An Extended No Bet Theorem

Zhen Liu *†

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Abstract

Given two players holding a common prior and distinct information partitions, the No Bet theorem says that when at a state it is common knowledge that one's conditional expectation is no less than a certain number but the other one's is not greater than it, their conditional expectations must be the same. The extended theorem generalizes the result by taking away the separating number which is sometimes not available in practice. We also generalize the extended theorem to the case where priors are heterogeneous, and find it can not be generalized with nonpartitional information structures. As the applications, we show if the ranking of each player's expectation, or just the identity of the highest(lowest) one is common knowledge, players must agree in the logic of the extended theorem, but may not agree under the original No Bet theorem.

1 Introduction

Initiated by Aumann's "agreeing to disagree" [1976], lots of literatures has shown consensus among people at common knowledge of certain information under a set of assumptions. An intuitive surprise behind it is that with incomplete information about states of nature, players can derive more information from known differences in their opinions or behavior, without exchanging private information; and commonly known differences ultimately eliminate differences. Theoretical economists use this notion to simplify the complication of incomplete information games, see Geanakoplos [1994].

*SUNY at Stony Brook, zheliu@ic.sunysb.edu.

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These results rely on common knowledge about individuals' private estimates or statistics of them. Existing examples are conditional posterior in Aumann [1976], conditional expectation in Polemarchakis and Geanakoplos [1982], strictly monotonic aggregate statistic of above estimates in McKelvey and Page [1986] and Nielsen et al. [1990]. Meanwhile, some results only require common knowledge about the inequality of individuals' private estimates. An interesting example in this category is the No Bet theorem in Sebenius and Geanakoplos [1983], which presents that if it is common knowledge that one player's expectation is not less than a separating number and the other one's expectation is not greater than it, then their expectations must be the same.[1] The other example is the No Trade result by Milgrom and Stokey [1982], which is noted in Rubinstein and Wolinsky [1990] as a special case of No Bet theorem.

We extend the No Bet theorem by taking away the separating number. If it is common knowledge at a state that one individual's expectation is not less than the other's expectation, then these expectations should be the same. We also generalize the theorem to the case where priors are heterogeneous; however in the case where information structures are nonpartitional, even though the original No Bet theorem is hold, the extended one is not.

After all, our result provides an insight into the situations where less information about players' opinions are available thus the No Bet theorem is not applicable, e.g., given the ranking of each player's expectation or just the identity of the highest(lowest) expectation being common knowledge.

In section 2 we adopt a semantic formalization of common knowledge by Aumann [1976] and give an alternative proof of No Bet theorem. Then in section 3 we use similar method to prove the extended version. We discuss the cases when priors are different and information structures are nonpartitional in section 4. Implications which can be applied in auctions or speculation are introduced in section 5.

While deriving these results, we discovered that similar results have already been obtained by Hanson [1998]. We compare the differences between our work and his, as well as the conclusion, in the last section of this paper.

[1] This theorem is called the No Bet theorem because it can be interpreted as saying that it cannot be common knowledge between two risk-neutral individuals who both expect to profit from a bet. If it is common knowledge that they both expect to gain from the bet, then it is common knowledge that for A the expectation of his payoff is positive and for B the expectation of her payoff is negative, while the sum of both is zero.

2 Common Knowledge and the No Bet Theorem

We follow the definitions as Aumann [1976]. Consider two players A and B in a closed small world, meaning that the state space Ω is a finite set. An event is a subset of Ω. We assume that A and B both believe in a common prior probability distribution P over Ω, which is positive for every state; and they hold different information partitions Π_A and Π_B. Members of each partition are exhaustive and disjoint. So if an arbitrary state $\omega \in \Omega$ is observed by A, she knows that any event containing $\Pi_A(\omega)$ happens, because she can not exclude any state ω' in $\Pi_A(\omega)$ other than ω. Meanwhile B also knows any event containing $\Pi_B(\omega)$ happens.

We call an event *common knowledge at* ω when everyone in the scenario knows it, everyone knows that everyone knows it, and so on ad infinitum. We want to emphasize that in this case *knowing* something means logically deriving something. Thus if some news is literally public, the content of news is not necessarily common knowledge, since it might contradict logical consequences of some players' knowledge.

