A Folk Theorem on Equilibrium Selection

Wei-Torng Juang*and Hamid Sabourian†

February 13, 2004

Abstract

We studies behaviours of evolutionary systems in the long run. The platform consists of a class of systems, each of them converges to some steady state within finite time. For any arbitrarily prescribed stage game equilibrium, we can always add one rule in each population of the system such that the newly added rule combination is universally dominant and can support the given equilibrium. Therefore, a folk theorem results and any Nash equilibrium of the stage game can be selected in the long run. Journal of Economic Literature Classification Numbers: C72, C73, D81.

1 INTRODUCTION

When playing a game, a player usually chooses his/her actions according to some explicitly or implicitly specified rules. A rule prescribes an action (pure or mixed) for each given history. Rules are quite common in our society, sometimes in different forms. Schools adopt some policies in processing applications from prospective students and in recruiting teachers. Although school administrations do not change their admission or recruiting policies very often, they do revise these policies intermittently. In playing a game, players not only change their actions, but also revise rules that direct them how to choose actions. We construct a model in which players may revise their rules in playing a game and study the asymptotic behavior of the play.

*Institute of Economics, Academia Sinica, Taipei 115, Taiwan

†Fellow of King's College and Reader of Faculty of Economics and Politics, University of Cambridge

One major distinction of our model from the literature is that we allow multiple rules and agents may revise their rules. Thus, this paper centers not only on the evolution of actions, but also on the evolution of rules. There are largely two motivations for this line of work.

The first reason concerns the rationality of economic agents. Most literature focuses on models with fixed rules. The fact is that, sometimes agents may shop around for different rules, pick up one of them and use it to help them make decisions. We attempt to make a small step from action updating towards rule adopting in an evolutionary framework.

The second reason concerns equilibrium refinements. In games with multiple equilibria, different rules may be prone to select different equilibria after introducing mutations in choosing among actions. Kandori, Mailath and Rob (1993) (hereafter KMR) consider a 2×2 coordination game where agents adopt the myopic best response. This rule best responds to the frequency of actions at the last period. They conclude that under this setup, players tend to coordinate on the risk dominant equilibrium. On the other hand, Robson and Vega-Redondo (1996) study a model similar to KMR's except that players use a naive imitation rule. The naive imitation rule prescribes the action that received the highest average payoff at the last period. Under their framework, the efficient equilibrium, rather than the risk-dominant one, is selected. Juang (2002) studies a model in which both the myopic best response rule and the naive imitation rule coexist and agents in the population can revise their rules. He finds that in the long run, the states in which all agents coordinate on the efficient outcome dominate most of the time. It seems that rules prevailing in a population play an important role in selecting among multiple equilibria. Consequently, we try to accommodate a class of rules as general as possible and examine the relation between rule profiles and equilibrium selection. Such an attempt to learn more about evolutionary systems with multiple rules might shed a little light on the selection of multiple equilibria in the long run.

The evolution of biological and economical systems has been receiving more and more attention. After Maynard Smith and Price (1973) and Maynard Smith (1982), there has been abundant literature on this line of work. Some examples are Binmore and Samuelson (1993, 1995), Binmore, *et al.* (1994), Foster and Young (1990), Fudenberg and Harris(1992), Young (1993), Ellison (1993, 1994), Bergin and Lipman (1996), Blume (1995) and Kandori and Rob (1995), to name a few. The above papers mainly employ the concept of evolution on the level of actions. On the other hand, there is little literature concerning the evolution of rules. See Blume and Easley(1992), Anderlini and Sabourian(1995), Kaniovski, Kryazhimskii and Young (2000) and Juang (2002) for reference. This paper extends Juang's work in two ways. Firstly, we study a multi-player, finite-action, normal form game with at least two strict Nash equilibria, which is more general than a 2×2 coordination game. Secondly, agents can choose among finitely many rules, instead of only two specific ones. The current paper also differentiates from the other papers mentioned above in that it assumes away the problem of learning to play Nash Equilibria and emphasizes the selection among multiple equilibria.

It is thought that the introduction of rare action mutations or rule mutations can produce selection power among multiple equilibria. In this paper, we find that the selection power created by mutations cannot be guaranteed and a "*folk theorem*" on equilibrium selection is restored. Our main conclusion is as follows. *A single rule profile* is a vector of rules with exactly one rule for each role population. For any prescribed stage game equilibrium, we always design a single rule profile such that from any other stage game equilibrium, a single deviation will trigger each rule in the single rule profile switch to the action corresponding to that prescribed stage game equilibrium. Therefore, among all transitions between stage game equilibria, the ones to the prescribed equilibrium under the newly designed single rule profile will encounter the least resistance: one action mutation suffices to achieve such transitions. In short, given any evolutionary system and any arbitrary stage equilibrium, we can always add a set of rules which exhibits the least resistance from any other stage equilibrium to this specific one. Note that this stage equilibrium and the newly added set of rules still constitute a steady state in the modified system. In addition, this steady state minimizes the stochastic potential among all steady states. If this state is the unique one that minimizes the stochastic potential, then with probability one the adjusted system will stay at the prescribed stage equilibrium in the long run.

The rest of the paper is organized as follows: The formal model is discussed in Section 2, followed by Section 3 with some initial results. We then proceed with the main propositions in Section 4. The discussion in Section 5 concludes this paper.

2 MODEL DESCRIPTION AND DISCUSSION

2.1 Basic Model

The main stage is an N-player, normal form game denoted by $G = \{A_n, \pi_n\}_{n=1}^{N}$. Here $A_n = \{a_{n,1}, a_{n,2}, ..., a_{n,J_n}\}$ is the set of finite pure actions for the nth player with finite J_n representing the cardinality of A_n and π_n is the corresponding payoff function for the nth player. Let E be the set of all pure Nash equilibria in the stage game and $|E|$ the cardinality of E. In this paper we deal with cases where all pure Nash equilibria are strict and $|E| \geq 2$ with e a representative element of E.

There are N populations in the society, with agents of the nth population playing the role of the nth player in the stage game. For simplicity we assume that each population consists of M agents. At each period t, each agent in population n is randomly paired with $N-1$ agents, one from each of the remaining populations, to play the stage game G described above. When choosing an action for the coming stage game, each agent is assumed to adopt a rule to help her make decisions. Let R_n be the space of rules available for agents of population n, $n = 1, ..., N$. These rules map past histories to actions. By assuming that players are of finite-period memory, hereafter we shall only focus on rules of T-period memory. Before playing the stage game, a fixed number g_r of agents of each population, $1 \leq g_r \leq M$, are randomly chosen to revise their rules, according to some rule selection criterion. A rule selection criterion maps past histories to distributions of rules. Again, we only focus on criteria of T-period memory. After these agents revise their rules, a fixed number g_a of agents of each population, $1 \leq g_a \leq M$, choose the action (pure or mixed) prescribed by her response rule. After playing the stage game, each agent receives the realized payoffs and observes the realized outcomes. Then the society moves into period $t+1$, random matching occurs, revisions on rules and actions take place and each agent plays another stage game, and so on. Formal definitions of rules and criteria will be specified later.

Note that we have included cases in which there is inertia in agents' revision processes on two levels: inertia in action revision (g_a) and inertia in rule revision (g_r) respectively. The assumption is made to accommodate asynchrony in agents' revision behaviors. It is probable that agents do not revise choice very often, or the time interval between two consecutive periods is too short for all of them to revise choices simultaneously.

