THE COMPETITIVE MULTIPLE-PARTNERS ASSIGNMENT GAME

By

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ABSTRACT

Multiple-partners assignment game is the name used by Sotomayor (1992, 1999) to describe the cooperative approach to the many-to-many matching market with separable and additive utilities. The competitive approach explores a new way of studying this game. The question is to know whether competitive equilibria always exist, and if so, how they can be obtained. One confirms their existence and proves that the minimum competitive equilibrium price, as well as the two optimal stable outcomes, can be obtained through dynamic mechanisms that generalize the auction of Demange, Gale and Sotomayor (1986). Several properties of interest to the cooperative and competitive markets are derived.

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INTRODUCTION

In the multiple-partners assignment game there are two finite and disjoint sets of players, *B* and *Q*. The players of a set may form more than one partnership with the players of the other set. Every participant has a quota representing the maximum number of partners. As soon as *b* and *q* become partners, they undertake an activity together that produces income v_{bq}, which is splitted between them at $u_{bq}\geq 0$ for *b* and $w_{bq}=v_{bq}-u_{bq}\geq 0$ for *q*. An outcome for this game is a matching, that is, any set of partnerships that does not exceed the quotas of the players, along with payoffs u_{bq}'s and w_{bq}'s. A widely known special case is the Assignment Game of Shapley and Shubik (1972), in which each agent is allowed to form one partnership at most.

There are two ways of studying such a game. The first one consists in looking at it as a model for a cooperative market, for instance, a job market where players are firms and workers. Each firm *b* can hire at most *r(b)* workers and each worker *q* can accept at most *s(q)* jobs from different firms. The number v_{bq} represents the productivity of worker *q* at firm *b*. This approach is dealt with in Sotomayor (1992 and 1999), and a different version is considered in Crawford and Knoer (1981).

Sotomayor (1992) proves that the appropriate concept of cooperative equilibrium is not the concept of the core, but that of pairwise-stability instead, and shows that stable outcomes always exist. An outcome *x* will be denominated pairwise-stable unless there are agents *b* and *q* that do not form a partnership at *x*, but that can increase their total payoff, becoming partners and at the same time leaving some of these partners, if necessary, in order to remain within their quotas. In a stable outcome, *B*-players can discriminate *Q*-players and vice-versa. This is what usually occurs between firms and workers after negotiations: offers and counter-offers are made in such a way that, in the final allocation, a worker can receive different wages from different firms and one firm can obtain different profits from different workers.

Sotomayor (1999) uses a convenient vector representation of a player's stable payoffs, which, after ordering the agents in *B* (respectively, *Q*), allows immersing these players' set of stable payoffs in a Euclidean space. Therefore, the natural partial order of this space produces a partial order in the set of stable payoffs for players in *B* (respectively, *Q*). That paper proves that, according to this representation, the set of stable payoffs has a complete lattice structure. If we compare any two stable payoffs by comparing the two payoff vectors of *B*-players, for example, there is a stable payoff vector that is optimal for these agents, since it is larger than any other stable payoff vector. Due to the symmetry of the model, there is also an optimal stable payoff for *Q*-players.

Another way of studying the multiple partners assignment game, still unexplored in the literature, is to use it to model a competitive market. In this approach, players are conveniently assumed to be buyers and sellers. Every seller *q* owns *s(q)* identical and indivisible objects with zero reservation prices. Every buyer *b* has a quota *r(b)*, which represents the maximum number of objects she can buy. The value of object *q* for buyer *b* is $v_{bq} \geq 0$. If buyer *b* purchases object *q* at price $w_{bq} \geq 0$ then *b* receives the individual payoff $u_{bq}=v_{bq}-w_{bq} \geq 0$. The total payoff of a buyer is the sum of her individual payoffs. No buyer is interested in purchasing more than one object from the same seller. Thus, we may think of every buyer *b* as a broker instead of a final consumer, who has in hand an offer of \$$v_{bq}$ for every seller *q*, from a client who is interested in purchasing *r(b)* objects at those prices, if *b* obtains them in the market.

The solution concept used is that of competitive equilibrium: allocation (w, μ) is a competitive equilibrium if (i) μ is an allocation of the objects to buyers that respects the quotas of the agents such that no buyer will receive more than one object from the same seller; (ii) prices w_q's are non-negative; (iii) every buyer is assigned by μ to a set of objects in her demand set, and (iv) every unsold object has zero price.

Unlike the cooperative market, as corroborated in the text, sellers cannot discriminate the buyers in a competitive equilibrium: Every seller sets the same price for each of his objects[2]. Moreover, every seller with a positive price will sell all of his objects and the number of objects in the market will be enough to meet the demand of all buyers.

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[2] In 1981, Stherby Parke Bernet ran a sequential auction to sell seven identical licenses to use RCA's communications satellite for cable television broadcasts. The winning bids varied widely. The highest was \$14.4 million and the lowest was \$10.7 million. The FCC nullified the auction saying the procedure was "unjustly discriminatory" in charging different prices for the same service and ordered RCA to charge the same price to all. This story is told in PR Newswere, November 9, 1981; Cristian Science Monitor, June 29, 1982 and Time, December 13, 1982, pp. 148.

The purpose of the present paper is to determine whether competitive equilibria always exist and, in the affirmative case, to find an algorithm to obtain them.

We prove an existence theorem of competitive equilibria, show that every competitive payoff is pairwise-stable, and we confirm several properties that are inherent to such allocations, of interest to the market. One of them is the existence of minimum and maximum competitive equilibrium prices. An unexpected result is that, unlike the one-to-one and many-to-one matching models, the set of competitive payoffs can be smaller than the set of stable payoffs: the buyer-optimal stable payoff is a competitive payoff and sells the objects for the minimum competitive equilibrium price. However, the competitive payoff corresponding to the maximum competitive equilibrium price is a pairwise-stable payoff that is not always optimal for sellers. As a matter of fact, the seller-optimal stable payoff may not even allocate the objects in accordance with a competitive equilibrium price. This result seems to be related to the fact that sellers own identical objects.

It is easy to build examples in which arbitrary competitive equilibria sell similar objects, belonging to different sellers, for different prices. Nevertheless, one of our results shows that, when these sellers have the same quota, the prices of such objects, under a minimum competitive equilibrium, are the same.

Knowing that competitive equilibria always exist, we address the question of obtaining a procedure that allows us to find competitive equilibria. Bearing this goal in mind, we propose a dynamic mechanism that yields the minimum competitive equilibrium price for the competitive market, so it can be used to obtain the optimal stable payoff for the buyers in the cooperative market. By reversing the roles between B and Q players, the mechanism can be used to produce the Q-optimal stable payoff for the cooperative market. This procedure resembles the English auction when the market has a single object and, restricted to the Assignment game of Shapley and Shubik, it coincides with the dynamic auction of Demange, Gale and Sotomayor (1986). To our knowledge, it is the first generalization of these auctions to the many-to-many matching model. Its main feature is that it is simple enough to be implemented in real auctions: In any step of the auction, given the prices announced by the auctioneer, the buyers indicate their most preferred sets of objects according to those prices (their demand sets). The auctioneer replicates each buyer the number of times of her quota and finds all the demand structures (this will be defined further ahead in the text) corresponding to the buyers' demands. This includes precisely associating each copy of a buyer with one object or with a set of objects. The demand sets of any two copies are disjoint. If a demand structure exists, in such a way that it is possible to assign each copy of a buyer to an object in her demand set (which is equivalent to assigning every buyer to her most preferred set of objects at the given prices), respecting the sellers' quotas, then the auction ends. Otherwise, Hall's theorem[3] implies that, for every demand structure there is a minimal overdemanded set of objects, that is, an overdemanded set with the property that none of its proper subsets is overdemanded. A set of objects is overdemanded if the number of buyers who demand only items from this set is greater than the number of items in the set. Among all demand structures, the auctioneer chooses one that has the minimum number of minimal overdemanded sets. Afterwards, he chooses a minimal overdemanded set from the selected demand structure and raises the price of each object in the set by one unit. All other prices remain the same as those in the previous step.

The present article is structured as follows. Section 2 describes the cooperative market and proves some preliminary results related to this market. Section 3 outlines the competitive approach and proves the existence of competitive equilibria and several properties of such outcomes. Section 4 describes the mechanism, with an illustrative example in section 4.1, and the main results in Section 4.2. Section 5 concludes and discusses related works. Some of the proofs are presented in the Appendix.

2.THE COOPERATIVE FRAMEWORK AND PRELIMINARY RESULTS

There are two finite and disjoint sets of players, B and Q. The B-players may form more than one partnership with Q-players, and Q-players may form more than one partnership with B-players. The set B has m elements and the set Q has n elements. Each $b \in B$ has a quota $r(b)$ and each $q \in Q$ has a quota $s(q)$, representing the maximum number of partnerships they can form. For each pair (b,q) there is a non-negative number v_{bq} which is splitted between b and q if both form a partnership. Dummy players,

[3] Let B and C be two finite disjoint sets. For each b in B, let D_b be a subset of C. A simple assignment is an assignment of C to B, such that each b is assigned exactly one element j of C, such that j is in D_b, and each j in C is assigned to at most one element of B. Then, the Theorem of Hall is:
A simple assignment exists, if and only if, for every subset B' *of* B, *the number of objects in* $D(B')$ *is at least as great as the number of buyers in* B'.

denoted by 0, are included for technical convenience in both sides of the market. We have that $v_{b0}=v_{0q}=0$ for all $b\in B$ and $q\in Q$. As for the quotas, a dummy player can form as many partnerships as needed to fill up the quotas of the non-dummy players. If b and q form a partnership then b receives the individual payoff $u_{bq}\geq 0$ and q receives the individual payoff $w_{bq}=v_{bq}-u_{bq}\geq 0$. The game is then given by $M\equiv(B,Q,v,r,s)$.

This model, which is presented in Sotomayor (1992) and is a version of the one of Crawford and Knower (1981), is an extension of the Assignment game of Shapley and Shubik (1972) to the case of multiple partners.

