An alternative characterization of the nucleolus in airport problems

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Abstract

We consider the problem of sharing the cost of a public facility among agents who have different needs for it. We show that the nucleolus is the only rule satisfying efficiency, equal treatment of equals, last-agent cost additivity, and last-agent consistency. Our result reveals the importance of the last agent in characterizing the rule and generalizes that of Potters and Sudhölter (1999). *Journal of Economic Literature* Classification Numbers: C71; D30; D63.

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1 Introduction

We consider a class of cost sharing problems in which agents are ordered in terms of their needs for a public facility, and satisfying a given agent

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implies satisfying all agents with smaller needs than his. An example is the so-called airport problem: different airlines need airstrips of different lengths. The larger a plane, the longer the airstrip it needs. Serving a given plane implies serving all smaller planes. To accommodate all planes, the airstrip must be long enough for the largest plane. A "rule" is a function that associates an allocation of the total cost of the airstrip, called a "contribution vector," with each airport problem. A number of properties of good behavior of rules have been formulated from different perspectives such as fairness criteria or operational principles. The literature devoted to the search for existence of rules satisfying these properties, singly and in various combinations, is initiated by Littlechild and Owen (1973).[1]

To any solution defined on the class of TU games can be adapted to airport problems. First, we transform an airport problem into a TU game, called an "associated airport problem," by defining the worth of each coalition as the cost of satisfying the agent with the largest need in that coalition. We then apply a TU game solution to solve the associated airport game. This yields a payoff vector for the game. Finally, the contribution vector for the airport problem is determined by taking the inverse image of this payoff vector. A well-known example is the "nucleolus" (Schmeidler, 1969), which lexicographically maximizes the "welfare" of the worst-off coalitions.

When the nucleolus is adapted to airport problems, it satisfies the following properties. To condense the verbal definitions of the properties, suppose that there are n agents indexed by $\{1, \ldots, n\}$, agent i's cost parameter c_i represents the cost of satisfying his need, and these cost parameters are ordered as $c_1 \leq \cdots \leq c_n$. Agent n is referred to as the "last agent." The first property is *equal treatment of equals*: agents with equal cost parameters should contribute equal amounts. Second is *homogeneity*: if all cost parameters are multiplied by the same positive number, so should the contribution vector. Third is *others-oriented cost additivity*:

[1]For a survey of this literature, see Chun and Thomson (2001).

if c_n increases by some positive amount δ, then the contribution of the last agent should increase by δ, and all other agents should contribute the same amounts as they did initially.[2] Last is "cost complement consistency,"[3] which is the expression for airport problems of a general principle of "consistency."[4] Applying this principle requires that certain conceptual ideas be discussed. We describe them as follows.

Consider a problem and suppose that a contribution vector x is chosen for it. Then, imagine that agent i pays his contribution x_i and leaves, and reassess the situation from the viewpoint of the remaining agents. Instead of thinking of x_i as covering an abstract part of c_n, it is natural to impute it to the cost parameters of the remaining agents. The question is how these imputations are calculated. One way to do so is to consider x_i as being used to cover c_i. Thus, agent j with $c_j \geq c_i$ benefits by x_i no matter which part of c_i is covered first. Agent j's revised cost parameter is then $c_j - x_i$. What if $c_j < c_i$? Suppose that x_i is mainly intended to cover $c_i - c_j$. In this case, agent j may or may not benefit from agent i's contribution. It depends on the difference between x_i and $c_i - c_j$. If $x_i \leq c_i - c_j$, agent j does not benefit; otherwise, his benefit is $x_i - (c_i - c_j)$. Thus, his revised cost parameter is defined as the difference between c_j and the maximum of $x_i - (c_i - c_j)$ and zero. In other words, it is the minimum of $c_i - x_i$ and c_j. *Cost complement consistency* says that the components of x pertaining to the remaining agents should still be chosen for the problem just defined.

Sönmez (1994) provides a "recursive" formula for the contribution vector chosen by the nucleolus. In the formula, the contribution of the last agent is defined as the difference between his cost parameter and the sum of the contributions of all other agents. Note that the rule satisfies *cost complement consistency.* If we restrict attention to the departure of the last agent, each agent becomes the last agent after several reductions.

[2]Potters and Sudhölter (1999) combine *others-oriented cost additivity* and *homogeneity* as a property, referred to as *covariance.*

[3]Potters and Sudhölter (1999) refer to it as *ν-consistency.*

[4]For a survey of the literature on *consistency* and its converse, see Thomson (2000).

