Non-walrasian equilibria and the law of one price: the wash-sales assumption*

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July 1, 2005

Abstract

The paper discusses the generality of the failure of the law of one price highlighted by Koutsougeras (2003). This author introduces a market game with multiple trading posts for each commodity and presents an example with price dispersion in equilibrium. We show that such a striking result does not hold when agents are not allowed to buy the goods they are selling on a same post. The failure of the law of one price finally relies on an assumption that is not very intuitive. We explain why and how the result does not come from the possibility for the agent to arbitrage prices difference whenever he faces one.

1 Introduction

General laws in economics are not so common. The idea that, on an homogenous market, commodities are exchanged at the same price, could be seen as one of them. Koutsougeras (2003) proposes a non-competitive strategic market game with multiple trading posts for each single good. Inside this framework, he shows, through an example, that non-cooperative equilibria are compatible with non-uniform prices accross different posts for the same commodity. The fact that his proof consists in providing an example, questions the very generality of the proposition. Is the "law" of one price exclusivelly related to the assumption that

*I would like to deeply thank Regis Breton for the discussions whe had when I was working on that project, and for his very useful remarks on the first version of the paper. I also thank Jean Cartelier and Vincent Bignon for their helpful comments on a first version of this work, as well as Yannick Kalantzis. All errors are mine.

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agents have direct influences on prices? The present paper answers that this is not the case[1].

In the example built by Koutsougeras, some agents are buying units of the good they are selling, or vice versa, on a same tradint post. These specific transactions are called wash-sales. They consist of purchases and sales from an agent cancelling each other on a same post. In this paper, we show that the presence of wash-sales is a necessary condition for the price dispersion in equilibrium. Therefore, non-walrasian hypothesis and the possibility of a strategic influence on prices are not sufficient to explain a prices difference for a same commodity. This partly restricts the very striking result of Koutsougeras (2003).

Let us, in an aside, precise that forbidding wash-sales does not preclude agents from being at the same time buyer and seller for a same commodity, but on different posts. An agent is simply not allowed to buy and sell on both sides of a single post. As a consequence, wash-sales do not reflect explicitly refer to a common or intuitive behaviour. They are not directly linked to an arbitrage behaviour or to speculation, that usually consist in buying goods in one place in order to sell them on another one at a different price. A wash-sale is more likely related to what can be considered as a *price manipulation*. It does not change neither the prices nor the final allocation of the agent, but, the relation between bids and asks is changed. These trades artificially increase the thickness of the market[2] and diminish the influences other agents can have on prices. This is an amusing point, that the prices dispersion is related to *manipulations* that neutralise other agents' influence on prices, when influence on prices is the first and necessary origin for this result.

We can build a more detailed explanation of this intuition, that runs as follows. Wash-sales influence arbitrage opportunities when prices diverge. They can restrict them. Consider an agent that, at the same time, is a buyer and a seller on a trading post (i, s). If this post is the most expensive one, the agent could want to diminish his bid on it and transfer units of account to another cheapest post, in order to get more commodities. This would however lower the price on the post (i, s), and diminish the value of his sales (effect that does not exist when wash sales are not allowed). This second effect can outweigh the first

[1] We ask the question this way, because this is how the break down of the uniform price law in the multiple trading post market game is presented, by Koutsougeras but also Gael Giraud (2003) in his introduction.

[2] Peck and Shell (1990) show in the offered constraint version of the one post per commodity market game, that increasing the liquidity through wash sales can bring Non-cooperative equilibria close to competitive equilibria, even if the number of agents is small. The impact of wash-sales is not identical here.

one so that the agent is worst of in the end.

The paper is organised as follows. The section 2 presents the model and the main result. Section 3 is devoted to the proof of the mean result and the last section concludes.