For simplicity of description, we hereafter denote by rule (n, i) or $r^{n,i}$ the ith rule available to agents of population n. We also refer to group (n, i, j) as the collective of the nth players who adopt rule $r^{n,i}$ and choose action $a_{n,j}$. Similarly, group (n, i) refers to the collective of the nth players adopting rule $r^{n,i}$.

2.2 Histories and the State Space

Define $A \equiv A_1 \times ... \times A_N$ as the space of all feasible action profiles for each stage game. Let A^M be the space of outcomes of total M stage games played by all players in one period. Suppose that all players are anonymous. Thus the order of M outcomes of any element in A^M is irrelevant. Denoted by $\Theta \equiv A^{M \times T}$ the space of all T-period action-histories. We define a rule as follows.

Definition 1 *A rule (n, i), denoted by $r^{n,i} : A^{M \times T} \longrightarrow A_n$, is a map which, for each element $\theta \in \Theta$, assigns an element $a_{n,j} \in A_n$. That is, $r^{n,i}(\theta) \in A_n$, $\forall \theta \in \Theta$.*

It is quite natural that different rules may extract different information from any θ. Note that we only outline the information available for players and each player does not necessarily take into account all the information when playing games. By the fact that the action set for each player is finite, it follows that the space of all feasible action profiles A and thus the space Θ of all T-period action-histories are all finite. Since each rule we consider in the model always prescribes a pure action for any action history θ, it is straightforward that the space R_n of rules for the nth population is also finite.

For descriptive simplicity, we hereafter refer to $R = R_1 \times ... \times R_N$ as the rule space for the underlying system. The following is an example of two rules common in the literature.

Example 2 *Suppose all agents are of one-period memory and the stage game is a 2×2 coordination game with a payoff matrix depicted as below.*

	A	B
A	a, a	b, c
B	c, b	d, d

Figure1. Coordination Game: $a > c$, $d > b$.

Suppose that $a > d$ and $a - c < d - b$. Thus (A, A) is efficient and (B, B) is risk dominant Given any state ω realized in the last period, consider the following two rules.

Let $r^{1,1}$ be a naive imitation rule and $r^{1,2}$ be the myopic best response rule in population 1. We then have[1]

$$r^{1,1}(\theta) = \begin{cases} (1,0) \; if \; \pi^A(\theta) > \pi^B(\theta); \\ (0,1) \; if \; \pi^A(\theta) < \pi^B(\theta), \end{cases}$$

and

$$r^{1,2}(\omega_t) = \begin{cases} (1,0) \; if \; \frac{n^A(\theta)}{M} > \frac{d-b}{a+d-b-c}; \\ (0,1) \; if \; \frac{n^A(\theta)}{M} < \frac{d-b}{a+d-b-c}, \end{cases}$$

where $\pi^A(\theta)$ and $\pi^B(\theta)$ are average payoffs for using action A and action B respectively in θ and $n^A(\theta)$ is the number of players using action A in θ.

Apart from revising actions, there are a fixed number of agents randomly chosen from each population to revise their rules at the beginning of each period. To incorporate information on rules into players' information set, denoted by $F \equiv (A_1 \times R_1) \times ... \times (A_N \times R_N)$ as the space of all feasible action-rule profiles for each stage game. Let $H \equiv F^M$ be the space of outcomes of total M stage games played by all players in one period. Define $\Omega \equiv H^T$ the space of all T-period action-rule histories. We To be specific, any $f \in F$ refers to some outcome from an N-player stage game. It contains the action and the rule chosen by each player. Therefore, any $h_t \in H$ refers to what have happened in period t, including outcomes of all M stage games played by all players and the corresponding rule profiles. Each element $\omega_t \in \Omega$ can be regarded as a state of the system. Hereafter we shall denote by ω_t the state of the underlying system at the end of period t and refer Ω as the state space for the system. Since the aggregate population, the number of rules, the number of actions and agents' memory are all finite, it follows straightforward that the state space Ω is finite.

When revising rules, each agent is guided by some rule selection criterion defined as follows.

[1]When there is a draw,the rule may randomoise between both actions by assigning a probability distribution$(f, 1 - f)$ with $f \in [0, 1]$.

Definition 3 *Let R_n be a set of rules available for agents of the n-population and ΔR_n the set of probability distributions on elements of R_n. A rule selection criterion $c_n : \Omega \to \Delta R_n$ is a map which assigns a probability distribution (possibly a degenerate one) on R_n to any given $\omega \in \Omega$. That is, $c(\omega) \in \Delta R_n, \forall \omega \in \Omega$.*

Hereafter we shall assume that agents of the same population use a common criterion to revise rules. Such an assumption is made for the sake of analytical simplicity and will not affect the conclusion qualitatively. Denote by c_n the criterion used by agents of population n and by $c = (c_1, c_2, ..., c_N)$ the criteria vector of the system. So far we have discussed a multi-rule evolutionary system without action mutations or rule mutations. For descriptive simplicity, we summarize such a system as below.

Definition 4 *A multi-rule evolutionary system without mutation, denoted by $\{G, R, c, g_a, g_r\}$, is a system with stage game $G = \{A_n, \pi_n\}_{n=1}^{N}$, a rule space R, a vector of rule selection criteria $c = (c_1, c_2, ..., c_N)$, inertia in actions g_a, and inertia in rules g_r. There is no action mutations or rule mutations in such a system.*

Given an evolutionary system$\{G, R, c, g_a, g_r\}$, let $\Delta\Omega$ be the set of all probability distributions on elements of Ω. When considering stochastic situations, we shall also regard each element of $\Delta\Omega$ as a state of the underlying system. We can then define $\Gamma : \Delta\Omega \to \Delta\Omega$ as the dynamics of the system. For any state of the system $\Delta\omega_t \in \Delta\Omega$ in period t, $\Gamma(\Delta\omega_t)$ describes how the system will evolve at the end of period $t+1$. Suppose $\Gamma(\Delta\omega_t)$ puts a positive probability on some $\omega_{t+1} = (h_{t-T+2}, ..., h_t, h_{t+1}) \in \Omega$, then there must be some $\omega_t = (h_{t-T+1}, h_{t-T+2}, ..., h_t)$ on which $\Delta\omega_t$ puts a positive probability. There are two possible sources of randomness in Γ: randomization from rule selection criteria and matching processes between agents. Thus the transition process of the system is random in general.

Definition 5 *Given a system $\{G, R, c, g_a, g_r\}$ and its dynamics Γ, an invariant distribution of the system is a probability distribution on Ω , denoted by $\Delta\omega \in \Delta\Omega$, such that $\Gamma(\Delta\omega) = \Delta\omega$. When an invariant distribution puts all its mass on a single state $s \in \Omega$, we call s a steady state of the system. In other words, $\Gamma(s) = s$. Let S be the set of all steady states of the system.*

Note that we have changed notations here by referring to $s \in \Omega$ a steady state and $\omega \in \Omega$ a general state which is not necessarily a steady state. Since we attempt to study the behavior of a system in the long run, there is one class of states that plays an important role in our analysis.