The meet $\Pi \equiv \Pi_A \wedge \Pi_B$ is the finest common coarsening of Π_A and Π_B; the join $\hat{\Pi} \equiv \Pi_A \vee \Pi_B$ is the coarsest common refinement of Π_A and Π_B. According to the insightful characterization of common knowledge by Aumann [1976], given a state $\omega \in \Omega$, an event E is common knowledge at ω if and only if E contains the member of the meet Π that contains ω or in short, $\Pi(\omega) \subseteq E$. As usually defined, the conditional expectation of random variable X for player A is

$$f_A(\omega) = E(X|\Pi_A(\omega)) = \sum_{\omega' \in \Pi_A(\omega)} x(\omega')P(\omega'|\Pi_A(\omega)),$$

Clearly, $f_B(\omega)$ is similarly defined over Π_B.

Here we prove the No Bet theorem, the proof is different from the original one in Sebenius and Geanakoplos [1983].

Theorem 2.0.1 *If it is common knowledge at* ω *that* $f_A(\omega) \geqslant a$ *and* $f_B(\omega) \leqslant a$*, then* $f_A(\omega) = f_B(\omega) = a$*.*

Proof. Let $\{\Pi_A^i\}$ be the set of all members of Π_A contained in $\Pi(\omega)$, the $\{\Pi_B^j\}$ be the set of all members of Π_B contained in $\Pi(\omega)$. By definition for every member of Π_A, $f_A(\omega)$ is constant for any ω in it, then for any $\omega' \in \Pi_A^i$

$$\sum_{\omega' \in \Pi_A^i} f_A(\omega') P(\omega' | \Pi(\omega)) = f_A(\omega') P(\Pi_A^i | \Pi(\omega)),$$

thus we can derive

$$\begin{aligned} E(X|\Pi(\omega)) &= \textstyle\sum_{\omega' \in \Pi(\omega)} x(\omega') P(\omega' | \Pi(\omega)) \\ &= \textstyle\sum_{\Pi_A^i \subseteq \Pi(\omega)} \left[\sum_{\omega' \in \Pi_A^i} x(\omega') P(\omega' | \Pi_A^i) \right] P(\Pi_A^i | \Pi(\omega)) \\ &= \textstyle\sum_{\Pi_A^i \subseteq \Pi(\omega),\, \Pi_A^i \ni \omega'} f_A(\omega') P(\Pi_A^i | \Pi(\omega)) \\ &= \textstyle\sum_{\Pi_A^i \subseteq \Pi(\omega)} \sum_{\omega' \in \Pi_A^i} f_A(\omega') P(\omega' | \Pi(\omega)) \\ &= \textstyle\sum_{\omega' \in \Pi(\omega)} f_A(\omega') P(\omega' | \Pi(\omega)) \end{aligned}$$

Similarly $E(X|\Pi(\omega)) = \sum_{\omega' \in \Pi(\omega)} f_B(\omega') P(\omega' | \Pi(\omega))$.

It is common knowledge at ω that $f_A(\omega) \geqslant a$, then the meet

$$\Pi(\omega) \subseteq \{\omega' | f_A(\omega') \geqslant a\},$$

meaning $f_A(\omega') \geqslant a$ for every $\omega' \in \Pi(\omega)$.

So it must be true that

$$E(X|\Pi(\omega)) = \sum_{\omega' \in \Pi(\omega)} f_A(\omega') P(\omega' | \Pi(\omega)) \geqslant a,$$

and by the same reason,

$$E(X|\Pi(\omega)) = \sum_{\omega' \in \Pi(\omega)} f_B(\omega') P(\omega' | \Pi(\omega)) \leqslant a.$$

Then $E(X|\Pi(\omega)) = a$. Because $f_A(\omega') \geqslant a$ for all $\omega' \in \Pi(\omega)$, we must have $f_A(\omega') = a$ for all $\omega' \in \Pi(\omega)$. Naturally $f_A(\omega) = a$. Similarly $f_B(\omega) = a$, so $f_A(\omega) = f_B(\omega) = a$. ■

3 An Extended Version.

Indeed, we can go one step further to remove the separating number, thus we have an extended No Bet theorem.