Thus, the contribution of the last agent determines the entire contribution vector. We ask whether the rule can be characterized on the basis of properties that concern the contribution of the last agent.

We weaken *others-oriented cost additivity* to "last-agent cost additivity" by dropping the requirement that all other agents should contribute the same amounts as they did initially, and *cost complement consistency* to "last-agent consistency" by restricting attention to the last agent. Besides, we replace *homogeneity* with *efficiency*: the sum of all contributions should be equal to the cost parameter of the last agent. We show that the nucleolus is the only rule satisfying *efficiency*, *equal treatment of equals*, *last-agent cost additivity*, and *last-agent consistency* (Theorem 1).

The result is appealing on two accounts. First, in contrast to other models of fair allocation for which the departure of a particular agent does not play any role, our result points out the importance of the last agent in characterizing the nucleolus. If *last-agent consistency* is replaced with "first-agent consistency," obtained by restricting attention to the first agent (the agent with the smallest cost parameter), the nucleolus is no longer the only acceptable rule. An alternative is the "sequential equal contributions" rule (Littlechild and Owen, 1973), whose contributions vector is identical to that prescribed by the Shapley value (Shapley, 1953) of the associated airport game.[5] Second, Potters and Sudhölter (1999) show that the nucleolus is the only rule satisfying *equal treatment of equals*, *homogeneity*, *other-oriented cost additivity*, and *cost complement consistency*. Note that *homogeneity*, *others-oriented cost additivity*, and *cost complement consistency* altogether imply *efficiency*. In addition, *others-oriented cost additivity* and *cost complement consistency* are stronger than *last-agent cost additivity* and *last-agent consistency*, respectively. Thus, our result generalizes that of Potters and Sudhölter.

The rest of this paper proceeds as follows. Section 2 introduces the model, the nucleolus, and the properties. Sections 3 and 4 present the main

[5] Aadland and Kolpin (1998) refer to it as the "serial cost-share" rule. We follow Chun and Thomson (2001)'s terminology.

result and show the independence of the properties. Section 5 concludes.

2 Notation and definitions

There is an universe of "potential" agents, denoted by $\mathcal{I} \subsetneqq \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers. Let $\mathcal{N}$ be the class of finite subsets of $\mathcal{I}$. Given $N \in \mathcal{N}$ and $i \in N$, let $c_i \in \mathbb{R}_+$ be **agent *i*'s cost parameter** and $c \equiv (c_i)_{i \in N}$ the profile of cost parameters.[6] An airport problem for N, or simply a **problem for *N***, is a list $c \in \mathbb{R}_+^N$.[7] Let $\mathcal{C}^N$ be the class of all problems for N. A **contribution vector** for $c \in \mathcal{C}^N$ is a vector $x \in \mathbb{R}^N$. Let $X(c)$ be the set of all contribution vectors for $c \in \mathcal{C}^N$. A **rule** is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ that associates with each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$ a vector in $X(c)$. Let S be our generic notation for rules. For simplicity, we assume that $N \equiv \{1, \ldots, n\}$ and $c_1 \leq \cdots \leq c_n$ where n is the cardinality of N. Thus, agent n is referred to as the "last agent." For each coalition $N' \subset N$, we denote $(c_i)_{i \in N'}$ by $c_{N'}$, $(S_i(c))_{i \in N'}$ by $S_{N'}(c)$, and so on.

We now introduce the nucleolus. Since the contribution vector chosen by the rule is calculated by a sequence of linear programs, it is not easy to compute in general. However, Littlechild (1974) and Sönmez (1994) provide explicit formulae. For our purpose, we adopt Sönmez's formula.

Nucleolus, *Nu*: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$,

$$\begin{aligned} Nu_0(c) &\equiv 0 \\ Nu_i(c) &\equiv \min\left\{\frac{c_k - \sum_{p=0}^{i-1} Nu_p(c)}{k-i+2} \mid i \leq k \leq n-1\right\} \quad \text{where } 1 \leq i \leq n-1 \\ Nu_n(c) &\equiv c_n - \sum_{p=0}^{n-1} Nu_p(c). \end{aligned}$$

The rule satisfies the following properties. First, each agent should

[6] By $\mathbb{R}_+$ we denote the set of real numbers, $\mathbb{R}_+ \equiv \{x \in \mathbb{R} \mid x \geq 0\}$.