A matching μ is a set of partnerships of the kind (b,q), $(b,0)$ or $(0,q)$, for $(b,q)\in B\text{x}Q$. If b and q are matched under μ, we write $b\in\mu(q)$ or $q\in\mu(b)$. A dummy player may be matched to more than one player of the opposite side and more than once to the same player. We will say that a subset $S\subseteq Q$ is an **allowable set of partners** for $b\in B$, if $|S|=r(b)$. We will extend this terminology to include the sets S with k non- dummy players and $r(b)-k$ repetitions of the dummy player. Analogously, we define an allowable set of partners for $q\in Q$. In order to simplify our notation, we will also write $S\subseteq B$ or $S\subseteq Q$ for any allowable set S of B-players or Q-players, respectively.

If we consider that a player may be assigned to the dummy player as many times as needed to fill up his/her/its quota, we can define **a matching μ to be feasible, if each player is matched to an allowable set of partners.** The value of μ is $\sum_{q\in Q,b\in\mu(q)} v_{bq}$. The matching μ is **optimal** if it attains the maximum value among all feasible matchings.

Definition 1. *A **feasible outcome for M,** denoted by $(u,w;\mu)$, is a feasible matching μ and a pair of payoffs (u,w), where the individual payoffs of each $b\in B$ and $q\in Q$ are given by the arrays of numbers u_{bq}, with $q\in\mu(b)$, and w_{bq}, with $b\in\mu(q)$, respectively, such that $u_{bq}+w_{bq}=v_{bq}$, $u_{bq}\geq 0$ and $w_{bq}\geq 0$. Consequently, $u_{b0}=u_{0q}=w_{b0}=w_{0q}=0$ in case these payoffs are defined.*

If $(u,w;\mu)$ is a feasible outcome, we say that μ is compatible with payoff (u,w) and vice-versa.

The key notion is that of stability. We will be assuming that agents' preferences are *separable across pairs*, in the sense that the payoff of the partnership (b,q), v_{bq}, does not depend on which other partnerships are formed by players b and q. If S is an allowable set of partners for $b\in B$, the payoff of the coalition $S\cup\{b\}$ is the sum of the numbers v_{bq}'s with $q\in S$. Similarly, we define the payoff of $S\cup\{q\}$, where S is an allowable set of partners for $q\in Q$. Under this structure of preferences it is proved in Sotomayor (1992) that the essential coalitions for this game are the pairs (b,q)'s, and so the appropriate concept of stability is that of pairwise-stability: A feasible outcome is stable if, for all pairs b and q, such that (b,q) is not a partnership, the sum of any b's individual payoff with any q's individual payoff is not less than v_{bq}. The interpretation of this condition is the natural one. If it is not satisfied then it would pay b and q to break up one of their present partnerships, if necessary, and form a new one together, because this could give to both of them a higher payoff. By defining $u_b(min)=min\{u_{bq};\ q\in\mu(b)\}$ and $w_q(min)=min\{w_{bq};\ b\in\mu(q)\}$ for a feasible outcome $(u,w;\ \mu)$, the definition of stability is equivalent to:

Definition 2. *The feasible outcome* $(u,w;\mu)$ *is* ***stable*** *for* M *if* $u_b(min) + w_q(min)\geq v_{bq}$ *for all pairs* (b,q) *with* $b\notin\mu(q)$..

If $(u,w;\mu)$ *is* a stable outcome we say that **(u,w) is a stable payoff.** It is proved in Sotomayor (1999) that, by using a convenient vectorial representation of the stable payoffs, there exists a stable payoff which is weakly preferred by any player in B (respectively, Q) to any other stable payoff. This payoff is called *B-optimal stable payoff* (respectively, *Q-optimal stable payoff*). Furthermore, if, for example, $(u',w';\mu')$ is a B-optimal stable outcome and $(u,w;\mu)$ is any other stable outcome, then, each component of u' *(represented as a vector)* is greater than or equal to the corresponding component of u *(represented as a vector).* In particular, if $\mu=\mu'$ then for every $b\in B$, $u'_{bq}\geq u_{bq}$ for all $q\in\mu(b)$.

The following proposition asserts that, under the B-optimal stable payoff, no Q-player discriminates his partners: He gets the same individual payoff with any partner.

Proposition 1. *Let* (u',w') *be the B-optimal stable payoff for* M*. Let* μ *be a matching compatible with* (u',w')*. Then,* $w'_{bq}=w'_{b'q}$ *for all* $q\in Q$ *and all* b *and* b' *in* $\mu(q)$*. (*b *and* b' *might be dummy players).*

Proof. For all $q\in Q$ define $w''_{bq}=w'_q(min)$, if $b\in\mu(q)$. Let u'' be the corresponding payoff for the *B*-players. We claim that $(u'',w'';\mu)$ is stable. In fact, if $b\in\mu(q)$ for some q, $u''_{bq}=v_{bq} - w''_{bq} = v_{bq} - w'_q(min) \geq v_{bq} - w'_{bq}=u'_{bq}$. Then,

$$u''_{bq}\geq u'_{bq}, \text{ for all pairs } (b,q) \text{ with } b\in\mu(q), \tag{1}$$

so $u''_b(min) \geq u'_b(min)$ for all b. Therefore, for every pair (b,q) with $b\notin\mu(q)$, $u''_b(min)+w''_q(min)\geq u'_b(min)+w'_q(min)\geq v_{bq}$ by stability of (u',w'), *so* $(u'',w'';\mu)$ is stable. The *B*-optimality of (u',w',μ) and (1) imply that $u'_{bq}=u''_{bq}$, for all pairs (b,q) with $b\in\mu(q)$, which implies that $w''_{bq}=w'_{bq}$ for all (b,q) with $q\in\mu(b)$. Hence, $w'_{bq}=w'_q(min)$ *for all* $b\in\mu(q)$, and the proof is complete.ζ

Symmetrical conclusions apply by reverting the roles between *B*-agents and *Q*-agents in Proposition 1. The following propositions, from Sotomayor (1992) and Sotomayor (1999), respectively, will be necessary in the next sections:

Proposition 2. *Let* $(u,w;\mu)$ *be a stable outcome for* M*. Then* μ *is an optimal matching.*

Proposition 3. *If* $(u,w;\mu)$ *is a stable outcome for* M *and* μ' *is an optimal matching, then* $(u,w;\mu')$ *is also stable for* M*.*

3. THE COMPETITIVE FRAMEWORK AND PRELIMINARY RESULTS

In this section we will define a competitive market related to the cooperative market *M*. The concept of competitive equilibrium is the adequate solution concept. This framework is more appropriate for a buyer-seller market than for a labor market. In fact, as we will see later, Q-players do not discriminate their partners under competitive equilibrium prices. That is, it is natural that every seller sells his objects for the same price, but it is reasonable that, some times, a worker does not get the same salary in all her jobs or a firm does not get the same profit with all workers it hires. Thus, *B*-players and *Q*-players will be conveniently assumed to be buyers and sellers, respectively. Quota $s(q)$ of seller q means that q owns $s(q)$ identical and indivisible objects, and quota $r(b)$ of buyer b represents the maximum number of objects b is allowed to buy. No buyer is interested in acquiring more than one object of a given seller. An allocation of the objects to the buyers will be called a matching.

Every object has a reservation price of 0 *(which can be obtained after normalization).* Generically, we will denote buyers by b, b', and sellers and objects by q, q'. The value of any object of seller q to buyer b is $v_{bq} \geq 0$. That is, v_{bq} is the gain of trade when any of the objects of seller q is sold to buyer b. If buyer b acquires some object of q for price $\pi \geq 0$ then b will receive an individual payoff $u_{bq} = v_{bq} - \pi \geq 0$. Without loss of generality we can consider $r(b) \leq n$. We will also include the dummies: an artificial "null-object", 0, whose value is zero to all buyers and whose price is always zero and a fictitious buyer, 0, whose value is zero for all objects and whose payoff is always zero. If an object is matched to the dummy buyer we say that it is left unsold. An outcome is a matching plus a price for each object. This market will be denoted by M^*.

An allowable set of objects for buyer b contains $r(b)$ objects, some of which may be repetitions of the null-object. Furthermore, it does not contain more than one object of the same seller. **A feasible matching matches a buyer to an allowable set of objects and each non-null object to one buyer** (who might be the dummy buyer).

A vector of prices $p \in R^N_+$, with $N \equiv \sum_{q \in Q} s(q)$, is called a **feasible price vector for M^***. Since the quota restriction imposes a limit on the number of objects that a buyer may acquire, and a buyer is not interested in buying more than one object from the same seller, her preferences under a feasible price vector are defined only over allowable sets of objects. The value of an allowable set of objects S to buyer b is the sum of the values of the objects in S to b. Then, given a feasible price vector, the preferences of buyers are completely described by the numbers v_{bq}: For any two allowable sets of objects S and S', buyer b prefers S to S' at prices p, if her total payoff when she buys S is greater than her total payoff when she buys S'. She is indifferent between these two sets, if she gets the same total payoff with both sets. Object q is *acceptable* to buyer b at prices p if, under these prices, b likes q at least as well as the null-object.

Under the structure of preferences we are assuming, each buyer b can determine which allowable sets of objects she would most prefer to buy at a given price vector p. We denote the set of all such allowable sets by $\boldsymbol{D_b(p)}$ and call it the **demand set of** ***b*** **at prices** ***p*****.** (Note that $D_b(p)$ is never empty, because there is always the option of buying $r(b)$ copies of the null object. Note also that, if $S \in D_b(p)$, then every element of S is acceptable to b).

Definition 3. *The feasible price vector p is called* ***competitive*** *for M^*, if there is a feasible matching μ such that, if $\mu(b)=S$ then S is in $D_b(p)$.*

Therefore, at competitive prices p, each buyer is assigned to a set of objects in her demand set. The matching μ is said to be *compatible* with the competitive price p. A matching μ is called **competitive** if it is compatible with a competitive price.