Theorem 3.0.2 *If it is common knowledge at ω that $f_A(\omega) \geqslant f_B(\omega)$, then $f_A(\omega) = f_B(\omega) = E(X|\Pi(\omega))$.*

Proof. As in the proof of theorem 2.0.1,

$$E(X|\Pi(\omega)) = \sum_{\omega' \in \Pi(\omega)} f_A(\omega')P(\omega'|\Pi(\omega)) = \sum_{\omega' \in \Pi(\omega)} f_B(\omega')P(\omega'|\Pi(\omega)).$$

So

$$\sum_{\omega' \in \Pi(\omega)} [f_A(\omega') - f_B(\omega')]P(\omega'|\Pi(\omega)) = 0. \tag{1}$$

Because it is common knowledge that $f_A(\omega) \geqslant f_B(\omega)$, the meet

$$\Pi(\omega) \subseteq \{\omega' | f_A(\omega') - f_B(\omega') \geqslant 0\}.$$

Because the possibility of every state is positive, we know to hold equation 1, for every $\omega' \in \Pi(\omega)$ it must be true that

$$f_A(\omega') = f_B(\omega').$$

It is then obvious that $f_A(\omega) = f_B(\omega)$.

Moreover, let's define that two different sets Π_A^i and $\Pi_A^{i'}$ are *connected* via a set Π_B^j, if $\Pi_A^i \cap \Pi_B^j \neq \phi$ and $\Pi_A^{i'} \cap \Pi_B^j \neq \phi$. When Π_A^i and $\Pi_A^{i'}$ are connected via Π_B^j, there are two elements $\omega \in \Pi_A^i$ and $\omega' \in \Pi_A^{i'}$, both in Π_B^j, then it must be true that

$$f_A(\omega) = f_B(\omega) = f_B(\omega') = f_A(\omega').$$

For $\Pi(\omega)$, all the members of Π_A are connected via some members of Π_B and vice versa. Therefore $f_A(\omega) = f_B(\omega) = C$, C is a constant, for every $\omega \in \Pi(\omega)$. Also $C = f_A(\omega) = f_B(\omega) = E(X|\Pi(\omega))$. ■

4 Discussions

4.1 On Heterogeneous Priors

In our framework, we assume players hold a common prior though it is subjective. The effect of heterogeneous priors has been well investigated by Morris [1994]. Similar to his report, within the context of our paper, the no bet result still holds with limited heterogeneous priors. Before the rigorous statement, an example can present the idea.

Let the space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\Pi_A = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}$, $\Pi_B = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$. Let A and B's prior probability distributions be

State	ω_1	ω_2	ω_3	ω_4
P_A	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
P_B	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$

Given $\Pi(\omega_2)$ is common knowledge, player A and player B hold the same conditional probability $\frac{1}{3}$ for every state in $\Pi(\omega_2)$, though their priors are different.

Remark 4.1.1 *With everything else unchanged, as long as there is no difference in the beliefs about the member of meet which contains the state, i.e., the minimum event that is common knowledge, players with heterogeneous prior beliefs can not bet.*

Proof. Recall that at state ω, every common knowledge event contains $\Pi(\omega)$. Although players A and B hold heterogeneous priors P_A and P_B respectively, since there is no difference in the beliefs about events that are common knowledge, i.e. $P_A(\omega'|\Pi(\omega)) = P_B(\omega'|\Pi(\omega))$, $\forall \omega' \in \Pi(\omega)$. Hence

$$\sum_{\omega' \in \Pi(\omega)} x(\omega') P_A(\omega'|\Pi(\omega)) = \sum_{\omega' \in \Pi(\omega)} x(\omega') P_B(\omega'|\Pi(\omega)),$$

which can be transformed into

$$\sum_{\omega' \in \Pi(\omega)} f_A(\omega') P_A(\omega'|\Pi(\omega)) = \sum_{\omega' \in \Pi(\omega)} f_B(\omega') P_B(\omega'|\Pi(\omega)).$$

Now because there is no difference in $P_A(\omega'|\Pi(\omega))$ and $P_B(\omega'|\Pi(\omega))$, $\forall \omega' \in \Pi(\omega)$ and also $f_A(\omega') \geqslant f_B(\omega')$, $\forall \omega' \in \Pi(\omega)$, we obtain that $f_A(\omega) = f_B(\omega)$.■

4.2 On Nonpartitional Information Structure

In general an information structure can be taken as a function I which tells the player at state ω that all elements in $I(\omega)$ are possible. As a basic tool to model cognition, or intuitively the process deriving information from environmental signals, this function should be able to describe not only logically correct cognition, but also cognitive errors. It is well-known that information partition is logically correct since it entails three important properties of function I:

(P-1) For all $\omega \in \Omega$, $\omega \in I(\omega)$. That is, if the player knows the set X then X is true.

