Linear and Symmetric Allocation Methods for Partially Defined Cooperative Games

Running Title: Allocation Methods for Partially Defined Games

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Abstract

A partially defined cooperative game is a coalition function form game in which some of the coalitional worths are not known. An application would be cost allocation of a joint project among so many players that the determination of all coalitional worths is prohibitive. This paper generalizes the concept of the Shapley value for cooperative games to the class of partially defined cooperative games. Several allocation method characterization theorems are given utilizing linearity, symmetry, formulation independence, subsidy freedom, and monotonicity properties. Whether a value exists or is unique depends crucially on the class of games under consideration.

Key words: allocation method, value, cooperative game, axioms, incomplete information

1. Introduction

There is a growing literature on the applications of cooperative game theory to the allocation of costs or benefits of a joint endeavor [for example, see Curiel et. al (1993), Driessen (1994), Skorin-Kapov (1993), and Young (1994) which provides an extensive review of earlier literature]. This paper is motivated by such applications when the determination of all coalitional worths is prohibitively expensive. For example, each coalitional worth may require an extensive engineering or accounting study, and the number of such studies increases exponentially with the number of players. This latter problem is sometimes alleviated if the game has a precise underlying structure (e.g., airport landing fees, minimum cost spanning tree games, assignment games, and network flow games). When there is no precise underlying structure, accountants often use *ad hoc* methods based upon only a small number of the coalitional worths. The purpose of this paper is to present axiomatic rationales for allocation methods when not all coalitional worths are known.

Letscher (1990) introduced the idea of partially defined games. Some of the results in this paper were first reported in Housman (1992). Willson (1993) characterized the reduced Shapley value using the axioms of linearity, symmetry, and margin monotonicity. Willson's work is in the spirit of Young's (1985) characterization of the Shapley value for cooperative games. In keeping with Shapley's (1953) original characterization of the Shapley value for cooperative games, we use the axioms of linearity, symmetry, and subsidy freedom (sometimes called the null player axiom). We take the viewpoint that a partially defined game is an incomplete representation of an unknown "fully defined" game. Often we have some *a priori* knowledge about relationships among the coalitional worths (e.g., superadditivity), and our allocation method should make use of this knowledge. So, special care is taken to examine special classes of games in addition to the class of all games. Similar attention to special classes of games in the context of classical value theory includes Monderer (1988) showing that every semivalue on a subspace of games can be extended to a semivalue on all games and Gilboa and Monderer (1991) showing a variety of characterizations of quasi-values on subsets of games.

In section 2, we define partially defined games, extensions, and the reduced Shapley value. By way of an example, we show why the reduced Shapley value may not be an appropriate allocation method. In section 3, we characterize all linear and symmetric allocation methods as weighted Shapley values. In section 4, we characterize all linear and symmetric allocation methods having one of three different monotonicity properties. In particular, we generalize Willson's (1993) characterization of the reduced Shapley value as the unique linear, symmetric, and margin monotone allocation method. In section 5, we characterize all linear, symmetric, and formulation independent (a property equivalent to the better known covariant with respect to strategic equivalence) allocation methods. We also show the general incompatibility of margin monotonicity and formulation independence, and we argue the intuitive primacy of formulation independence. In section 6, we characterize all linear, symmetric, and subsidy free (often called the null player axiom) allocation methods on the classes of zero monotonic, size monotonic, superadditive, and convex games. We close the paper with a few concluding remarks.

2. Partially Defined Games and Allocation Methods

Throughout this paper, we let $N = \{1, 2, ..., n\}$ be the fixed set of *players*. A nonempty subset S of N is called a *coalition*, and we write $|S|$ for the number of players in the coalition S. A *cooperative game* is a real-valued function w defined on the coalitions. The real number $w(S)$ is called the *worth* of coalition S and is interpreted as the total benefit available to the members of the coalition S if they cooperate with each other in the most efficient possible manner. In the context of a joint cost allocation problem, $w(S)$ is the cost savings obtained through cooperation as opposed to each member working alone. A partially defined cooperative game is a cooperative game in which only some of the coalitional worths are known. In this paper, whether a coalitional worth is known will depend only on the number of members in the coalition. Formally, we call M a *set of known coalition sizes* if M is a subset of N containing n. A *(symmetric) partially defined cooperative game* with respect to the set of known coalition sizes M, abbreviated as an M*-game*, is a real-valued function w defined on coalitions whose sizes are in M, that is, $w(S)$ is defined if and only if $|S| \in M$. Note that we assume that the worths of the grand coalition N is always known ($n \in M$). Note also that an N-game is a "fully defined" cooperative game.