Definition 4. *The pair (p, μ) is a* ***competitive equilibrium*** *for M^*, if p is competitive, μ is compatible with p, and $p_q=0$ if object q is left unsold. If (p, μ) is a competitive equilibrium, p will be called* ***an equilibrium price vector****.*

An important difference between the cooperative behavior and competitive behavior of the sellers is that **under competitive equilibrium prices, every seller sells all their objects for the same price.** In fact, if a seller has two identical objects, q and q', and $p_q > p_{q'}$ for some price vector p, then no buyer b will demand, at prices p, a set S of objects that contain object q. This is because , by replacing q by q' in S, b gets a more preferable allowable set of objects. But then, q will remain unsold with a positive price, and so p cannot be competitive. Thus, **if a seller does not complete his quota under a competitive equilibrium, then some of his objects will have a price of zero, so the price of any of his objects must be zero. This implies that, if $p_q>0$ then the owner of q has sold all his objects.**

Remark 1. Since a seller sells his identical objects for the same competitive price and no buyer is allowed to buy more than one object of a seller, we do not cause any confusion by using the same notation for a seller and any of his objects. At competitive prices, since the array of payoffs for any seller q in M is given by the array of prices of his objects, so the array of individual payoffs for any seller q in the related cooperative market M is given by $s(q)$ repetitions of a same number and so, in order to represent such payoffs, we do not need to make any reference to the buyers who are matched to q. For example, $(p_q, p_{q,\dots,} p_q)$ will denote the array of payoffs of seller q and p_q will denote the price of any of his objects. If μ^* is a feasible matching for M^* we can define a corresponding matching μ for the related cooperative market M such that seller $q\in\mu(b)$ if and only if one of his objects is allocated to b at μ^*. Thus, $q\in\mu^*(b)$ means that object q is allocated to buyer b and $q\in\mu(b)$ means that buyer b and seller q are partners at μ.. We say that μ and μ^* **correspond** to each other. Clearly, μ and μ^* have the same value, so **μ is optimal for M if and only if μ^* is optimal for M^*.** If (p^*, μ^*) is a competitive equilibrium, we can define the corresponding outcome (u,p^*,μ), called **competitive equilibrium outcome**, where the u_{bq}'s, given by $u_{bq}=v_{bq} - p^*_q$ if $q\in\mu^*(b)$, define the array of payoffs for buyer b. The array of payoffs for seller q is given by $s(q)$ repetitions of the number p^*_q. According to our previous definitions, $\mu^*(b)$ belongs to $D_b(p^*)$ for every b. Then, for all $b\in B$, $(v_{bq} - p^*_q) \geq (v_{bk} - p^*_k)$ for all $q\in\mu^*(b)$ and $k\notin\mu^*(b)$, from which follows that $u_b(min) = min\{(v_{bq} - p^*_q); q\in\mu^*(b)\} \geq (v_{bk} - p^*_k)$ for all $b\in B$ and $k\notin\mu^*(b)$. That is, $u_b(min) + p^*_k \geq v_{bk}$ for all (b,k) with $k\notin\mu(b)$, and so **the competitive equilibrium outcome $(u,p^*; \mu)$ is stable for the related cooperative market M.** With an analogous argument we can show that, if $(u,w;\mu)$ is stable for M and the array of payoffs for any seller q is given by $s(q)$ repetitions of the number w_q, then **$(u,w;\mu)$ is a competitive equilibrium outcome.**ν

Definition 5. *The competitive equilibrium price* p *is the* ***minimum (respectively, maximum) competitive equilibrium price*** *for* M^***,** *if* $p_q \leq p'_q$ *(respectively, if* $p_q \geq p'_q$*) for all objects* q *and all competitive equilibrium price* p'.

Proposition 4 . *If* (p^*,μ^*) *is a competitive equilibrium then* μ^* *is an optimal matching for* M^*.

Proof. Let $(u,p^*;\ \mu)$ be the stable outcome, for the related cooperative market M, corresponding to (p^*, μ^*). By Proposition 2, μ is optimal for M . The result follows from Remark 1.ν

Proposition 5. Let *(p^*,μ^*) be a competitive equilibrium for M^*. Let λ^* be an optimal matching for M^*. Then (p^*,λ^*) is a competitive equilibrium for M^*.*

Proof. Let $(u,p^*;\ \mu)$ be the stable outcome, for the related cooperative market M, corresponding to (p^*, μ^*). Let λ be the corresponding matching to λ^*, which is optimal for M by Remark 1. By Proposition 3, $(u,p^*;\ \lambda)$ is stable for M, so (p^*,λ^*), is a competitive equilibrium for M^*, by Remark 1.ν

The existence of stable outcomes for M is proved in Sotomayor (1992, 1999). For the competitive equilibria it is an immediate consequence of:

Proposition 6. *Let $(u,p^*;\mu)$ be a B-optimal stable outcome for the related cooperative market M. Let μ^* be the corresponding matching to μ.. Then (p^*,μ^*) is a competitive equilibrium for the competitive market M^*.*

Proof. It is immediate from Proposition 1 and Remark 1. ν

The existence of a minimum competitive equilibrium is proved in the proposition below:

Proposition 7. *Let (u,p^*,μ) be a B-optimal stable outcome for the related cooperative market M. Let μ^* be the corresponding matching to μ. Then, (p^*,μ^*) is a minimum competitive equilibrium for M^*.*

Proof. In fact, by Proposition 6, (p^*,μ^*) is a competitive equilibrium. Then, suppose by way of contradiction that there is some competitive equilibrium (w',λ^*), such that $0\leq w'_q<p^*_q$, for some object q . Let $(u',w';\ \lambda)$ be the stable outcome for M corresponding to $(w',\ \lambda^*)$, as defined in Remark 1. Proposition 3 asserts that every optimal matching for M is compatible with any stable outcome. Then, $(u',w';\mu)$ is stable for M. The fact that

$p^*_q>0$ and (p^*,μ^*) is a competitive equilibrium implies that q is not left unsold at μ^*, and so at μ.. Take $b\in\mu.(q)$. Then, $u'_{bq}=v_{bq}-w'_q > v_{bq} - p^*_q= u_{bq}$. Thus, $u'_{bq} > u_{bq}$ for at least one buyer b, contradicting the B-optimality of $(u,p^*;\mu)$. Hence, (p^*,μ^*) is a minimum competitive equilibrium.ν

We can also prove the existence of the maximum competitive equilibrium price. This allocation does not necessarily correspond to a Q-optimal stable outcome for M as the following simple situation illustrates. Consider two buyers, each with quota of 1, and one seller with two objects. The values are given by 3 and 2, respectively. It is clear that the outcome which allocates one object to each buyer at prices 3 and 2, and gives payoff zero to the buyers, is the Q-optimal stable outcome. However it is not competitive. The maximum competitive equilibrium price is $p=(2,2)$.

Proposition 8. *There is a maximum competitive equilibrium for M^*.*

Proof. Let $(u,w;\mu)$ be a Q-optimal stable outcome for the related cooperative market M. For all $q\in Q$ define $p^*_{bq}=w_q(min)$, if $b\in\mu(q)$. Set $p^*_q\equiv p^*_{bq}$ for al $q\in Q$. Then, $p^*_q(min)=w_q(min)$. We claim that $(u',p^*;\mu)$ is stable for M, where u' is the corresponding payoff for the buyers. In fact, $(u',p^*;\mu)$ is clearly feasible and if $b\in\mu(q)$ for some q, $u'_{bq}=v_{bq} - p^*_q =v_{bq} - w_q(min) \geq v_{bq} - w_{bq}=u_{bq.}$ Then,

$$u'_{bq}\geq u_{bq}, \text{ for all } b \text{ and all } q\in\mu(b), \qquad (1)$$

so $u'_b(min) \geq u_b(min)$ for all b. Therefore, for every pair (b,q) with $b\notin\mu(q)$, $u'_b(min)+p^*_q(min)\geq u_b(min)+w_q(min)\geq v_{bq}$ by stability of (u,w), *so* $(u',p^*;\mu)$ is stable for M, and so the corresponding allocation (p^*,μ^*) is a competitive equilibrium for M^*, by Remark 1. We claim that (p^*,μ^*) is a maximum competitive equilibrium. Suppose not. Then, there is some competitive equilibrium (w',λ^*) such that $w'_q>p^*_q\geq 0$, for some q . Let $(u',w';\lambda)$ be the corresponding stable outcome for M. Proposition 3 asserts that every optimal matching is compatible with any stable outcome. Then, μ is compatible with $(u',w';\lambda)$, so $(u',w';\mu)$ is stable for M and $w'_q>p^*_q$ for all q, contradicting the Q-optimality of $(u,p^*;\mu)$. Hence, (p^*,μ^*) is a maximum competitive equilibrium and the proof is complete.ν

Objects owned by different sellers, but which are not distinguishable among buyers may be sold for different prices, even under a minimum competitive equilibrium price. See example below.

Example 1. $B=\{1,2,3,0\}$, $Q=\{j,k,q,0\}$, $r(1)=r(2)=r(3)=2$, $s(j)=2$, $s(k)=3$, $s(q)=1$.The values of the buyers are given by: $v_1=(6,6,1,0)$, $v_2=(4,4,,2,0)$ and $v_3=(3,3,1,0)$, where the first coordinate is the value of any object of seller j, the second coordinate is the value of any object of seller k, and so on. It is a matter of verification that the minimum equilibrium price allocates the objects of j to 1 and 2 at price $p_j=2$; the three objects of k to the three buyers at price $p_k=0$ and the object of q to buyer 3 at price $p_q=0$. Hence, the prices of the objects of sellers j and k are not the same although all them have the same value to any buyer. ν

The fact illustrated in the example above does not occur when the quotas of the sellers of the non-distinguishable objects are the same.