2 The model

The model uses the framework introduced by Koutsougeras (2003). There are a finite set H of agents in the economy, who exchange L commodity types $i = 1;...;L$. Each agent $h \in H$ is characterised by his preferences represented by a utility function $u_h : \mathbb{R}_+^L \to \mathbb{R}$ and an initial endowment $e_h = \left\{e_h^1;...;e_h^L\right\} \in \mathbb{R}_+^L$.

The economy is organised as follow. There are $K_i > 0$ trading-posts for each commodity i. We will notice (i,s) the post s where the commodity i is exchanged. On each trading post individuals $h \in H$ make bids $b_h^{i,s}$ in terms of a unit of account and offers in terms of quantity of commodity $q_h^{i,s}$. Exchanges are supposed to be costless.

An agent $h \in H$ cannot sell more that his initial endowment and his strategy set is:

$$S_h = \left\{(b_h; q_h) \in \prod_{i=1}^{L} \mathbb{R}_+^{K_i} \times \prod_{i=1}^{L} \mathbb{R}_+^{K_i} : \sum_{s=1}^{K_i} q_h^{i,s} \leq e_h^i, i = 1, 2, ..., L\right\}$$

Given a set of strategies $S = \{S_h, h \in H\}$ let:

$$B^{i,s} = \sum_{h \in H} b_h^{i,s} \text{ and } Q^{i,s} = \sum_{h \in H} q_h^{i,s}$$

and for each $h \in H$:

$$B_h^{i,s} = \sum_{\substack{n \neq h \\ n \in H}} b_n^{i,s} \text{ and } Q_h^{i,s} = \sum_{\substack{n \neq h \\ n \in H}} q_n^{i,s}$$

On each trading post the transactions clear according to the followng price rule:

$$p^{i,s} = \begin{cases} \frac{B^{i,s}}{Q^{i,s}} \text{ if } Q^{i,s} \neq 0 \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

In order to insure that agents do not go bankrupt, for each commodity l and each agent h the final allocations are determined as follows:

$$x_h^i = \begin{cases} e_h^i - \sum_{s=1}^{K_i} q_h^{i,s} + \sum_{s=1}^{K_i} \frac{b}{p^{i,s}} \text{ if } \sum_{i=1}^{L} \sum_{s=1}^{K_i} b_h^{i,s} \leq \sum_{i=1}^{L} \sum_{s=1}^{K_i} q_h^{i,s} p^{i,s} \\ e_h^i - \sum_{s=1}^{K_i} q_h^{i,s} \text{ otherwise} \end{cases} \tag{2}$$

where it is postulated that $\frac{1}{p^{i,s}} = 0$ if $p^{i,s} = 0$. The allocation rule (2) means that, if the agent $h \in H$ is not able to sell as much as he buys, his purchases are confiscated. As a consequence, in equilibrium the budgetary constraint is necessary checked. It follows that each consumer is considered as solving the following program:

$$\begin{aligned} &\max_{(b_h,q_h)\in S_h} u_h\left(x_h\right) \\ &s.t \begin{cases} \sum_{i=1}^{L}\sum_{s=1}^{K_i} b_h^{i,s} \leq \sum_{i=1}^{L}\sum_{s=1}^{K_i} q_h^{i,s} p^{i,s} \\ \sum_{s=1}^{K_i} q_h^{i,s} \leq e_h^i \end{cases} \end{aligned} \tag{3}$$

This model is the Bid and Offer kind of the Cournot-type market game. It implicitly allows each agent $h \in H$ to be active on both sides of a same post (i, s), that is to say $b_h^{i,s} > 0$ and $q_h^{i,s} > 0$. In that case agent h simultaneously buys and sells on the trading-post (i, s) and makes wash-sales: some sales and buyings cancel each others.

Now, it is possible to build a version of this model inside which wash sales are forbidden: agents are not allowed to act as a buyer and as a seller on the same post. The model is based on the Buy and Offer type one with additional constraints on each trading post and for each agent:

$$b_h^{i,s} q_h^{i,s} = 0,\ i = 1, ..., L;\ s = 1, ..., K_i \text{ and } h \in H \tag{4}$$

We will refer to this model as the the Bid or Sell type market game.