Example 2.1. *Let* $n = 6$ *and* $M = \{1, 2, 5, 6\}$. *To conserve space, we will remove parentheses and commas in the notation for coalitional worths. For example,* $w(\{1, 4, 5\})$ *will be shortened to* $w(145)$. *Define* w *by* $w(N) = w(12345) = w(12346) = w(12356) = w(12456) = w(13456) = 120$, $w(23456) = 60$, $w(12) =$

$w(13) = w(14) = w(15) = 90$, $w(16) = 60$, $w(23) = w(24) = w(25) = w(34) = w(35) = w(45) = 30$, *and* $w(26) = w(36) = w(46) = w(56) = w(1) = w(2) = w(3) = w(4) = w(5) = w(6) = 0$. *Note that the worths of coalitions having three or four members are not known.*

Since our viewpoint is that partially defined games arise when we have insufficient resources to determine all coalitional worths, it is important to know what "fully defined" games could underlie a given partially defined game. Let Ω be a collection of N-games. An Ω-*extension* of the M-game w is an N-game $\hat{w} \in \Omega$ satisfying $\hat{w}(S) = w(S)$ for all $|S| \in M$. Define Ω_M to be the set of M-games w that have an Ω-extension $\hat{w}$, and whatever word is used to describe an N-game in Ω (e.g., convex) will also be used to describe an M-game in Ω_M. Collections of games often cited in the literature include convex, superadditive, and zero-montonic games. The N-game w is *convex* if $w(S)+w(T) \leq w(S\cup T)+w(S\cap T)$ for all coalitions S and T. The N-game w is *superadditive* if $w(S) + w(T) \leq w(S \cup T)$ for all disjoint coalitions S and T. The N-game w is *zero-monotonic* if $w(S)+w(\{i\}) \leq w(S\cup\{i\})$ for all coalitions S and players $i \notin S$.

Example 2.2. *Let w be the M-game described in Example 2.1. The M-game w has no convex extension. Indeed, if $\hat{w}$ were a convex extension of w, then* $180 = \hat{w}(12) + \hat{w}(13) - \hat{w}(1) \leq \hat{w}(123) = \hat{w}(123) + \hat{w}\,(4) + \hat{w}\,(5) + \hat{w}\,(6) \leq \hat{w}(N) = 120$, *which is impossible. The M-game w has a unique superadditive extension defined by*

$$\hat{w}(S) = \begin{cases} 30, \textit{ if } |S| = 3 \textit{ and } 1 \notin S \\ 90, \textit{ if } |S| = 3 \textit{ and } 1 \in S \\ 30, \textit{ if } |S| = 4,\ 6 \in S, \textit{ and } 1 \notin S \\ 60, \textit{ if } |S| = 4,\ 6 \notin S, \textit{ and } 1 \notin S \\ 90, \textit{ if } |S| = 4,\ 6 \in S, \textit{ and } 1 \in S \\ 120, \textit{ if } |S| = 4,\ 6 \notin S, \textit{ and } 1 \in S \\ w(S), \textit{ if } |S| \in M \end{cases}.$$

The proof is a straight-forward, but tedious, application of the superadditivity inequalities. For example, if $S = \{1, 2, 3, 6\}$, *then* $90 = \hat{w}(12) + \hat{w}(36) \leq \hat{w}(S)$ *and* $\hat{w}(S) \leq \hat{w}(N) - \hat{w}(45) = 90$. *The N-game w has many zero-monotonic extensions: $\hat{w}$ is a zero-monotonic extension of w if and only if the following* *conditions hold:*

$$\begin{array}{l} 30 \leq \hat{w}(S) \leq 60, \text{ if } |S| \notin M \text{ and } 1 \notin S \\ 90 \leq \hat{w}(S) \leq 120, \text{ if } |S| \notin M \text{ and } 1 \in S \\ \hat{w}(R) \leq \hat{w}(S), \text{ if } |R| = 3,\ |S| = 4, \text{ and } R \subset S \\ \hat{w}(S) = w(S), \text{ if } |S| \in M. \end{array}$$

The proof is a straight-forward application of the zero-monotonicity inequalities. In summary, w is zero-monotonic and superadditive but not convex.