Proposition 9. *Let p be the minimum competitive equilibrium price for M^*. Let $j, k\in Q$, such that $s(j)=s(k)$ and $v_{bj}=v_{bk}$ for every $b\in B$. Then, $p_j=p_k$.*

Proof. Let μ^* be some optimal matching for M^*. Then, (p,μ^*) is a minimum competitive equilibrium for M^*. Suppose by way of contradiction that $p_j\neq p_k$. Without loss of generality, it can be assumed that $p_j>p_k\geq 0$. Then, seller j must have sold all his objects at μ^* and for all b assigned to an object of j we have that $v_{bj}- p_j < v_{bj}-p_k = v_{bk} - p_k$. The competitivity of p then implies that if b is assigned to an object of j then she is also assigned to an object of k. Since $s(j)=s(k)$ we have that $\mu^*(j)= \mu^*(k)$. We claim that the price vector p^* is also competitive with matching μ^*, where $p^*_j=p^*_k=p_k$ and $p^*_q = p_q$ for all objects of seller $q\notin\{j,k\}$. We have to show that, if $q\in\mu^*(b)$, then $v_{bq}- p^*_q \geq v_{bq'} - p^*_{q'}$, for every $q'\notin\mu^*(b)$. (This is equivalent to requiring that $\mu^*(b)$ is in $D_b(p^*)$, by Remark 1). We have two cases.

Case 1. $q\in\{j,k\}$. Then, for all $q'\notin\mu^*(b)$, we have that $v_{bj}\text{-}p^*_j = v_{bk} - p^*_k = v_{bk} - p_k \geq v_{bq'} - p_{q'} = v_{bq'} - p^*_{q'}$, where the inequality follows from the competitivity of p.

Case 2. $q\notin\{j,k\}$. If $q'\notin\{j,k\}$, the result is immediate from the competitivity of p. If $q'\in\{j,k\}$, then $b\notin\mu^*(j)=\mu^*(k)$. We have that,

$$v_{bq}- p^*_q= v_{bq}- p_q\geq v_{bk} - p_k = v_{bj} - p^*_j = v_{bq'} - p^*_{q'},\ \text{if}\ q' = j,\ \text{and}$$

$$v_{bq}- p^*_q = v_{bq} - p_q \geq v_{bk} - p_k = v_{bk} - p^*_k = v_{bq'} - p^*_{q'},\ \text{if}\ q' = k,$$

where the inequalities follow from the competitivity of p.

Then, in any case, $v_{bq}\text{-}\ p^*_q \geq v_{bq'} - p^*_{q'}$. Therefore, p^* is competitive. However, $p^*<p$, which contradicts the minimality of p. Hence, $p_j=p_k$ and the proof is complete. ■

4. THE DYNAMIC MECHANISM

In order to describe the dynamic mechanism some preliminaries are necessary. For our purposes, if $p^*\in R^N_+$, with $s(q)$ repetitions of p^*_q for each $q\in Q$, it will be more convenient to work with the price vector in R^n obtained as follows: We replace the $s(q)$ repetitions of p^*_q by the number p^*_q, by keeping in mind that the coordinate p^*_q is the price of each one of the $s(q)$ objects of seller q. Thus, we obtain a competitive market associated to M^*, $A(M^*)$, where a feasible price vector is a vector of R^n_+ whose q-*th* coordinate is the price of each one of the $s(q)$ objects of seller q. By introducing a dummy-seller, who can be repeated in order to fill up the quotas of the buyers, a feasible matching will assign a buyer to an allowable set of sellers for her (instead of an allowable set of objects for her), and a seller to an allowable set of buyers for him. Then, the feasible matchings for $A(M^*)$ are the feasible matchings for M. The demand set of b at prices p is the set of favorite allowable sets of sellers for buyer b at prices p. The allocation (p, μ) is a competitive equilibrium for $A(M^*)$ if each buyer is assigned to an allowable set of sellers in her demand set and $p_q=0$ whenever seller q does not have completed his quota. We will keep the same notation as before, $D_b(p)$, for the demand set of b at prices p. It is immediate that **(p, μ) is a competitive equilibrium for $A(M^*)$ if and only if (p^*,μ^*) is a competitive equilibrium for M^*, where μ^* is the matching of M^* corresponding to μ and p^* is the extension of p to M^* as described above.**

We will describe a dynamic mechanism for the competitive market *A(M*),* and will see that it produces, in a finite number of steps, a competitive price. It will be proved in the next section that the resulting allocation is the minimum competitive equilibrium for *A(M*),* which corresponds to the minimum competitive equilibrium for *M**, as remarked above. It will follow from Proposition 7 that the dynamic mechanism also produces a *B*-optimal stable outcome for the cooperative market *M*.

Given a feasible price p for *A(M*)*, buyers have preferences over individual sellers and over allowable sets of sellers. The preferences of each buyer $b\in B$ over individual sellers can be represented by an ordered list of preferences $L_b(p)$ of the following form:

$L_b(p)= q_1, q_2, q_3, 0, q_4$

indicates that, at prices p, buyer b prefers any object of seller q_1 to any object of seller q_2, any object of seller q_2 to any object of seller q_3, any of these objects to 0, but prefers not to fill her quota to buy any object of seller q_4 at price p . The objects of q_1, q_2 and q_3 are acceptable to b while the ones of q_4 are unacceptable. Buyer b may also be indifferent among several potential partners. Brackets in the preference list will denote this, so, for example, the list

$L_{b'}(p)=[q_1,q_2], q_3,q_4, [q_5,q_6,q_7], 0$

indicates that b' is indifferent between any of the objects of seller q_1 and seller q_2, prefers any object of these agents to any object of q_3, and so on.

The following terminology will be needed for the description of the mechanism. The **bid of b at p, $\mathbf{B_b(p)}$,** is a truncation of the preference list $L_b(p)$ so that it contains exactly all $S\in D_b(p)$. To make this clear, consider that $L_b(p)=[q_1,q_2],q_3,q_4,[q_5,q_6,q_7],...$ and b has quota $r(b)=1$. Then, $D_b(p)=\{q_1,q_2\}$, so $B_b(p)=[q_1,q_2]$; if $r(b)=2$, $D_b(p)=\{\{q_1,q_2\}\}$, so $B_b(p)=[q_1,q_2]$; if $r(b)=4$, $D_b(p)=\{\{q_1,q_2,q_3,q_4\}\}$, so $B_b(p)=[q_1,q_2],q_3,q_4$; if $r(b)=6$, $D_b(p)=\{\{q_1,q_2,q_3,q_4,q_5,q_6\},\{q_1,q_2,q_3,q_4,q_5,q_7\},\{q_1,q_2,q_3,q_4,q_6,q_7\}\}$, so $B_b(p)=[q_1,q_2],q_3,q_4,[q_5,q_6,q_7]$, and so on. Notice that, if $L_b(p)=[q_1,q_2,0]$, q_3 and $r(b)=3$, for example, then $D_b(p)=\{\{q_1,q_2,0\},\{q_1,0,0\},\{q_2,0,0\}, \{0,0,0\}\}$. In this case, we will write $B_b(p)=[q_1,q_2,0,0,0]$.

Remark 2. We will say that $q\in B_b(p)$ if and only if q is listed by b in $B_b(p)$. Therefore, $q\in B_b(p)$ if and only if the number of elements of $L_b(p)$ strictly preferred to q by b at prices p is less than $r(b)$. Of course, the number of elements listed in $B_b(p)$ (this set may include several copies of the dummy seller), is greater than or equal to $r(b)$. ζ

Now, for each non-dummy buyer b, break the ties of $B_b(p)$ in any way desired. The resulting set will be denoted by $B^*_b(p)$. Then, we can define:

$A_{b,i}(p)$ is the set formed by the i-th element in $B^*_b(p)$, if $i<r_b$;

$A_{b,i}(p)$ is the set formed by the elements in $B^*_b(p)-[A_{b,1}(p)\cup...\cup A_{b,r(b)-1}(p)]$, if $i=r_b$.

The set $\mathbf{A_b(p)=\{A_{b,1}(p),..., A_{b,r(b)}(p)\}}$ will be called a **demand structure for b at prices p** (corresponding to the given tie-breaking rule). The set of all $A_{b's}(p)$ will be called a **demand structure at p** and will be denoted by $A(p)$.

To illustrate these definitions, consider that buyer b has quota $r(b)=6$ and that $B_b(p)=[q_1,q_2],q_3,q_4,[q_5,q_6,q_7]$. If we break ties to get $B^*_b(p)=q_1,q_2,q_3,q_4,q_6,q_5,q_7$, the corresponding demand structure for b at prices p is: $A_{b,1}(p)=\{q_1\}$, $A_{b,2}(p)=\{q_2\}$, $A_{b,3}(p)=\{q_3\}$, $A_{b,4}(p)=\{q_4\}$, $A_{b,5}(p)=\{q_6\}$ and $A_{b,6}(p)=\{q_5,q_7\}$. Notice that, if $L_b(p)=[q_1,q_2,0]$, q_3 and $r(b)=3$, for example, then $B_b(p)=[q_1,q_2,0,0,0]$, so we can break ties to get, for example, $B^*_b(p)=0,0,0,q_1,q_2$. In this case, the corresponding demand structure for b at prices p is : $A_{b,1}(p)=\{0\}$, $A_{b,2}(p)=\{0\}$ and $A_{b,3}(p)=\{q_1,q_2,0\}$.

The pair (b,i), with $b\in B$ and $1\leq i\leq r(b)$, is a **loyal demander of S** if $A_{b,i}(p)\subseteq S$.