Let us define the law of one price for a commodity i as the price uniformity accross all active trading posts (i, s) with $s = 1; ...; K^i$. An active post (i, s) is defined as a trading post with a strictly positive price $\left(B^{i,s} > 0 \text{ and } Q^{i,s} > 0\right)$. The result of the paper is the following theorem.

Theorem 1 *When agents are not allowed to be active on both sides of a same trading post, all non-cooperative equilibria satisfies the law of one price.*

The proof is proposed in the next section. This theorem shows that, if we agree that wash-sales can be seen as market or price *manipulation*, rather than speculation or intermediation[3], the law of one price fails only in very unusual

[3] In a different framework, with demand uncertainty, Peck (2002) says that situations of particular interest are when consummers trade on on one side of the market, because it is unusual to see a consummer on both sides of a single market. He adds that "this statement excludes financial intermediaries or market makers, who attempt to buy law and sell high" (Peck, 2003, page 294). This is our stand on wash sales taking place on a same post, because all

occasions. In that sense, it does not destroy the idea that the price rule (1) is both interesting and relevant to investigate exchange in many markets, because it satisfies market clearing and the uniqueness of prices[4].

3 Proof of the result

The proof runs as follows. It shows that prices dispersion is not compatible with equilibria in the Bid or Sell game. We proceed by contradiction. We suppose that there is an equilibrium with price dispersion and show that the condition required such that an agent bids on the most expensive post cannot be jointly satisfied for all the agents buying on this post.

Now, the proof of the theorem is not fully intuitive. It surprisingly requires arguments that, in a way, confirm the no-arbitrage opportunity result of Koutsougeras (2003) when prices are different. In order to clarify the sketch of the proof, we propose to give its intuition through the picture of a very simple example.

3.1 The Sketch of the proof from a picturesque example

Let us consider two posts 1 and 2 for a commodity i. The quantities of good deposited on each post Q^1 and Q^2 are supposed to be identical and normalised to 1. The quantities of money are $B^1 = 6$ and $B^2 = 4$. As $Q^1 = Q^2 = 1$, the prices are respectively $p^1 = 6$ and $p^2 = 4$, so that $p^1 > p^2$. As it is never an optimal strategy to buy on post 1 in order to sell on post 2 (it is a quite obvious principle that can be proven, see below) we look at an agent $h \in H$ that simultaneously buys on both posts.

The purpose is to build a specific case with price dispersion, agent h cannot arbitrage[5]. This hinges on a specific and necessary asumption about the initial strategy of the agent, we are able to picture. Then, it appears that this condition cannot be jointly satisfied for all agents that together bid the 6 units of account.

Let us suppose that agent h deposits respectively $b^1 = b^2 = 1$ unit of account on each post. The first intuition is that the agent could switch the unit of account

trades take place at the same price. The ban of wash-sales on a single post does not preclude the attempt to buy law and sell high, that is possible to do with actions on different posts, specifically in the mutliple tranding posts per commodity framework.

[4] In that sense, the market clearing does not necessary match with the competitive equilibrium, which needs more than a market clearing rule.

[5] This is why the proof is not intuitive. Given strategies of others, there are situations where the isolated agent cannot arbitrage a prices difference. But, in that case, strategies of other cannot be optimal. This is what we try to picture here.

from post 1 to post 2 in order to buy at a less expensive price and get more commodities. This is not possible here because if he does switch the unit of account, the quantity of commodity bought is smaller. In the first case he buys $\frac{1}{6}+\frac{1}{4}=\frac{5}{12}$ units of good which is greater than $\frac{2}{5}$ the quantity he gets if he changes his strategy[6].