Definition 6 *An equilibrium state is a state in which all agents of each population choose the same action corresponding to the same pure Nash equilibrium in the last T periods.*

For any stage equilibrium e, let $e(n)$ be the index of the action in A_n corresponding to e. Thus the action profile for the stage game equilibrium e is $(a_{1,e(1)}, a_{2,e(2)}, ..., a_{N,e(N)})$. Let $\omega(e)$ be any equilibrium state in which the equilibrium e is played by all combinations of players in the last T periods. Denote by $\theta(e)$ the corresponding action-histories. In terms of action histories, $\omega(e)$ is any state in which $\theta(e)$ occurs. In general, there are more than one equilibrium state $\omega(e)$'s for each $\theta(e)$. At any equilibrium state $\omega(e)$, we can conclude that all agents of population n are choosing action $a_{n,e(n)}$ in the last T periods for $n = 1, 2, ..., N$. Denote by $\Omega(e)$ the set of equilibrium states corresponding to the stage equilibrium e. Let $\Omega(E)$ be the union of all equilibrium states. In other words,

$$\Omega(E) = \cup_{e \in E} \Omega(e) \tag{1}$$

We are interested in systems which will reach some elements of $\Omega(E)$ within finite periods. Systems possessing such a property are called synchronizable.

Definition 7 *A system $\{G, R, c, g_a, g_r\}$, or the corresponding dynamics Γ, is said to be synchronizable if, starting from any initial state ω_0, with probability one the system will reach an equilibrium state within finite periods. To be specific, the dynamics Γ is synchronizable if and only if*

$$Pr(\omega_t \in \Omega(E) \text{ for some } t < \infty \mid \Gamma, \omega_0) = 1, \text{ for any } \omega_0 \in \Omega. \tag{2}$$

If (2) is violated, we say that the system is non-synchronizable. Note that non-synchronizability only implies that synchronizing on equilibrium states is not guaranteed globally. Thus a non-synchronizable system may still reach some equilibrium states within finite periods. A natural question is that what kinds of systems are synchronizable. To answer this question, let us look at systems that are non-synchronizable.

Example 8 *Consider a system in which two populations are playing the* 2×2 *coordination game illustrated in Figure1. Suppose that only one rule is available for each population. If these two rules are "always A" for population one and "always B" for population two respectively, then we will observe that each pair of agents persistently plays the same off-diagonal action pair.*

Naturally, a rule profile plays an important role in determining the synchronizability of the underlying system. In the above example, the non-synchronizability of a system comes from rigidity of rules and the fact that no revisions of rules occur. If there is more than one rule available in at least one population, then the system may be synchronizable. The following example illustrates how revision on rules affects the synchronizability of a system.

Example 9 *Follow the previous example except that both rules "always A" and "always B" are available in population one. Suppose that the criterion adopted by agents of population one is that "choose the rule that performed better". Note that "both players play B" is a strict Nash equilibrium. It follows that agents of population one adopting the rule "always B" always receive higher payoff than agents adopting the rule "always A". Consequently, any agent of population one revising rule will adopt the rule "always B". Therefore with probability one the rule "always A" will be excluded within finite periods. In this case, the system is still synchronizable.*

The above example demonstrates that if we can combine a rule profile with an appropriate criterion vector, then it is not difficult to construct a synchronizable system. In fact, inertia in actions and rules can also serve to eliminate inefficient cycles and thus synchronize a system. The next example illustrates this point.

Example 10 *Follow Example 8 except that the only rule available in both populations is the myopic best response. Let* m^* *be the minimal integer such that* A *best responds to the action profile where at least* m^* *agents playing* A*. Consider the following* three *cases.*

(i) $m^* \leq g_a \leq M$*. At least* m^* *agents revise actions at each period. If the system starts at the state such that all agents of population one play* A *and all agents of population two play* B*, then the system will play the two off-diagonal action pair alternately. Consequently such system is not synchronizable.*

(ii) $M - m^* + 1 \leq g_a \leq m^* - 1$. *This implies* $M - m^* + 1 \leq M - g_a \leq m^* - 1$. *Suppose at some period* t*, all agents of population one play* A *and all agents of population two play* B*. Therefore at period* $t + 1$*,* g_a *agents of population one choose* B *and the other* $M - g_a$ *agents choose* A*. On the other hand,* g_a *agents of population two choose* A *and the other* $M - g_a$ *agents choose* B*. At period* $t + 2$*, agents of both populations, in revising actions, will choose* B*. Thus it is not difficult to obtain the result that with probability one the system will coordinate on the risk dominant equilibrium within finite periods.*

(iii) $g_a \leq M - m^*$. *This implies* $m^* \leq M - g_a$ *and at each period at least* m^* *agents repeat what they did in the previous period. By a similar argument in (ii), we can have that with probability one the system will coordinate on one of the two equilibria within finite time.*

When there is inertia, the mechanism that randomly chooses agents to revise choices produces stochastic outcomes on the action profile of each population. Such stochastic outcomes may break inefficient cycles. Depending on the rule profile, this in turns imposes on the system the pressure which finally forces all agents to coordinate on one equilibrium. Even though there is not sufficient inertia, as that in (i) of Example 10, the system can still be synchronizable. The idea is to add in a new appropriate rule and a suitable criterion, similar to that proposed in Example 8.

The above discussion demonstrates that synchronizability is a general property of a large class of evolutionary systems $\{G, R, c, g_a, g_r\}$.

Without imposing any conditions on how each rule behaves under equilibrium states, a synchronizable system itself cannot guarantee the stability of its dynamics in equilibrium states. Suppose a system reaches a equilibrium state $\omega \in \Omega(e)$ at period t. Without any further assumption, there might exist at least one rule prescribing a different action at period $t + 1$ so that the system escapes from the current equilibrium state to another non-equilibrium state outside $\Omega(e)$ at period $t + 1$. Such phenomenon is quite disturbing since it violates the concept of rationality. One feasible solution is to impose the following assumption on rules: when the underlying system reaches an equilibrium state in which the same Nash equilibrium has been played by all players in the past T periods, a rule should prescribe the same action corresponding to the same Nash equilibrium in the current period. From this point of view, we are particularly interested in the following class of rules.

Definition 11 *A rule* r *is said to be* weakly rationalizable *if and only if it* *continues to prescribe the same corresponding pure action under any equilibrium state $\omega \in \Omega(E)$. Specifically, a rule $r^{n,i}$ available in population n is* weakly rationalizable *if, for any stage equilibrium $e \in E$ and action history $\theta(e) = e^{M\times T}$, $r^{n,i}(\theta(e))$ puts all its mass on action $a_{n,e(n)}$.*

Therefore, if a synchronizable system consists of *weakly rationalizable* rules, it will eventually reach one of the equilibrium states and all agents will continue to choose the same pure actions from then on. Immediate examples of *weakly rationalizable* rules include the myopic best response rule and the naive imitation rule. When all agents played the same stage game equilibrium in the last T period, these rules will prescribe pure actions corresponding to the same stage game equilibrium in the current period. Notice that the weak rationality assumption imposes no restrictions at all on how a rule should behave in any states other than equilibrium states.

