Efficient Equilibria and Information Aggregation in Common Interest Voting Games*

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Abstract

We characterize *efficient* equilibria of common interest voting games with privately informed voters and study the implications of efficient equilibrium selection for Condorcet jury theorems. We show that larger juries can do no worse than smaller ones and derive a simple necessary and sufficient condition for asymptotic efficiency of different voting rules. This condition implies that the unanimity as well as near unanimity rules are asymptotically inefficient *regardless* of equilibrium selection. However, if the signal distribution fails a non-degeneracy condition, the unanimity rule dominates any other rule. Finally, if signals are conditionally independent, full information equivalence can be exactly achieved for *any* rule that allows the divisibility of individual votes, and for any finite number of voters.

JEL Classification No.: C7, D7.

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1 Introduction

Condorcet (1785) pioneered the study of elections by showing that under certain conditions, decisions made by a group (such as a jury, a committee or an electorate) are superior on average to those made by an individual (such as a judge or a dictator), thus providing an instrumental rationale for participatory democracy. In Condorcet's framework, differences of opinion among individuals arise not due to a fundamental clash of values or interests but due to different (and imperfect) information that they may have regarding various options. Social choice is therefore seen as an exercise in *information aggregation* in order to uncover an underlying 'truth'. In general, different decision (or voting) rules will differ in their ability to aggregate information. In this paper we compare alternative voting rules in this light.[1]

There are two sets of questions that we ask with respect to the information aggregation properties of different voting rules. The first of these relate to Condorcet's original results that have subsequently come to be known as Condorcet's jury theorems (CJTs). There are at least two different versions. The first claims that in an election with two alternatives and under the majority rule, a jury or an electorate consisting of several members is more likely to make the correct choice than any single decision maker (we refer to this as CJT1). A second and distinct claim is that as the number of voters grows towards infinity, the probability of making the wrong choice vanishes under the majority rule. We refer to this asymptotic version as CJT2. In addition, one can also imagine a generalization of the statement of CJT1, saying that larger juries have a higher probability of making the correct choice than smaller ones. We call this CJT3.

Following Condorcet, various papers have established conditions under which the jury theorems are valid, extending Condorcet's results to rules other than the majority rule, e.g., super-majority or unanimity rules, under Condorcet's original assumption of sincere voting (i.e., voters vote the way they would had they been called upon to make the decision alone).[2] As pointed out by Austen-Smith and Banks (1996), sincere voting

[1]It may be argued that in most applications of interest, both private information and some degree of heterogeneity of preferences will be present. Nevertheless, the pure common interest case serves as an useful theoretical benchmark by focusing on the purely coordination aspect of information aggregation.

[2]See Miller (1988), Ladha (1992) and Berend and Paroush (1992). Young (1986, 1988) analyzes the

is far from an innocuous assumption. While voting, voters will not only take into account their own information, but also the information that can be inferred from their being *pivotal*, which is the only situation in which their votes matter. Since being pivotal may contain significant information, individuals may not vote sincerely even when they expect others to do so. In other words, sincere voting often may not constitute a Nash equilibrium of the voting game, even though all voters have identical interests. This observation has led to an emerging literature on equilibrium voting behavior and its consequences. Following this literature, we also focus on the case with strategically sophisticated voters.

We begin by reestablishing a result, first shown by McLennan (1998), that in common interest games, the strategy profile which maximizes the (common) payoff function also constitutes a Bayesian Nash equilibrium.[3] We call this profile the efficient equilibrium of the voting game. We show further that there exists an efficient equilibrium in pure strategies that is typically asymmetric, even in symmetric environments. It follows that under very general conditions, a (weak) version of CJT3 (hence CJT1) must obtain: for the class of voting rules satisfying monotonicity with respect to jury size, given efficient equilibrium selection, larger juries can do no worse than smaller ones.

Next, we establish a necessary and sufficient condition on the voting rule that guarantees that the equilibrium probability of a wrong decision approaches zero as the jury size approaches infinity. For non-degenerate conditionally independent signal distributions with finite support, and in efficient equilibrium, error probabilities approach zero in the limit if and only if the voting rule is such that the number of votes required for *each* decision (conviction as well as acquittal) grows unboundedly as the jury size increases. Thus, the asymptotic properties of the equilibrium outcome (i.e., obtaining a CJT2) is closely related to the asymptotic properties of the voting rule. An immediate corollary is that the unanimity rule (which requires a single vote for the defendant's acquittal by any jury) always produces asymptotic inefficiency. On the other hand, for the sequence of efficient equilibria, any proportional rule (such as simple majority rule)

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extension of 'Condorcet's Rule' to more than two alternatives.

[3]The same is true if attention is restricted to only symmetric strategy profiles and symmetric Nash equilibrium. See McLennan (1998). Myerson (1998) proves a similar result.

is informationally efficient in the limit.

In an interesting and provocative paper, Feddersen and Pesendorfer (1998) have shown that in a model with conditionally independent and symmetric binary signals, unanimity rule could lead to a *higher* probability of convicting an innocent defendant than less demanding majority or super-majority rules. Furthermore, for the symmetric equilibrium that they study, unanimity rule produces asymptotic inefficiency—the probabilities of convicting the innocent and acquitting the guilty remain bounded away from zero even as the jury size goes to infinity. Duggan and Martinelli (2001) and Meirowitz (2001) generalize the result to richer signal spaces, including a continuum of signals, and show that, as long as the signal distribution satisfies a bounded likelihood ratio condition, unanimity rule is asymptotically inefficient *if attention is restricted to symmetric equilibria*. In a recent paper, Martinelli (2002) has shown (by methods different from ours) that the asymptotic inefficiency of unanimity rule is robust to equilibrium selection, provided likelihood ratios are bounded. On the other hand, Feddersen and Pesendorfer (1998 &1997) (see also Wit (1998)) show that, for any interior proportional rule (i.e. majority as well as super-majority rules), asymptotic efficiency obtains along the sequence of symmetric equilibria. Our necessary and sufficient condition on voting rules that deliver asymptotic efficiency unifies and generalizes these results, for any finite number of signals and common interests.[4] These necessary and sufficient conditions are reminiscent of the "double–largeness" condition in Pesendorfer and Swinkels (2000) that is necessary and sufficient for efficient information aggregation in multi-object common value auctions.

Finite non degenerate signal distributions imply that likelihood ratios are bounded, and this is critical for our results. We demonstrate this by considering a binary signal model in which one of the signals can be received only if the defendant is innocent, but not otherwise. Conceptually, this may be thought of as a case where there exists some 'proof' of innocence (which may nevertheless escape the detection of individual jurors), as opposed to merely noisy 'evidence' which still leaves some residual doubt about the defendant's innocence in the juror's mind. We show that in this simple model, unanimity

[4] Feddersen and Pesendorfer (1997) and Martinelli (2002) also consider the case of heterogenous juries for the case of any interior proportional rule and the unanimity rule respectively.

rule is the most efficient voting rule for *any* jury size; indeed, it always produces full information equivalence, i.e., the outcome is the same as would have been the case if all the signals were common knowledge. Moreover, if the full information equivalent outcome is sensitive to the realized vector of signals (i.e. it is not optimal to always convict or always acquit the defendant), unanimity rule is the *unique* efficient rule. This result complements those in Duggan and Martinelli (2000) and Martinelli (2002), who show that unanimity rule can be approximately efficient *in the limit* when arbitrarily strong evidence in favor of innocence may be obtained.

The second question that we ask with respect to information aggregation properties of voting rules is the following. For a fixed jury size, are there any voting rules that, in equilibrium, would implement the same outcome as would be obtained when the jurors could freely share their private information? Our results imply that this full information equivalence is generally not achievable, except for specific voting rules in a binary signal model. Similarly, Austen–Smith and Banks (1996) provide an example where each juror has two binary signals, and where no voting rule can lead to full information equivalence. Put differently, communication among voters (which, in a common interest game, can lead to the optimal utilization of all the available information) will generally have a strictly positive value. Our final set of results goes to show that a simple extension of the voting mechanism can perfectly substitute for the need for communication in a large class of situations, although not always. We consider voting rules that allow jurors to 'split' their votes, i.e., each juror can give a fraction of her vote to one option (say conviction) and the remaining fraction to the other. We show that when the signal distribution satisfies conditional independence, the divisibility of votes can exactly deliver full information equivalence— there always exists an equilibrium satisfying that property. Moreover, this is true for *any* monotonic voting rule and any number of voters, and even when the signal distribution is not identical across voters. Intuitively, divisible votes allow the voters to convey the intensity of their information with respect to different states of the world. If the signal distribution satisfies conditional independence, then voters can precisely convey their information, as well as make it count in the decision in an efficient manner, even though the *additivity* of the voting rule imposes a non-trivial constraint *a priori*. The restrictiveness of this constraint is illustrated in a counter-example where the signal distribution is not conditionally independent, and full information equivalence can be shown to be unattainable.

Other notable contributions to the literature on voting under private information include the following. Dekel and Piccione (2000) analyze sequential, rather than simultaneous, voting procedures. Persico (2002) studies a voting model in which voters must spend resources to acquire information, creating a possible free rider problem. He compares different voting rules in terms of the incentives generated for information acquisition, as well as their information aggregation properties. Coughlan (2000), Doraszelski, Gerardi and Squintani (2002) and Gerardi and Yariv (2002) study voting behavior when voters can communicate. While these are interesting issues in themselves, our focus in the current paper is elsewhere.

The rest of the paper is organized as follows. In section 2, the model is presented. In section 3, we analyze efficient equilibria and characterize voting rules that yield CJTs 1,2 and 3. Section 4 takes up the case of binary signals while Section 5 traces the implication of allowing votes to be divisible. Section 6 concludes while the Appendix contains some of the proofs.

2 The Model

There is a countable set of individuals $\{1, 2, ...\}$, indexed by j. The individuals have to take a joint decision $d \in \mathcal{D} = \{A, C\}$ (A stands for 'acquit', C stands for 'convict') by forming a jury J consisting of a finite subset of $|J|$ individuals.[5]

There are two states denoted by $s \in \mathcal{S} = \{I, G\}$ (I represents 'innocence' and G represents 'guilt'). All individuals have a common state-dependent payoff function over states s and decisions d that is given by:

$$u(s,d) = \begin{cases} -q & \text{if } s = I, d = C, \\ -(1-q) & \text{if } s = G, d = A, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

[5]Throughout we use the notation $|.|$ to denote the cardinality of a set.

where $q \in (0,1)$.[6]

The state s is not known but each individual j has private information represented by a signal or type $t_j \in \mathcal{T}_j = \{1, ..., m_j\}$. Let $\Omega = \mathcal{S} \times \Pi_{j=1}^{\infty} \mathcal{T}_j$ be the set of possible states and type profiles with typical element $\omega = (s, t_1, t_2, ...)$. Let the function $S : \Omega \to \mathcal{S}$ defined by $S(\omega) = s$ represent the unknown state and the function $T_j : \Omega \to \mathcal{T}_j$ defined by $T_j(\omega) = t_j$ represent the privately known type of individual j. Let $\mathcal{F}$ be the sigma–algebra on Ω generated by the collection of sets of the form $\{\omega | S(\omega) = s\}$ and $\{\omega | T_j(\omega) = t_j\}$ for $s \in \mathcal{S}$, $t_j \in \mathcal{T}_j$, $j \in \{1, 2, ...\}$. Let P be a probability measure on $(\Omega, \mathcal{F})$. We assume that the probability triple $(\Omega, \mathcal{F}, P)$ is common knowledge among all (potential) jurors.