(P-2) For all $\omega \in \Omega$ and for all $\omega' \in I(\omega)$, $I(\omega') \subseteq I(\omega)$. That is, if the player knows X he also knows that he knows X.

(P-3) For all $\omega \in \Omega$ and for all $\omega' \in I(\omega)$, $I(\omega) \subseteq I(\omega')$. That is, if the player does not know X he also knows that he does not know X.

If any one of above properties is relaxed, I is no longer a partition structure.

By a counter example, Geanakoplos [1989] shows if I only satisfy **(P-1,2)**, the no bet result can not hold. Let the space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, where all states are equally likely. Assume $I_A = \{\omega_1, \omega_2, \omega_3\}$; $I_B(\omega_1) = \{\omega_1, \omega_2\}$, $I_B(\omega_2) = \{\omega_2\}$, $I_B(\omega_3) = \{\omega_2, \omega_3\}$. Note here I_B is not partitional.

Then let random variable X be valued as $X(\omega_2) = 1$ and $X(\omega_1) = X(\omega_3) = 0$. For all ω it is common knowledge that player 1 knows his expectation is $\frac{1}{3}$ and player 2 knows her expectation is 0.5 or 1, which is strictly greater than $\frac{1}{3}$.

Let's define common knowledge with general information structures. An event S is called *self-evident at* ω for a player with information structure I if $\omega \in S$ and for all $\omega' \in S$, $I(\omega') \subseteq S$.

Proposition 4.2.1 *If both I_A and I_B satisfy **(P-1)**, for players A and B the event E is common knowledge at ω if and only if it contains a self-evident event S such that $\omega \in S$.*

Geanakoplos [1989] and Rubinstein [1998] show it is equivalent to the definition of Aumann [1976]. [2] Using this definition, Geanakoplos [1989] shows the no bet result with a separating number is reserved under nonpartitional structure and certain restrictions.

Proposition 4.2.2 *(Geanakoplos:1989) If I_A and I_B satisfy **(P-1,2)** and is nested, i.e., if $\omega \neq \omega'$ either $I(\omega) \subseteq I(\omega')$ or $I(\omega') \subseteq I(\omega)$ or $I(\omega) \cap I(\omega') = \phi$, together with the common prior assumption, it can not be common knowledge at ω that $f_A(\omega) > a$ and $f_B(\omega) < a$.*

However, with the same assumption, our result is no longer true. It can be demonstrated in a short example.

Let $S = \{\omega_1, \omega_2\}$, each state happens with equal probability. Let $I_A(\omega_1) = \{\omega_1\}$, $I_A(\omega_2) = S$, $I_B(\omega_1) = S$, $I_B(\omega_2) = \omega_2$. As to the random variable X,

[2] We have a better result which claims that without **(P-1)**, we can also define common knowledge by self-evident event. But given common knowledge it is not necessary to obtain self-evident event. We will show the proof and use this finding later.

$X(\omega_1) = 1$, $X(\omega_2) = 0$. We see $f_A(\omega_1) = 1$, $f_A(\omega_2) = f_B(\omega_1) = 0.5$, $f_B(\omega_2) = 0$. So it is common knowledge at both states that $f_A(\omega) > f_B(\omega)$.

5 A Few Implications

The main contribution of our result is to show that players need less information to reach agreement. Apparently knowing whose expectation is greater or smaller is weaker than knowing the existence of a bet with a known separating number. This implies some interesting observations which may be applied in analysis of auction, speculation trade or other incomplete information games. Please note that in this section we come back to the assumption of common prior and information partition.

To present the implications better we define a pair (I, Q) as a common knowledge graph, where $I = \{i\}, i = 1, \cdots, N$ is the set of individuals, Q is a set of 2-elements subsets of I. Each element of I, an individual, is a *vertex* in the graph. Each element of Q is called an common knowledge *edge*, satisfying the property that for two individuals in the same edge, it is common knowledge at a state that one individual's conditional expectation is not less than the other's. So by theorem 3.0.2, for every two individuals as two vertices in the same edge in Q, their conditional expectations must be the same. Consequently we can say that

Corollary 5.0.3 *In a common knowledge graph (I, Q), for all N vertices in I, if any two of them can be either directly or indirectly connected by edges in Q, then the conditional expectations of all N individuals must be equal.*

Proof. Just apply theorem 3.0.2. We know for any i,j , $i \neq j \in I$ connected by one edge, $f_i(\omega) = f_j(\omega)$. Then for all $i \in N$ connected by edges, $f_i(\omega)$ must be equal. ■

Then we can show what will happen if the ranking of people's conditional expectations are common knowledge.