Suppose M is a set of known sizes and Ω is a collection of N-games. An *allocation method* on M and Ω is a function φ that to every M-game $w \in \Omega_M$ assigns an *allocation* $x = (x_1, x_2, ..., x_n) \in R^n$ satisfying $\sum_{i \in N} x_i = w(N)$. We will usually write $\varphi_i(w)$ for x_i. We interpret $\varphi_i(w)$ as the fair share to player i if all the players cooperate to obtain the total benefit $w(N)$. Thus an allocation method provides a method for dividing the total benefit of cooperation among the players.

Willson (1993) defines the *reduced Shapley value* ψ on M-games by

$$\psi_i(w) = \frac{1}{n} \sum_{m \in M} \left(\binom{n-1}{m-1}^{-1} \sum_{\substack{|S|=m \\ i \in S}} w(S) - \binom{n-1}{m}^{-1} \sum_{\substack{|S|=m \\ i \notin S}} w(S) \right) \tag{2.1}$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is the standard binomial coefficient. If $M = N$, then this formula can be interpreted as the average, over coalition sizes, of the differences between the average worth of coalitions containing the player and the average worth of coalitions not containing the player. If the differences for coalition sizes not in M are taken to be zero, then this interpretation carries over to general M. If $M = N$, the reduced Shapley value is the Shapley (1953) value defined on N-games. In general, the reduced Shapley value agrees with the Shapley value if all unknown coalitional worths are set equal to some constant (a different constant may be chosen for each coalition size).

The formula given by Shapley (1953) for the Shapley value defined on N-games involves a weighted average of marginal contributions, $\Delta(S - \{i\}, S; w) = w(S) - w(S - \{i\})$. In order to generalize this approach to M-games, we consider marginal contributions, $\Delta(R, S; w) = w(S) - w(R)$, for coalitions satisfying $R \subseteq S - \{i\}$ and $|R|$ is the largest number in M that is strictly less than $|S|$. An equivalent formula for the reduced Shapley value is given by

$$\psi_i(w) = \frac{1}{n} \sum_{m \in M} \binom{n-1}{m-1}^{-1} \sum_{\substack{|S|=m \\ i \in S}} \binom{m-1}{\rho(m)}^{-1} \sum_{\substack{|R|=\rho(m) \\ R \subseteq S-\{i\}}} (w(S) - w(R)) \tag{2.2}$$

where $\rho(m) = \max\{0, r : r \in M \text{ and } r < m\}$ is the next smaller size for which coalitional worths are known, and we let $w(\phi) = 0$ for notational convenience. That formulas 2.1 and 2.2 are equivalent follows from comparing coefficients for each $w(S)$ in the two formulas. Indeed, suppose S is a coalition satisfying $i \in S$ and $|S| = s \in M$. Then the coefficient of $w(S)$ in formula 2.2 is $\frac{1}{n}\binom{n-1}{s-1}^{-1}\binom{s-1}{\rho(s)}^{-1}\binom{s-1}{\rho(s)} = \frac{1}{n}\binom{n-1}{s-1}^{-1}$ which is the coefficient of $w(S)$ in formula 2.1. Suppose R is any other coalition. Then R is a coalition satisfying $i \notin R$ and $|R| = r = \rho(s) \in M$ for some $s \in M$. It follows that the coefficient of $w(R)$ in formula 2.2 is $-\frac{1}{n}\binom{n-1}{s-1}^{-1}\binom{s-1}{r}^{-1}\binom{n-r-1}{s-r-1} = -\frac{1}{n}\binom{n-1}{r}^{-1}$ [after some algebra] which is the coefficient of $w(R)$ in formula 2.1.

Given the preceeding discussion, the reduced Shapley value for partially defined cooperative games seems to be a natural generalization of the Shapley value for cooperative games. The example challenges this intuition.

Example 2.3. *Let w be the M-game described in Example 2.1. The reduced Shapley value of our example M-game w is $\psi_i(w) = (41, 17, 17, 17, 17, 11)$. Note that the Shapley value of the unique superadditive extension $\hat{w}$ is $\psi_i(\hat{w}) = (62, 14, 14, 14, 14, 2)$. So, the reduced Shapley value may not equal the Shapley value of its unique extension. In fact, the Shapley value of* no *zero-monotonic extension yields the reduced Shapley value for our example! Indeed, suppose $\hat{w}$ is a zero-monotonic extension of w. Notice that $\psi_i(\hat{w})$ is an increasing function of $\hat{w}(S)$ if $i \in S$, is a decreasing function of $\hat{w}(S)$ if $i \notin S$, and is an increasing function of $\hat{w}(S) - \hat{w}(S - \{i\})$ if $i \in S$. So, $\psi_1(\hat{w})$ will be minimized by setting $\hat{w}(S) = 90$ if $|S| = 3$ and $1 \in S$, $\hat{w}(S) = 60$ if $|S| = 4$ and $1 \notin S$, and $\hat{w}(S) = \hat{w}(S - \{1\})$ if $|S| = 4$ and $1 \in S$. Hence, $\psi_1(\hat{w}) - \psi_1(w) \geq 5$. In summary, the reduced Shapley value for our example partially defined game can be equal to the Shapley value of a corresponding "fully defined" game only if the corresponding game is not zero-monotonic.*

The example shows us that the reduced Shapley value is sometimes an inappropriate choice for an allocation method for partially defined games if we believe that our partially defined game corresponds to some "fully defined" game for which we only know some of the coalitional worths.