Nevertheless, the weak rationality property seems less natural for a rule which always prescribes a mixed action at any non-equilibrium state. The weak rationality requires that once the underlying system reaches some equilibrium state, such rule must switch to the corresponding pure action immediately, even though such equilibrium state was hit by chance. Since we assume that agents' memory is of finitely many T periods, any state of such system would be a history of T consecutive periods. Notice that an equilibrium state is a history that the all agents played the same equilibrium for T consecutive periods. Consequently, the weak rationality assumption requires that if the underlying system has played the same equilibrium for T periods, then a weakly rationalizable rule should prescribe the corresponding pure action with certainty. By equipping agents with longer memory, we could be more confident in the weak rationality assumption.

When making choice among a set of rules, agents may use the performance in the past as the criterion and choose the rule that received the highest payoff at the previous period. They may as well choose the rule that has the highest expected payoff for the coming period, given the current state. Alternatively, they may choose any other criteria which they think are appropriate. In this paper, we shall deal with a class of criteria defined as below.

Definition 12 *A rule selection criterion c is moderate if it prescribes each existing rule with a positive probability under any equilibrium state.*

We have to emphasize that no explicit restrictions are imposed on a moderate rule at states other than equilibrium states. A moderate rule selection criterion requires only that each existing rule be selected with a positive probability under any equilibrium state. Under an equilibrium state, all agents in the same population choose the same action and receive the same payoff for T periods. Thus a criterion which prescribes the rule receiving the highest payoff at the previous period is moderate. If all rules are weakly rationalizable, each of them will prescribe for the coming game the pure action corresponding to the same Nash equilibrium that has been played so far. Such mechanism will also incur the same expected payoff for each rule at the coming period. Hence a criterion which prescribes the rule with the highest expected payoff for the coming period is moderate as well. Note that the criteria that prescribe with a positive probability to each existing rule that received the highest payoff in the histories are also moderate, but not vice verse.

Note that a moderate criterion does not necessarily explicitly allocate a positive probability for each existing rule under an equilibrium state. For example, a player may sample one (or more than one) player, compare the rule adopted by that player to her own, and then decide whether to switch or not. Such a rule selection criterion is also moderate by our definition. Consequently, the assumption for moderate rule selection criteria is not very restrictive for a T-period memory model.

Note that we assume that under equilibrium states, only existing rules can be candidates. Rules which are extinct have been erased from the memory of agents using a moderate criterion. As a result, they can only be adopted through mutations which will be dealt with in Section 4.

3 MULTIPLE RULES WITHOUT MUTATIONS

From the above discussions we know that a system $\{G, R, c, g_a, g_r\}$ could be synchronizable or non-synchronizable. We shall defer discussions of non-synchronizable cases till Section 5. From now on, we focus on synchronizable systems with weakly rationalizable rules and moderate rule selection criteria. The proposition below states that, for any such system, with probability one it will reach an equilibrium state, then drift to a steady state within finite periods and finally stay there forever as long as there are no mutations on actions or rules.

Proposition 13 *Consider a multi-rule evolutionary system without mutation, $\{G, R, c, g_a, g_r\}$. With probability one the system will reach one steady state within finite periods. At this steady state the same Nash equilibrium is played by all agents and all agents within the same population adopt the same rule.*

Proof. By the assumptions of synchronizability and weak rationality, with probability one such a system will reach an equilibrium state $\omega \in \Omega(e)$ in finite periods and stay within the set $\Omega(e)$ from then on. Suppose this equilibrium state corresponds to the action profile $a^e = (a_{1,e(1)}, a_{2,e(2)}, ..., a_{N,e(N)})$ in the stage game. Since all rules are weakly rationalizable, all agents will continue to play the same action profile a^e at each of the later periods. By the assumption that each rule selection criterion c_n is moderate, each existing weakly rationalizable rule will be chosen with strictly positive probability when an agent revises her rule. This implies in the following M periods, there is a strictly positive probability that some existing rule(s) in each population will vanish. The iterated elimination of existing rules will continue till exactly one rule prevail in one population. Thus after the system reaches a state in which all agents of the same population adopt the same rule, it stays at this steady state. This completes the proof of this proposition. ■

According to Proposition 13, we know that with probability one a system $\{G, R, c, g_a, g_r\}$ will reach one of the steady states within finite periods. For a system with $|E|$ Nash equilibria in the stage game, N populations, $|R_n|$ rules in any population n, we have a list of $|E| \times |R_1| \times |R_2| \times ... \times |R_N|$ steady states. With some abuse of terminology, denote by $s = (e, i_1, i_2, ..., i_N)$ as the steady state in which the stage game equilibrium e is played and all agents of population n adopt the same rule r^{n,i_n} for $n = 1, 2, ..., N$. We name any such $r = (i_1, i_2, ..., i_N)$ as a single rule profile and denote by $\Re$ the set of all feasible single rule profiles.

4 MULTIPLE RULES WITH MUTATIONS

There are two sources of mutations considered in this section: action mutations and rule mutations. The former has been widely applied in evolutionary systems where action mutations perturb equilibrium states such that only the most stable states can be selected in the long run. Such mutation may come from players' intentional experiments, involuntary trembles or mistakes, pure fantasy, or other relevant sources. When agents are allowed to revise rules, the same arguments apply to rule mutations as well. Although agents use some criterion to revise rules, sometimes and somehow they might adopt rules not prescribed by their criterion. Such defects may be caused by similar sources as those of action mutations. To distinguish these two mutations, we make the following conventions. First, when taking actions, with probability ε one mutation will emerge. At each period, agents randomly chosen to revise actions will take the actions prescribed by their rules; otherwise, they take the same actions they used at the last period. The first convention says that, with probability $1-\varepsilon$, an agent will do as described above, while with probability ε, she randomly picks one action from the action set; each action has an equal opportunity of being chosen. Second, when revising rules, a mutation will occur with probability δ. That is, with probability $1-\delta$, an agent will choose the rule prescribed by her criterion, while with probability δ, she randomly pick one rule from the rule set; rule has an equal opportunity of being adopted. For each agent who does not revise rules, we suppose that with the same probability δ she will make mistakes and simply randomly pick one rule from the rule set with each rule being chosen with an equal probability.

With the above set-up, we can define a multi-rule evolutionary system with mutation as the following.

Definition 14 *A multi-rule evolutionary system with mutation* $\{G, R, c, g_a, g_r, \varepsilon, \delta\}$ *is a system with stage game* $G = \{A_n, \pi_n\}_{n=1}^N$ *, a rule space* R*, a vector of rule selection criteria* $c = (c_1, c_2, ..., c_N)$*, inertia in action revision* g_a*, inertia in rule revision* g_r*, the probability of action mutation* ε *and the probability of rule mutation* δ*.*

A state ω is called *recurrent* if $\Pr(\omega_t = \omega \text{ for some } t \geq 1 \mid \omega_0 = \omega) = 1$.[2] Note that every state ω of the system is recurrent if $\varepsilon, \delta > 0$. It is straightforward to see that the set of recurrent states and the set of steady states are equivalent in our A multi-rule evolutionary system without mutation $\{G, R, c, g_a, g_r, \}$. Thus we shall hereafter use the two terms interchangeably and represent such a state by s rather than ω. To characterize states where

[2] In general, we need to characterize recurrent communication classes (see Young, 1993, pp.68) which may consist of one or more than one recurrent state. In our model each recurrent communication class consists of exactly one recurrent state, so we can just focus on recurrent states.

the system is likely to stay in the long run, we need to locate all recurrent states of the system without mutations ($\varepsilon, \delta = 0$) and pick up the one(s) that minimizes the stochastic potential (γ hereafter) which will be defined as follows. The concept is due to Young (1993) and Fredlin and Wentzell (1984).