For any non-empty finite set J of individuals, let T_J denote the collection $\{T_j\}_{j \in J}$, t_J denote a particular realization $\{t_j\}_{j \in J}$ of T_J and let $\mathcal{T}_J$ be the set of such realizations. Thus $T_{J \setminus \{j\}}$ denotes the signals of those in J other than j and $t_{J \setminus \{j\}}$ a realization of $T_{J \setminus \{j\}}$. We assume that[7]

$$P(S = s) > 0 \text{ for all } s \in \mathcal{S} \tag{2}$$

and

$$P[T_J = t_J] > 0 \text{ for all } t_J \text{ and finite, non–empty } J. \tag{3}$$

The joint decision d is taken by forming a finite jury J. Members of J then play a non–cooperative simultaneous move Bayesian game.[8] Specifically, after observing their respective private signals, each juror $j \in J$ takes an action $x_j \in \{0, 1\}$, simultaneously and independently of other jurors. The action $x_j = 1$ is interpreted as a vote for the decision C while $x_j = 0$ is interpreted as a vote for the decision A. Let $\sigma_j : \mathcal{T}_j \to [0, 1]$ denote the (behavior) strategy of j with $\sigma_j(t_j)$ denoting the probability with which juror

[6] The parameter q captures the relative importance of committing two different kinds of errors (convicting an innocent defendant and acquitting a guilty one). It will become apparent later that q can also be interpreted as a threshold of 'reasonable doubt', i.e., a juror would want to convict a defendant if and only if she comes to believe that the latter is guilty with probability q or more.

[7] Following usual convention, in what follows we denote the events $\{\omega | S(\omega) = s\}$ and $\{\omega | T_J(\omega) = t_J\}$ by the more convenient $\{S = s\}$ and $\{T_J = t_J\}$, etc.

[8] Since juries frequently engage in communcation, the model applies more to a two-candidate election with a dispersed electorate such as, say, a local council race. We follow the literature in using the terminology of jury trials, although the best application of the model lies elsewhere, in our view.

$j \in J$ chooses the action 1 given a type realization $t_j \in \mathcal{T}_j$. Denote by $\sigma_{J\setminus\{j\}}$ the strategies of jurors in J other than j and let $\sigma_J = (\sigma_j, \sigma_{J\setminus\{j\}})$ be a strategy profile for the jury J and Σ_J, the set of such profiles.

Given a non–empty set of jurors J and a realization of types t_J, a strategy profile σ_J generates a probability distribution over action profiles $x_J = \{x_j\}_{j\in J}$. Let $\mathcal{X}_J$ be the set of action profiles x_J and define

$$\Pr_{\sigma_J}(x_J|T_J = t_J) = \prod_{j:x_j=1} \sigma_j(t_j) \prod_{j:x_j=0} (1 - \sigma_j(t_j)) \tag{4}$$

to be the probability with which the action profile x_J is generated by σ_J given $T_J = t_J$. For any set $X \subset \mathcal{X}_J$, let $\Pr_{\sigma_J}(X|T_J = t_J) = \sum_{x_J\in X} \Pr_{\sigma_J}(x_J|T_J = t_J)$ if X is non–empty and equal to 0 otherwise. For any $B \in \mathcal{F}$ with $P(B) > 0$, denote by $\Pr_{\sigma_J}(X|B) = \sum_{t_J\in \mathcal{T}_J} \Pr_{\sigma_J}(X|T_J = t_J)P(T_J = t_J|B)$ the probability with which action profiles in X are generated by σ_J given the event B.

Individual votes are aggregated into a decision d by a decision or voting rule. Such a rule is any function $\mathbf{d}(.)$ that takes the action profile $x_J \in \mathcal{X}_J$ into a decision $\mathbf{d}(x_J) \in \mathcal{D}$. For any $d \in \mathcal{D}$, let $X_{\mathbf{d}=d} = \{x_J \in \mathcal{X}_J | \mathbf{d}(x_J) = d\}$ and denote by $\Pr_{\sigma_J}(\mathbf{d} = d|B) = \Pr_{\sigma_J}(X_{\mathbf{d}=d}|B)$ the probability with which the decision $d \in \mathcal{D}$ is generated by σ_J and $\mathbf{d}$ given any $B \in \mathcal{F}$ with $P(B) > 0$. Define

$$U(\sigma_J, \dot{\mathbf{d}}) = \sum_{s\in\mathcal{S}} P(S = s) \sum_{d\in\mathcal{D}} \Pr_{\sigma_J}(\mathbf{d} = d|S = s)u(s, d) \tag{5}$$

to be the ex–ante expected payoff of any individual from σ_J given $\mathbf{d}$. We will look for Bayesian Nash equilibria of the simultaneous move voting game defined by the voting rule $\mathbf{d}$ and the set of jurors J.

Definition 1 *Given a jury J and a decision rule $\mathbf{d}$, a strategy profile σ_J is a Bayesian Nash Equilibrium (BNE), if for each $j \in J$, $\sigma_j \in \arg\max_{\sigma'_j} U(\sigma'_j, \sigma_{J\setminus\{j\}}, \dot{\mathbf{d}})$.*

Before we proceed to an analysis of such equilibria we consider the benchmark case where all private signals are public and introduce the notion of full information equivalence. For a non–empty jury J, consider the decision problem when the realization

t_J of T_J is common knowledge. From (1), note that the expected payoff from the decision C is $-qP(S = I|T_J = t_J)$ whereas the expected payoff from the decision A is $-(1-q)P(S = G|T_J = t_J)$. Thus, the optimal full information decision rule is to choose C whenever t_J is such that $P(S = G|T_J = t_J) > q$ and choose A whenever t_J is such that $P(S = G|T_J = t_J) < q$, choosing any probability of conviction in $[0, 1]$ when $P(S = G|T_J = t_J) = q$. Let $V^*(J)$ be the ex–ante expected payoff of all individuals (or value function) whenever the jury J makes its decision according to this full information decision rule. With private information and in the absence of communication, whether or not a jury is able to implement the full information decision rule depends, among other things, on the properties of the voting rule $\mathbf{d}$. This motivates the following definition.

Definition 2 *A decision rule* $\mathbf{d}$ *satisfies full–information equivalence for a jury* J *if there exists a BNE* σ_J *such that* $Pr_{\sigma_J}(\mathbf{d} = C|T_J = t_J) = 1$ *whenever* $P(S = G|T_J = t_J) > q$ *and* $Pr_{\sigma_J}[\mathbf{d} = C|T_J = t_J] = 0$ *whenever* $P(S = G|T_J = t_J) < q$.

Notice that if $\mathbf{d}$ satisfies full information equivalence for a jury J then there exists a BNE σ_J such that $U(\sigma_J, \mathbf{d}) = V^*(J)$.

3 Efficient Equilibria & Condorcet Jury Theorems

Our first theorem[9] shows that, for any jury J and voting rule $\mathbf{d}$, there is a pure strategy BNE that attains the maximum feasible *ex ante* payoff among all strategy profiles. We will call such a payoff-optimal profile an efficient equilibrium.

Theorem 1 *Fix* J *and* $\mathbf{d}$. *If* $\sigma_J \in \arg\max_{\sigma'_J \in \Sigma_J} U(\sigma'_J, \mathbf{d})$ *then* σ_J *is a BNE. Further, there exists a pure strategy profile in the set* $\arg\max_{\sigma'_J \in \Sigma_J} U(\sigma'_J, \mathbf{d})$. *If this pure strategy profile is the unique maximizer of* $U(\sigma'_J, \mathbf{d})$ *in the class of pure strategy profiles, then it is the unique maximizer of* $U(\sigma'_J, \mathbf{d})$ *in the class of all strategy profiles (pure and mixed).*

[9] The main part of the result that follows was proved in McLennan (1998, Theorem 1). Since it is central to the subsequent arguments, and since the proof is a one-liner, we reproduce it here for the reader's convenience.

Proof. Suppose, contrary to claim, that σ_J is not a BNE. Then there exists some j and some $\sigma'_j \neq \sigma_j$ such that $U(\sigma'_j, \sigma_{J\setminus\{j\}}, \mathbf{d}) > U(\sigma_J, \mathbf{d})$, contradicting the definition of σ_J.

Since the voting game is finite, by Kuhn's theorem[10], the behavior strategy profile σ_J has an outcome equivalent mixed strategy profile, denoted by μ_J. It follows that

$$U(\sigma_J, \mathbf{d}) = U(\mu_J, \mathbf{d}) = \sum_{\xi'_J \in \Xi_J} \mu_J(\xi'_J) U(\xi'_J, \mathbf{d}) = U(\xi_J, \mathbf{d}),$$

where Ξ_J is the set of pure strategy profiles with ξ'_J its generic element, $\mu_J(\xi'_J)$ is the probability assigned to ξ'_J by μ_J and ξ_J is an element of $\mathrm{argmax}_{\xi'_J \in \Xi_J} U(\xi'_J, \mathbf{d})$. Then ξ_J must be a BNE. Moreover, if ξ_J is the unique element of $\mathrm{argmax}_{\xi'_J \in \Xi_J} U(\xi'_J, \mathbf{d})$ it must also be the unique element of $\arg\max_{\sigma'_J \in \Sigma_J} U(\sigma'_J, \mathbf{d})$. ■

For a jury J and a decision rule $\mathbf{d}$ let $V(J, \mathbf{d})$ denote the value function or the expected payoff from any efficient equilibrium:

$$V(J, \mathbf{d}) = \max_{\sigma_J \in \Sigma_J} U(\sigma_J, \mathbf{d}) \tag{6}$$

A primary concern of this paper is to establish properties of the function $V(J, \mathbf{d})$. In this section we ask the following two questions. First, for what kind of voting rules $\mathbf{d}$ is $V(J, \mathbf{d})$ monotonic, i.e., $V(J, \mathbf{d}) \geq V(J', \mathbf{d})$ whenever $J' \subset J$. In other words, what voting rules deliver CJT3 (and so CJT1)? Second, what kind of voting rules deliver asymptotic efficiency (CJT2), i.e., $V(J, \mathbf{d})$ approaches 0 when J becomes large? In Section 5 we look for voting rules that exactly deliver full information equivalence, for a fixed jury J.