Corollary 5.0.4 *If at ω the ranking of $f_i(\omega)$, every $i \in N$, is common knowledge for all individuals in I, then they must be equal.*

Proof. If the ranking is common knowledge at ω for all individuals, then for arbitrary two individuals it is common knowledge at ω that one's expectation is not less than the other's. This is equivalent to a common knowledge graph where arbitrary two vertices are connected by one edge. By corollary 5.0.3, all individuals' conditional expectations must be equal. ■

Thus in auction if the ranking of all bidders' expected values is common knowledge, and they have a common prior about the value of object, then their valuations should be equal.

Hanson [1998] stated and proved a more interesting result saying the identification of "extremists" implies consensus — if the identity of the individual who gives the highest or the lowest estimate is common knowledge at ω, all individuals should agree. This result is just an implication of corollary 5.0.3.

Corollary 5.0.5 *If at ω the identity of the individual whose conditional expectation is the maximum or the minimum is common knowledge for all individuals in I, then their conditional expectations must be equal.*

Proof. If it is individual i whose estimate is the maximum, and this is common knowledge at ω, then it is common knowledge for individuals i and j that $f_i(\omega) \geqslant f_j(\omega)$ for all $j \in I, j \neq i$. So in a common knowledge graph, i's vertex is connected by edges to all other vertices. It can be seen any two vertices are therefore connected by edges. By corollary 5.0.3, all individuals' conditional expectation must be equal. A similar argument can be made when individual i's estimate is the minimum.

So in auction if it is common knowledge who is the bidder with the highest or the lowest expected value, all bidders must agree, there is no real "extremist" actually!

6 A Comparison As A Conclusion

It is necessary to compare the difference between this paper and Hanson's paper. In both of them, the extended No Bet theorem has been proved, the identification of the player with the maximum or minimum expectation deduces consensus is also obtained as an application. The difference is his work also emphasizes the aspects of communication and learning process via which players' opinions converge. He shows that for n players to do so at most $\log_2 n$ bits information about the extremist is required to broadcast, which is a bounded and moderate message length in contrast to other types of message.

What is only shown in our paper is many details about the assumptions, the major findings are: 1) generalization with differences in players' priors; 2) not being able to generalize with nonpartitional information structure; 3) common knowledge of the ranking of conditional expectations also leads to consensus.

References

R. Aumann. Agreeing to disagree. *Annal of Statistics*, 4:1236–1239, 1976.

J. Geanakoplos. Game theory without partitions, and applications to speculations and consensus. *Cowles Foundation Discussion Paper*, (914), 1989.

J. Geanakoplos. *Handbook of Game Theory*, volume 2, chapter 40, pages 1439–1496. Elsevier Science B.V., 1994.

R. D. Hanson. Consensus by identifying extremists. *Theory and Decision*, 44 (3):293–301, 1998.

R. McKelvey and T. Page. Common knowledge, consensus and aggregate information. *Econometrica*, 54:109–127, 1986.

P. Milgrom and N. Stokey. Information, trade and common knowledge. *Journal of Economic Theory*, 26:17–27, 1982.

S. Morris. Trade with heterogeneous prior beliefs and asymmetric information. *Econometrica*, 62:1327–1347, 1994.

L. T. Nielsen, A. Brandenburger, J. Geanacopolos, R. McKelvey, and T. Page. Common knowledge of an aggregate expectations. *Econometrica*, 58(5):1235–1239, 1990.

H. Polemarchakis and J. Geanakoplos. We can't disagree forever. *Journal of Economic Theory*, 28:192–200, 1982.

A. Rubinstein. *Modelling Bounded Rationality*, chapter 4, page 59. The MIT Press, 1998.

A. Rubinstein and A. Wolinsky. On the logic of "agreeing to disagree". *Journal of Economic Theory*, 51(1):184–193, 1990.

J. Sebenius and J. Geanakoplos. Don't bet on it: Contingent agreements with asymmetric information. *Journal of the American Statistic Association*, 78: 424–426, 1983.