[7] By $\mathbb{R}_+^N$ we denote the Cartesian product of $|N|$ copies of $\mathbb{R}_+$, indexed by the elements of N.

contribute a non-negative amount and at most as much as his cost parameter.

Reasonableness: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $0 \leqq S(c) \leqq c$.[8]

Next are the properties informally defined in the introduction.

Efficiency: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{i \in N} S_i(c) = \max_{i \in N} c_i$.

Equal treatment of equals: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $S_i(c) = S_j(c)$.

Last-agent cost additivity: For each $N \in \mathcal{N}$, each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$, and each $\delta \in \mathbb{R}_+$, if $c'_n = c_n + \delta$ and for each $j \in N \backslash \{n\}$, $c'_j = c_j$, then $S_n(c') = S_n(c) + \delta$.

A variable-population property follows. To introduce it, we need the following notation. Let $N \in \mathcal{N}$, $c \in \mathcal{C}^N$, $i \in N$, and $x \in X(c)$. The **reduced problem of c with respect to $N' \equiv N \backslash \{i\}$ and x**, $r^x_{N'}(c)$, is defined by

(i) for each $j \in N'$ such that $c_j < c_i$, $\left(r^x_{N'}(c)\right)_j \equiv \min\{c_j, c_i - x_i\}$, and

(ii) for each $j \in N'$ such that $c_j \geq c_i$, $\left(r^x_{N'}(c)\right)_j \equiv c_j - x_i$.

Cost complement consistency: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv S(c)$, then $r^x_{N'}(c) \in \mathcal{C}^{N'}$ and $x_{N'} = S\left(r^x_{N'}(c)\right)$.

A weaker version of *cost complement consistency* is obtained by restricting attention to the last agent.[9]

[8] Vector inequalities: $x \geqq y$, $x \geq y$, and $x > y$.

[9] Our definition of the "last-agent reduced problems" is the same as the one used by Potters and Sudhölter (1999). Intuitively, the definition makes sense only when $x_n \geq c_n - c_{n-1}$. Although it is more appropriate to define such reduced problems only when $x_n \geq c_n - c_{n-1}$, our result does not change essentially even if we use this alternative definition.

Last-agent consistency: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, if $x \equiv S(c)$, then $r^x_{N\backslash\{n\}}(c) \in \mathcal{C}^{N\backslash\{n\}}$ and $x_{N\backslash\{n\}} = S\left(r^x_{N\backslash\{n\}}(c)\right)$.

Remark: *Efficiency* and *last-agent consistency* together imply *reasonableness.*

3 The result

We begin with a lemma. It says that the nucleolus satisfies the following monotonicity property: if all cost parameters increase in such a way that of two agents, the larger cost parameter increases by at least as much as the smaller one, then the last agent should contribute at least as much as he did initially.[10]

Last-agent marginal cost monotonicity: For each $N \in \mathcal{N}$ and each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$, if for each pair $\{i, j\} \subseteq N$ with $i < j$, $0 \leq c'_i - c_i \leq c'_j - c_j$, then $S_n(c) \leq S_n(c')$.

Lemma 1 The nucleolus satisfies *last-agent marginal cost monotonicity*.

Proof. Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Without loss of generality, we assume that $N \equiv \{1, \ldots, n\}$ and $c_1 \leq \cdots \leq c_n$. Let $c' \in \mathcal{C}^N$ be such that $0 \leq c'_1 - c_1 \leq \cdots \leq c'_n - c_n$. Let $x \equiv Nu(c)$ and $y \equiv Nu(c')$. We show that $x_n \leq y_n$. The proof is in two steps. Step 1 shows that for each agent $j \in N\backslash\{n\}$, the sum of the contributions of those agents whose cost parameters are at most as large as agent j's increases by at most $c'_{n-1} - c_{n-1}$. For each $i \in N$, let $\delta_i \equiv c'_i - c_i$. Thus, $0 \leq \delta_1 \leq \cdots \leq \delta_n$.

Step 1: **For each $\boldsymbol{j \in \{1, \ldots, n-1\}}$, $\boldsymbol{0 \leq \sum_{k=1}^{j} y_k - \sum_{k=1}^{j} x_k \leq \delta_{n-1}}$.**

The proof is by induction on j.