In principle, the voting rule $\mathbf{d}$ could take a very complicated form. However, most voting protocols observed in practice take the form of a *cutoff rule*: an option is selected if it receives more than a certain number of votes. Any such rule can be summarized by a function $\mathbf{k} : \mathbb{N} \to \mathbb{N}$ such that $1 \leq \mathbf{k}(n) \leq n$. The interpretation is that for a jury with n members, the decision C is selected if and only if the number of votes cast in favor of C is $\mathbf{k}(n)$ or more. Let $\mathbf{K}$ denote the set of such rules. In the rest of this paper, we focus on decision rules $\mathbf{d}$ identified by such a cutoff function $\mathbf{k}$, using the notation $\mathbf{k}$ to denote a decision rule. Some well–known examples of such rules are $\mathbf{k}(n) = n$

[10] See Kuhn (1953).

(unanimity rule, unanimity being needed to convict the defendant); $\mathbf{k}(n) = 1$ (a veto required for conviction); and $\mathbf{k}(n) = \lceil \frac{n}{2} \rceil$ (majority rule, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x). All these voting rules are monotonic in the following sense.

Definition 3 *A voting rule $\mathbf{k} \in \mathbf{K}$ is monotonic if $\mathbf{k}(n)$ as well as $n - \mathbf{k}(n)$ is non-decreasing in n.*

Our next result shows that a larger jury can do no worse than any smaller jury composed of a subset of its members, for any monotonic voting rule $\mathbf{k} \in \mathbf{K}$. Thus, all such rules deliver CJT3 (hence CJT1).

Theorem 2 *For every monotonic voting rule $\mathbf{k} \in \mathbf{K}$, and for any two juries J, J' such that $J' \subset J$, $V(J, \mathbf{k}) \geq V(J', \mathbf{k})$.*

Proof. Let $\sigma^*_{J'}$ be an efficient equilibrium for the jury J'. Let $Y = J \backslash J'$. Construct a strategy profile σ_Y for members of Y as follows: any subset of Y with $\mathbf{k}(|J|) - \mathbf{k}(|J'|)$ members vote for A regardless of the signal received, while the remaining members vote for C, again irrespective of their private signal. Since $\mathbf{k}$ is monotonic, both these numbers are non–negative.

Consider the profile $\sigma_J = (\sigma^*_{J'}, \sigma_Y)$ for the jury J. Clearly, the strategy profiles σ_J and $\sigma^*_{J'}$ are outcome equivalent so that $U(\sigma_J, \mathbf{k}) = V(J', \mathbf{k})$. Since $V(J, \mathbf{k}) \geq U(\sigma_J, \mathbf{k})$, by Theorem 1, the result follows. ∎

Theorem 2 is a direct consequence of Theorem 1. As long as the voting rule is monotonic, it is always possible to neutralize the effect of additional jury members by instructing them to always vote one way or the other. Then, by Theorem 1, in efficient equilibrium, an expanded jury can do no worse than any of its sub-juries. Feddersen and Pesendorfer (1998) have presented an example in which the probability of one type of error (convicting an innocent) increases with jury size, for the unanimity rule. Though they do not show it, in their example (and for the symmetric equilibrium they focus on) the expected payoff of each juror is decreasing in the size of the jury. In light of

Theorem 2, this is clearly an effect of selecting symmetric equilibria.[11]

What class of voting rules are capable of delivering CJT2, i.e., efficient information aggregation in the limit when the jury size becomes large? For the question to be non–trivial, we must impose some condition on the signal technology that guarantees asymptotically perfect information (among the electorate at large) and then ask whether the constraints imposed by particular voting rules and the lack of communication possibilities permit enough usage of this information so that asymptotic efficiency is achieved. For the next result we suppose that for all j and j', conditional on S, the random variables T_j, $T_{j'}$ are identically and independently distributed. Specifically, we assume that for all j, $\mathcal{T}_j = \{1, ..., m\}$ with $m > 1$, and that for all J and t_J,

$$P[T_J = t_J | S = s] = \prod_{j \in J} P(T_j = t_j | S = s) \text{ for all } s \in \mathcal{S}. \tag{7}$$

Furthermore, we assume that the type distributions are *non-degenerate*, i.e., each signal has a strictly positive probability of occurrence in each state:

$$P(T_j = t_j | S = s) > 0 \text{ for all } t_j \in \mathcal{T}_j \text{ and } s \in \mathcal{S}, \tag{8}$$

Finally, we assume that the types for each j are strictly ordered by their likelihood ratios:

$$\frac{P(T_j = t_j | S = G)}{P(T_j = t_j | S = I)} \text{ is strictly increasing in } t_j. \tag{9}$$

Given these assumptions on the type distribution, and given that we restrict attention to voting rules in the class $\mathbf{K}$, a jury J is uniquely identified by its size $|J|$. Accordingly, we will denote by n the size of a jury and let $V(n, \mathbf{k})$ be the value function of the game with n jurors and rule $\mathbf{k}$, from an associated efficient equilibrium. Observe that by Theorem 2, $V(n, \mathbf{k})$ is non–decreasing in n.

Theorem 3 *Fix a monotonic rule* $\mathbf{k} \in \mathbf{K}$ *and assume that the signal distributions satisfy (7), (8) and (9). Then,* $\lim_{n\to\infty} V(n, \mathbf{k}) = 0$ *if and only if* $\lim_{n\to\infty} \mathbf{k}(n) = \lim_{n\to\infty}[n -$

[11] While efficient equilibria may be asymmetric even in symmetric models and so require greater 'coordination' than symmetric equilibria, a focus on efficient equilibria allows us to isolate those properties of strategic voting that depend on equilibrium selection and those which do not.

$\mathbf{k}(n)] = \infty$. *Further, if either* $\lim_{n\to\infty} \mathbf{k}(n) < \infty$ *or* $\lim_{n\to\infty}[n - \mathbf{k}(n)] < \infty$*, then there exists a* $\overline{n}$ *such that for all* $n > \overline{n}$*,* $V(n+1, \mathbf{k}) = V(n, \mathbf{k})$.

Theorem 3 identifies the set of monotonic rules in $\mathbf{K}$ that deliver a CJT2 and thus unifies and extends a number of results obtained previously in the literature. In particular, it implies that the unanimity rule is asymptotically inefficient for *any* sequence of equilibria considered. The same is true for near-unanimity rules.[12] This generalizes a result in Feddersen and Pesendorfer (1998), where the sequence of *symmetric* equilibria in a binary signal model is shown to have this property. As we demonstrate in the next section, such equilibria are inefficient whenever they involve mixed strategies. Theorem 3 nevertheless establishes that the inferiority of unanimity rule is fairly robust to model and equilibrium selection.[13] Another corollary of our result is that for any interior proportional rule (such as simple or supermajority rules) we must have asymptotic efficiency if the sequence of efficient equilibria are considered. Given Theorem 1, this result also follows directly from Feddersen and Pesendorfer (1997, 1998) and Duggan and Martinelli (2000), who have previously proved this property for symmetric equilibria. More importantly, for non-degenerate signal distributions, Theorem 3 establishes a precise connection between the asymptotic properties of the *rule* and the asymptotic properties of the *outcome*. For any rule satisfying monotonicity, however complicated, the limiting properties can be inferred by straightforward application of this result.

To prove Theorem 3 it will be helpful to establish some basic properties of pure strategy equilibria. For any *pure* strategy profile σ_J, let piv_j be the event that juror j is

[12]For example, a rule that stipulates that for a motion to be taken up by a committee or legislature, it must be supported by one member, and seconded by another. This may be thought of as a rule with $\mathbf{k}(n) = 2$ in favour of admissibility of the motion.

[13]Martinelli (2002) also shows the asymptotic inefficiency of the unanimity rule, in a very general setting and independent of equilibrium selection. For a finite signal space, this result requires not having any signal t_j such that $P(T_j = t_j | S = G) = 0$, i.e., a failure of (8). While our approach is quite different, our results suggest that (8) is required for the asymptotic inefficiency of all rules failing the "double largeness condition" of Theorem 3. In Section 4.1 we show in the context of a binary signal model that if (8) fails, then the unanimity rule is in fact *optimal*, even for a fixed jury size.

pivotal, i.e., his vote affects the outcome:[14]

$$piv_j = \{\omega | \sum_{h \in J \backslash \{j\}} \sigma_h(t_h) = k - 1\} \in \mathcal{F}. \tag{10}$$

Furthermore, we say that a pure strategy σ_j satisfies the *cutoff property* if there exists $\bar{t}_j \in \{0\} \cup \mathcal{T}_j$ such that $\sigma_j(t_j) = 1$ iff $t_j > \bar{t}_j$. A strategy profile $\sigma_J = \{\sigma_j\}_{j \in J}$ satisfies the cutoff property if all component strategies σ_j do. Our first lemma shows that in any pure strategy equilibrium, if juror j's vote affects the outcome, then σ_j must satisfy the cutoff property.

Lemma 1 *Assume that the signal distributions satisfy (7), (8), (9) and let* $\mathbf{k}(n) = k$. *For a jury* J *of size* n, *in any pure strategy BNE* σ_J, *for each juror* j *with* $piv_j \neq \emptyset$,

$$\begin{aligned} \sigma_j(t_j) = 1 &\Rightarrow P(S = G | piv_j, T_j = t_j) \geq q, \\ \sigma_j(t_j) = 0 &\Rightarrow P(S = G | piv_j, T_j = t_j) \leq q. \end{aligned} \tag{11}$$

and σ_j *satisfies the cutoff property.*

Proof. Let σ_J be a pure strategy BNE and consider j such that $piv_j \neq \emptyset$ and $T_j = t_j$. Since $piv_j \neq \emptyset$, we obtain from (3) that $P(piv_j | T_j = t_j) > 0$ for all t_j. Conditional on piv_j and $T_j = t_j$, the expected payoff of j from voting for conviction is equal to $-qP(S = I | piv_j, T_j = t_j)$ whereas the expected payoff from voting for acquittal instead is $-(1-q)P(S = G | piv_j, T_j = t_j)$. Since j's vote affects the outcome if and only if piv_j occurs and $P(piv_j | T_j = t_j) > 0$, it follows that if $\sigma_j(t_j) = 1$ (respectively, $= 0$) then $P(S = G | piv_j, T_j = t_j) \geq q$ (resp., $\leq q$), establishing (11). Furthermore, the event piv_j depends only on the realization of $T_{J \backslash \{j\}}$ so that by (7), conditional on $S = s$, it is independent of the event $T_j = t_j$ for each $t_j \in \mathcal{T}_j$ and $s \in \mathcal{S}$. By Bayes Rule, (8) and (9) it follows that $P(S = G | piv_j, T_j = t_j)$ is increasing in t_j. From (11), σ_j must then satisfy the cutoff property. ■

[14] If piv_j occurs then a vote for conviction (respectively, acquittal) from j results in the decision C (resp., A) being chosen. If piv_j does not occur then j's vote does not affect the decision. Since piv_j can be generated by unions and intersections of sets of the form $\{\omega | T_h \in \mathcal{T}_h^1\} \in \mathcal{F}$ and $\{\omega | T_h \in \mathcal{T}_h \backslash \mathcal{T}_h^1\} \in \mathcal{F}$, where $h \in J \backslash \{j\}$ and $\mathcal{T}_h^1 = \{t_h \in \mathcal{T}_h | \sigma_h(t_h) = 1\}$, piv_j is measurable.