3. Linear and Symmetric Allocation Methods

In this section, we characterize all linear and symmetric allocation methods on classes of partially defined cooperative games that are convex cones and symmetric. We begin by defining our conditions and interpreting them in the context of allocation problems.

For the remainder of this paper, M is a set of known coalition sizes, Ω is a collection of N-games, and φ is an allocation method on M and Ω. We will interpret Ω as the possible allocation problems that could arise *a priori*, and so Ω_M will be interpreted as the possible partially defined allocation problems that could arise *a priori*. We will interpret $\varphi_i(w)$ as the fair share given to player i in the partially defined allocation problem w. We now define and interpret two conditions on collections of N-games and allocation methods.

Suppose that v and w are M-games and a and b are real numbers. Define the M-game $av + bw$ by the formula $(av + bw)(S) = av(S) + bw(S)$ for all coalitions S satisfying $|S| \in M$. The set Ω is a *convex cone* if $av + bw \in \Omega$ whenever $v, w \in \Omega$ and a, b are positive real numbers. Note that if Ω is a convex cone, then Ω_M is a convex cone. The convex cone condition can be interpreted as saying that changing the currency and combining possible allocation problems should result in other possible allocation problems. If w is an allocation problem and b is a positive real number, then bw is really the same allocation problem expressed with a different currency (e.g., francs instead of dollars). If v and w are allocation problems (e.g., municipal waste collection and sewage treatment), then it should make sense to combine the two allocation problems into the single allocation problem $v + w$.

Suppose that w is an M-game and σ is a permutation of N. If S is a coalition, then let $\sigma(S)$ be the set $\{\sigma(i) : i \in S\}$. Define the M-game σw by the formula $(\sigma w)(S) = w(\sigma^{-1}(S))$ for all coalitions S satisfying $|S| \in M$. The set Ω is *symmetric* if $\sigma w \in \Omega$ whenever $w \in \Omega$ and σ is a permutation of N. Note that if Ω is symmetric, then Ω_M is symmetric. The symmetry condition can be interpreted as saying that relabeling the players in a possible allocation problem should result in another possible allocation problem.

Collections of cooperative games that are convex cones and symmetric include the collections of all games, zero-monotonic games, superadditive games, and convex games.

An allocation method φ is *linear* if $\varphi(av + bw) = a\varphi(v) + b\varphi(w)$ whenever $v, w \in \Omega_M$ and a, b are positive real numbers. Our interpretation is that the fair share to a player should not depend on the unit of currency used to state the allocation problem nor whether the allocation problem is first divided into separate and additive allocation problems. Note that if Ω_M is a convex cone, then $av + bw \in \Omega_M$ making the definition meaningful. Linearity is slightly stronger than Shapley's additivity axiom in which $a = b = 1$. The additional, relatively innocuous, proportionality assumption, $\varphi(bw) = b\varphi(w)$ whenever $w \in \Omega_M$ and b is a positive real number, made here is necessary to rule out highly discontinuous allocation methods. The reader could also substitute additivity for linearity in all that follows if we restrict ourselves to rational rather than real numbers.

an allocation method φ is *symmetric* if $\varphi_{\sigma(i)}(\sigma w) = \varphi_i(w)$ whenever $w \in \Omega_M$ and σ is a permutation of N. The fair share to a player should not depend on the label given to represent that player. Note that if Ω_M is symmetric, then $\sigma w \in \Omega_M$ making the definition meaningful. This is the standard definition of symmetry used in the literature, although sometimes it is called anonymity. Two players i and j are called *substitutes* in the M-game w if $w(S - \{i\}) = w(S - \{j\})$ whenever S is a coalition satisfying $i, j \in S$ and $|S| - 1 \in M$. Symmetry implies the weaker property *equal treatment*: $\varphi_i(w) = \varphi_j(w)$ whenever i and j are substitutes in w. In example 2.1, players 2, 3, 4, and 5 are substitutes and so any symmetric allocation method should assign the same payoffs as did the reduced Shapley value.

