A stochastic game model for a FCFS queue with load-increasing service rate

Anthony C. Brooms, **Birkbeck College*

March 29, 2005

Abstract

This short note reviews some of results contained in Brooms [4] (to which the reader is directed for a fuller exposition of the results mentioned here). We consider customer joining behaviour for a *first come first served* single server queueing system where the service rate responds to changes in the queue size. Customers are prepared to join the system only if their expected time there is projected to be not too high. In order to make an assessment as to whether or not quality of service requirements will be met upon joining such a system, assumptions regarding the form of the joining decisions taken by other customers need to be taken into account. The joining decision that is to be implemented by each customer is taken on the basis of quantities which may depend on the number of customers observed on arrival to the system.

We treat this problem as an infinite-player non-cooperative (stationary) game, where the expected sojourn times at particular entry states are considered. The aim is to characterize the conditions under which Nash equilibrium joining policies exist, and to explore the structure of such policies.

The seminal and most relevant work on the game-theoretic analysis of this class of queueing system was carried out by Altman and Shimkin [2], in which a processor sharing system was investigated. They established the existence and uniqueness of a symmetric Nash equilibrium joining policy for the stationary game; it was demonstrated via simulation methods (and, in [1], using the theory of the Stochastic Approximations algorithm) that it can be used to characterize the convergent behaviour of the system when customers base their joining decisions on a certain class of dynamic learning rule. Some of the game-theoretic results for the processor sharing system were later extended, in Ben-Shahar et. al. [3], for the class-heterogeneous case.

We take $\mathbb{Z}^+ = \{1, 2, \ldots\}$, and $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$ in our discussion.

An arriving customer has to choose between either joining a shared service system, which is a FCFS queue (denoted by Q_S), or balking.

It is assumed that Q_S has a buffer size B, which may be finite or infinite. Any customer which arrives when the buffer is full is not permitted to enter the system.

*Postal address: School of Economics, Mathematics, & Statistics, Birkbeck College, Malet Street, London WC1E 7HX, U.K.

The departure process in Q_S at queue length x forms a Poisson process of rate $\mu(x)$, where $\mu(x)$ is a *strictly increasing* and *bounded* function on $x \in \{1, 2 \ldots, B\}$, with $\mu(0) = 0$. Set $\overline{\mu} = \sup\{\mu(x) : x = 1, 2, \ldots\}$.

Let θ be the quality of service requirement. If an arriving customer perceives that the expected sojourn time in Q_S is greater than this value, then it will be reluctant to enter the system.

It is assumed that $\mu(1)^{-1} < \theta$. This condition ensures that it is always worthwhile for a customer to enter Q_S if the system is empty upon arrival.

Let A_k be the arrival time of the k-th customer at (although not necessarily into) the system, where $0 = A_0 < A_1 < A_2 < \ldots$; denote this k-th customer by the label C_k, $k \in \mathbb{N}$, where it is assumed that C_0 arrives at time A_0 (i.e. at time 0).

A *decision rule*, $u(\cdot) : \{0, 1, \ldots, B{-}1\} \longmapsto [0, 1]$, is defined to be a function that specifies the probability that a customer adhering to it enters Q_S, which is equal to $u(x)$ if the number of customers in Q_S is equal to x just prior to its arrival. The decision rule for C_k is represented by $u_k(\cdot)$, and the collection of decision rules used by each customer, a *policy*, is given by $\pi = (u_0(\cdot), u_1(\cdot), u_2(\cdot), \ldots, \ldots)$.

Let $v_k(x, \pi)$, $x \in \{0, 1, \ldots, B-1\}$, be the sojourn time of C_k in Q_S, given that x customers were present in Q_S just prior to its arrival, and that any customer arriving in the future adheres to its decision rule inferred by π. Further define $V_k(x, \pi)$ to be the expected value of $v_k(x, \pi)$.

A decision rule $u_k(\cdot)$ for the k-th customer is said to be *optimal* against the policy π if $u_k(x) = \mathbf{1}\{V_k(x, \pi) < \theta\} + q_x \mathbf{1}\{V_k(x, \pi) = \theta\}$ for $x \in \{0, 1, \ldots, B-1\}$ and arbitrary $q_x \in [0, 1]$, where $\mathbf{1}\{\cdot\}$ is the indicator function.

A policy $\pi = (u_0(\cdot), u_1(\cdot), u_2(\cdot), \ldots)$ is said to be a Nash equilibrium policy if, for every $k \in \mathbb{N}$, the decision rule of the k-th customer, $u_k(\cdot)$, is optimal against π.