For a jury J and any pure strategy profile σ_J, let $J_c(\sigma_J) = \{j \in J | \sigma_j(t_j) = 1 \forall t_j\}$ and $J_a(\sigma_J) = \{j \in J | \sigma_j(t_j) = 0 \forall t_j\}$. Jurors in J_c (respectively, J_a) vote to convict (resp., acquit) regardless of their signal. Let $J_i(\sigma_J) = J \backslash \{J_c(\sigma_J) \cup J_a(\sigma_J)\}$ be the set of jurors who vote *informatively*, i.e., their vote depends on their signal. If $|J_c(\sigma_J)| \geq k$ then the jury always votes to convict while if $|J_a(\sigma_J)| > n-k$ then the jury always votes to acquit. When $|J_c(\sigma_J)| < k$ and $|J_a(\sigma_J)| \leq n-k$, the set $J_i(\sigma_J)$ of informative voters is non–empty, piv_j is non–empty for all $j \in J$, and the jury votes *informatively*, i.e., sometimes votes to convict and sometimes votes to acquit depending on the realization of T_J. Note that when σ_J satisfies the cutoff property, each $j \in J_i(\sigma_J)$ has an interior cutoff $\bar{t}_j \in \{1, ..., m-1\}$.

For the rest of the proof it will be helpful to consider monotonic rules $\mathbf{k}$ satisfying $\mathbf{k}(n) = k$, for all $n \geq k$ and some non–negative integer k. Given such a rule, we show now that for a large enough jury, some juror must vote to acquit with probability 1, given that the jury does not convict with probability 1.

Lemma 2 *Assume that the signal distributions satisfy (7), (8), (9) and that* $\mathbf{k}(n) = k$ *for all* $n \geq k$. *For each* n *and a jury* J_n *of size* n, *let* σ_{J_n} *be a pure strategy BNE. If there exists* $\overline{n}_1(k)$ *such that for all* $n > \overline{n}_1(k)$, $|J_c(\sigma_{J_n})| < k$, *then there exists* $\overline{n}_2(k)$ *such that for all* $n > \overline{n}_2(k)$, $|J_a(\sigma_{J_n})| \geq 1$.

Proof. Pick any jury J_n of size $n > \overline{n}_1(k)$ and suppose that there does not exist $\overline{n}_2(k)$ such that for all $n > \overline{n}_2(k)$, $|J_a(\sigma_{J_n})| \geq 1$. Pick $n > \overline{n}_1(k)$ such that $|J_a(\sigma_{J_n})| = 0$ so that $J_i(\sigma_{J_n}) \neq \emptyset$ and $piv_j \neq \emptyset$ for all $j \in J$. By Lemma 1, σ_{J_n} satisfies the cutoff property. Pick $j \in J_i(\sigma_{J_n})$. The event piv_j tells juror j that exactly $k - |J_c(\sigma_{J_n})| - 1$ of the $n - |J_c(\sigma_{J_n})| - 1$ other jurors $j' \in J_i(\sigma_{J_n}) \backslash \{j\}$ have signals that are above their cutoffs $\bar{t}_{j'} \in \{1, ..., m-1\}$. Thus, when piv_j occurs, at most $k - |J_c(\sigma_{J_n})| - 1$ of these jurors j' have signals that take the *highest* possible value m. Since $t_j \leq m$, by (7), (8) and (9), the event $\{piv_j, T_j = t_j\}$ is weaker evidence for $\{S = G\}$ than the event $B \in \mathcal{F}$, where B occurs when exactly $k - |J_c(\sigma_{J_n})|$ jurors among the $n - |J_c(\sigma_{J_n})|$ jurors in $J_i(\sigma_{J_n})$ have signals equal to m. That is,

$$P(S = G | piv_j, T_j = t_j) \leq P(S = G | B) = \frac{P(S = G)}{P(S = G) + l(|J_c(\sigma_{J_n})|, n, k) P(S = I)}$$

where

$$l(|J_c(\sigma_{J_n})|, n, k) = \left[\frac{P(T_j = m|S = I)}{P(T_j = m|S = G)}\right]^{k-|J_c(\sigma_{J_n})|} \left[\frac{P(T_j < m|S = I)}{P(T_j < m|S = G)}\right]^{n-k-|J_c(\sigma_{J_n})|}.$$

Using (9) observe that $\frac{P(T_j=m|S=I)}{P(T_j=m|S=G)} < 1 < \frac{P(T_j<m|S=I)}{P(T_j<m|S=G)}$. Then, for any positive scalar $M > 0$ there exists $n > \overline{n}_1(k)$ large enough, with $|J_a(\sigma_{J_n})| = 0$ and $J_i(\sigma_{J_n}) \neq \emptyset$, such that $l(|J_c(\sigma_{J_n})|, n, k) > M$. But this implies that for n and M large enough, $P(S = G|piv_j, T_j = t_j) < q$ for any $t_j \in \mathcal{T}_j$, a contradiction with the fact that $j \in J_i(\sigma_{J_n})$, i.e., $1 \leq \bar{t}_j < m$, by (11). ■

The next lemma contains a result that is similar to, but more restricted than Theorem 3. It states that for a rule $\mathbf{k}(n) = k$ for all $n \geq k$, there exists an upper bound on the size of a jury, beyond which a larger jury can do no better than a smaller one.

Lemma 3 *Assume that the signal distributions satisfy (7), (8), (9) and that* $\mathbf{k}(n) = k$ *for all* $n \geq k$. *There exists* $\overline{n}(k)$ *such that for all* $n \geq \overline{n}(k)$, $V(n, \mathbf{k}) = V(\overline{n}(k), \mathbf{k}) < 0$.

Proof. For each n and a jury J_n of size n, let σ_{J_n} be an *efficient* pure strategy BNE and suppose first that there exists $\overline{n}_1(k)$, such that for all $n > \overline{n}_1(k)$, $|J_c(\sigma_{J_n})| < k$. By Lemma 2, there exists $\overline{n}_2(k)$ such that for all $n > \overline{n}_2(k)$, $|J_a(\sigma_{J_n})| \geq 1$. Pick any jury J_n of size $n > \overline{n}_2(k)$ and note that it can achieve the same outcome as any jury J_{n+1} of size $n+1$ by setting $|J_c(\sigma_{J_n})| = |J_c(\sigma_{J_{n+1}})|$, $|J_a(\sigma_{J_n})| = |J_a(\sigma_{J_{n+1}})| - 1$ and setting identical thresholds $\bar{t}_j \in \{1, ..., m-1\}$ for the identical number of remaining jurors. But then, from Theorem 2 we must have $V(n+1, \mathbf{k}) = V(n, \mathbf{k})$ for all $n > \overline{n}_2(k)$ implying that $V(n, \mathbf{k}) = V(\overline{n}(k), \mathbf{k})$ for all $n \geq \overline{n}(k) = \overline{n}_2(k) + 1$. Moreover, in at least one efficient equilibrium, the jury of size $\overline{n}(k)$ chooses the decision A, with probability at least as high as that of all jurors in $J_{\overline{n}(k)} \backslash J_c(\sigma_{J_{\overline{n}(k)}})$ receiving the lowest signal 1 and voting for acquittal. Since, conditional on $S = G$, this latter probability is strictly positive by (8), and since $P(S = G) > 0$, it follows that $V(\overline{n}(k), \mathbf{k}) < 0$.

Suppose next that for each jury J_n of size $n \geq k$ there exists a jury $J_{n'}$ of size $n' > n$ such that $|J_c(\sigma_{J_{n'}})| \geq k$ in any efficient equilibrium $\sigma_{J_{n'}}$ of $J_{n'}$. Then the decision C is chosen with probability 1 by the jury $J_{n'}$. Since choosing this decision is always feasible for the jury J_n, we obtain via Theorem 2 that $V(n+1, \mathbf{k}) = V(n, \mathbf{k})$ for all $n \geq k$

implying that $V(n, \mathbf{k}) = V(\overline{n}(k), \mathbf{k})$ for all $n \geq \overline{n}(k) = k$. Since, in at least one efficient equilibrium, the jury of size $\overline{n}(k)$ chooses the decision C even when $S = I$, and since $P(S = I) > 0$, it follows that $V(\overline{n}(k), \mathbf{k}) < 0$. ■

Proof of Theorem 3.

For the 'only if' part, assume (without loss of generality) $\lim_{n\to\infty} \mathbf{k}(n) = \overline{k} < \infty$. Since $\mathbf{k}$ is assumed to be a non-decreasing function, and increases only by integer values, there must be a $\widehat{n}$ such that for all $n > \widehat{n}$, $\mathbf{k}(n) = \overline{k}$. Let $\mathbf{k}'$ be the rule $\mathbf{k}'(n) = \overline{k}$ for all $n \geq \overline{k}$. Then for all $n > \widehat{n}$, $V(n, \mathbf{k}) = V(n, \mathbf{k}')$ and further, by Lemma 3, there exists $\overline{n}(\overline{k})$ such that for all $n \geq \overline{n}(\overline{k})$, $V(n, \mathbf{k}') = V(\overline{n}(\overline{k}), \mathbf{k}') < 0$, implying $\lim_{n\to\infty} V(n, \mathbf{k}) = V(\overline{n}(\overline{k}), \mathbf{k}') < 0$.