Definition 6. *Given the feasible price vector p for $A(M^*)$, we will say that the set $S\subseteq Q$ is **overdemanded** for the demand structure $A(p)$, if there is a set T of loyal demanders of S, such that $|T|>\sum_{q\in S}S(q)$, where $S(q)=\min\{s(q)$, number of $(b,i)\in T$ with $q\in A_{b,i}(p)\}$.*

The overdemanded set S is said to be **minimal,** if no proper subset of S is overdemanded. Thus if, for example, b has quota $r(b)=1$ and $A_{b,1}(p)=\{q_3,q_4\}$; b' has quota $r(b')=2$ and $A_{b',1}(p)=\{q_3\}$ and $A_{b',2}(p)=\{q_4\}$; and b'' has quota $r(b'')=2$ and $A_{b'',1}(p)=\{q_1\}$, $A_{b'',2}(p)=\{q_3\}$, then, $T=\{(b,1),(b',1),(b',2),(b'',2)\}$ is a set of loyal demanders of $S=\{q_3,q_4\}$. If $s(q_3)=1$ and $s(q_4)=3$ then $S(q_3)=1$ and $S(q_4)=2$. Then $4=|T|>3=S(q_3)+S(q_4)$. Set S is overdemanded, but it is not minimal. In fact, the set

$S'=\{q_3\}$ is overdemanded by $T'=\{(b',1),(b'',2)\}$. Indeed, S' is a minimal overdemanded set.

Remark 3. It follows from the definition of competitive equilibrium that, if $p\in R^n$ is a competitive equilibrium price for *A(M*)*, then each buyer b can be matched to her most preferred allowable set of sellers at prices p (this may include copies of the dummy-seller). Therefore, there is some demand structure $A(p)$ for which each pair (b, i) can be matched to exactly one seller q, with $q\in A_{b,i}(p)$ (q might be the dummy-seller). In addition, every non-dummy seller q is matched $s(q)$ times at most. Hence, there is no overdemanded set for $A(p)$.

If $p\in R^n$ is not a competitive equilibrium price for *A(M*)*, then there is no way to match each buyer to her $r(b)$ most preferred sellers under a feasible matching. This means that every demand structure $A(p)$ has an overdemanded set. This is a consequence of a simple adaptation of Hall's Theorem to the case where $s(q)$ can be greater than one. (See P. Hall (1935), Gale (1960)).ζ

Now, we can describe the dynamic mechanism. It can be thought of as being an auction procedure to sell the objects of the market *A(M*)*. Thus, we call the matchmaker "auctioneer". We will take all prices and valuations to be integers.

Step (1): The auctioneer announces an initial price vector, $p(1)= (0,...,0)\in R^n_+$. Each buyer b "bids" by announcing $B_b(1)\equiv B_b(p(1))$.

Step (*t*+1): After bids $B_b(t)$ are announced, the auctioneer determines all the demand structures at $p(t)$, using all possible tie-breaking rules. If there is some demand structure $A(t)\equiv A(p(t))$, for which it is possible to match each pair (b, i) to a seller $q\in A_{(b,i)}(t)$, so that no real seller is matched more times than his quota, the algorithm stops. If no such demand structure exists, Hall's Theorem implies that there is some overdemanded set for every demand structure. Then, the auctioneer chooses some demand structure that has the minimum number of minimal overdemanded sets, among all demand structures. (This corresponds to the choice of a tie-breaking rule). Next, he selects a minimal overdemanded set for the demand structure chosen and raises the price of all objects belonging to each seller in the set by one unit. All other prices remain at level $p(t)$. This defines $p(t+1)$.

It is clear that the algorithm stops at some step, because, as soon as the price of the objects of a given seller becomes higher than any buyer's valuation for them, the seller will not be in the bid of any buyer. It follows from the construction of the algorithm, that the final price is a competitive price vector for *A(M*)*. What is less clear is that this algorithm yields the same price, independent of the demand structures selected by the auctioneer. We will prove this fact in section 4.2, by showing that the price obtained in the algorithm is the minimum equilibrium price vector for *A(M*)*. Before we will illustrate the mechanism with an example.

4. 1 EXAMPLE

The following example illustrates the dynamic mechanism. There are four non-dummy-buyers, 1, 2, 3 and 4, and six non-dummy sellers, q_1, q_2, ..., q_6. Seller q_1 has two identical objects and the other sellers have a quota of one. The maximum number of objects that each buyer can purchase is given by 3, 2, 1 and 1, respectively. These numbers define the quotas of the buyers. The values of the buyers to the non-null objects are given by the following vectors: v_1=(7,5,4,4,2,1,0), v_2=(5,4,1,0,2,1,0), v_3=(2,0,0,0,0,1,0) and v_4=(3,1,1,1,1,2,0), where the j-th coordinate of v_i is the value of any object of seller q_j to buyer i.

Step 1. $p(1)$=(0,0,...,0). The matrix of numbers v_{bq}-$p_q(1)$ is given in the table below, with the entries corresponding to the sellers of the demand sets in boldface

	q_1	q_2	q_3	q_4	q_5	q_6	0
1	**7**	**5**	**4**	**4**	2	1	0
2	**5**	**4**	1	0	2	1	0
3	**2**	0	0	0	0	1	0
4	**3**	1	1	1	1	2	0

The bids of the buyers at $p(1)$ are: $B_1(1)=q_1,q_2,[q_3,q_4]$; $B_2(1)=q_1,q_2$; $B_3(1)=B_4(1)=q_1$. There is only one demand structure given by: $A_{1,1}(1)=\{q_1\}$, $A_{1,2}(1)=\{q_2\}$, $A_{1,3}(1)=\{q_3,q_4\}$; $A_{2,1}(1)=\{q_1\}$, $A_{2,2}(1)=\{q_2\}$, $A_{3,1}(1)=\{q_1\}$ and $A_{4,1}(1)=\{q_1\}$. It is not possible to find a competitive matching. The set $S=\{q_1,q_2\}$ is overdemanded. In fact, the set of loyal demanders of S is T={(1,1),(1,2),(2,1),(2,2),(3,1),(4,1)}; $S(q_1)=min\{2,4\}=2,$

$S(q_2)=min\{1,2\}=1$. Then $6=|T|>2+1=3$. However $\{q_1,q_2\}$ is not minimal, because $\{q_1\}$ as well as $\{q_2\}$ are overdemanded: The set of loyal demanders of q_1 has 4 elements, which is greater than $S(q_1)$, and the set of loyal demanders of q_2 has 2 elements, which is greater than $S(q_2)$. It is a matter of verification that there are only two minimal overdemanded sets: $\{q_1\}$ and $\{q_2\}$. Suppose the auctioneer chooses $\{q_2\}$. Then, he raises the price of the objects of q_2 by one unit (actually q_2 has only one object).

Step 2. $p(2)=(0,1,...,0)$. The matrix of numbers $v_{bq}-p_q(2)$ is given in the table below, with the entries corresponding to the sellers of the demand sets in boldface . The numbers $v_{bq}-p_q(2)<0$ will be replaced by Z:

	q_1	q_2	q_3	q_4	q_5	q_6	0
1	**7**	**4**	**4**	**4**	2	1	0
2	**5**	**3**	1	0	2	1	0
3	**2**	Z	0	0	0	1	0
4	**3**	0	1	1	1	2	0

Then, $B_1(2)=q_1,[q_2,q_3,q_4]$; $B_2(2)=q_1,q_2$; $B_3(2)=B_4(2)=q_1$. Because of the indifferences of buyer 1 we have three demand structures. The first one is given by: $A_{1,1}(2)=\{q_1\}$, $A_{1,2}(2)=\{q_2\}$, $A_{1,3}(2)=\{q_3,q_4\}$; $A_{2,1}(2)=\{q_1\}$, $A_{2,2}(2)=\{q_2\}$, $A_{3,1}(2)=q_1$ and $A_{4,1}(2)=q_1$. It is not possible to find a competitive matching. There are two minimal overdemanded sets: $\{q_1\}$ and $\{q_2\}$. The second demand structure is given by: $A'_{1,1}(2)=\{q_1\}$, $A'_{1,2}(2)=\{q_3\}$, $A'_{1,3}(2)=\{q_2,q_4\}$; $A'_{2,1}(2)=\{q_1\}$, $A'_{2,2}(2)=\{q_2\}$, $A'_{3,1}(2)=q_1$ and $A'_{4,1}(2)=q_1$. It is not possible to find a competitive matching. There is one minimal overdemanded set: $\{q_1\}$. The third demand structure is given by: $A''_{1,1}(2)=\{q_1\}$, $A''_{1,2}(2)=\{q_4\}$, $A''_{1,3}(2)=\{q_2,q_3\}$; $A''_{2,1}(2)=\{q_1\}$, $A''_{2,2}(2)=\{q_2\}$, $A''_{3,1}(2)=q_1$ and $A''_{4,1}(2)=q_1$. It is not possible to find a competitive matching. There is one minimal overdemanded set: $\{q_1\}$. The auctioneer must choose a demand structure with the minimum number of minimal overdemanded sets. Suppose the auctioneer chooses A'. As a result, he raises the price of both objects of q_1 by one unit. (Observe that both objects of q_2 are overdemanded).

Step 3. $p(3)=(1,1,...,0)$. The matrix of numbers v_{bq}-$p_q(3)$ is given in the table below, with the entries corresponding to the sellers of the demand sets in boldface .

	q_1	q_2	q_3	q_4	q_5	q_6	0
1	**6**	**4**	**4**	**4**	2	1	0
2	**4**	**3**	1	0	2	1	0
3	**1**	Z	0	0	0	**1**	0
4	**2**	0	1	1	1	**2**	0

Then, $B_1(3)=q_1,[q_2,q_3,q_4]$; $B_2(3)=q_1,q_2$, $B_3(3)=B_4(3)=$ $[q_1,q_6]$. There are three demand structures. The first one is: $A_{1,1}(3)=\{q_1\}$, $A_{1,2}(3)=\{q_2\}$, $A_{1,3}(3)=\{q_3,q_4\}$; $A_{2,1}(3)=\{q_1\}$, $A_{2,2}(3)=\{q_2\}$, $A_{3,1}(3)=\{q_1,q_6\}$ and $A_{4,1}(3)=\{q_1,q_6\}$. It is not possible to find a competitive matching. There are two minimal overdemanded sets: $\{q_2\}$ and $\{q_1,q_6\}$. The second demand structure is: $A'_{1,1}(3)=\{q_1\}$, $A'_{1,2}(3)=\{q_3\}$, $A'_{1,3}(3)=\{q_2,q_4\}$, $A'_{2,1}(3)=\{q_1\}$, $A'_{2,2}(3)=\{q_2\}$, $A'_{3,1}(3)=\{q_1,q_6\}$ and $A'_{4,1}(3)=\{q_1,q_6\}$. The only minimal overdemanded set is $\{q_1,q_6\}$. The third demand structure is given by: $A''_{1,1}(3)=\{q_1\}$, $A''_{1,2}(3)=\{q_4\}$, $A''_{1,3}(3)=\{q_2,q_3\}$, $A''_{2,1}(3)=\{q_1\}$, $A''_{2,2}(3)=\{q_2\}$, $A''_{3,1}(3)=\{q_1,q_6\}$ and $A''_{4,1}(3)=\{q_1,q_6\}$. As before, it is not possible to find a competitive matching. The only minimal overdemanded set is $\{q_1,q_6\}$. The auctioneer must choose a demand structure with the minimum number of minimal overdemanded sets. Suppose the auctioneer chooses *A'*. As a result, he raises the price of all objects of q_1 and q_6 by one unit.