[10] Aadland and Kolpin (1998) define a stronger version of *last-agent marginal cost monotonicity* under the name of "cost monotonicity" obtained by adding the requirement that all other agents should also contribute at least as much as they did initially.

Case 1: $j = 1$. Note that for each $j \in \{1, \ldots, n-1\}$, $j\delta_{n-1} \geq 0$. Then the formulae for y_1 and x_1 give

$$\begin{aligned} y_1 &\equiv \min\left\{\frac{c_1 + \delta_1}{2}, \cdots, \frac{c_{n-1} + \delta_{n-1}}{n}\right\} \\ &\geq \min\left\{\frac{c_1}{2}, \cdots, \frac{c_{n-1}}{n}\right\} \\ &\equiv x_1, \end{aligned}$$

and

$$\begin{aligned} y_1 &\equiv \min\left\{\frac{c_1 + \delta_1}{2}, \cdots, \frac{c_{n-1} + \delta_{n-1}}{n}\right\} \\ &\leq \min\left\{\frac{c_1 + \delta_{n-1}}{2}, \cdots, \frac{c_{n-1} + \delta_{n-1}}{n}\right\} \\ &\leq \min\left\{\frac{c_1}{2}, \cdots, \frac{c_{n-1}}{n}\right\} + \delta_{n-1} \\ &\equiv x_1 + \delta_{n-1}. \end{aligned}$$

Thus, $0 \leq y_1 - x_1 \leq \delta_{n-1}$.

Case 2: $j > 2$. The induction hypothesis is that for each $j \in \{1, \ldots, t\}$, $0 \leq \sum_{k=1}^{j} y_k - \sum_{k=1}^{j} x_k \leq \delta_{n-1}$, where $t \in \mathbb{N}$ is such that $t < n-1$. We show that $0 \leq \sum_{k=1}^{t+1} y_k - \sum_{k=1}^{t+1} x_k \leq \delta_{n-1}$. Let $\beta_t \equiv \sum_{k=1}^{t} y_k$ and $\gamma_t \equiv \sum_{k=1}^{t} x_k$. Since $x \geqq 0$ and $y \geqq 0$, then $\beta_t \geq 0$ and $\gamma_t \geq 0$. By the induction hypothesis, $\gamma_t - \beta_t \leq 0$. It follows that for each $j \in \{t+1, \ldots, n-1\}$, $(j-t)(\gamma_t - \beta_t) \leq 0$. Note that for each $j \in \{t+1, \ldots, n-1\}$, $\delta_j \geq 0$. Then, the formulae for y_{t+1} and x_{t+1} give

$$\begin{aligned} y_{t+1} &\equiv \min\left\{\frac{c_{t+1} + \delta_{t+1} - \beta_t}{2}, \cdots, \frac{c_{n-1} + \delta_{n-1} - \beta_t}{n-t}\right\} \\ &\geq \min\left\{\frac{c_{t+1} - \beta_t}{2}, \cdots, \frac{c_{n-1} - \beta_t}{n-t}\right\} \\ &\geq \min\left\{\frac{c_{t+1} - \gamma_t}{2}, \cdots, \frac{c_{n-1} - \gamma_t}{n-t}\right\} + \gamma_t - \beta_t \\ &\equiv x_{t+1} + \gamma_t - \beta_t. \end{aligned}$$

Thus, $\sum_{k=1}^{t+1} y_k - \sum_{k=1}^{t+1} x_k \geq 0$.

We show next that $\sum_{k=1}^{t+1} y_k - \sum_{k=1}^{t+1} x_k \leq \delta_{n-1}$. Let $\alpha_t \equiv \gamma_t - \beta_t + \delta_{n-1}$. By the induction hypothesis, $\alpha_t \geq 0$. It follows that for each $j \in \{t+1, \ldots, n-1\}$, $(j-t)\,\alpha_t \geq 0$. Note that for each $j \in \{t+1, \ldots, n-1\}$, $0 \leq \delta_j \leq \delta_{n-1}$. Then, the formulae for y_{t+1} and x_{t+1} give

