Theory of Quantum Games

Vladislav Kargin
(Cornerstone Research, 599 Lexington Avenue floor 43, New York, NY 10022; skarguine@cornerstone.com, 212-605-5018)

Recent progress in quantum information theory has stimulated a surprising development in game theory. [See references at the end]. In this development game theory has been applied to the analysis of conflict situations, where the outcome depends on a quantum state. For example in quantum detection theory, it is natural to define games in which Nature chooses a quantum state and the researcher chooses the measurement of the state. This leads to questions about the min-max and equilibrium solutions of these games. In this talk I will explain what are the quantum games and what are the questions that arise in their study.

States of quantum-mechanical objects -- electrons, photons, atoms, molecules, etc. -- are described by density matrices. A density matrix is a self-adjoint, non-negative, trace-class operator of a complex Hilbert space with the trace of 1. A particular case is formed by projectors on one-dimensional subspaces, which are called pure states. They play a role of elementary objects in quantum theory.

A measurement of a quantum state can be represented by a measure that takes values in non-negative operators. For examples, a measure on a finite space of events gives a probability of outcome i as trace(rho*M_i) where rho is the state and operator M_i represent the outcome. In problems of quantum detection, the researcher devises a measurement to find out the most information about what is the quantum state.

For example, one possible game of quantum detection is as follows. Consider the situation in which a particular pure quantum state is chosen (with known probabilities) from a set of pure quantum states. This state is given to player N ("Nature") who can add limited amount of noise to it. Player R ("Researcher") chooses a measurement on the state. The realization of the measurement is random and the task of the R is to guess the initial quantum state with the minimal probability of error. This setup extends Wald's approach to hypothesis testing to the quantum situation. Does the min-max theorem hold? Is there an equilibrium? What are min-max optimal strategies in particular situations?

Another similar example arises when nature can select either a given quantum state rho_0 or an arbitrary state rho_1 that have a suitably defined distance of at least D from rho_0. The researcher can perform K measurements on K identical instances of the chosen state and his task is to guess whether the state is rho_0 or not. The question is how the maxmin detection probability is related to K and D. This is a generalization of classical statistical problem of testing whether a sample is drawn from a particular distribution. The quantum version of the problem is relevant for quantum cryptography, where an eavesdropper wants to interfere with a signal transmission but have to minimize the probability of detection.

This setup provides an interesting application of the usual game-theoretic methods to situations with very peculiar strategy spaces. In some of the games, it is natural to model the choice variable of each player as a quantum state, that is, as a linear operator of a Hilbert space. In others, each player can choose an operation on a given quantum state, that is, an operator that acts on the Hilbert space operators. Building a sufficiently general theory of quantum games seems to be an interesting task.

In this theory we would want to have explicit solutions to games and theorems like the following:
"Theorem": If the set of players' strategies in a two-player zero-sum quantum game is convex, then minmax and maxmin solutions of the game exist and are equal.

Another interesting topic arises in quantum games with shared quantum entanglement. Entanglement is a uniquely quantum concept that allows two remote particles exhibit correlated behavior. Recall that the concept of the Nash equilibrium is based on players choosing a randomized strategy independently of each other. The correlated equilibrium gives the players an additional opportunity for interaction so that the players can observe an outcome of a joint random variable. In the quantum setting each player can have an access to a part of an entangled quantum state. Consequently, measurements on this state are correlated even if the two players cannot communicate between themselves. This begs for comparison with classical situation.

Finally, a completely undeveloped topic is whether quantum computer can be used for solving classical combinatorial games. Quantum algorithms were proved to be very effective in some situations where classical algorithms failed: for example for factoring large numbers and for searching in a large database. It is an open question whether these methods can be useful for analysis of classical combinatorial games like chess.

S C Benjamin and P M Hayden 2001 Multiplayer quantum games Phys. Rev. A 64 030301
J Du et al. 2002 Experimental realization of quantum games on a quantum computer Phys. Rev. Lett. 88 137902
J Eisert, M Wilkens and M Lewenstein 1999 Quantum games and quantum strategies Phys. Rev. Lett. 83 3077-3080
N F Johnson 2001 Playing a quantum game with a corrupt source Phys. Rev. A 63 020302
R Kay, S C Benjamin and N F Johnson 2001 Evolutionary quantum game J. Phys. A 34 L547
S E Landsburg 2004 Quantum game theory, Notices of AMS 51 394-399
C F Lee and N F Johnson 2002 Quantum games: a theoretical formalism arXiv.org/quant-ph/abs/0207012
D A Meyer 1999 Quantum strategies Phys. Rev. Lett. 82 1052-1055