The reduced Shapley value is linear and symmetric on all collections of games that are convex cones and symmetric. However, there are other such allocation methods. Suppose $b \in R^M$ satisfies $b_n = 1$. The *b-weighted Shapley value* ψ^b is defined by

$$\psi_i^b(w) = \frac{1}{n} \sum_{m \in M} b_m \left(\binom{n-1}{m-1}^{-1} \sum_{\substack{|S|=m \\ i \in S}} w(S) - \binom{n-1}{m}^{-1} \sum_{\substack{|S|=m \\ i \notin S}} w(S) \right). \quad (3.1)$$

Note that the weighting is with respect to the size of the coalitions instead of with respect to the player indices and is therefore distinct from the literature on linear but nonsymmetric allocation methods for cooperative games (see Kalai and Samet (1988) and Nowak and Radzik (1995)). The reduced Shapley value is the special case when $b_m = 1$ for all $m \in M$. The first theorem is that b-weighted Shapley values are linear and symmetric and are the only linear and symmetric allocation methods. This Theorem generalizes Theorems 3.6 and 3.7 of Willson (1993) because there Ω is taken to be the collection of all N-games.

Theorem 3.1. *Suppose Ω_M is a convex cone and a symmetric collection of games. The allocation method φ on Ω_M is linear and symmetric if and only if φ is a b-weighted Shapley value.*

Proof. The function ψ^b is clearly linear and is symmetric because $\sum_{|S|=m,\sigma(i)\in S} \sigma w(S) = \sum_{|S|=m,i\in\sigma^{-1}(S)} w(\sigma^{-1}(S)) = \sum_{|S|=m,i\in S} w(S)$ and similarly $\sum_{|S|=m,\sigma(i)\notin S} \sigma w(S) = \sum_{|S|=m,i\notin S} w(S)$. That ψ^b is an allocation method, that is, yields allocations, follows from the following calculation:

$$\sum_{i\in N} \psi_i^b(w) - w(N)$$

$$= \frac{1}{n} \sum_{i\in N} \sum_{m\in M-\{n\}} b_m \left(\binom{n-1}{m-1}^{-1} \sum_{\substack{|S|=m \\ i\in S}} w(S) - \binom{n-1}{m}^{-1} \sum_{\substack{|S|=m \\ i\notin S}} w(S) \right)$$

$$= \frac{1}{n} \sum_{m\in M-\{n\}} b_m \left(\binom{n-1}{m-1}^{-1} \sum_{|S|=m} \sum_{i\in S} w(S) - \binom{n-1}{m}^{-1} \sum_{|S|=m} \sum_{i\notin S} w(S) \right)$$

$$= \frac{1}{n} \sum_{m\in M-\{n\}} b_m \left(\binom{n-1}{m-1}^{-1} m - \binom{n-1}{m}^{-1} (n-m) \right) \sum_{|S|=m} w(S)$$

$$= \frac{1}{n} \sum_{m\in M-\{n\}} b_m (0) \sum_{|S|=m} w(S) = 0.$$

Conversely, suppose φ is linear and symmetric. We need to show that there exist constants b_m, $m \in M$, satisfying $b_n = 1$, for which $\varphi(w) = \psi^b(w)$ for all $w \in \Omega_M$.

As a special case, suppose first that Ω is the collection of all games. For each coalition T satisfying $|T| \in M$, define the M-game e^T by $e^T(T) = 1$ and $e^T(S) = 0$ otherwise. Clearly, i and j are substitutes in e^T if both $i, j \in T$ or both $i, j \notin T$. By the equal treatment property of φ, there are constants a_T and a'_T for which $\varphi_i(e^T) = a_T$ if $i \in T$ and $\varphi_i(e^T) = a'_T$ if $i \notin T$. Since $\varphi(e^T)$ is an allocation, $e^T(N) = \sum_{i\in N} \varphi_i(e^T) = |T| a_T + (n - |T|) a'_T$ which implies $a_N = \frac{1}{n}$ and $a'_T = -|T| a_T / (n - |T|)$ if $T \neq N$. Suppose T and T' are coalitions satisfying $|T| = |T'| \in M$, and let σ be a permutation of N satisfying $\sigma(T) = T'$. By the symmetry of φ, if $i \in T$, then $a_{T'} = \varphi_{\sigma(i)}(e^{T'}) = \varphi_{\sigma(i)}(\sigma e^T) = \varphi_i(e^T) = a_T$.