Let $\mathbb{T}^\infty$ be the class of policies in which the decision rule for each customer is a non-increasing function of $x \in \{0, 1, \ldots, B-1\}$.

The following result describes the behaviour of the sojourn time, for each customer, as the entry queue length, x, varies.

Lemma 1 *For all $\pi \in \mathbb{T}^\infty$ and $k \in \mathbb{N}$, $V_k(x, \pi)$ is strictly increasing in x, in the sense that for $x \in \{0, 1, \ldots, B-2\}$,*

$$V_k(x+1, \pi) - V_k(x, \pi) \geq \delta_x > 0$$

uniformly in π.

For $L \in \mathbb{N}$ and $q \in [0, 1)$, an $[L, q]$-threshold decision rule, $u(\cdot)$, takes the form $u(x) = \mathbf{1}\{x < L\} + q\mathbf{1}\{x = L\}$ for $x \in \{0, 1, \ldots, B{-}1\}$, where $\mathbf{1}\{\cdot\}$ is the indicator function. This may be represented more compactly by $[L, q]$, or indeed $[g]$, where $g = L + q$. Of course, for $B < \infty$, $[B]$ is equivalent to $[g]$ whenever $g > B$.

A policy π in which the decision rule for each customer is given by $[g]$, is denoted by $[g]^\infty$: we call this a *symmetric threshold policy.*

The next two lemmas describe the behaviour of the sojourn time, for each customer, and for each entry queue length, as a function of g.

Lemma 2 *For each $k \in \mathbb{N}$, and $x \in \{0, 1, \ldots, B-1\}$*
(i) $V_k(x, [g]^\infty)$ is constant in g on $[0, 1]$;
(ii) $V_k(x, [g]^\infty)$ is strictly decreasing in g on $[1, B]$.

Lemma 3 *Suppose that the inter-arrival times are exponential. Then for every $k \in \mathbb{N}$, $x \in \{0, 1, \ldots, B-1\}$, $V_k(x, [g]^\infty)$ is continuous in g on $[0, B]$.*

A *symmetric Nash equilibrium policy* (SNEP) is a policy which i) consists of identical decision rules for each and every customer, ii) is a Nash equilibrium.

The proof of the following theorem is constructed by invoking the previously stated lemmas.

Theorem *Suppose that the inter-arrival times are exponential. Then in the class of policies $\mathbb{T}^\infty$,*
(i) there exist a finite number of SNEPs;
(ii) at least one of the SNEPs is characterized by a non-randomized threshold.

It has been argued that Nash equilibria can give an indication of the likely 'operating points' of these kinds of systems. This claim is explored for our system using a simulation procedure similar to that proposed in [2]. Consider a dynamic learning scheme in which each customer bases their joining decision on data collected by a central entity prior to its arrival to the system. Here it is assumed that the buffer size, B, is finite, and that a proportion, $p > 0$, of the arrivals always enter the system whenever there is room in the buffer. All other arriving customers follow the decision rule

$$\mathrm{join\ Q_S\ with\ probability\ S_\epsilon(\theta - \hat{V}_t(X_t))}, \tag{1}$$

where ϵ is a small positive parameter, S_ϵ is an increasing function, with $S_\epsilon(x) = 0$ for $x \leq -\epsilon$ and $S_\epsilon(x) = 1$ for $x \geq \epsilon$, $\hat{V}_t(x)$ is the empirical average (sample mean) sojourn time of *all* customers who have exited Q_S by time t, but who entered it when the queue length was x, and X_t represents the number in Q_S just before time t.

Simulation experiments suggest that when a unique (non-randomized) SNEP of the stationary game exists, in the class $\mathbb{T}^\infty$, then quantities such as the empirical averages and simulated entrance probabilities, under the learning rule, show a close correspondence to expected sojourn times and entrance probabilities under the SNEP in the associated stationary game. Convergence and stability properties in the case of multiple Nash equilibria are currently under investigation; however, simulation experiments appear to suggest that the SNEPs are viable poles of attraction and still provide a rough guide to the operating points of the system.

References

[1] E. Altman and N. Shimkin. Individual Equilibrium and Learning in a Processor Sharing System. Technical report, INRIA, July 1996.

[2] E. Altman and N. Shimkin. Individual Equilibrium and Learning in Processor Sharing Systems. *Operations Research*, **46**(6):776–784, 1998.

[3] I. Ben-Shahar, A. Orda, and N. Shimkin. Dynamic Service Sharing with Heterogeneous Preferences. *Queueing Systems*, **35**:83–103, 2000.

[4] A. C. Brooms. On the Nash equilibria for the FCFS queueing system with load-increasing service rate. To appear in *Advances in Applied Probability*, 2005.