This impossibility to profit from the prices dispersion hinges on a specific assumption, the agent is relatively more active on the less expensive post. This means that his relatively weight $\frac{b^1}{B^1}$ on market 1 is smaller than his weight on market 2, equal to $\frac{b^2}{B^2}$. This condition is obviously checked in the example ($\frac{b^1}{B^1}=\frac{1}{6}<\frac{1}{4}=\frac{b^2}{B^2}$). Now, it is easy to build an other case with $\frac{b^1}{B^1}>\frac{b^2}{B^2}$ and show that there is an arbitrage opportunity. If the agent deposits $b^1=2$ units of account on the $B^1=6$ that are put on the post 1, we have $\frac{b^1}{B^1}=\frac{1}{3}>\frac{1}{4}=\frac{b^2}{B^2}$ and the quantity of good he gets is $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$. If he tries to take advantage of the price dispersion and swtiches 1 unit of account from post 1 to post 2, he finally gets $\frac{1}{5}+\frac{2}{5}=\frac{3}{5}>\frac{7}{12}$ unit and profits from the arbitrage opportunity.

Let us now come back to relative weight condition $\frac{b^1}{B^1}<\frac{b^2}{B^2}$ if $p^1>p^2$. It is a necessary condition that applies to each agent that is buying on post 1. It can be represented as follows. For each fraction of B^1 left on post 1, there must be a greater one left on post 2, which is by definition impossible. If it is assumed, for simplying the analysis, that the smallest unit of money is 1, when an agent deposits one unit of account on post 1 he has to, if he plays a best response strategy, deposit at least one unit of money on post 2 (as $B^1>B^2$ the relative weight on post 1 is greater rather than the one on post 2). It is then quite obvious that this condition cannot be checked for each agent that makes a one unit of account bid on post 1. This is exactly how the general proof is built, proof it is now easier to establish.

3.2 Proof of the theorem

We will focus on a market i. Let us consider a set of prices $p=\left\{p^{i,s}, s=1,...,K_i\right\}$ and a set of strategies $S=\left\{S^h : h\in H\right\}$. Assume that all agents play best response strategies and that there is a price $p^{i,s}$ such that $p^{i,s}>p^{i,r}\ \forall\ r=1,...,K_i$ with $r\neq s$. The proof shows that these conditions cannot be checked together.

[6] This is related to the *marginal unit* effect (Gobillard, 2005). The intuition is that an agent that switches, as an example, one unit of account from an expensive market to a cheapest one, will not only be affected from the fact he changes the use this marginal unit. The change of the marginal unit also modifies the prices and by the way the entire set of allocations. From the point of view of the agent, the price on the cheap market increases and he gets less commodity from its previous action on this market. At the end, this effect can compensate the first one.

In order to simplify the presentation of the proof we consider posts that have strictly positive prices. It is clear that if only $p^{i,s}$ is strickly positive the law of one price does not fail. Next, the proof only requires that at least one other post (i,r) is active ($p^{i,r}>0$). We will follow five steps. (1) The problem and its first order conditions; (2) The multiplier associated to the budgetary constraint is different from zero; (3) It is not a best response to buy on the most expensive post in order to sell on a less expensive one; (4) Description of the situation of an agent that buys on both posts (i,s) and (i,r); (5) There is a relative weight necessary condition for agents buying on the most expensive post that cannot be checked for all of them.

Step 1: the problem and its first order conditions. The proof is based on the resolution of the individual program. In the Bid or Sell version of the Cournot-type model, the problem that defines the non-cooperative equilibria is given by the equations (3) and (4). The program can be summarised as follows:

$$\begin{array}{l}\max\limits_{(b_h,q_h)\in S_h} u_h\left(x_h^i\right)\\ s.t\left\{\begin{array}{l}\sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} b_h^{i,s}-\sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} q_h^{i,s}p^{i,s}\leq 0\\ \sum\limits_{s=1}^{K_i} q_h^{i,s}-e_h^i\leq 0, \forall i=1,...,L\\ b_h^{i,s}q_h^{i,s}=0, \forall s=i=1,...,K_i \text{ and } i=1,...,L\end{array}\right.\\ x_h^i=e_h^i-\sum\limits_{s=1}^{K_i} q_h^{i,s}+\sum\limits_{s=1}^{K_i}\frac{b_h^{i,s}}{p^{i,s}}; i=1,...,L\\ p^{i,s}=\frac{B_h^{i,s}+b_h^{i,s}}{Q_h^{i,s}+q_h^{i,s}}\end{array} \tag{5}$$