Let the resistance $r(s, s')$ be the least number of mutations needed for the transition from recurrent state s to recurrent state s'. Define a directed graph Γ as follows: each vertex s corresponds to a recurrent state s, and for each distinct $s \neq s'$, the directed edge (s, s') has resistance $r(s, s')$.

s-TREE: an *s-tree* in Γ is a spanning tree such that from every vertex $s' \neq s$, there is a unique path directed from s' to s.

For every vertex s, let T_s be the set of all s-trees on Γ. The resistance of an s-tree $\tau \in T_s$ is the sum of the resistances of its edges,

$$r(\tau) = \Sigma_{(s", s') \in \tau} r(s", s').$$

The stochastic potential γ_s of a recurrent state s is the least resistance among all s-trees:

$$\gamma(s) = \min_{\tau \in T_s} r(\tau). \tag{3}$$

Since there are two kinds of mutations in this model, we shall incorporate them into the same framework. For ease of comparison, we introduce a parameter η such that

$$\delta = \varepsilon^{\eta}. \tag{4}$$

Note that η can be any positive number. In term of probability, a rule mutation is equivalent to η action mutations. The following formalizes the above discussion on the stochastic potential under the framework of a multi-rule evolutionary model.

Let $r = (i_1, i_2, ..., i_N)$ and $r' = (i'_1, i'_2, ..., i'_N)$ be any two single rule profiles. To compute the stochastic potential for any given steady state $s = (e, r)$, we need to construct a set of all s-trees and characterize the one(s) with the least resistance among them. Since for any steady state s, the number of all feasible s-trees is enormous in general, it would be a tough job to work it all out, not to mention the whole demanding computation for all $|E| \times I_1 \times I_2 \times ... \times I_N$ steady states.

A few observations may simplify the task substantially, however. First, the transition between any steady state s' to another s can always be summarized at most as two types of transition: the switch from equilibrium e'

to equilibrium e and the switch from a single rule profile r' to another r. Since all rules are weakly rationalizable, there must be at least one action mutation to ignite the first type of switch. The least number of mutations necessary for such switch depends on the underlying single rule profile under which such switch occurs. On the other hand, since agents can only choose among existing rules, rule mutations are indispensable for the second type of switches. To differentiate these two types of transition, we name the one with only action mutations as *equilibrium transition* and the other with only rule mutations as *rule transition*. Second, in constructing an s-tree, any steady state s' other than s needs exactly one edge (an equilibrium transition or a rule transition) being directed to one of the other steady states. It is easy to see that any tree containing any single directed edge from s' to s'' with $e' \neq e''$ and $r' \neq r''$ will not be the one with the least stochastic potential. Third, when considering the least stochastic potential, the type of transition through which the steady state is connected to another depends on the relative resistance between the two types of transition. In other words, each steady state will be connected to another through either an equilibrium transition or a rule transition that is of the least resistance.

To facilitate later discussion, we need to define a few terms as follow.

Definition 15 *The direct resistance of a single rule profile r from Nash equilibrium e' to Nash equilibrium e, denoted by $DR_r(e', e)$, is the least action mutations necessary for the system to switch directly from e' to e, while keeping r fixed.*

$DR_r(e', e)$ can serve as an indicator of the resistance from e' to e, while keeping r fixed. Obviously $DR_r(e, e)$ is zero since we do not need any mutation for a system to stay at e. Note that $DR_r(e', e)$ is greater than or equal to $r((e', r), (e, r))$ since the latter may go through the equilibrium transition under other single rule profile r' while the former can only do it with r fixed. That is, switches of indirect routes should be admissible as well and should be preferred in particular when such indirect route can provide paths with less resistance. For example, if the direct equilibrium switch from e' to e needs too many action mutations under r, then it may be beneficial to switch to other equilibria $e^1, e^2, ..., e^H$ before reaching e. In addition, it may also pay off to undergo the equilibrium switch under another single rule profile r". However, it will be shown in the lemma below that we only need to consider direct resistance in constructing an s-tree with the least stochastic potential.

Recall that $\mathcal{R}$ is the set of all feasible single rule profiles in a multi-rule evolutionary system. Define a map $DIF : \mathcal{R} \times \mathcal{R} \rightarrow \{0, 1, 2, ..., N\}$ as a function indicating the number of elements which are different between two vectors. For example, let $N = 3$, $r = (1, 2, 3)$ and $r' = (3, 2, 1)$, then $DIF(r, r') = 2$ since there are two elements (the first and the third ones) different between r and r'. Therefore we need at least $DIF(r, r')$ rule mutations for the system to switch from r' to r or the other way round. This is so because, for each element i'_n of r' such that $i'_n \neq i_n$, there must be at least one rule mutation in population n and such a mutation leads to the adoption of rule (n, i_n). The discussion in the above several paragraphs can be summarized as follows.

Lemma 16 *Given any s-tree* $\tau_s^* = \arg\min_{\tau \in T_s} r(\tau)$. *Any steady state* $s' = (e', r')$ *is connected to the s-tree* τ_s^* *through a directed edge* $((e', r'), (e", r"))$ *with properties as follows:* (i) *If* $\min_{e''' \in E \backslash e'} DR_r(e', e''') < \eta$, *then* $e" \in \arg\min_{e''' \in E \backslash e'} DR_r(e', e''') \neq e'$ *and* $r" = r'$; (ii) *If* $\min_{e''' \in E \backslash e'} DR_r(e', e''') > \eta$, *then* $e" = e'$ *and* $DIF(r', r") = 1$; (iii) *If* $\min_{e''' \in E \backslash e'} DR_r(e', e''') = \eta$, *then both types of transition in* (i) *and* (ii) *may occur.*

Proof. It is easy to see that any steady state s' must connect to another through either one rule transition or one equilibrium transition within an s-tree with the least stochastic potential. If it connects to another through rule transition, then such transition must need exactly one rule mutation. The remaining proof is just to compare the relative resistance between feasible rule transitions and equilibrium transitions and follows straightforward. ■

Having constructed an s-tree with the least stochastic potential for each $s \in S$, the steady state(s) which will be selected in the long run is (are) the one(s) with the smallest stochastic potential. To be specific, any steady state s^* such that

$$s^* = \arg\min_{s \in S} \gamma(s), \tag{5}$$

will be selected in the long run with a positive probability. Denote by S^* the set of all such steady states fulfilling (5) and name it as the long-run equilibrium set.

For any given system $\{G, R, c, g_a, g_r, \varepsilon, \delta\}$, the long-run equilibrium set S^* depends on the parameters of the system. Although we can locate the long-run equilibrium set for any explicitly specified system, it is not straightforward to do so under an arbitrary system. However, we can locate a set of steady states which, if not empty, will be equivalent to the long-run equilibrium set. Define $S^{**} \equiv \{(e^{**}, r^{**}) \in S \mid DR_{r^{**}}(e, e^{**}) \leq \ DR_{r''}(e', e'')$ for any e, r'', and (e', e'') where $e' \neq e''\}$. Suppose there exists a steady state (e^{**}, r^{**}) in S^{**}. We can infer that, under r^{**}, the system needs the least mutations to switch from any e to e^{**}, compared to any other transition by r" from e' to e" with $e' \neq e$". Therefore, among all switches between different stage equilibria, the switch from e to e^{**} under r^{**} is the easiest. It follows that the steady state (e^{**}, r^{**}) minimizes the stochastic potential and will be selected in the long run. The next proposition states that, if there exists any non-empty set S^{**}, then this set is equivalent to the long-run equilibrium set S^*. In other words, with probability one the system will stay at one of steady states in S^{**} in the long run. If the set S^{**} is a singleton, then we can conclude that with probability one the system will stay at that state in the long run.