Step 4. $p(4)=(2,1,0,0,0,1,0)$. The matrix of numbers v_{bq}-$p_q(4)$ is given in the table below, with the entries corresponding to the objects of the demand sets in boldface .

	q_1	q_2	q_3	q_4	q_5	q_6	0
1	**5**	**4**	**4**	**4**	2	0	0
2	**3**	**3**	1	0	2	0	0
3	**0**	Z	**0**	**0**	**0**	**0**	**0**
4	**1**	0	**1**	**1**	**1**	**1**	0

Then, $B_1(4)=q_1,[q_2,q_3,q_4]$; $B_2(4)=[q_1, q_2]$; $B_3(4)=[q_1,q_3,q_4,q_5,q_6,0]$ and $B_4(4)=[q_1,q_3,q_4,q_5,q_6]$. There are several demand structures. Under $A_{1,1}(4)=\{q_1\}$, $A_{1,2}(4)=\{q_2\}$, $A_{1,3}(4)=\{q_3,q_4\}$; $A_{2,1}(4)=\{q_1\}$, $A_{2,2}(4)=\{q_2\}$; $A_{3,1}(4)=\{q_1,q_3,q_4,q_5,q_6,0\}$ and $A_{4,1}(4)=\{q_1,q_3,q_4,q_5,q_6\}$, for example, it is not possible to find a competitive matching and the minimal overdemanded set is $\{q_2\}$. However, under $A'_{1,1}(4)=\{q_1\}$, $A'_{1,2}(4)=\{q_3\}$, $A'_{1,3}(4)=\{q_2,q_4\}$ and $A'_{2,1}(4)=\{q_1\}$, $A'_{2,2}(4)=\{q_2\}$; $A'_{3,1}(4)=\{q_1,q_3,q_4,q_5,q_6,0\}$ and $A'_{4,1}(4)=\{q_1,q_3,q_4,q_5,q_6\}$, there is a competitive matching that matches buyer 1 to $\{q_1,q_3,q_4\}$, buyer 2 to $\{q_1,q_2\}$, buyer 3 to q_5 and buyer 4 to q_6. Therefore, the final price is (2,1,0,0,0,1,0).ζ

Remark 4. It is not hard to prove that a set S is minimal overdemanded for $A(M^*)$ if and only if the set of all objects of the sellers in S is minimal overdemanded for M^*. This is because for all $q\in S$, the number of (b,i)'s, loyal demanders of S with $q\in A_{b,i}(p)$, is strictly greater than $s(q)$, so $S(q)=s(q)$. Therefore, our mechanism is able to operate in the market M^*: it is enough to change the vector of prices in R^n by their extension in R^N_+.ν

4.2 MAIN RESULTS

In this section, we demonstrate that the dynamic mechanism yields the minimum competitive equilibrium price for $A(M^*)$.

Theorem 1. *Price vector p is the minimum competitive price for $A(M^*)$.*

Proof: Suppose by way of contradiction that p is not the minimum competitive price. Then, there is some competitive price y such that $p\neq y$ and p is not smaller than y. For each step t of the auction denote $U(t)\equiv\{q\in Q; p_q(t)=y_q\}$. We have that $p(1)=(0,...,0)$, so $p(1)\leq y$. Therefore, since we are working with all integers, there is at least one-step t of the auction such that $U(t)\neq\phi$ and $p(t)\leq y$. From the competitivity of y it follows that there is some demand structure for y with no overdemanded set. Choose one of such demand structures and call it $A^*(y)$.

We need the following technical results whose proofs are left to the Appendix.

Lemma 1. *Let t be some step of the auction at which $U(t)\neq\phi$ and $p(t)\leq y$. Let $A(t)$ be any demand structure at step t under price $p(t)$. Let $T'=\{(b,i); A_{b,i}(t)\cap U(t)\neq\phi\}$. Suppose that $T'\neq\phi$. Then, there is some demand structure $A'(t)$, such that for each $(b,i)\in T'$, there exists some (b,j), with $A^*_{b,j}(y)\subseteq U(t)$, and such that $A^*_{b,j}(y)=A'_{b,i}(t)$, if $i\neq r(b)$ and $A^*_{b,j}(y)\subseteq A'_{b,i}(t)$, otherwise. Furthermore, $A'_{b,i}(t)= A_{b,i}(t)$ for all $(b,i)\notin T'$.*

Lemma 2. *Let t be some step of the auction at which $U(t)\neq\phi$ and $p(t)\leq y$. Let $A(t)$ be any demand structure at step t under price $p(t)$. Let T' and $A'(t)$ be defined as in Lemma 1. Then, a) $A'(t)$ has no minimal overdemanded set containing elements of $U(t)$; b) every minimal overdemanded set for $A'(t)$, if any, is a minimal overdemanded set for $A(t)$.*

Proof of Theorem 1 *(continued).* Let t be the last step of the auction at which $p(t)\leq y$ and let $S_1=\{q\in Q; p_q(t+1)>y_q\}$. Then, $S_1\neq\phi$. Since we are working with all integers, $S_1\subseteq U(t)$. **Let $A(t)$ be the demand structure chosen by the auctioneer at prices $p(t)$ which has the minimum number of minimal overdemanded sets.** Let S be the minimal overdemanded set for $A(t)$ whose prices are raised at stage t+1. Thus, $S=\{q\in Q; p_q(t+1)>p_q(t))\}$, so $S_1= S\cap U(t)$, and so $S\cap U(t)\neq\phi$.

By Lemma 1 and Lemma 2-a, there is some demand structure $A'(t)$, defined from $A(t)$ and $A^*(y)$, that has no minimal overdemanded set containing some element of $U(t)$. Then, S is not a minimal overdemanded set for $A'(t)$. On the other hand, Lemma 2-b asserts that every minimal overdemanded set for $A'(t)$, if any, is a minimal overdemanded set for $A(t)$. Therefore, $A'(t)$ has less minimal overdemanded sets than $A(t)$, contradiction. Hence, p is the minimum competitive price for $A(M^*)$.ζ

Given a competitive price vector p for $A(M^*)$, it is not true that there is an optimal matching which is compatible to it. (To see this, consider one buyer b, two sellers *1* and *2*, every agent with quota one and $v_b=(4, 5)$. Price vector $p=(1, 3)$ is competitive. Buyer b demands only the object of seller *1*, which is allocated to her, and the object of seller *2* is unsold. Price p is not a competitive equilibrium price because the price of the unsold object is not zero. We can also observe that the only optimal matching assigns the buyer to seller 2 and this matching is not compatible with p). Hence, not always competitive prices are competitive equilibrium prices. However,

Theorem 2. *If p is the minimum competitive price for $A(M^*)$, then there is a matching μ^* such that (p, μ^*) is an equilibrium.*

Proof. Let μ be a matching corresponding to p. Call the objects of seller q *overpriced* if q does not complete his quota under μ but $p_q>0$. Suppose (p, μ) is not a competitive equilibrium, so there is at least one seller k whose objects are overpriced. We will give a procedure for altering μ so as to eliminate the overpriced objects of seller k. For this purpose we construct a direct graph whose vertices are $B \cup Q$. There are two types of arcs. If $q \in \mu(b)$ and b likes q' at least as well as q for all $q' \in \mu(b)$ there is an arc from b to q. If q is in $B_b(p)-\mu(b)$ there is an arc from q to b. (Observe that, since every buyer is matched under μ to her favorite set of allowable sellers, it follows that if there is an arc from q to b and an arc from b to q' then b is indifferent between q and q'). Then k is in $B_b(p)$ for some $b \notin \mu(k)$, for if not we could decrease p_k a little bit and still have competitive prices, which contradicts the minimality of p. Let $B^* \cup Q^*$ be all vertices that can be reached by a directed path starting from k, followed by $b_1 \notin \mu(k)$.

Case 1: B^* contains a buyer b such that $\mu(b)$ contains the dummy-seller. Then, there is an arc from b to 0. Let $(k=q_1, b_1,q_2,b_2,q_3,...,q_t,b,0=q_{t+1})$ be a path from k to 0. Then, we may change μ by replacing q_2 by k in $\mu(b_1)$; q_3 by q_2 in $\mu(b_2)$; ..., the dummy-seller q_{t+1} by q_t in $\mu(b)$. Since each b_j is indifferent between q_j and q_{j+1}, for all $j=1,...,t$, the matching is still competitive and k has less one unsold object, and hence he has less one overpriced object.