If $\left(\gamma_h,\lambda_h^i,\mu_h^{i,s}, i=1,...,L; s=1,...,K_i\right)\in\mathbb{R}^2\times\prod\limits_{i=1}^{L}\mathbb{R}^{K_i}$ are the multipliers for each constraint, the Lagrangian L_h associated to the program of each agent h is:

$$L_h=u\left(x_h\right)+\left\{\begin{array}{c}\gamma_h\left(\sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} b_h^{i,s}-\sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} q_h^{i,s}p^{i,s}\right)\\ +\sum\limits_{i=1}^{L}\left[\lambda_h^i\left(\sum\limits_{s=1}^{K_i} q_h^{i,s}-e_h^i\right)\right]\\ +\sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i}\mu_h^{i,s}b_h^{i,s}q_h^{i,s}\end{array}\right. \tag{6}$$

It is clear that the first constraint binds. Then, we have, for each agent $h\in H$

and $\forall\ i = 1, ..., L$ and $s = i = 1, ..., K_i$:

$$\begin{array}{rcl}
\sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} b_h^{i,s} - \sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} q_h^{i,s} p^{i,s} & = & 0 \\
\gamma_h & \leq & 0 \\
\lambda_h^i & \leq & 0 \\
\lambda_h^i \sum\limits_{s=1}^{K_i}\left(q_h^{i,s} - e_h^i\right) & = & 0 \\
b_h^{i,s} q_h^{i,s} & = & 0
\end{array}$$

and the first order conditions:

$$\frac{\partial L_h}{\partial b_h^{i,s}} = \frac{\partial u_h}{\partial x_h^i} \cdot \left[\frac{Q_h^{i,s}+q_h^{i,s}}{B_h^{i,s}+b_h^{i,s}} \frac{B_h^{i,s}}{B_h^{i,s}+b_h^{i,s}}\right] + \gamma_h \frac{Q_h^{i,s}}{Q_h^{i,s}+q_h^{i,s}} + \mu_h^{i,s} q_h^{i,s} = 0 \quad (7)$$

$$\frac{\partial L_h}{\partial q_h^{i,s}} = -\frac{\partial u_h}{\partial x_h^i} \cdot \left[\frac{B_h^{i,s}}{B_h^{i,s}+b_h^{i,s}}\right] - \gamma_h \frac{\left(B_h^{i,s}+b_h^{i,s}\right)}{Q_h^{i,s}+q_h^{i,s}} \frac{Q_h^{i,s}}{Q_h^{i,s}+q_h^{i,s}} + \lambda_h^i + \mu_h^{i,s} b_h^{i,s} = 0 \quad (8)$$

From equations (1) and (7), we have:

$$\frac{\partial u_h}{\partial x_h^i} = -\gamma_h \left(p^{i,s}\right)^2 \frac{Q_h^{i,s}}{B_h^{i,s}} - \mu_h^{i,s} p^{i,s} q_h^{i,s} \frac{B_h^{i,s}+b_h^{i,s}}{B_h^{i,s}} \quad (9)$$

and from equations (1) and (8):

$$\frac{\partial u_h}{\partial x_h^i} = -\gamma_h \left(p^{i,s}\right)^2 \frac{Q_h^{i,s}}{B_h^{i,s}} + \mu_h^{i,s} b_h^{i,s} \frac{B_h^{i,s}+b_h^{i,s}}{B_h^{i,s}} + \lambda_h^i \frac{B_h^{i,s}+b_h^{i,s}}{B_h^{i,s}} \quad (10)$$