Proposition 17 *Given any multi-rule evolutionary system with mutation* $\{G, R, c, g_a, g_r, \varepsilon, \delta\}$*, if the set* S^{**} *is not empty, then* $S^{**} = S^*$*.*

To facilitate the proof of the proposition, recall that $\tau^*(s^*) \in \arg\min_{\tau \in T s^*} r(\tau)$ is the s^*-tree with the least stochastic potential. With some abuse of notations, let $\tau^*_{s^*}(s^1, s^j) \equiv \{(s^1, s^2), (s^2, s^3), ..., (s^{j-2}, s^{j-1}), (s^{j-1}, s^j) \in \tau^*(s^*)\}$. To be specific, $\tau^*_{s^*}(s^1, s^j)$ is the path from s^1 to s^j that consists of a set of directed edges in s^*-tree $\tau^*(s^*)$. Given such s^*-tree $\tau^*(s^*)$ and any two distinct recurrent states s^i and s^j, define the distance $d(s^i, s^j \mid \tau^*(s^*))$ of as the cardinality of $\tau^*_{s^*}(s^i, s^j)$. That is, s^i is connected to s^j through $d(s^i, s^j \mid \tau^*(s^*))$ directed edges. We take the convention that $d(s^i, s^i \mid \tau^*(s^*)) = 0$ and $d(s^i, s^j \mid \tau^*(s^*)) = \infty$ if $\tau^*_{s^*}(s^i, s^j)$ is empty. For any vertex $s = (e, r)$, define the leading state of s as $LS(s, \tau^*(s^*)) \equiv \arg\min_{s^i}\{d(s, s^i \mid \tau^*(s^*)) \mid (s^i, s^j) \in \tau^*_{s^*}(s, s^*)$, $s^i = (e^i, r)$ *and* $s^j = (e^i, r^j)$ *with* $r^j \neq r\}$. That is, the leading state s^i of s is the vertex in $\tau^*_{s^*}(s, s^*)$ which is reached by s through equilibrium transition and the leading state itself is directed to another steady state through rule transition. It is obvious that a steady state can be the leading state of itself. The proof of the proposition is constructed as follows. For any given steady state s^0 and its given s^0-tree, $\tau^*(s^0)$, we can always change the s^0-tree into another s^{**}-tree with $s^{**} \in S^{**}$ by trimming and adding rule transitions on the s^0-tree, the so called tree operations. Furthermore, the constructed s^{**}-tree has the stochastic potential less than or equal to the original s^0-tree. Since the state s^0 is arbitrarily given, it follows that s^{**} is also an element of S^*.

Proof. Suppose there exists a steady state $s^{**} = (e^{**}, r^{**}) \in S^{**}$, then for any $s^0 = (e^0, r^0) \in S$, we shall prove that $\gamma(s^{**}) \leq \gamma(s^0)$. Given any s^0-tree, $\tau^*(s^0) \in \arg\min_{\tau \in T_{s^0}} r(\tau)$. Since each edge in $\tau^*(s^0)$ involves either rule transition or equilibrium transition, we can always rearrange the indices of all single rule profiles and stage game equilibria such that the path of the directed edges from s^{**} to s^0 is as follows.

$$
\begin{aligned}
s^{**} &= (e^{**}, r^M) \to ... \to (e^{n(M)}, r^M) \to (e^{n(M)}, r^{M-1}) \to ... \\
&\to (e^{n(M-1)}, r^{M-1}) \to (e^{n(M-1)}, r^{M-2}) \to ... \\
&\to (e^{n(M-2)}, r^{M-2}) \to (e^{n(M-2)}, r^{M-3}) ... \\
&\to (e^{n(2)}, r^2) \to (e^{n(2)}, r^1) \to ... \\
&\to (e^{n(1)}, r^1) \to (e^{n(1)}, r^0) \to ..(e^{n(0)}, r^0) = s^0.
\end{aligned}
$$

Here $n(\cdot)$ is some index function. Thus $LS((e^{m'}, r^m), \tau^*(s^0)) = (e^{n(m)}, r^m)$ for any $m \in \{1, 2, ...M\}$. Note that each rule transition $(e^{n(m)}, r^m) \to (e^{n(m)}, r^{m-1})$ needs exactly one rule mutation by Lemma 16. Moreover, any steady state (e, r^m) with $m \in \{1, 2, ...M\}$ in this tree is directed either to another state (e, r') with $r' \neq r^m$ if it is a leading state or to another state (e', r^m) with $e' \neq e$ if it is not. Since we focus exclusively on $\tau^*(s^0)$ in the proof, we shall omit $\tau^*(s^0)$ from $LS((e^{m'}, r^m), \tau^*(s^0))$ whenever there is no risk of misunderstanding. We now construct an s^{**}-tree as follows:

(1) Given $s^0 = (e^{n(0)}, r^0)$, set $(e^{i(0)}, r^0) \equiv (e^{n(0)}, r^0)$. Here $i(\cdot)$ is another index function.

(2) Start from $k = 0$. For any given $(e^{i(k)}, r^k)$, construct an edge $((e^{i(k)}, r^k), (e^{i(k)}, r^{k+1}))$. For this given state $(e^{i(k)}, r^{k+1})$, trace its leading state. Set this leading state $LS((e^{i(k)}, r^{k+1})) \equiv (e^{i(k+1)}, r^{k+1})$. Since it is a leading state in $\tau^*(s^0)$, it must be directed by rule transition to another state $(e^{i(k+1)}, r^{k'})$ with $r^{k'} \neq r^{k+1}$. Delete that original rule transition, that is, the edge $((e^{i(k+1)}, r^{k+1}), (e^{i(k+1)}, r^{k'}))$.

(3) Iterate step (2) with k from 0 to $M-1$. For example, for $k = M-1$, we obtain state $(e^{i(M-1)}, r^{M-1})$. We then construct an edge $((e^{i(M-1)}, r^{M-1}), (e^{i(M-1)}, r^M))$, trace the leading state $LS((e^{i(M-1)}, r^M)) \equiv (e^{i(M)}, r^M)$ of the state $(e^{i(M-1)}, r^M)$. Since the leading state $(e^{i(M)}, r^M)$ must be directed to another state, say, $(e^{i(M)}, r^{M'})$ with $r^{M'} \neq r^M$, through rule transition. We then locate and delete this original rule transition $((e^{i(M)}, r^M), (e^{i(M)}, r^{M'}))$.

(4) If $(e^{i(M)}, r^M) = (e^{**}, r^M)$, end the construction; otherwise construct the edge $((e^{i(M)}, r^M), (e^{**}, r^M))$.

(5) If there exists any edge $((e^{**}, r^M), (e', r^M))$ in $\tau^*(s^0)$, then delete that edge and end the construction; otherwise perform step (6).