Hence, there exist constants c_m, $m \in M$, satisfying $c_n = \frac{1}{n}$ and $\varphi_i\left(e^T\right) = c_{|T|}$ if $i \in T$ and $\varphi_i\left(e^T\right) = -|T|\,c_{|T|}/\left(n - |T|\right)$ if $i \notin T$. Now given any $w \in \Omega_M$, we may write $w = \sum_{|T| \in M} w(T)\, e^T$. By the linearity of φ, it follows that

$$\begin{aligned}
\varphi_i(w) &= \sum_{|T| \in M} w(T)\,\varphi_i\left(e^T\right) \\
&= \sum_{m \in M} \left(\sum_{\substack{|T|=m \\ i \in T}} w(T)\,\varphi_i\left(e^T\right) + \sum_{\substack{|T|=m \\ i \notin T}} w(T)\,\varphi_i\left(e^T\right) \right) \\
&= \sum_{m \in M} \left(\sum_{\substack{|T|=m \\ i \in T}} w(T)\,c_m + \sum_{\substack{|T|=m \\ i \notin T}} w(T)\,\tfrac{-m}{n-m} c_m \right) \\
&= \sum_{m \in M} c_m \left(\sum_{\substack{|T|=m \\ i \in T}} w(T) - \tfrac{m}{n-m} \sum_{\substack{|T|=m \\ i \notin T}} w(T) \right).
\end{aligned}$$

Set $b_m = n\binom{n-1}{m-1} c_m$, $m \in M$, to obtain $\varphi(w) = \psi^b(w)$. In summary, the theorem holds when Ω is the collection of all N-games.

Now consider the general case in which Ω is a convex cone and symmetric, but Ω need not contain all games. Again suppose φ is linear and symmetric on Ω_M. Our approach will be to extend the definition of φ to all M-games preserving linearity and symmetry. We will then make use of our special case result.

Define $Span(\Omega_M)$ to be the set of M-games spanned by Ω_M, that is, $Span(\Omega_M)$ is the collection of M-games $\sum_{i \in I} a_i w^i$ where a_i is a real number and $w^i \in \Omega_M$ for all i in some finite set I. Since Ω_M is symmetric, $Span(\Omega_M)$ is symmetric. Indeed, suppose $w \in Span(\Omega_M)$ and σ is a permutation of N. Then $w = \sum_{i \in I} a_i w^i$ for some finite set I and some real number a_i and $w^i \in \Omega_M$ for all $i \in I$. By symmetry of Ω_M, it follows that $\sigma w^i \in \Omega_M$ and $\sigma w = \sum_{i \in I} a_i\left(\sigma w^i\right) \in Span(\Omega_M)$.

We will now extend the definition of φ to $Span(\Omega_M)$. For each $w \in Span(\Omega_M)$, there exist a finite set I and real number a_i and $w^i \in \Omega_M$ for $i \in I$ satisfying $w = \sum_{i \in I} a_i w^i$; define $\bar{\varphi}(w) = \sum_{i \in I} a_i \varphi\left(w^i\right)$. We must check that $\bar{\varphi}$ is well-defined. Suppose w does not have a unique representation as a linear combination of M-games in Ω_M, that is, suppose I and J are finite sets, a_i is a real number and $w^i \in \Omega_M$ for all $i \in I$, b_i is a real number and $w^j \in \Omega_M$ for all $j \in J$, and $w = \sum_{i \in I} a_i w^i = \sum_{j \in J} b_j w^j$. Rearranging the last equality, we obtain $\sum_{i \in I^+} a_i w^i + \sum_{j \in J^-} \left(-b_j\right) w^j = \sum_{i \in I^-} \left(-a_i\right) w^i + \sum_{j \in J^+} b_j w^j$ where $I^+ = \{i \in I : a_i \geq 0\}$, $I^- = \{i \in I : a_i < 0\}$, $J^+ = \{j \in J : a_j \geq 0\}$, and $J^- =$

$\{j \in J : a_j < 0\}$. Both sides of this last equality are positive (zero coefficients may be ignored) linear combinations of games in Ω_M, and so the linearity of φ implies $\sum_{i\in I^+} a_i\varphi(w^i) + \sum_{j\in J^-}(-b_j)\varphi(w^j) = \sum_{i\in I^-}(-a_i)\varphi(w^i) + \sum_{j\in J^+} b_j\varphi(w^j)$. Rearranging this equality, we obtain $\sum_{i\in I} a_i\varphi(w^i) = \sum_{j\in J} b_j\varphi(w^j)$. Hence, the value of $\bar{\varphi}(w)$ does not depend on the representation chosen for w, that is, $\bar{\varphi}$ is well-defined. It is now straight-forward to show that $\bar{\varphi}(w) = \varphi(w)$ for all $w \in \Omega_M$, that φ is an allocation method implies $\bar{\varphi}$ is an allocation method, and that the linearity and symmetry of φ implies that $\bar{\varphi}$ is linear and symmetric.