Step 2: he multiplier γ_h associated to the budgetary constraint is different from 0. It is firstly intuitive that γ_h cannot be equal to zero because it is the multiplier associated to the budgetary constraint. Then, if it is equal to zero, the marginal utilities are all supposed to be equal to zero. We can prove it very quickly. Let us suppose that $\gamma_h = 0$ and show there is a contradiction. From equation (9) we have for all $i \in 1;...;L$ and $s \in 1;...;K^i$:

$$\frac{\partial u_h}{\partial x_h^i} = -\mu_h^{i,s} p^{i,s} q_h^{i,s} \frac{B_h^{i,s}+b_h^{i,s}}{B_h^{i,s}} \quad (11)$$

So, if $q_h^{i,s} = 0$, the marginal utility $\frac{\partial u_h}{\partial x_h^i}$ has to be null, which is impossible. But, from the budgetary constraint $\sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} b_h^{i,s} = \sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} q_h^{i,s} p^{i,s}$ and if some $q_h^{i,s}$ are stricly positive it is necessary that there exist $b_h^{i,r} > 0$. Since $b_h^{i,s} q_h^{i,s} = 0$ it must exist $q_h^{i,r} = 0$, contradicting.

Step 3: $b_h^{i,s} q_h^{i,r} > 0$ is not a best response strategy. Let us check that if $p^{i,s} > p^{i,r}$, an agent $h \in H$ will not play the strategy $b_h^{i,s} > 0$ and $q_h^{i,r} > 0$ as a best response. If $b_h^{i,s} > 0$ and $q_h^{i,r} > 0$ we have $b_h^{i,r} = q_h^{i,s} = 0$ and $B^{i,r} = B_h^{i,r}$ and $Q^{i,s} = Q_h^{i,s}$. In that case, from equation (1) the condition (9) becomes:

$$\frac{\partial u_h}{\partial x_h^i} = -\gamma_h p^{i,s} \frac{B^{i,s}}{B_h^{i,s}} \tag{12}$$

and the condition (10) becomes:

$$\frac{\partial u_h}{\partial x_h^i} = -\gamma_h \left(p^{i,r}\right) \frac{Q_h^{i,r}}{Q^{i,r}} + \lambda_h^i \tag{13}$$

From these last equations and as $\gamma_h \neq 0$ we have:

$$p^{i,s} \frac{B^{i,s}}{B_h^{i,s}} = \left(p^{i,r}\right) \frac{Q_h^{i,r}}{Q^{i,r}} - \frac{\lambda_h^i}{\gamma_h} \tag{14}$$

Next, as $\frac{\lambda_h^i}{\gamma_h} \geq 0$[7], $\frac{p^{i,s}}{p^{i,r}} > 1$ and $\frac{B^{i,s}}{B_h^{i,s}} \frac{Q^{i,r}}{Q_h^{i,r}} > 1$, the equality (14) cannot hold. Then, if $p^{i,s} > p^{i,r}$ the agent will not buy on post (i,s) and sell on post (i,r).

Step 4: Being a buyer on both posts (i,s) and (i,r). Let us consider the equation (9) for both posts (i,r) ans (i,s):

$$\begin{aligned} \frac{\partial u_h}{\partial x_h^i} &= -\gamma_h \left(p^{i,s}\right)^2 \frac{Q_h^{i,s}}{B_h^{i,s}} - \mu_h^{i,s} p^{i,s} q_h^{i,s} \frac{B_h^{i,s} + b_h^{i,s}}{B_h^{i,s}} && (15) \\ \frac{\partial u_h}{\partial x_h^i} &= -\gamma_h \left(p^{i,r}\right)^2 \frac{Q_h^{i,r}}{B_h^{i,r}} - \mu_h^{i,r} p^{i,r} q_h^{i,r} \frac{B_h^{i,r} + b_h^{i,r}}{B_h^{i,r}} && (16) \end{aligned}$$