(6) If we reach here, it implies there exists the edge $((e^{**}, r^M), (e^{**}, r^{M-1}))$. Trace back along the path from s^{**} to s^0 till the first equilibrium transition is reached. This equilibrium transition may be represented by an edge $((e^{**}, r^{m^*}), (e', r^{m^*}))$ for some $m^* \in [1, M-1]$. Delete this edge. To be specific, delete the edge $((e^{**}, r^{m^*}), (e', r^{m^*}))$ with $m^* = \arg\max\{m \mid ((e^{**}, r^m), (e', r^m)) \in \tau^*_{s^0}(s^{**}, s^0), e' \neq e^{**}\}$. Note that for the edges along the path from s^{**} to s^0 just before the first equilibrium transition is reached, all of them are rule transition. Reverse the direction of these edges

$$((e^{**}, r^M), (e^{**}, r^{M-1})), ..., ((e^{**}, r^{m^*+1}), (e^{**}, r^{m^*}))$$

into

$$((e^{**}, r^{M-1}), (e^{**}, r^M)), ..., ((e^{**}, r^{m^*}), (e^{**}, r^{m^*+1})).$$

By the above steps, we have constructed an s^{**}-tree from the given s^0-tree. Note that in each step (2) or step (3), the edge we construct has the same resistance as the edge we delete. In step (4), we have $r((e^{i(M-1)}, r^M), (e^{**}, r^M)) \leq ((e^{**}, r^M), (e', r^M))$ in step (5) or $((e^{**}, r^{m^*}), (e', r^{m^*}))$ in step (6) by the fact $s^{**} = (e^{**}, r^M) \in S^{**}$ and the definition of S^{**}. Therefore we have $\gamma(s^{**}) \leq \gamma(s^0)$. Since s^0 is arbitrarily chosen from S, this completes the proof. ■

The above Proposition characterizes a class of steady states which always minimizes the stochastic potential and thus will be selected in the long run. An explicit example for such a steady state (e^{**}, r^{**}) is that $DR_{r^{**}}(e, e^{**}) = 1$ for all $e \neq e^{**}$. We give a brief explanation as the following. Suppose there is a single rule profile $r^{**} = (i_1^{**}, i_2^{**}, ..., i_N^{**})$ such that whenever a deviation occurs and destroys the current steady state, each rule (n, i_n^{**}) prescribes an action or a sequence of actions that leads to e^{**}. Thus under r^{**}, an action mutation is sufficient to induce the system to transit from any $e \neq e^{**}$ to e^{**}. By Proposition 18, we know that with a positive probability such a steady state (e^{**}, r^{**}) will be selected in the long run. Note that if the set S^{**} is a singleton, then the unique steady state (e^{**}, r^{**}) will be selected with probability one. The above discussion implicitly proposes the potential existence of a class of single rule profiles which are defined as universally dominant in Definition 19.

Definition 18 *Given a multi-rule evolutionary system* $\{G, R, c, g_a, g_r, \varepsilon, \delta\}$, *a single rule profile r^{**} from R is universally dominant if there exists an equilibrium $e(r^{**})$ such that $DR_{r^{**}}(e, e^{**}) = 1$ for all $e \neq e(r^{**})$.*

Corollary 19 *Given a synchronizable system $\{G, R, c, g_a, g_r, \varepsilon, \delta\}$, suppose there exists a single rule profile r^{**} which is universally dominant. Then, with a positive probability the steady state $(r^{**}, e(r^{**}))$ will be selected in the long run. Furthermore, only universally dominant single rule profiles can be selected in the long run.*

Proof. For any r^{**} which is universally dominant, we have $DR_{r^{**}}(e, e^{**}) = 1$ for all $e \neq e(r^{**})$. Applying the previous proposition completes the proof. ■

Since universally dominant single rule profiles play a very important role in our model, it is natural to ask what exactly such rules are. An answer to this may resort to some "trigger" rules. This kind of rules may behave just like any other rule such as best response or imitation in the literature except in states which are reached from any equilibrium state through deviation from any player. When there occurs a deviation and the current equilibrium state has been destroyed, then each rule will switch to an action or a sequence of actions leading to the prescribed equilibrium, say, e. In the latter case when a sequence of actions leading to e is prescribed, this sequence of actions may be interpreted as a phase of punishments and rewards before entering the new equilibrium e. This mechanism is very common in designing strategies for repeated games to support some prescribed payoff profiles. From the viewpoint of efficiency, such universally dominant single rule profiles serve to restore the system into balance or equilibrium without too much try and error whenever a current equilibrium is destroyed by either intentional deviations or exogenous shocks.

Note that we can always add a universally dominant single rule profile into a multi-rule evolutionary system. Consider any synchronizable system $\{G, R, c, g_a, g_r, \varepsilon, \delta\}$, with weakly rationalizable R and moderate c. To support any arbitrary stage equilibrium e in the long run, we can add a universally dominant single rule profile into the system without affecting the synchronizability. This can always be done since we may add this single rule profile "only into players' memory" so that it will only emerge through mutations. Suppose the added single rule profile calls for play of the underlying stage equilibrium e. It follows that with a positive probability the newly adjusted system will stay at a steady state corresponding to the stage equilibrium e in the long run.

Proposition 20 *Consider any synchronizable system* $\{G, R, c, g_a, g_r, \varepsilon, \delta\}$ *with weakly rationalizable* R *and moderate* c*. For any prescribed stage equilibrium* e^{**}*, we can add in the system a single rule profile* r^{**} *such that with a positive probability the system will stay at the steady state* (e^{**}, r^{**}) *in the long run.*

Proof. For any given e^{**}, we can add in the rule space R a rule profile r^{**} which is universally dominant and supports the stage equilibrium e^{**} in the following way. Without loss of generality, denote by $\{G, R^{*}, c, g_a, g_r, \varepsilon, \delta\}$ the modified system, Γ^{*} its dynamics and assume that $r^{**} = (r^{1,1}, r^{2,1}, ..., r^{N,1})$. For each $e \in E$, define $\Omega^1(e) \equiv \{\omega \in \Omega \mid \Sigma_j x(n', 1, j) = M, x(n', 1, e(n')) = M - 1$ for some n'; $x(n, 1, e(n')) = M$ for all $n \neq n'\}$. In other words, under any state in $\Omega^1(e)$, the system adopts r^{**} and all agents except one play actions corresponding to the stage equilibrium e. Define $\Omega^1 \equiv \cup_{e \in E} \Omega^1(e)$. Each newly added rule $r^{n,1}, n = 1, 2, ..., N$, works in the following way. It imitates an arbitrary existing rule , say $r^{n,2}$, under any state $\omega \in \Omega \setminus \Omega^1$. When under states in Ω^1, it prescribes the action $a_{n,e^{**}(n)}$ corresponding to the stage equilibrium e^{**}. It is easy to check that the newly updated rule profile is still synchronizable, weakly rationalizable and $DR_{r^{**}}(e, e^{**}) = 1$ for all $e \neq e^{**}$. By Proposition 18 we have that with a positive probability the system will stay at (e^{**}, r^{**}) in the long run. ■

Proposition 21 demonstrates a shadow impact on equilibrium selection power introduced by mutations: any stage equilibrium could be supported by adding a specific single rule profile. Thus we cannot obtain a general conclusion on which equilibrium should be selected in the long run without specifying explicitly the exact rule profile. However, this result does not contradict those in KMR, Robson and Vega-Redondo (1996) and Juang (2002). As, in the above papers, agents are restricted to choose among a specific set of rules, we can still select among the equilibria in the long run. Proposition 21 implies that a general result cannot be obtained without explicitly specifying the rule profile.