For the 'if' part, for each jury J_n of size n we will construct a strategy profile σ_{J_n} with the property that $\lim_{n\to\infty} U(\sigma_{J_n}, \mathbf{k}) = 0$. The result will then follow from Theorem 1. For each n, define the functions $c(n) = \max[0, 2\mathbf{k}(n) - n]$ and $a(n) = \max[0, n - 2\mathbf{k}(n)]$. Then $\lim_{n\to\infty}(n - c(n) - a(n)) = \infty$. For each jury J_n of size n define a strategy profile $\sigma_{J_n} = \{\sigma_j^n\}_{j\in J_n}$ as follows. Let $c(n)$ individuals always vote to convict and $a(n)$ individuals always vote to acquit, regardless of their signals. For each remaining individual j, let $\sigma_j^n(t_j) = \alpha$ if $t_j = 1$ and $\sigma_j^n(t_j) = \alpha'$ otherwise, for some $\alpha, \alpha' \in [0, 1]$ such that

$$\begin{aligned}
\alpha_G &\equiv P(T_j = 1|S = G)\alpha + P(T_j > 1|S = G)\alpha' > \frac{1}{2} \\
\alpha_I &\equiv P(T_j = 1|S = I)\alpha + P(T_j > 1|S = I)\alpha' < \frac{1}{2}
\end{aligned}$$

From (9), it is immediate that such α, α' exist and are independent of n. Note that for each n, conditional on $S = G$, with probability 1 there are $c(n)$ votes for conviction and $a(n)$ votes for acquittal; and each of the remaining $n - c(n) - a(n)$ voters vote for conviction with probability $\alpha_G > \frac{1}{2}$, independently across such voters by (7). For any $\varepsilon > 0$, by the (weak) law of large numbers,[15] it follows that for n large enough,

$$\Pr_{\sigma_{J_n}}[\{x_{J_n} \in \mathcal{X}_{J_n} | \frac{\sum_{j\in J_n} x_j - c(n)}{n - a(n) - c(n)} \geq \alpha_G - \varepsilon\}|S = G] > 1 - \varepsilon$$

Pick ε small enough such that $\alpha_G - \varepsilon > \frac{1}{2}$, so that $(\alpha_G - \varepsilon)(n - a(n) - c(n)) + c(n) > \mathbf{k}(n)$. Then, $\Pr_{\sigma_{J_n}}[\{x_{J_n} \in \mathcal{X}_{J_n} | \sum_{j\in J_n} x_j \geq \mathbf{k}(n)\}|S = G]$, the probability of conviction

[15]See, e.g., Billingsley (1995).

given $S = G$, is arbitrarily close to 1 for n large enough. Analogously, since $\alpha_I < \frac{1}{2}$, the probability of acquittal given $S = I$ is also arbitrarily close to 1, for n large enough. Hence, $\lim_{n \to \infty} U(\sigma_{J_n}, \mathbf{k}) = 0$. ■

4 Binary Signals

Several previous papers have used a binary signal model for simplicity and tractability. However, the results have generally been derived based on the *symmetric* equilibrium (usually involving mixed strategies). We now give a complete characterization of the structure of pure strategy efficient equilibria in such models. It will become apparent in the process that efficient equilibria are usually asymmetric. This in turn will enable us to characterize the optimal voting rule $\mathbf{k} \in \mathbf{K}$ and show that in the binary signal model, the optimal voting rule induces full information equivalence for any jury of size n. Throughout this section we will maintain assumptions (7), (8) and (9). In Section 4.1 we will investigate the effect of relaxing (8).

Suppose each juror can receive one of two signals so that $\mathcal{T}_j = \{1, 2\}$ for all j. For ease of exposition, we introduce the notation $\pi = P(S = G) \in (0, 1)$ and, using (7), let $p_G = P(T_j = 2|S = G)$ and $p_I = P[T_j = 2|S = I]$ for each j. By (8) and (9), $p_G, p_I \in (0, 1)$ with $p_G > p_I$. It will be convenient to define $\lambda(z, y)$ as the posterior probability on $S = G$ if it is known that exactly y out of z signals ($0 \leq y \leq z$) have turned out to be equal to 2. By Bayes' Rule:

$$\lambda(z, y) = \frac{(p_G)^y (1 - p_G)^{z-y} \pi}{(p_G)^y (1 - p_G)^{z-y} \pi + (p_I)^y (1 - p_I)^{z-y} (1 - \pi)} \tag{12}$$

Note that by (9), for fixed y, $\lambda(z, y)$ is decreasing in z while $\lambda(z, z - y)$ is increasing in z.

For any integer $y \geq 1$, let $n_a^*(y)$ be the largest integer $n' \in [y, \infty)$ satisfying:

$$\lambda(n', y - 1) < q \leq \lambda(n', y) \tag{13}$$

The interpretation is as follows. For a jury of size $n' > n_a^*(y)$, with at least y votes required for conviction, a juror would strictly prefer to vote for acquittal even if his signal is equal to 2 and he knows that exactly $y-1$ other jurors have also received a signal equal to 2 and have voted for conviction. If $\lambda(y,y)\geq q$, then $n_a^*(y)$ is well–defined as it equals the largest value of n' for which $\lambda(n',y)\geq q$. If $\lambda(y,y)<q$ there exists no integer satisfying the inequality above. In that case, we define $n_a^*(y)=0$.

Similarly, for $y\geq 0$, let $n_c^*(y)$ be the largest integer $n'\in[y+1,\infty)$ satisfying:

$$\lambda(n',n'-y-1)\leq q<\lambda(n',n'-y) \tag{14}$$

For a jury of size $n'>n_c^*(y)$, with at least $n'-y$ votes required for conviction, a juror would strictly prefer to vote for conviction even if his signal is equal to 1 and he knows that exactly $n'-y-1$ other jurors have received a signal equal to 2 and have voted for conviction. If $\lambda(y+1,0)\leq q$ then $n_c^*(y)$ is well–defined as it equals the largest value of n' for which $\lambda(n',n'-y-1)\leq q$. If $\lambda(y+1,0)>q$, there will not exist any integer satisfying the above inequality. In that case, we define $n_c^*(y)=0$.

Now consider the problem with n jurors with at least k votes required for conviction. Observe that $n_a^*(k)<n$ if $\lambda(n,k)<q$, $n_a^*(k)=n$ if $\lambda(n+1,k)<q\leq\lambda(n,k)$ and $n_a^*(k)>n$ if $q\leq\lambda(n+1,k)$. Similarly, $n_c^*(n-k)<n$ if $q<\lambda(n,k-1)$, $n_c^*(n-k)=n$ if $\lambda(n,k-1)\leq q<\lambda(n+1,k)$ while $n_c^*(n-k)>n$ if $\lambda(n+1,k)\leq q$. Let

$$n^*(n,k)=\min\{n,n_a^*(k),n_c^*(n-k)\} \tag{15}$$

so that

$$n^*(n,k)=n \text{ iff } \lambda(n,k-1)\leq q\leq\lambda(n,k) \tag{16}$$

Since n and k are fixed in the rest of this section, we drop the arguments of the functions n_a^*, n_c^* and n^* in what follows in order to minimize on notation.

Consider a pure strategy profile σ_J satisfying the cutoff property that has the following additional features. Suppose $|J_i(\sigma_J)|\in\{n^*-1,n^*\}$ if $n^*=n_a^*>k$ and $q=\lambda(n_a^*,k)$, or if $n^*=n_c^*>n-k+1$ and $q=\lambda(n_c^*,n_c^*-(n-k)-1)$; with $|J_i(\sigma_J)|=n^*$ otherwise. Let $|J_a(\sigma_J)|=n-|J_i(\sigma_J)|$ if $n^*=n_a^*$ and equal to 0 otherwise, while $|J_c(\sigma_J)|=n-|J_i(\sigma_J)|$ if $n^*=n_c^*$ and equal to 0 otherwise. Let Σ_J^* be the set of such strategy profiles.

For any strategy profile in Σ_J^*, n^* (or one less) of the jurors vote informatively, i.e., with $\sigma_j(2)=1$ and $\sigma_j(1)=0$. The remaining voters all vote for acquittal if $n^*=n_a^*$

and all vote for conviction if $n^* = n_c^*$. We show below that strategy profiles in Σ_J^* are efficient equilibria. Finally, note from (7) that the full information value function $V^*(.)$ is determined entirely by the size n of a jury, allowing us to denote the full information value function as $V^*(n)$. We extend the domain of $V^*(n)$ as follows:

$$V^*(0) = \max[-(1-q)\pi, -q(1-\pi)]. \tag{17}$$

We are now in a position to characterize efficient equilibria.[16]

Proposition 1 *Assume $\mathcal{T}_j = \{1, 2\}$ for all j, (7), (8) and (9). Consider a jury J of size n and a rule $\mathbf{k} \in \mathbf{K}$ with $\mathbf{k}(n) = k$.*

1. *If $\lambda(n-k+1, 0) < q < \lambda(k,k)$ and σ_J is an efficient pure strategy equilibrium then $\sigma_J \in \Sigma_J^*$.*

2. *If $\sigma_J \in \Sigma_J^*$ then it is an efficient pure strategy equilibrium.*

3. *$V(n, \mathbf{k}) = V^*(n^*(n, k))$.*

Proof. See the Appendix. ■

Proposition 1 effectively establishes a simple algorithm for computing efficient equilibria in a binary model. This can be described in words as follows. Imagine for a moment that all jurors vote informatively, i.e. for conviction if their signal is equal to 2, and in favor of acquittal if their signal is equal to 1. If it is incentive compatible for each individual juror to vote in such a way then informative voting by every juror is the efficient equilibrium. If not, then an individual best response to informative voting by all others is either to always vote for conviction or to always vote for acquittal. Then, let any one juror do so and let everyone else vote informatively. If this is incentive compatible for the remaining informative voters, we have found the efficient equilibrium. If not, decrease the number of voters voting informatively by one, and repeat the same

[16]While Proposition 1 characterizes efficient equilibria in pure strategies, in most cases of interest, mixed strategy equilibria are not efficient. Moreover, when the additional condition on λ in the necessity part of the result is not satisfied, efficient equilibria involve either always choosing C or always choosing A. In such cases, there are many strategy profiles outside Σ_J^* which are efficient equilibria.

exercise till an equilibrium is found. The resultant equilibrium will be optimal among all equilibria.

So far, we have discussed the efficiency of equilibria, *given* the voting rule. It is instructive to ask: (assuming jurors are always able to coordinate on the best possible equilibrium) which voting rules are best? Formally, for a jury of size n, a voting rule $\mathbf{k}^* \in \mathbf{K}$ is efficient if

$$\mathbf{k}^* \in \arg\max_{\mathbf{k}\in\mathbf{K}} V(n, \mathbf{k}) \tag{18}$$

for each n. From Proposition 1.3, since $V^*(n)$ is a non–decreasing function (more information is better), $\mathbf{k}^*(n)$ is the value of $k \in \{1, ..., n\}$ which maximizes $n^*(n, k)$. As long as $\lambda(n, 0) \leq q \leq \lambda(n, n)$ (so that the decision problem is non–trivial), there exists a $k \in \{1, ..., n\}$ such that (16) holds and the maximized value of $n^*(n, k)$ is n. Hence we get the following characterization of the optimal voting rule $\mathbf{k}^*$:

$$\mathbf{k}^*(n) \in \{k \in \{1, ..., n\} |\ n^*(n, k) = n\} \tag{19}$$

The most efficient voting rule is such that under such a rule, it is an equilibrium for all jurors to vote informatively. By Lemma 4 (see the proof of Proposition 1), it also satisfies full information equivalence. However, the last property is a special feature of binary signal models. With a richer signal structure, full information efficiency is generally not achievable under *any* voting rule (see Austen-Smith and Banks (1996)) that only permits indivisible votes.