It follows that:

$$\left(p^{i,s}\right)^2 \frac{Q_h^{i,s}}{B_h^{i,s}} \gamma_h - \mu_h^{i,s} q_h^{i,s} p^{i,s} \frac{B_h^{i,s} + b_h^{i,s}}{B_h^{i,s}} = \left(p^{i,r}\right)^2 \gamma_h \frac{Q_h^{i,r}}{B_h^{i,r}} + \mu_h^{i,r} q_h^{i,r} p^{i,r} \frac{B_h^{i,r} + b_h^{i,r}}{B_h^{i,r}} \tag{17}$$

If we suppose that the agent is a buyer on both posts, $b_h^{i,s}$ and $b_h^{i,r}$ are strictly positive, $q_h^{i,s} = q_h^{i,r} = 0$ and $Q_h^{i,s} = Q^{i,s}$ and $Q_h^{i,r} = Q^{i,r}$. Equation (17) becomes:

$$\left(p^{i,s}\right) \frac{B^{i,s}}{B_h^{i,s}} = \left(p^{i,r}\right) \frac{B^{i,r}}{B_h^{i,r}} \tag{18}$$

since $\gamma_h \neq 0$ (see the following proof).

Step 5: the relative weight condition and its impossibility in equilibrium. From the equation (18), as we suppose $p^{i,s} > p^{i,r}$, it is necessary that $\frac{B^{i,s}}{B_h^{i,s}} < \frac{B^{i,r}}{B_h^{i,r}}$, that is to say that

$$\frac{b_h^{i,s}}{B^{i,s}} < \frac{b_h^{i,r}}{B^{i,r}} \tag{19}$$

This is what we call the *relative weight* condition. An agent that buys on the most expensive post (i,s) will play a best response strategy if he is a buyer

[7] It is in fact possible to show that $\lambda_h^i = 0$ if the utility functions respect the inada's conditions.

on other active posts (i, r) and that the relative weights of his bids on those posts are greater than the one on post (i, s). Let us focus on posts (i, s) and (i, r). ${}_bH \subset H$ is the set of agents that are buyers on post (i, s) and ${}_sH$ its complemetary: ${}_sH = H -_b H$. By definition we have:

$$\sum_{h \in H} b_h^{i,s} = \sum_{h \in \{{}_sH\}} b_h^{i,s} = B^{i,s} \tag{20}$$

From the relative weight condition (19), we must have:

$$\sum_{h \in \{{}_sH\}} \frac{b_h^{i,s}}{B^{i,s}} < \sum_{h \in \{{}_sH\}} \frac{b_h^{i,r}}{B^{i,r}} \tag{21}$$

and, if conditions (20) and (21) are satisfied, $\sum_{h \in \{{}_sH\}} \frac{b_h^{i,r}}{B^{i,r}}$ is supposed to be greater that one, contridacting $\sum_{h \in H} b_h^{i,r} = B^{i,r}$, because $\sum_{h \in H} b_h^{i,r} \geqslant \sum_{h \in \{{}_sH\}} b_h^{i,r}$ by definition. The proof follows from the contradiction.

4 Conclusion

We have shown that when agents are not allowed to act on both sides of a trading post, in equilibrium, the law of uniform price remains. Wash sales influence and restrict the arbitrage opportunities. When an agent tries to take advantage of a seeming arbitrage opportunity, his new situation is affected in a double way if he trades wash-sales. When he switches money from the expensive post to an other one, he gets less commodities. But, if he is also a seller on this post, he gets less money from his sales. This effect diminishes the arbitrage opportunity and gives rise to the possibility of equilibria with different prices for a same commodity, if wash sales are allowed, even if this cannot happen without wash-sales. According to the very specificity of wash-sales, the failure of the law of one price finally fails only in very unusual occasions.

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