$$\begin{aligned} y_{t+1} &\equiv \min\left\{\frac{c_{t+1}+\delta_{t+1}-\beta_t}{2}, \frac{c_{t+2}+\delta_{t+2}-\beta_t}{3}, \ldots, \frac{c_{n-1}+\delta_{n-1}-\beta_t}{n-t}\right\} \\ &\leq \min\left\{\frac{c_{t+1}+\delta_{n-1}-\beta_t}{2}, \frac{c_{t+2}+\delta_{n-1}-\beta_t}{3}, \ldots, \frac{c_{n-1}+\delta_{n-1}-\beta_t}{n-t}\right\} \\ &\leq \min\left\{\frac{c_{t+1}-\gamma_t}{2}, \frac{c_{t+2}-\gamma_t}{3}, \ldots, \frac{c_{n-1}-\gamma_t}{n-t}\right\} + \alpha_t \\ &\equiv x_{t+1} + \gamma_t - \beta_t + \delta_{n-1}. \end{aligned}$$

It follows that $\sum_{k=1}^{t+1} y_k - \sum_{k=1}^{t+1} x_k \leq \delta_{n-1}$.

Step 2: Completion of the proof.

Note that $y_n \equiv c_n + \delta_n - \sum\limits_{k=1}^{n-1} y_k$ and that $x_n \equiv c_n - \sum\limits_{k=1}^{n-1} x_k$. By Step 1, $0 \leq \sum\limits_{k=1}^{n-1} y_k - \sum\limits_{k=1}^{n-1} x_k \leq \delta_{n-1}$. Since $\delta_{n-1} \leq \delta_n$, it follows that $x_n \leq y_n$.

Q.E.D.

With the help of Lemma 1, we are now ready to prove our main result.

Theorem 1 The nucleolus is the only rule satisfying *efficiency*, *equal treatment of equals*, *last-agent cost additivity*, and *last-agent consistency*.

Proof. Clearly, the nucleolus satisfies *efficiency* and *last-agent cost additivity.* It also satisfies *equal treatment of equals* and *last-agent consistency* (Potters and Sudhölter, 1999).

Conversely, let S be a rule satisfying the four properties. Let $N \in \mathcal{N}$, $c \in \mathcal{C}^N$, $x \equiv S(c)$, and $y \equiv Nu(c)$. Without loss of generality, we assume that $N \equiv \{1, \ldots, n\}$ and $c_1 \leq \cdots \leq c_n$. We show that $x = y$. The proof is by induction on $|N|$.

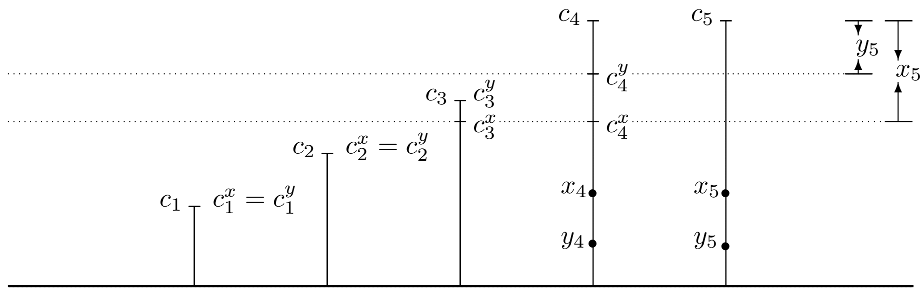

Figure 1: **Illustration of the proof of Theorem 1 for $|N| > 2$, $c_n = c_{n-1}$, and $x_n > y_n$.** Here $N \equiv \{1, \ldots, 5\}$, $x \equiv S(c)$, and $y \equiv Nu(c)$. If agent 5 pays x_5 and leaves. The reduced problem with respect to $N' \equiv \{1, \ldots, 4\}$ and x, $r^x_{N\setminus\{5\}}(c)$, is denoted by $(c_1^x, c_2^x, c_3^x, c_4^x)$. If the agent pays y_5 and leaves. The reduced problem with respect to $N' \equiv \{1, \ldots, 4\}$ and y, $r^y_{N\setminus\{5\}}(c)$, is denoted by $(c_1^y, c_2^y, c_3^y, c_4^y)$. In the figure, $c_1 = c_1^x = c_1^y$, $c_2 = c_2^x = c_2^y$, and $c_3 = c_3^y$.

Case 1: $|N| = 1$. By *efficiency*, $x = y$.