4.1 Degenerate Signals: 'Proof' versus 'Evidence'

We now relax assumption (8) by considering the case where $p_G = 1$ but $p_I \in (0, 1)$. This means that whenever a single signal equals 1, the defendant's innocence is established beyond doubt. However, any number of signals taking value 2, while making guilt more likely, does not make it a certainty. Intuitively, whenever the defendant is innocent, there exists a definitive *proof* of this fact (e.g. an alibi which establishes innocence beyond doubt). However, individual jurors may not always be able to detect such proof. This is in contrast to noisy *evidence*, which is inconclusive beyond a point.

Proposition 2 *In the binary signal model with (7) but $p_G = 1$ and $p_I \in (0,1)$ the unanimity rule for conviction $\mathbf{k}(n) = n$ is the only rule that satisfies full information equivalence for all n.*

Proof. Fix a jury J of size n, let y be the number of signals that take the value 2 and let $\mathbf{d}_n^*(y) \in \mathcal{D}$ be an optimal full information decision rule as a function of y. We consider two cases. First, suppose n is such that $\lambda(n,n) \leq q$. Then, $\mathbf{d}_n^*(y) = A$ for all y is an optimal full information full information decision rule. For any rule $\mathbf{k}$, this can be achieved by setting $\sigma_j(2) = \sigma_j(1) = 0$ for all $j \in J$. Hence, any rule satisfies full information equivalence.

Next, consider the case where $\lambda(n,n) > q$. Using $p_G = 1$ in (12), we get $\lambda(n,y) = 0$ for all $y < n$. Hence, $\mathbf{d}_n^*(y) = C$ iff $y = n$. Take the unanimity rule. It is immediate that the strategy profile defined by $\sigma_j(2) = 1$ and $\sigma_j(1) = 0$ for all $j \in J$ is a BNE that has the same outcome as the full information decision rule. Thus the unanimity rule satisfies full information equivalence. Now take any other rule $\mathbf{k}$ with $\mathbf{k}(n) < n$ and suppose there exists a BNE σ_J such that $\mathbf{k}$ satisfies full information equivalence. Then the set $J' = \{j \in J | \sigma_j(2) = 1\}$ must have at least $\mathbf{k}(n)$ elements. If not, for the case where $y = n$ and all members of the set $J \backslash J'$ vote for acquittal, the decision chosen will be A whereas the full information decision $\mathbf{d}_n^*(n) = C$. But then for the case where $y < n$ but $T_j = 2$ for all $j \in J'$, the decision chosen will be C whereas the full information decision is A, a contradiction. Finally, since $\lim_{n\to\infty} \lambda(n,n) = 1 > q$, for large enough n, the unanimity rule is the only rule that satisfies full information equivalence. ■

Duggan and Martinelli (2000) have shown in a continuous signal framework that the asymptotic inefficiency of unanimity rule is obtained if and only if there is no arbitrarily strong signal in favor of guilt or innocence. Complementing this result, our binary signal example shows that in the presence of the possibility of 'perfect' evidence of innocence, the unanimity rule is the most efficient rule for *any* jury size and uniquely so for a large enough jury.

5 Full Information Equivalence with Divisible Votes

So far we have restricted attention to voting rules that grant a single *indivisible* vote to each voter. For a large class of such voting rules, information is aggregated efficiently as the jury becomes very large. Yet for a jury of any fixed size, the decentralized information is not efficiently aggregated into the decision, unless the information structure is very simple (binary). It is natural to ask whether efficiency can be improved by adopting alternative rules in such environments.

It is easy to see that there exists an optimal direct mechanism which allows for perfect information aggregation. By selecting the optimal action (from the jurors' view-point) for every reported vector of signals, such a mechanism eliminates any incentive to misreport.[17] However, it will, in general, be sensitive to the fine details of the problem—the utility function and the probability distribution over signals. This is obviously impractical for the applications we have in mind: procedural rules for organizations, judiciaries or legislatures need to be laid down in advance and are meant to apply to a variety of recurrent problems. Constitutions cannot be rewritten every time the system encounters a new and quantitatively dissimilar decision problem. It is therefore useful to restrict attention to *indirect* mechanisms that are, in their construction, independent of the features mentioned above, i.e., context free. The pre-specified single indivisible voting rules we have considered so far belong to this class. We now turn to the question: are there other context free mechanisms that can deliver better results?

It turns out that a simple amendment to the usual voting rules can substantially improve their efficiency properties. This consists of allowing the votes to be *divisible*, i.e. rules that allow jurors to cast a fraction of their votes for one alternative, and the remaining fraction for the other. This mechanism is somewhat in the spirit of approval voting, in that voters can express some degree of approval for either option. The difference lies in the fact that it allows continuously divisible votes and also enforces

[17]The problem of designing an optimal mechanism with respect to some social welfare function is less trivial when either the social objective conflicts with that of jurors, or jurors have conflicting preferences themselves. Since we only consider *common* interest collective choice problems in this paper, such complications do not arise.

a budget constraint of votes for each juror. Hence, it combines the spirit of approval voting with the 'one person, one vote' principle.[18]

Fix a jury J with n members. With divisible votes, the players' pure strategies can be described by a function $\xi_j : \mathcal{T}_j \to [0,1]$, describing the fraction of votes juror $j \in J$ casts in favor of C, as a function of her signal. Let ξ_J be a strategy profile. Pick any voting rule $\mathbf{k} \in \mathbf{K}$ such that the jury J can take the decision C if and only if $\sum_{j\in J} \xi_j \geq \mathbf{k}(n)$.[19] It is straightforward to check that, given $\mathbf{k}$, all the equilibria of the case when votes are indivisible remain when votes are divisible. However, new equilibria generally arise. We show now that in the special case when signals are conditionally independently (though not necessarily identically) distributed, there exists an equilibrium that satisfies full–information equivalence.

Suppose that (8) holds and denote by $l_j(t_j) = \frac{P[T_j=t_j|S=G]}{P[T_j=t_j|S=I]}$ the likelihood ratio of signal T_j. By conditional independence, for any realized vector of signals $T_J = t_J$, the posterior on $s = G$ can be written using Bayes' Rule as:

$$P[S = G|T_J = t_J] = \frac{\pi \prod_{j\in J} l_j(t_j)}{\pi \prod_{j\in J} l_j(t_j) + (1-\pi)}$$

where $\pi = P[S = G]$. Then $P[S = G|T_J = t_J] \geq q$ if and only if $\prod_{j\in J} l_j(t_j) \geq L \equiv \frac{q(1-\pi)}{\pi(1-q)}$ or

$$\sum_{j\in J} \log l_j(t_j) \geq \log L. \tag{20}$$

Let $\bar{l}_J = \max_{j,t_j} l_j(t_j)$ and $\underline{l}_J = \min_{j,t_j} l_j(t_j)$. To focus on the interesting cases, we assume $n \log \underline{l}_J < \log L < n \log \bar{l}_J$. If this is not satisfied, then the optimal decision is to always convict or always acquit the defendant, regardless of available information. It is trivial to achieve full information equivalence in that scenario.

[18]The budget constraint on vote totals does not affect our results. For example, consider the following *scoring rule*: each juror must assign a score belonging to the interval [0, 1] for each alternative. The alternative receiving the highest total score is chosen. It can be shown that the set of equilibrium outcomes for this scoring rule coincides with that for the divisible voting rule, so that these rules are equivalent. Notice that the scoring rule is a natural generalization of approval voting from discrete to continuous votes.

[19]Since votes are divisible, $\mathbf{k}(n)$ can be allowed to take non-integer values.

Theorem 4 *Fix a jury J with n members and assume that conditional on the state, the distribution of signals is independent, (8) holds and $n \log \underline{l}_J < \log L < n \log \bar{l}_J$. Then, for each voting rule $\mathbf{k} \in \mathbf{K}$ with $\mathbf{k}(n) = k$, there exists a pure strategy BNE ξ_J described by:*

$$\xi_j(t_j) = \begin{cases} \frac{k[\log l_j(t_j) - \log \underline{l}_J]}{\log L - n \log \underline{l}_J} & \text{if } k \leq \frac{\log L - n \log \underline{l}_J}{\log \bar{l}_J - \log \underline{l}_J} \quad (1) \\ \\ 1 - \frac{(n-k)\left[\log \bar{l}_J - \log l_j(t_j)\right]}{n \log \bar{l}_J - \log L} & \text{if } k > \frac{\log L - n \log \underline{l}_J}{\log \bar{l}_J - \log \underline{l}_J} \quad (2) \end{cases} \tag{21}$$

for $j \in J$, such that $\mathbf{k}$ satisfies full information equivalence for the jury J.

Proof. First consider the case described in (21.1). Since $n \log \underline{l}_J < \log L$ and since $\log l_j(t_j) \geq \log \underline{l}_J$ by definition, it follows that $\xi_j(t_j) \geq 0$. Further, since $\xi_j(t_j)$ is a positive affine function of $\log l_j(t_j)$, $\xi_j(t_j) \leq \frac{k\left[\log \bar{l}_J - \log \underline{l}_J\right]}{\log L - n \log \underline{l}_J} \leq 1$. We now show that under ξ_J, the full information outcome is implemented. But from (21.1) it is immediate that

$$\sum_{j \in J} \xi_j(t_j) \geq k \Longleftrightarrow \sum_{j \in J} \log l_j(t_j) \geq \log L, \tag{22}$$

so that from (20) we conclude that under ξ_J the sum of votes exceeds k if and only if it is optimal for the jury J to convict the defendant under full information. From Theorem 1, it is then immediate that ξ_J is a BNE.[20] The proof is identical for the case described in (21.2) and is therefore omitted. ■

The requirement that signals be conditionally independent is only a sufficient condition for full information equivalence to be possible. We have been unable to determine a tighter necessary and sufficient condition on the signal distribution that allows informational equivalence.[21,22] However, the following example demonstrates that *some*

[20] Theorem 1, though stated for finite strategy spaces, can easily be extended to infinite strategy spaces as in this case.

[21] We conjecture that if the marginal distributions on each player's signals satisfy the monotone likelihood property, Theorem 4 will still be true. The problem with showing this result is that a simple constructive proof as in Theorem 4 is no longer available. We will investigate the issue further in future research.

[22] Theorem 4 can be extended to the case where there are more than two alternatives, given conditional independence and a symmetry condition on preferences. Proof available on request.

restriction is necessary—the result of Theorem 4 will not be true for arbitrary probability distributions over signals.

