, define y such that, if i is in S, then he is assigned according to the cycle; if i is not in S, then assign i to his partner under x. Since every player in S gets his favorite house among the houses of the players in A, then, no element of S belongs to a blocking coalition of y, because such coalition would block x, and so would be contained in A. Then y is simple and the proof is complete.ζ

Definition 2. The allocation x is called a **Pareto optimal simple allocation (PS for short)** if it is simple and there is no simple allocation y such that:

(i) all players weakly prefers y to x, and

(ii) at least one player prefers y to x.

Therefore, if x is PS, then if some player prefers a simple allocation y to x, there is some other player who prefers x to y. The existence of such an allocation x is guaranteed by the fact that the set of simple allocations is non-empty and finite and the preferences are transitive.

Theorem. *The Housing Market has a non-empty core.*

Proof. Let x be some PS allocation. We claim that x is in the core. In fact, suppose not. By Lemma 1 there is a simple allocation y which weakly dominates x via the coalition of all players. This contradicts the Pareto optimality of x. Hence, x is in the core and the proof is complete.ζ

3. FINAL REMARKKS

The key point of our existence proof is that, if a simple allocation x is not in the core, then one must be able to find a blocking coalition S and an allocation y such that y dominates x via S, no player is worse off under y than under x and y is simple. That is, no agent in the blocking coalition S can be part of a blocking coalition of y. Then, S must be a cycle, where every agent gets his favorite house among those of the non-trading agents who are part of blocking coalitions of x.

That such a coalition S always exists for the Housing Market is asserted by Lemma 1. In some sense, this coalition is a "top" trading cycle. However, it is important to point out that the proof presented here is not related to the algorithm of Gale. Its non-constructiveness can be reinforced by the following example due to Jun Wako. In this example the set of PS allocations does not contain any of the core allocations yielded by the Top Trading Cycles algorithm.

Example 1. Consider the Housing Market where the set of players is N={1,2,3}. Player 1 prefers w_2 to w_3 and w_3 to w_1; player 2 is indifferent between w_1 and w_3 and prefers any of these houses to w_2; player 3 prefers w_2 to w_1 and w_1 to w_3.

The core allocations are given by:

	1	2	3
x_1:	w_2	w_1	w_3
x_2:	w_1	w_3	w_2
x_3:	w_2	w_3	w_1
x_4:	w_3	w_1	w_2

The allocations yielded by the Top Trading Cycles algorithm are x_1 and x_2. None of them is a PS allocation: x_1 is weakly dominated by x_3 via N and x_2 is weakly dominated by x_4 via N. It is a matter of verification that x_3 and x_4 are the only PS allocations.ζ

The role played by Lemma 1 in other markets seems to be fundamental for the non-emptiness of the core, as the example below suggests. It considers the roommate problem introduced by Gale and Shapley in 1962. In this market, if player i gets the house of player j then j gets the house of i. Thus, the set of feasible allocations of the roommate problem can be smaller than the set of feasible allocations of the corresponding Housing market. Under this restriction, we can expect that not all core allocations of a given Housing market are in the core of the corresponding roommate problem.

Example 2 (Gale and Shapley, 1962). Consider the set of players $N=\{a,b,c,d\}$. Player *a* prefers *b* to *c* and *c* to *d;* player *b* prefers *c* to *a* and *a* to *d*; player *c* prefers *a* to *b* and *b* to d. The preference of player d is arbitrary. The core of this game as a Housing market is given by the coalition in which *a* gets the house of *b*, *b* gets the house of *c*, *c* gets the house of *a* and *d* is non-trading. However this allocation is not feasible for the game considered as a roommate problem, so it is not in the core of this game. It is easy to see that the core of this game is empty.

Lemma 1 does not apply to this example. The allocation where every player is non-trading is not in the core and is not weakly dominated by a simple allocation via the coalition of all agents, since it is the only simple allocation.

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