Case 2: The dummy-seller is not in $\mu(b)$ for every $b \in B^*$. Then, we claim that there must be some q in Q^* such that $p_q=0$, for suppose not. By definition of $B^* \cup Q^*$ we know that if $b \notin B^*$ then $Q^* \cap [B_b(p)- \mu(b)] = \phi$. On the other hand, if $b \in B^*$, $q \notin Q^*$, $q' \in Q^*$ and q and q' are in $\mu(b)$, then b prefers q to q'. Therefore we can decrease the price of the objects of each seller in Q^* by some positive ε and still have competitiveness, contradicting the minimality of p. So choose q in Q^* such that $p_q=0$ and let $(k=q_1, b_1,q_2,b_2,q_3,...,q_t,b_t,q)$ be a path from k to q where $b_1 \notin \mu(k)$. Again change

μ by replacing q_2 by k in $\mu(b_1)$, q_3 by q_2 in $\mu(b_2)$, ..., q by q_t in $\mu(b)$ and leaving one object of q unsold. The resulting matching is still competitive. Again the number of unsold objects of k has been reduced and so does the number of overpriced objects.ν

We have proved that the matching obtained in the auction mechanism can be chosen so that the resulting allocation is a minimum competitive equilibrium for $A(M^*)$. This allocation corresponds to a minimum competitive equilibrium for the competitive market M^*. Hence the following corollary is immediate:

Corollary. *The outcome produced by the auction mechanism allocates the objects to the buyers according to the minimum competitive equilibrium for M^*, which corresponds to the B-optimal stable outcome for M.*

Proposition 1 implies that, under the Q-optimal stable outcome $(u,w;\mu)$, $u_{bq}=u_{bq'}$ for all q and q' in $\mu(b)$. Then every buyer b gets the array of payoffs $(u_b,...,u_b)$ with $r(b)$ repetitions of u_b. Thus, if we change the roles between buyers and sellers in the mechanism we get the Q-optimal stable payoff for the cooperative market M.

5. CONCLUDING REMARKS AND RELATED WORK

In the present paper, we developed a generalization of the buyer-seller market game of Shapley and Shubik to the case where buyers are interested in sets of different objects and sellers own identical objects. For this model we have a nonempty and complete lattice formed by the set of stable payoffs (Sotomayor, 1999). Therefore, there is always an optimal stable payoff for the buyers and another one for the sellers. We used the game to model a related competitive market, proved the existence of competitive equilibria, investigated some of their properties and then provided a dynamic procedure in order to obtain such allocations.

The competitive and noncooperative approaches in Shapley and Shubik model are equivalent because a stable payoff is a competitive equilibrium payoff and vice-versa. This is not the case of the multiple partners assignment game: the optimal stable payoff for the buyers corresponds to the minimum competitive price equilibrium payoff, but the maximum competitive equilibrium payoff cannot be obtained, in general, from the optimal stable payoff for the sellers.

The allocation mechanism presented herein is a generalization of the multi-item auction mechanism of Demange, Gale and Sotomayor (1986). We showed that this mechanism converges, in a finite number of steps, to the minimum competitive equilibrium price that corresponds to the optimal stable payoff for the buyers. By inverting the roles of buyers and sellers, we obtain the optimal stable payoff for the sellers.

Competitive equilibria have been used to produce allocations with desirable properties of fairness and efficiency. For the one-to-one matching market, a generalization of the single-item second-price auction, first described by Vickrey (1961), for the buyer-seller market game proposed by Shapley and Shubik, is considered by Demange (1982) and Leonard (1983). Each buyer submits a sealed bid, listing her valuation for all the items. The auctioneer assigns the objects according to the minimum competitive equilibrium price. For the same market game, a three-step mechanism is presented by Sotomayor (2003): the sellers indicate their reservation prices in the first step, and each buyer, knowing the sellers' choices, selects a value for all the items. The auctioneer then assigns objects according to some preset competitive equilibrium price rule. If no ties exist in the selection of optimal matching, we will have the final allocation. Otherwise, the buyers participate, in the third step, in an auction with a minimum competitive equilibrium price rule, having the price obtained in the second step as the reservation price for the objects. The strategic behavior of buyers and sellers is analyzed under complete information.

The dynamic mechanism of Demange, Gale and Sotomayor (1986) yields a minimum competitive equilibrium price. It is a version of the Hungarian algorithm (see Dantizig, 1963) and a generalization of the English auction.

Alkan (1988) presents a dynamic mechanism for the one-to-one matching market in which the utility functions are piecewise linear. He shows that this mechanism finds an equilibrium price in finitely many steps and approximates an equilibrium price for general continuous utilities.

For a model in which buyers can purchase more than one item, all goods are homogeneous and the consumers have decreasing marginal utilities, Ausubel (1995) presents an auction that yields the minimum competitive equilibrium price. For the case with heterogeneous goods, a generalization of the mechanism of Demange, Gale and Sotomayor (1986) for the many-to-one model (every seller owns only one object and the buyers can purchase more than one item) is presented in Gul and Stacchetti (2000). This generalization applies to models where the utilities of the buyers satisfy the monotonicity property and the gross substitute condition of Kelso and Crawford (1982). Consequently, the mechanism proposed by Gul and Stacchetti can be used in many-to-one markets with additive utilities. These authors use several different concepts and new definitions to determine an overdemanded set, so that its computation, even in quite simple situations with additive utilities, may involve a large number of steps. Such difficulties restrict its implementation in real auctions. Restricted to these markets, our mechanism represents an enormous simplification of the algorithm presented in Gul and Stacchetti. In fact, consider for instance four buyers 1, 2, 3 and 4, and six sellers q_1, q_2, ..., q_6, each of whom owns only one object. The quotas of the buyers are given by the following vectors: v_1=(7,5,4,4,2,1,0), v_2= (5,3,1,0,2,1,0), v_3=(2,0,0,0,0,1,0) and v_4=(3,1,1,1,1,2,0), where the ith coordinate of v_j is the value of the object q_i for buyer j. The initial price is $p(0)$=(0,0,...,0). Then, buyer 1 is indifferent between $\{q_1,q_2,q_3\}$ and $\{q_1,q_2,q_4\}$; buyer 2 wishes $\{q_1,q_2\}$ and buyers 3 and 4 wish $\{q_1\}$. It is not possible to meet the demands of every buyer, so the auctioneer has to find an overdemanded set. By using our mechanism, every buyer is replicated the number of times of her quota. The demand sets, one for each copy, are obtained straightforwardly: $A_{11}(1)=\{q_1\}$, $A_{12}(1)=\{q_2\}$, $A_{13}(1)=\{q_3,q_4\}$, $A_{21}(1)=\{q_1\}$, $A_{22}(1)=\{q_2\}$, $A_{31}(1)=\{q_1\}$, $A_{41}(1)=\{q_1\}$. There are two overdemanded sets: $\{q_1\}$ and $\{q_2\}$, both minimal. The procedure used by Gul and Stacchetti to determine such sets initially involves the computation of four functions, which the authors call "requirement functions," one for each object. The specification of the domain of any of these functions requires the determination of 64 elements; for each of which it is necessary to compute two more numbers. Thus, at the end of the first step of the auction, (64x4)+(64x2)=384 computations will have already been made!

Nevertheless, the model used in the present paper is quite a special case. We believe that the dynamic mechanism proposed by us allows buyers a wider range of preferences. Our results suggest that for small enough bid increases, our mechanism will produce prices approximating the minimum competitive equilibrium price. However, our results concern only the linear surplus case.

APPENDIX

In this section we will demonstrate some of the results stated in section 4. The following remark will be useful for the proofs.

Remark 5. Let t be some step of the auction at which the set $U(t)\neq\phi$ and $p(t)\leq y$. Let $A(t)$ be any demand structure at step t. Write the set of elements of $\boldsymbol{B_b(t)}$ as $\boldsymbol{F(t)\cup G(t)}$ and the set of elements of $\boldsymbol{B_b(y)}$ as $\boldsymbol{F(y)\cup G(y)}$, with $G(t)\neq\phi$, $G(y)\neq\phi$, $F(t)\cap G(t)=\phi$, $F(y)\cap G(y)=\phi$, such that b is indifferent between any two sellers of $G(t)$ (respectively $G(y)$) at price $p(t)$ (respectively y). In addition, b strictly prefers any object of $F(t)$ (respectively $F(y)$), if any, to any object of $G(t)$ (respectively $G(y)$) at price $p(t)$ (respectively y) (Observe that $G(t)$ as well as $G(y)$ may have only one element). It is clear that, for all $j\neq r(b)$, **either $\boldsymbol{A_{b,j}(t)\subseteq F(t)}$ (respectively $\boldsymbol{A^*_{b,j}(y)\subseteq F(y)}$) or $\boldsymbol{A_{b,j}(t)\subseteq G(t)}$ (respectively $\boldsymbol{A^*_{b,j}(y)\subseteq G(y)}$). Also, $\boldsymbol{A_{b,r(b)}(t)\subseteq G(t)}$ (respectively, $\boldsymbol{A^*_{b,r(b)}(y)\subseteq G(y)}$) and if $\boldsymbol{A_{b,j}(t)\subseteq F(t)}$ (respectively $\boldsymbol{A^*_{b,j}(y)\subseteq F(y)}$) then $\boldsymbol{|A_{b,j}(t)|=1}$ (respectively $\boldsymbol{|A^*_{b,j}(y)|=1}$).**

Now, let $C\subseteq Q$. Suppose that $p_q(t)=y_q$ for all $q\in C$, b is indifferent between any two elements of C at prices $p(t)$ (and so is at prices y), $\boldsymbol{C\subseteq G(t)}$ **and** $\boldsymbol{C\subseteq G(y)}$. Then, $F(t)$ (respectively $F(y)$) is the set of all objects that are strictly preferred by b to any object of C at price $p(t)$ (respectively y). Also, $F(t)\cup G(t)$ (respectively $F(y)\cup G(y)$) is the set of all objects that are weakly preferred by b to any object of C at price $p(t)$ (respectively y). It can be shown that

$$\boldsymbol{F(y)\cup G(y) \subseteq F(t)\cup G(t) \text{ and } F(y)\subseteq F(t)} \qquad \textbf{(A.1)}$$

$$\boldsymbol{G(y)\cap U(t)=G(t)\cap U(t) \text{ and } F(y)\cap U(t)=F(t)\cap U(t).} \qquad \textbf{(A.2)}$$

Proof of (A.1). To prove the first inclusion, if $q''\in \boldsymbol{F(y)\cup G(y)}$, then $v_{bq''} - y_{q''} \geq v_{bq}-y_q \;\; \forall q\in C$. Using that $p_q(t)=y_q \;\; \forall q\in C$ and $y_{q''} \geq p_{q''}(t)$, we get that $v_{bq''} - p_{q''}(t) \geq v_{bq''}-y_{q''} \geq v_{bq}-y_q = v_{bq}-p_q(t) \;\; \forall q\in C$, so $q''\in$ $\boldsymbol{F(t)\cup G(t)}$. Hence, $\boldsymbol{F(y)\cup G(y) \subseteq F(t)\cup G(t)}$. The proof of the other inclusion is analogous.