Consider a jury J of size two, and a game with divisible votes and $0 < k \leq 2$. Suppose each juror j receives a binary signal $t_j \in \{1, 2\}$. The joint distribution on signals is as follows:

$$\begin{aligned} P[T_1 &= T_2 = 2|S = G] = P[T_1 = T_2 = 1|S = G] = \frac{1}{2} \\ P[T_1 &= 2, T_2 = 1|S = I] = P[T_1 = 1, T_2 = 2|S = I] = \frac{1}{2} \end{aligned}$$

Whenever the defendant is guilty, both jurors receive the same signal, while they receive different signals whenever the defendant is innocent.[23] Note that the full vector of signals *perfectly* reveals the true state, although the marginal distributions are pure noise. Obviously, the full information decision rule as a function of the signal realizations is as follows: $\mathbf{d}^*(2, 2) = \mathbf{d}^*(1, 1) = C$ and $\mathbf{d}^*(2, 1) = \mathbf{d}^*(1, 2) = A$. Suppose, if possible, that there is a strategy profile ξ_J that achieves the full information decision for each state and each realized vector of signals. Then, we must have $\xi_1(2)+\xi_2(2) \geq k$ and $\xi_1(1)+\xi_2(1) \geq k$ and $\xi_1(2) + \xi_2(1) < k$ and $\xi_1(1) + \xi_2(2) < k$ implying that $k \leq \sum_{j,t_j} \xi_j(t_j) < k$, a contradiction.

When players have rich signal spaces (i.e., there are more than two signals), it is not surprising that a binary instrument (a yes/no vote) fails to aggregate that information efficiently, since the available instruments are coarser than the information available to individual jurors. Allowing the divisibility of votes at least overcomes the problem of dimensionality by allowing each voter to not only express a preference for one option over the other, but also the *intensity* of that preference. However, as long as voters are unable to communicate, there still exists a problem of coordination. This is because we still restrict attention to voting rules in the set $\mathbf{K}$ which all have the constraint that the decision must be a function of the *sum* of the individual votes. As the above example demonstrates, whether this problem can be successfully solved depends on the fine

[23] The example presented here is not "knife-edge", i.e. it will continue to hold if there is positive weight on all signal realizations in every state. What matters is that the optimal decision be C whenever the signals are identical and D otherwise.

structure of the signal technology. Divisible voting rules (or scoring rules), though powerful, are not a *perfect* substitute for communication possibilities. Nevertheless, there are tantalizing parallels to models with indivisible votes and communication. Theorem 4, for example, states that as long as votes are divisible, the exact threshold for conviction, $\mathbf{k}$, does not matter. A similar invariance result is to be found in the model with communication analyzed by Gerardi and Yariv (2002).

6 Conclusion

We have characterized *efficient* equilibria of common interest voting games with privately informed voters. Efficient equilibria typically involve pure strategies and are asymmetric even in symmetric models. We studied the implications of efficient equilibrium selection for Condorcet jury theorems. Provided efficient equilibria are selected, larger juries can do no worse than smaller ones. We also derived a simple necessary and sufficient condition that relates the asymptotic efficiency of voting outcomes to the asymptotic properties of different voting rules. Mistakes are eliminated in the limit if and only if the number of votes required for each decision grows unboundedly. A corollary is that unanimity as well as near unanimity rules are asymptotically inefficient *regardless* of equilibrium selection. However, if the signal distribution fails a non-degeneracy condition, the unanimity rule dominates any other rule. Finally, if signals are conditionally independent, full information equivalence can be achieved for *any* rule that allows the divisibility of individual votes.

Several interesting questions remain open. What kind of equilibria exist when jurors have conflicting interests (different values q of in our model)? What is the effect of allowing communication among jurors in such a model? What are the outcomes of a divisible voting rule under more general signal structures, or when the jury is heterogeneous? What kind of mechanism should a social planner, who may not share the voters' values, design? We think these are fruitful questions for future research.

7 Appendix: Proof of Proposition 1

Lemma 4 *Assume $\mathcal{T}_j = \{1, 2\}$ for all j, (7), (8) and (9). Consider a jury J of size n and a rule $\mathbf{k} \in \mathbf{K}$ with $\mathbf{k}(n) = k$.*

1. *In the full information problem with $n'+1$ jurors with $n' \geq 0$ and $\lambda(n'+1, 0) \leq q \leq \lambda(n'+1, n'+1)$, $V^*(n'+1) = V^*(n')$ iff $\lambda(n'+1, y) = q$ for some $y \in \{0, ..., n'+1\}$.*

2. *Let σ_J be a pure strategy profile with $|J_c(\sigma_J)| < k$ and $|J_a(\sigma_J)| \leq n - k$. Then σ_J is an equilibrium iff σ_J satisfies the cutoff property and*

$$\lambda(|J_i(\sigma_J)|, k - |J_c(\sigma_J)| - 1) \leq q \leq \lambda(|J_i(\sigma_J)|, k - |J_c(\sigma_J)|). \tag{23}$$

$$\begin{aligned} &\lambda(|J_i(\sigma_J)| + 1, k - |J_c(\sigma_J)|) \geq q \text{ if } J_c(\sigma_J) \neq \emptyset \quad (1) \\ &\lambda(|J_i(\sigma_J)| + 1, k - |J_c(\sigma_J)|) \leq q \text{ if } J_a(\sigma_J) \neq \emptyset \quad (2) \end{aligned} \tag{24}$$

Furthermore, if σ_J is an equilibrium then $U(\sigma_J, \mathbf{k}) = V^*(|J_i(\sigma_J)|)$ and $|J_i(\sigma_J)| \leq n^*$.

Proof. (1) Let $\widetilde{y}_{n'}$ be the number of signals with value 2 among the n' jurors and note that the full information decision depends only on $\widetilde{y}_{n'}$. Denote by $\mathbf{d}^*_{n'}$ the full information decision rule with n' jurors. Then $\mathbf{d}^*_{n'} = C$ iff $\widetilde{y}_{n'} \geq y_{n'}$, for some $y_{n'} \in \{0, ..., n' + 1\}$ satisfying

$$\widehat{\lambda}(n', y_{n'} - 1) \leq q \leq \widehat{\lambda}(n', y_{n'}) \tag{25}$$

where, $\widehat{\lambda}(z, y) = \lambda(z, y)$ if $0 \leq y \leq z$, equal to 0 if $y < 0$ and equal to 1 if $y > z$. Consider now the full information problem with an additional $(n' + 1)$th juror and note that $V^*(n' + 1) \geq V^*(n')$ as the decision rule $\mathbf{d}^*_{n'}$ is also a feasible rule with n' jurors and the $(n' + 1)$-th juror's information is ignored.

Suppose that $V^*(n' + 1) = V^*(n')$. We must have $\lambda(n' + 1, y_{n'}) = q$. For if $\lambda(n' + 1, y_{n'}) < q$ so that $y_{n'} < n' + 1$ (respectively, $\lambda(n' + 1, y_{n'}) > q$ so that $y_{n'} > 0$), it will be strictly better to choose a rule that differs from $\mathbf{d}^*_{n'}$ only in picking the decision A (resp., C) as opposed to the decision $\mathbf{d}^*_{n'} = C$ (resp., $\mathbf{d}^*_{n'} = A$) , when $\widetilde{y}_{n'} = y_{n'}$ (resp., $\widetilde{y}_{n'} = y_{n'} - 1$) and the last juror has a signal equal to 1 (resp. 2).

Conversely, suppose $\lambda(n'+1, y) = q$ with $y \in \{0, ..., n'+1\}$. If $y = 0$ (resp., $y = n'+1$), then choosing C (resp. A) always is an optimal rule for both the size $n' + 1$ and the size n' jury so that $V^*(n' + 1) = V^*(n')$. So suppose $0 < y < n' + 1$ and note that the optimal rule $\mathbf{d}^*_{n'}$ for the size n' jury must have $y_{n'} = y$. Further, if for the size $n' + 1$ jury, the information of the last juror is ignored and $\mathbf{d}^*_{n'}$ is used, then when there are y (respectively, $y-1$) signals with value 2 of the first n' jurors and the $(n'+1)$-th juror has a signal equal to 1 (resp., 2) the decision C (resp., A) is chosen. Since $\lambda(n' + 1, y) = q$, such a decision is weakly optimal. In all other cases, choosing according to $\mathbf{d}^*_{n'}$ is strictly optimal. Thus, $V^*(n' + 1) = V^*(n')$.

(2) Let σ_J be a pure strategy equilibrium with $|J_c(\sigma_J)| < k$ and $|J_a(\sigma_J)| \leq n - k$ so that $|J_i(\sigma_J)| \geq k - |J_c(\sigma_J)|$ and $piv_j \neq \emptyset$ for all j. By Lemma 1, σ_J satisfies the cutoff property so that $\sigma_j(2) = 1$ and $\sigma_j(1) = 0$ for all $j \in J_i(\sigma_J)$. For $j \in J_i(\sigma_J)$ the event piv_j occurs when exactly $k - |J_c(\sigma_J)| - 1$ of the other $|J_i(\sigma_J)| - 1$ other jurors in $J_i(\sigma_J)$ have a signal equal to 2. From (11) we immediately obtain the left–hand side of (23) by considering the case where $T_j = 1$ and the right–hand side by considering the case where $T_j = 2$. Moreover if $J_c(\sigma_J) \neq \emptyset$, then for $j \in J_c(\sigma_J)$ the event piv_j occurs when exactly $k-|J_c(\sigma_J)|$ of jurors in $J_i(\sigma_J)$ have a signal equal to 2. From (11) we immediately obtain (24.1) by considering the case where $T_j = 1$. Similarly, if $J_a(\sigma_J) \neq \emptyset$ then from (11) for $j \in J_a(\sigma_J)$ and $T_j = 2$ we obtain (24.2).

Conversely, suppose that σ_J satisfies the cutoff property and that (23) and (24) hold. We show that σ_J is an equilibrium. Consider $j \in J_i(\sigma_J)$. Since σ_J satisfies the cutoff property and all jurors in $J_c(\sigma_J)$ (resp. $J_a(\sigma_J)$) vote for conviction (resp. acquittal) regardless of their signals, j knows that his vote matters (i.e., piv_j occurs) only when $k - |J_c(\sigma_J)| - 1$ of the $|J_i(\sigma_J)| - 1$ other jurors have a signal 2 and have voted (according to σ_J) for conviction. Since this event has strictly positive probability by (3), he would prefer to vote for conviction as long as $P(S = G|piv_j, T_j = t_j) \geq q$ and vote for acquittal otherwise. From (23) we conclude that $P(S = G|piv_j, T_j = 1) \leq q$ and $P(S = G|piv_j, T_j = 2) \geq q$ so that j would prefer to vote according to σ_J given that others are doing so. Furthermore, if $J_c(\sigma_J) \neq \emptyset$, then $j \in J_c(\sigma_J)$ knows that piv_j occurs only when $k - |J_c(\sigma_J)|$ jurors in $J_i(\sigma_J)$ have a signal 2 and have voted (according to σ_J) for conviction. Since this event has strictly positive probability by (3), he would prefer to vote for conviction as long as $P(S = G|piv_j, T_j = t_j) \geq q$. From (24.1) we conclude that this is true for $T_j = 1$ and so, by (9) when $T_j = 2$, so that j would prefer to vote according to σ_J. Similarly, if $J_a(\sigma_J) \neq \emptyset$, then considering $j \in J_a(\sigma_J)$ and using (24.2) it follows that such a juror would like to vote according to σ_J. Thus σ_J is an equilibrium profile.