Case 2: $|N| = 2$. Let $N \equiv \{i, j\}$ and $c \in \mathcal{C}^N$. If $c_i = c_j$, then by *equal treatment of equals* and *efficiency*, $x = y$. If $c_i \neq c_j$, then without loss of generality, we assume that $c_i < c_j$. Let $c' \in \mathcal{C}^N$ be such that $c'_i = c_i$ and $c'_j = c_i$. By the previous argument, $S(c') = Nu(c')$. By *last-agent cost additivity*, $x_j = S_j(c') + c_j - c_i$ and $y_j = Nu_j(c') + c_j - c_i$. Thus, $x_j = y_j$. By *efficiency*, $x_i = y_i$.

Case 3: $|N| > 2$. By the induction hypothesis, suppose that for each $N' \subset N$ with $|N'| \leq |N| - 1$ and each $c^* \in \mathcal{C}^{N'}$, $S(c^*) = Nu(c^*)$. We first show that $x_n = y_n$. We distinguish two subcases.

Subcase 3.1: $c_n = c_{n-1}$. (Figure 1) Suppose, by contradiction, that $x_n \neq y_n$. Thus, either $x_n > y_n$ or $x_n < y_n$. If $x_n > y_n$, then by *equal treatment of equals*, $x_{n-1} = x_n$ and $y_{n-1} = y_n$. It follows that $x_{n-1} > y_{n-1}$. Let $c^x \equiv r^x_{N\setminus\{n\}}(c)$ and $c^y \equiv r^y_{N\setminus\{n\}}(c)$. Recall that $x_n > y_n$. Thus, for each $i \in N\setminus\{n\}$, $c_i^x \leq c_i^y$. Since $c_1 \leq \cdots \leq c_n$, then $c_1^x \leq \cdots \leq c_n^x$. Moreover, $c_1^y - c_1^x \leq \cdots \leq c_{n-1}^y - c_{n-1}^x$. By Lemma 1, the nucleolus satisfies *last-agent marginal cost monotonicity*. Thus, $Nu_{n-1}(c^y) \geq Nu_{n-1}(c^x)$. By *last-agent consistency*, $x_{N\setminus\{n\}} = S(c^x)$ and $y_{N\setminus\{n\}} = Nu(c^y)$. Thus, $Nu_{n-1}(c^y) = y_{n-1}$ and $S_{n-1}(c^x) = x_{n-1}$. Note that $|N\setminus\{n\}| = |N| - 1$.

By the induction hypothesis, $S(c^x) = Nu(c^x)$. Thus, $x_{n-1} = Nu_{n-1}(c^x)$. Recall that $Nu_{n-1}(c^y) \geq Nu_{n-1}(c^x)$, and that $Nu_{n-1}(c^y) = y_{n-1}$. It follows that $y_{n-1} \geq x_{n-1}$, in violation of $y_{n-1} < x_{n-1}$. If $x_n < y_n$, then by a similar argument, we derive the desired contradiction.

Subcase 3.2: $\boldsymbol{c_{n-1} < c_n}$. Let c' be such that for each $j \in N \setminus \{n\}$, $c'_j \equiv c_j$, and $c'_n \equiv c_{n-1}$. Thus, $c'_n = c'_{n-1}$. By Subcase 3.1, $S_n(c') = Nu_n(c')$. Note that for each $i \in N \setminus \{n\}$, $c_i = c'_i$, and that $c_n = c'_n + c_n - c_{n-1}$. By *last-agent cost additivity*, $S_n(c) = S_n(c') + c_n - c_{n-1}$ and $Nu_n(c) = Nu_n(c') + c_n - c_{n-1}$. Thus, $x_n = y_n$.

By *last-agent consistency* and the induction hypothesis, we conclude that $x = y$. *Q.E.D.*

4 Independence of properties

We now show that the properties listed in Theorem 1 are logically independent. For this purpose, we introduce additional rules.

- **Example 1** A rule that satisfies *equal treatment of equals*, *last-agent cost additivity*, and *last-agent consistency*, but not *efficiency*.

  We distinguish two cases. Let $\delta \in \mathbb{R}_+$, $N \in \mathcal{N}$, $c \in \mathcal{C}^N$, and $c(\delta) \equiv (c_1 - \delta, \ldots, c_n - \delta)$. If $c(\delta) \in \mathcal{C}^N$, then the rule chooses the contribution vector recommended by the nucleolus for the problem $c(\delta)$ instead of the problem c; otherwise, it chooses the contribution vector recommended by the nucleolus.