We proceed to study a special case in which the stage game G is a 2×2 coordination game with one payoff dominant equilibrium e^1 and one risk dominant equilibrium e^2. Defining

$$DR_r = min \; \{ \; DR_r(e^2, e^1), DR_r(e^1, e^2)\}$$

yields the following result.

Proposition 21 *Consider any synchronizable system* $\{G, R, c, g_a, g_r, \varepsilon, \delta\}$ *with* R *weakly rationalizable and* c *moderate. In addition,* G *is a* 2×2 *coordination* *game with one risk-dominant equilibrium and one payoff-dominant equilibrium. In the long run, all agents will adopt the single rule profile with the smallest direct bond. In addition, all agents of the same population choose the same action corresponding to stage equilibrium which the underlying rule profile is inclined to choose in the long run.*

Proof. This is a direct application of Proposition 18 and is omitted. ■

Remark 22 *Suppose that there exists more than one rule with the smallest direct bond and they are inclined to choose the same action. Then, in the long run, these rules will dominate the whole system and all agents will play the stage equilibrium that these rules are inclined to choose. If they are not inclined to choose the same action, then each of the two Nash equilibria will prevail with a positive probability respectively in the long run.*

Corollary 23 *Given the same assumptions as those in proposition 4.5, suppose further that the direct bond of any single rule profile is at least two and the naive imitation rule is present in both populations. Then, in the long run, there is a positive probability that all agents will adopt the naive imitation rule and play the payoff dominant equilibrium. If the naive imitation rule is the only rule with the least bond, then the probability of adoption is one.*

Suppose that the above system is at the steady state in which all agents of both populations are adopting the naive imitation rule and playing the risk dominant equilibrium. Consider the case in which two agents, one from each population, mutate simultaneously to the other actions and they are matched to play the stage game at some period t. Then both defectors will receive the highest payoff at period t. Consequently all agents of both populations will switch to the action corresponding to the payoff dominant equilibrium at period $t+1$. In other words, two mutations in action plus some luck in matching process are sufficient for the system to switch from the risk dominant equilibrium to the payoff dominant one. Therefore the direct bond of the naive imitation rule is two. Since the direct bond of any other rule is at least two, the above corollary is immediate.

5 Non-Synchronizable Systems

The above discussions focus on the synchronizable cases. If a system is not synchronizable, then the underlying system will converge to one of recurrent classes by the standard theory of stochastic processes. The following lemma is straightforward by the finiteness of the state space.

Lemma 24 *Consider a multi-rule evolutionary system without mutation, $\{G, R, c, g_a, g_r\}$. Each of its recurrent classes is the one that consists of either (i) a steady state or (ii) a cycle of states with some period $t_c \in \mathbb{N}$ such that $t_c \leq K \equiv \left|A^{M\times T}\right| \times \left|R^{M\times T}\right|$.*

In a cyclical recurrent class, its period could be so long that t_c is close to $K \equiv \left|A^{M\times T}\right| \times \left|R^{M\times T}\right|$. Moreover, there could be so many feasible states in each stage of the cycle that any state $\omega \in \Omega$ is possible in part of the stages. The lemma below says that in a cyclical recurrent class, it cannot happen that all states are feasible in all stages of the cycle. In any cyclical recurrent class of period t_c, let $A_c^M(t) \subseteq A^M, 1 \leq t \leq t_c$, be the range of this cyclical recurrent class in stage t.

Lemma 25 *For any cyclical recurrent class, there must exist a stage t^*, such that $A^M \backslash A_c^M(t^*) \neq \varnothing$.*

Proof. Suppose not. We then have $A_c^M(t) = A^M$ for $1 \leq t \leq t_c$. This implies that it is possible that the system undergoes a path in which a specific equilibrium e be played by all players in the duration of T periods. If the system indeed reaches any of such equilibrium states, then by our assumption that all rules are weakly rationalizable, it follows that each rule will continue to prescribe the action corresponding to the equilibrium e from then on. It results in an escape from the underlying cyclical recurrent class to some steady state and destroy the cyclical recurrent class. ■

By the above lemma, we may conclude that there at least exists one stage t^* in any cyclical recurrent class such that $A^M \backslash A_c^M(t^*) \neq \varnothing$. This implies an action mutation may result in a deviation from the prescribed path. Thus, for any rule profile r^c (not necessarily single rule profile) in any cyclical recurrent class, we can always construct a rule profile $R(r^c)$ such that each rule in the profile $R(r^c)$ always mimics one of its counterparts within r^c except that whenever there is a deviation from the prescribed path, each rule in the rule profile $R(r^c)$ switches to some other prescribed path that leads to an equilibrium state $\omega(e)$ within finite periods. Since the rule selection criteria adopted by players are moderate, it follows that the system will evolve into some steady state (e, r) with $r \subseteq R(r^c)$ a single rule profile.

It is easy to see that the transition from such a cyclical recurrent class to the steady state $s = (e, r)$ costs one action mutation and thus has the stochastic potential less than or equal to the corresponding recurrent class. Therefore, any steady state in which its single rule profile is universally dominant is also stochastically stable even cyclical recurrent classes are considered. This can be summarized as the following proposition.

Proposition 26 *Consider any system* $\{G, R, c, g_a, g_r, \varepsilon, \delta\}$ *with weakly rationalizable* R *and moderate* c*. For any prescribed stage equilibrium* e^{**}*, we can add in the system a rule profile containing* r^{**} *such that with a positive probability the system will stay at the steady state* (e^{**}, r^{**}) *in the long run.*

Proof. Suppose there exist a cyclical recurrent class $C(r^c)$ with r^c be its rule profile and has the least stochastic potential. We can then construct a rule profile $R(r^c)$ as stated above such that e^{**} is the equilibrium supported by $R(r^c)$ and $r^{**} \subseteq R(r^c)$ be one of the single rule profiles that results. This means that one action mutation suffices for the underlying system to switch from the cyclical recurrent class $C(r^c)$ to the steady state (e^{**}, r^{**}). The remaining proof is similar to that for its counterpart of the synchronizable cases. ■

6 CONCLUSION

We have constructed a model illustrating the evolution of multiple rules and actions and find that the behavior of a system in the long run is determined by the interaction between available rules. Thus, the risk dominant equilibrium supported in KMR (1993), the efficient equilibrium prevailing in Robson and Vega-Redondo (1996), and the dominance of the imitation rule combined with the efficient equilibrium proposed in Juang (2002), can all be summarized and explained under the framework of our model. To be specific, depending on the initial sets of rules, we may observe the risk dominant equilibrium or the efficient one dominate in the long run. Moreover, we can always add a specific single rule profile into the current system to support an arbitrary stage equilibrium we choose. Consequently, we obtain a folk theorem on equilibrium selection and the equilibrium selection power exerted by mutations cannot be guaranteed. One possible way to solve this problem is to model noise or mutations explicitly, as proposed by Binmore *et al*. (1995).

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