For the last part, suppose σ_J is an equilibrium so that it satisfies the cutoff property and (23) holds. Under σ_J, the decision C is chosen iff at least $k - |J_c(\sigma_J)|$ signals of the $|J_i(\sigma_J)|$ signals of jurors in $J_i(\sigma_J)$ take the value 2. Comparing (23) with (25) we see that this is an optimal decision rule in the full information problem with $|J_i(\sigma_J)|$ jurors so that $U(\sigma_J, \mathbf{k}) = V^*(|J_i(\sigma_J)|)$. Finally, note that $|J_i(\sigma_J)| \leq n^*$ trivially if $n^* = n$. So suppose that $n^* = n_a^* < n$. Then $\lambda(k,k) \geq q$. For if not, then $n_a^* = 0$ as there exists no integer n' for which $\lambda(n', k) \geq q$. But from (23),

$$q \leq (|J_i(\sigma_J)|, k - |J_c(\sigma_J)|) \leq \lambda(|J_i(\sigma_J)| + |J_c(\sigma_J)|, k)$$

a contradiction. Thus, $n_a^* \geq k$ satisfies (13). If $|J_i(\sigma_J)| > n_a^*$, we obtain using the right–hand side of (23)

$$q \leq \lambda(|J_i(\sigma_J)|, k - |J_c(\sigma_J)|) \leq \lambda(|J_i(\sigma_J)|, k) \leq \lambda(n_a^* + 1, k)$$

contradicting the definition of n_a^* as the largest integer that satisfies (13). The proof for the case where $n^* = n_c^* < n$ is identical and so omitted. ■

Proof of the Proposition (parts (2) and (3))

We show first that any $\sigma_J \in \Sigma_J^*$ is an equilibrium profile and $U(\sigma_J, \mathbf{k}) = V^*(n^*)$. Consider first the case where $\sigma_J \in \Sigma_J^*$ is such that $|J_i(\sigma_J)| = n^*$. To begin with suppose that $\lambda(n - k + 1, 0) \leq q \leq \lambda(k, k)$ so that $n_a^* \geq k$ and $n_c^* \geq n - k + 1$. If $n^* = n$ then $|J_c(\sigma_J)| = 0 = |J_a(\sigma_J)|$ and further, by (16), $\lambda(n, k-1) \leq q \leq \lambda(n, k)$. Comparing with (23), by Lemma 4.2 we conclude that σ_J is an equilibrium and $U(\sigma_J, \mathbf{k}) = V^*(n)$. Next, if $n^* = n_a^*$ then $|J_c(\sigma_J)| = 0$ and $|J_a(\sigma_J)| = n - n_a^*$. Since n_a^* is the largest integer that satisfies (13), $\lambda(n_a^*, k-1) < q \leq \lambda(n_a^*, k)$ and further $\lambda(n_a^* + 1, k) < q$. Comparing these with (23) and (24.2) respectively, by Lemma 4.2 we conclude that σ_J is an equilibrium and $U(\sigma_J, \mathbf{k}) = V^*(n_a^*)$. The case where $n^* = n_c^* < n$ is identical and so its proof is omitted.

Next suppose $\lambda(k,k) < q$ so that $n^* = n_a^* = 0 = |J_i(\sigma_J)| = |J_c(\sigma_J)|$ and $|J_a(\sigma_J)| = n$. Since $\lambda(1,1) \leq \lambda(k,k) \leq q$, each juror would prefer to acquit so that σ_J is an equilibrium and the jury always acquits. Using $\lambda(k,k) \leq q$ and (9) we observe that the payoff from always acquitting is greater than the payoff from always convicting so that $U(\sigma_J, \mathbf{k}) = V^*(0)$. The proof for the case $\lambda(n-k+1, 0) \geq q$ and $n^* = n_c^* = 0$ is identical and so omitted.

Finally, consider $\sigma_J \in \Sigma_J^*$ such that $|J_i(\sigma_J)| = n^* - 1$. Then, by the definition of Σ_J^*, either $n^* = n_a^* > k$ and $\lambda(n_a^*, k) = q$, or $n^* = n_c^* > n-k+1$ and $\lambda(n_c^*, n_c^* - (n-k)-1) = q$. In the former case, $|J_c(\sigma_J)| = 0$ and $|J_a(\sigma_J)| = n - n_a^* + 1$. Furthermore, since $n_a^* > k$ and $\lambda(n_a^*, k) = q$, $\lambda(n_a^* - 1, k)$ is well–defined and $\lambda(n_a^* - 1, k-1) < q < \lambda(n_a^* - 1, k)$. Comparing this and $\lambda(n_a^*, k) = q$ with (23) and (24.2) respectively, by Lemma 4.2 we conclude that σ_J is an equilibrium and $U(\sigma_J, \mathbf{k}) = V^*(n_a^* - 1)$. Further, since $\lambda(n_a^*, k) = q$, by Lemma 4.1 we see $U(\sigma_J, \mathbf{k}) = V^*(n_a^* - 1) = V^*(n_a^*)$. The proof for the case where $n^* = n_c^* > n - k + 1$ and $\lambda(n_c^*, n_c^* - (n-k) - 1) = q$ is identical and so omitted.

To complete the proof of part (2) of the Proposition, it remains to show if $\sigma_J \in \Sigma_J^*$ then it is an *efficient* equilibrium. Note that if any pure strategy profile σ_J has $|J_a(\sigma_J)| \geq k$ or $|J_a(\sigma_J)| > n-k$ then either it always chooses the decision C or always the decision A. In either case the payoff obtained is at most equal to $V^*(0)$. Since $U(\sigma_J, \mathbf{k}) = V^*(n^*)$ for all $\sigma_J \in \Sigma_J^*$, it immediately follows by Lemma 4.2 and Theorem 1 that any $\sigma_J \in \Sigma_J^*$ is an efficient equilibrium and further that $V(n, \mathbf{k}) = V^*(n^*)$. This completes the proof of parts (2) and (3).

Proof of the Proposition (part (1))

Suppose that $\lambda(n-k+1, 0) < q < \lambda(k,k)$ so that $n_a^* \geq k$ and $n_c^* \geq n-k+1$ and $n^* > 0$. Let σ_J be an efficient pure strategy equilibrium. We wish to show that $\sigma_J \in \Sigma_J^*$.

Note first that we must have $|J_a(\sigma_J)| \leq n-k$ and $|J_c(\sigma_J)| < k$. For if $|J_a(\sigma_J)| > n-k$, then the jury votes to always acquit. Consider the strategy profile σ_J' with the property that a subset of exactly k voters vote informatively i.e., with $\sigma_j(2) = 1$ and $\sigma_j(1) = 0$, with the rest voting for acquittal. Under σ_J' the jury votes for conviction iff exactly k out of the k informative voters have a signal 2. Since all signal profiles have positive probability and $\lambda(k,k) > 0$ we conclude that the jury does strictly better under σ_J' than

σ_J, so that σ_J cannot be an efficient equilibrium. Similarly one obtains $|J_c(\sigma_J)| < k$. By Lemma 4.2 we see that $|J_i(\sigma_J)| \leq n^*$ and $U(\sigma_J, \mathbf{k}) = V^*(|J_i(\sigma_J)|)$.

If $V^*(n^*) > V^*(n^* - 1)$ then $|J_i(\sigma_J)| = n^*$ as otherwise any $\sigma_J^* \in \Sigma_J^*$ would dominate it, by parts (2) and (3) of Proposition 1. Furthermore, in such a case we must have $J_c(\sigma_J) = \emptyset$ if $n^* = n_a^*$ and $J_a(\sigma_J) = \emptyset$ if $n^* = n_c^*$. For if $n^* = n_a^*$ and $J_c(\sigma_J) \neq \emptyset$ then using Lemma 4.2 we obtain

$$q \leq \lambda(|J_i(\sigma_J)| + 1, k - |J_c(\sigma_J)|) = \lambda(n_a^* + 1, k - |J_c(\sigma_J)|) < \lambda(n_a^* + 1, k)$$

a contradiction with the definition of n_a^* as the largest integer that satisfies (13). Similarly, one shows that $J_a(\sigma_J) = \emptyset$ if $n^* = n_c^*$. We conclude that $\sigma_J \in \Sigma_J^*$ if $V^*(n^*) > V^*(n^* - 1)$.

So suppose that $V^*(n^*) = V^*(n^* - 1)$. Then $n \geq \min\{n_a^*, n_c^*)$. For if $n < \min\{n_a^*, n_c^*)$ then $n^* = n$ and, moreover, from (13) and (14),

$$\lambda(n, k-1) < \lambda(n_c^*, n_c^* - (n-k) - 1) \leq q \leq \lambda(n_a^*, k) < \lambda(n, k)$$

implying that there exists no $y \in \{0, ..., n\}$ such that $\lambda(n, y) = q$ contradicting, via Lemma 4.1, that $V^*(n^*) = V^*(n^* - 1)$. It follows that $n^* \in \{n_a^*, n_c^*\}$. Consider the case where $n^* = n_a^*$. From (13) and Lemma 4.1 we see that $\lambda(n_a^*, k) = q$. Since $\lambda(k, k) > q$ we must have $n_a^* > k$. Furthermore,

$$\lambda(n_a^* - 1, k - 1) < q < \lambda(n_a^* - 1, k)$$

so that using Lemma 4.1 again we obtain $V^*(n_a^* - 1) > V^*(n_a^* - 2)$ as there does not exist $y \in \{0, ..., n_a^* - 1\}$ such that $\lambda(n_a^* - 1, y) = q$. But then $|J_i(\sigma_J)| \geq n_a^* - 1$, as otherwise any $\sigma_J^* \in \Sigma_J^*$ would dominate it, by parts (2) and (3) of Proposition 1. Hence, $|J_i(\sigma_J)| \in \{n_a^* - 1, n_a^*\}$. Moreover, we must have $J_c(\sigma_J) \neq \emptyset$, for if not, using Lemma 4.2 we obtain

$$q \leq \lambda(|J_i(\sigma_J)| + 1, k - |J_c(\sigma_J)|) \leq \lambda(n_a^*, k - |J_c(\sigma_J)|) < \lambda(n_a^*, k) = q$$

a contradiction. We conclude that $\sigma_J \in \Sigma_J^*$ when $V^*(n^*) = V^*(n^* - 1)$ and $n^* = n_a^*$. The proof for the case $n^* = n_c^*$ is identical and therefore omitted. ■

8 References

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