  Inefficient nucleolus, $\boldsymbol{INu^{\delta}}$: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$,

$$INu^{\delta}(c) \equiv \begin{cases} Nu(c(\delta)) & \text{if } c(\delta) \in \mathcal{C}^N, \\ Nu(c) & \text{otherwise.} \end{cases}$$

- **Example 2** A rule that satisfies *efficiency*, *last-agent cost additivity*, and *last-agent consistency*, but not *equal treatment of equals*.

  The rule makes the agent with the largest cost parameter, if there is a unique such agent, pay the entire cost; otherwise, the agent with the largest index among them pays the entire cost (Potters and Sudhölter, 1999). This rule just defined is the "free-rider" solution.

  Last-agent rule, ***LA***: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$,

$$LA_i(c) \equiv \begin{cases} 0 & \text{if } i \neq n, \\ c_n & \text{otherwise.} \end{cases}$$

- **Example 3** A rule that satisfies *efficiency*, *equal treatment of equals*, and *last-agent consistency*, but not *last-agent cost additivity*.

  The rule is defined as follows: imagine that all agents are ordered in terms of their cost parameters. To simplify our explanation, we assume that all cost parameters are different. For each agent i, let S_i be the group of agents whose cost parameters are at most as large as agent i's. The rule requires that each agent contributes equally until there is an agent j such that the equal amount is equal to $\frac{c_j}{|S_j|}$. Then, each agent in S_j contributes $\frac{c_j}{|S_j|}$ and leaves. For each agent not in S_j, his cost parameter is decreased by c_j. Continue by requiring equal contributions from all agents not in S_j until there is an agent k such that the amount is equal to $\frac{c_k - c_j}{|S_k \backslash S_j|}$. Then, each agent in $S_k \backslash S_j$ contributes $\frac{c_k - c_j}{|S_k \backslash S_j|}$ and leaves. Continue in this way until the cost of the project is entirely covered (Aadland and Kolpin, 1998). It can be shown that the rule is, in fact, the "egalitarian rule" (Dutta and Ray, 1989) of the "associated airport game"[11].[12]

[11] Given $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$, the associated airport game is a TU game $v(c) \in \mathbb{R}^{2^{|N|}-1}$ defined by setting for each $\varnothing \neq S \subsetneqq N$, $v(c)(S) \equiv \max_{i \in S} c_i$.

[12] Aadland and Kolpin (1998) refer to it as the "restricted average cost-share" rule.

Constrained equal contributions rule, ***CEC***: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$,

$$\begin{aligned} CEC_0(c) &\equiv 0 \\ CEC_i(c) &\equiv \min\left\{\tfrac{c_k - \sum_{p=0}^{i-1} CEC_p(c)}{k-i+1} \mid i \leq k \leq n\right\} \quad 1 \leq i \leq n-1 \\ CEC_n(c) &\equiv c_n - \sum_{p=0}^{n-1} CEC_p(c). \end{aligned}$$

- **Example 4** A rule that satisfies *efficiency*, *equal treatment of equals*, and *last-agent cost additivity*, but not *last-agent consistency*.

Imagine that all agents are ordered in terms of their cost parameters. Each agent first contributes an equal amount, and the sum of their contributions of all agents is equal to the smallest cost parameter. Then, each agent whose cost parameter is greater than the smallest cost parameter continues to contribute an equal amount and the sum of these additional contributions is equal to the difference between the second smallest cost parameter and the first cost parameter, and so on (Littlechild and Owen, 1973). The rule just defined is the "Shapley value" (Shapley, 1953) of the associated airport game.[13]

Sequential equal contributions rule, ***SEC***: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$,

$$SEC_i(c) \equiv \frac{c_1}{n} + \frac{c_2 - c_1}{n-1} + \cdots + \frac{c_i - c_{i-1}}{n-i+1}.$$

5 Conclusion

We provided an alternative characterization of the nucleolus for airport problems. Namely, it is the only rule satisfying *efficiency*, *equal treatment*

Chun and Thomson (2001) provide a explicit formula to calculate the contribution vector chosen by the rule, and refer to it as the "constrained equal contributions" rule. We follow their terminology.

[13]Chun and Thomson (2001) refer to it as the sequential equal contributions rule. We follow their terminology.

of equals, *last-agent cost additivity*, and *last-agent consistency*. Our result can be regarded as an axiomatic explanation of the importance of the last agent for the nucleolus. It would be interesting to investigate whether this importance can be understood from a non-cooperative viewpoint.

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