Proof of (A.2). Observe that, if $q\in G(y)\cap U(t)$, then the facts that $p_q(t)=y_q$, b is indifferent between q and any element of C at price y, and $p_{q'}(t)=y_{q'}$ for all $q'\in C$, imply that $v_{bq}-p_q(t)= v_{bq}-y_q= v_{bq'} - y_{q'} = v_{bq'} - p_{q'}(t)$, so b is indifferent between q and any element of C at price $p(t)$, so $q\in G(t)$, so $G(y)\cap U(t)\subseteq G(t)\cap U(t)$. With an analogous argument we prove that $G(t)\cap U(t) \subseteq G(y)\cap U(\mathrm{t})$. We also have that $F(y)\cap U(t) \subseteq F(t)\cap U(t)$, by (A.1). Now, if $q\in F(t)\cap U(t)$ then q is strictly preferred by b to any object of C at price $p(t)$ (because $C\subseteq G(t)$). Since $p_q(t)=y_q$, then $v_{bq}-y_q = v_{bq}-p_q(t)>v_{bq'}-p_{q'}(t)=v_{bq'} - y_{q'}$ for all $q' \in C$, so b strictly prefers q to any element of C at prices y, so $q \in F(y)$ and so $F(t) \cap U(t) \subseteq F(y) \cap U(t)$.

It is also important to point out that,

if $A_{b,i}(t) \subseteq F(t) \cap U(t)$, for some (b,i), then $A_{b,i}(t) = A^*_{b,j}(y)$, for some (b,j). (A.3)

In fact, suppose that $A_{b,i}(t) \subseteq F(t) \cap U(t)$, for some (b,i). Then, $A_{b,i}(t) = \{q\}$ for some q and $q \in F(y) \cap U(t)$, by (A.2), so $A^*_{b,j}(y) = \{q\}$ for some (b,j). ζ

The original proof of Lemma 1 is very technical and long. The details of this proof are with the author, available to the interested readers. We present below a sketch of the proof that shows the main steps of it.

Sketch of the proof of Lemma 1. Define $A'(t)$ as follows. If $(b,i) \notin T'$, set $A'_{b,i}(t) \equiv A_{b,i}(t)$. If for all $(b,i) \in T'$ there is some (b,j), such that $A^*_{b,j}(y) \subseteq A_{b,i}(t) \cap U(t)$, define $A'_{b,i}(t) \equiv A_{b,i}(t)$ for all $(b,i) \in T'$ and we are done. Otherwise, there is some $(b,i) \in T'$, with $A_{b,i}(t) \equiv C \cup E$, where $C = A_{b,i}(t) \cap U(t)$, such that,

for all (b,j), $A^*_{b,j}(y)$ is not contained in C. (1)

We want to show that it is possible to define $A'(t)$ so that, for all $(b,j) \in T'$, with $j \neq r(b)$, there exists some (b,k) such that $A'_{b,j}(t) = A^*_{b,k}(y) \subseteq U(t)$; if $(b,r(b)) \in T'$, there exists some (b,k) such that $A^*_{b,k}(y) \subseteq A'_{b,r(b)}(t) \cap U(t)$. The plan of the proof is the following: By defining $F(t)$, $G(t)$, $F(y)$ and $G(y)$ as in Remark 5, we first show that $\boldsymbol{C \subseteq G(t)}$ and $\boldsymbol{C \subseteq G(y)}$, so we have satisfied all the hypothesis of Remark 5 and so (A.1) and (A.2) hold. From $\boldsymbol{C \subseteq G(y)}$ it also follows that $C \subseteq B_b(y)$, so all of C must be demanded by b at prices y, so every element of C must be in some $A^*_{b,j}(y)$ for some copy (b,j). Then, by (1), we conclude that such a copy of b is $(b,r(b))$. Then, $A^*_{b,r(b)}(y) = C \cup D$, with $D \cap C = \phi$ and $D \neq \phi$ by (1). It is clear that $A^*_{b,r(b)}(y) \subseteq G(y)$, because $(b,r(b))$ is the last copy of b. Now set:

$\Gamma \equiv \{ (b,j);\ A_{b,j}(t) \subseteq G(t) \text{ and } A_{b,j}(t) \cap U(t) \neq \phi\}$

$\Gamma' \equiv \{ (b,j);\ A^*_{b,j}(y) \subseteq G(y) \text{ and } A^*_{b,j}(y) \cap U(t) \neq \phi\}$

$\Im \equiv \{ (b,j);\ A_{b,j}(t) \subseteq G(t) \text{ and } A_{b,j}(t) \cap U(t) = \phi\}$

$\Im' \equiv \{ (b,j);\ A^*_{b,j}(y) \subseteq G(y) \text{ and } A^*_{b,j}(y) \cap U(t) = \phi\}$

We have that $\Gamma \neq \phi$, since $(b,i) \in \Gamma$. Also, $\Gamma' \neq \phi$, since $(b,r(b)) \in \Gamma'$.

The next step is to define a one-to-one map f from $\Gamma - \{(b,r(b)\}$ into $\Gamma' - \{(b,r(b)\}$. This can be done by establishing that $|\Gamma| \leq |\Gamma'|$ and $(b,r(b)) \in \Gamma$. Then, define

$\boldsymbol{A'_{b,j}(t) \equiv A_{b,j}(t)}$ **if** $\boldsymbol{A_{b,j}(t) \subseteq F(t)}$ **or** $\boldsymbol{(b,j) \in \Im}$**.**

$\boldsymbol{A'_{b,j}(t) \equiv A^*_{f(b,j)}(y)}$ **if** $\boldsymbol{(b,j) \in \Gamma - \{(b,r(b))\}}$**.**

$$\boldsymbol{A'_{b,r(b)}(t) \equiv G(t) - \bigcup_{j \neq r(b)} A'_{b,j}(t).}$$

To see that $A'(t)$ is well defined and is the desired demand structure, use (A.1) and (A.2) of Remark 5.ν

Proof of Lemma 2. For part a), suppose by way of contradiction that S is a minimal overdemanded set for $A'(t)$ and $S_1 \equiv S \cap U(t) \neq \phi$. Let T be the set of loyal demanders of S. The fact that S is overdemanded means exactly that

$$|\boldsymbol{T}| > \sum_{q \in S} S(q)| \qquad \textbf{(1)}$$

We will show that S-S_1 is non-empty and overdemanded for $A'(t)$, so S is not a minimal overdemanded set for $A'(t)$, which is a contradiction. To see this, define $T_1 = \{(b,i) \in T;\ A'_{b,i}(t) \cap S_1 \neq \phi\}$. Let T' be as defined in Lemma 1. Now, observe that $T_1 \subseteq T'$. In fact, if $(b,i) \notin T'$ then $A_{b,i}(t) = A'_{b,i}(t)$, so $A'_{b,i}(t) \cap U(t) = \phi$, and so $(b,i) \notin T_1$. By Lemma 1, for each $(b,i) \in T_1$ there is some (b,j), such that $A^*_{b,j}(y) \subseteq A'_{b,i}(t) \cap U(t)$. On the other hand, the fact that $(b,i) \in T$ implies that $A'_{b,i}(t) \subseteq S$, so $A'_{b,i}(t) \cap U(t) = A'_{b,i}(t) \cap S_1$. Then, $A^*_{b,j}(y) \subseteq A'_{b,i}(t) \cap S_1$, so $A^*_{b,j}(y) \subseteq S_1$. Thus, since $A'_{b,i}(t) \cap A'_{b,k}(t) = \phi$ if $i \neq k$,

$$|\boldsymbol{T_1}| \leq |\{(\boldsymbol{b,j});\ \boldsymbol{A^*_{b,j}(y) \subseteq S_1}\}| \leq \sum_{q \in S_1} S_1(q)\,, \qquad \textbf{(2)}$$

where the last inequality is due to the competitivity of y. But then, (1) and (2) imply that $|T$-$T_1| = |T| - |T_1| > \sum_{q \in S} S(q) - \sum_{q \in S_1} S_1(q) = \sum_{q \in S - S_1} S(q) \geq 0$, from which follows that T-$T_1 \neq \phi$. However, T-$T_1 = \{(b,i) \in T;\ A'_{b,i}(t) \subseteq S\text{-}S_1\}$, so S-S_1 is non-empty and overdemanded for $A'(t)$, as we wanted to show.

For part b), suppose that $A'(t)$ has overdemanded sets. Let S be some minimal overdemanded set for $A'(t)$. Let T be the set of loyal demanders of S. Let T' be as defined in Lemma 1. By part a), $S \cap U(t) = \phi$. Then, if $(b,i) \in T$, $A'_{b,i}(t) \subseteq S$, so $A'_{b,i}(t) \cap U(t) = \phi$, so $(b,i) \notin T'$. By Lemma 1, $A'_{b,i}(t) = A_{b,i}(t)$. Then, for all $(b,i) \in T$, $A_{b,i}(t) \subseteq S$. Then, T is a set of loyal demanders of S under $A(t)$ and $min\{s(q)$, number of $(b,i) \in T$ with $q \in A_{b,i}(t)\} = min\{s(q)$, number of $(b,i) \in T$ with $q \in A'_{b,i}(t)\}$. Hence, S is also minimal overdemanded for $A(t)$, and the proof is complete. ζ

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