In order to extend the definition of φ to all M-games, we need to consider the subspace orthogonal to $Span(\Omega_M)$. Define $Orth(\Omega_M)$ to be the collection of M-games v satisfying $\sum_{|S|\in M} v(S)\,w(S) = 0$ for all $w \in Span(\Omega_M)$. Since $Orth(\Omega_M)$ is a linear subspace, $Orth(\Omega_M)$ is a convex set. The collection $Orth(\Omega_M)$ is also symmetric. Indeed, suppose $v \in Orth(\Omega_M)$ and σ is a permutation of N. If $w \in Span(\Omega_M)$, then $w' = \sigma^{-1}w \in Span(\Omega_M)$ and

$$\begin{aligned}\sum_{|S|\in M}(\sigma v)(S)\,w(S) &= \sum_{|S|\in M}(\sigma v)(S)\,(\sigma w')(S)\\ &= \sum_{|S|\in M} v\left(\sigma^{-1}(S)\right) w'\left(\sigma^{-1}(S)\right)\\ &= \sum_{|R|\in M} v(R)\,w'(R) = 0.\end{aligned}$$

Let π_1 and π_2 be the projection maps from the vector space of all M-games to $Span(\Omega_M)$ and $Orth(\Omega_M)$, respectively. That is, for any M-game w the projection maps yield the unique M-games $\pi_1(w) \in Span(\Omega_M)$ and $\pi_2(w) \in Orth(\Omega_M)$ satisfying $w = \pi_1(w) + \pi_2(w)$. We can now define our extension of φ to all M-games. Define $\hat{\varphi}$ by the formula $\hat{\varphi}_i(w) = \bar{\varphi}_i(\pi_1(w)) + \frac{1}{n}(\pi_2(w))(N)$. We now show that $\hat{\varphi}$ has the desired properties. First, $\hat{\varphi}$ is an extension of $\bar{\varphi}$. Indeed, if $w \in Span(\Omega_M)$, then $\pi_1(w) = w$ and $\pi_2(w)$ is the M-game with all coalitional worths zero, and so $\hat{\varphi}_i(w) = \bar{\varphi}_i(w) + 0 = \varphi_i(w)$ since $\bar{\varphi}$ is an extension of φ. Second, $\hat{\varphi}$ is an allocation method because $\sum_{i\in N}\hat{\varphi}_i(w) = \sum_{i\in N}\bar{\varphi}_i(\pi_1(w)) + (\pi_2(w))(N) = (\pi_1(w))(N) + (\pi_2(w))(N) = (\pi_1(w) + \pi_2(w))(N) = w(N)$. Third, $\hat{\varphi}$ is linear because projection maps and compositions of linear maps are linear. Fourth, $\hat{\varphi}$ is symmetric. Indeed, suppose w is an M-game and σ is a permutation of N. Then

$$\begin{aligned}\hat{\varphi}_{\sigma(i)}(\sigma w) &= \bar{\varphi}_{\sigma(i)}(\pi_1(\sigma w)) + \tfrac{1}{n}(\pi_2(\sigma w))(N)\\ &= \bar{\varphi}_{\sigma(i)}(\sigma(\pi_1(w))) + \tfrac{1}{n}(\sigma(\pi_2(w)))(N)\\ &= \bar{\varphi}_i(\pi_1(w)) + \tfrac{1}{n}(\pi_2(w))(N)\\ &= \hat{\varphi}_i(w).\end{aligned}$$

In summary, $\hat{\varphi}$ is a linear and symmetric allocation method on the collection of all M-games which equals φ on Ω_M. By our earlier work, there exist constants $b_m,\ m \in M$, satisfying $b_n = 1$ such that $\hat{\varphi} = \psi^b$, and so $\varphi = \psi^b$. ■

When $M = N$ and Ω is the set of all games, Theorem 3.1 characterizes allocation methods satisfying all properties of a value except possibly the null player axiom. This characterization shows that each such allocation method is determined linearly by $n-1$ constants, one for each coalition size $m = 1, 2, \ldots, n-1$. This is technically similar to the Dubey *et al* (1981) characterization of semivalues (methods satisfying all properties of a value except possibly efficiency which we take as part of our definition of allocation method) that are uniquely defined by a vector of n weights, one for each coalition size $m = 1, 2, \ldots, n$.

We close this section with a uniqueness of representation theorem for weighted Shapley values.

Theorem 3.2. *Suppose Ω_M has a nonempty interior. The a-weighted and b-weighted Shapley values are equal on Ω_M if and only if $a = b$.*

Proof. Clearly, if $a = b$, then the a-weighted and b-weighted Shapley values are equal. Conversely, suppose the a-weighted and b-weighted Shapley values are equal. For $m \in M$, define the set functions v^m by $v^m(S) = 1$ if $S = \{1, 2, ..., m\}$ and $v^m(S) = 0$ otherwise. Let w be an M-game contained in the nonempty interior of Ω_M. Hence, for sufficiently small $\varepsilon > 0$, the M-games $w + \varepsilon v^m$ are contained in Ω_M for all $m \in M$. Since $\psi^a = \psi^b$, it follows that $0 = \psi_1^a(w + \varepsilon v^m) - \psi_1^b(w + \varepsilon v^m) = \psi_1^a(w) - \psi_1^b(w) + \varepsilon\left(\psi_1^a(v^m) - \psi_1^b(v^m)\right)$ [since ψ^a and ψ^b are linear] $= \varepsilon\left(\psi_1^a(v^m) - \psi_1^b(v^m)\right)$[since $\psi^a = \psi^b$ on Ω_M] $= \varepsilon \frac{1}{n}\binom{n-1}{m-1}^{-1}(a_m - b_m)$ [by formula 3.1]. Therefore, $a_m = b_m$ for all $m \in M$. ■

4. Monotonicity

In this section, we consider three natural monotonicity conditions for an allocation method on partially defined cooperative games, and characterize linear and symmetric allocation methods satisfying each of these monotonicity conditions.

Suppose M is a set of known sizes. Recall that we defined $\rho(m) = \max\{0, r : r \in M \text{ and } r < m\}$ to be the next smaller size for which coalitional worths are known, and we let $w(\phi) = 0$ for notational convenience. Given a player i, two coalitions R and S are *i-adjacent* if $|S| \in M$, $i \in S$, $|R| = \rho(|S|)$, and $R \subseteq S - \{i\}$. Given an M-game w, a *marginal contribution* of i is the quantity

$\Delta_i(R,S;w) = w(S) - w(R)$ for some i-adjacent coalitions R and S. Player i is *marginally favored* by M-game w over the M-game v if $\Delta_i(R,S;w) \geq \Delta_i(R,S;v)$ for all i-adjacent coalitions R and S. So, a player i is marginally favored by w over v if player i's marginal contributions in w are at least as great as in v. In such a circumstance, it is natural to assume that player i receives a higher payoff in w than v. The next condition formalizes this intuition. The allocation method φ is *margin monotone* on Ω_M if $\varphi_i(w) \geq \varphi_i(v)$ whenever a player i is marginally favored by an M-game $w \in \Omega_M$ over an M-game $v \in \Omega_M$.

The next theorem characterizes linear, symmetric, and margin monotone allocation methods. It is a generalization of Willson's (1993) main theorem in that he only considers Ω_M equal to the collections of all M-games. The proof given here is also much shorter and more transparent.

Theorem 4.1. *Suppose Ω_M is a convex cone, symmetric, and has a nonempty interior. The allocation method φ is linear, symmetric, and margin monotone on Ω_M if and only if φ is the reduced Shapley value.*

Proof. Suppose the allocation method φ is linear, symmetric, and margin monotone on Ω_M. Let k be the smallest number in M. For $m \in M - \{k\}$, define the set functions v^m by $v^m(S) = 1$ if $|S| = m$ and $1 \in S$, $v^m(S) = 1$ if $|S| = \rho(m)$ and $1 \notin S$, and $v^m(S) = 0$ otherwise. Let w be an M-game contained in the nonempty interior of Ω_M. Hence, for sufficiently small $\varepsilon > 0$, the M-games $w+\varepsilon v^m$ are contained in Ω_M for all $m \in M - \{k\}$. Because $v^m(T) - v^m(S) = 0$ for all 1-adjacent coalitions S and T, it follows that player 1 is marginally favored by w over $w+\varepsilon v^m$ and by $w+\varepsilon v^m$ over w. Because φ is margin monotone, it follows that $\varphi_1(w) = \varphi_1(w+\varepsilon v^m)$. Because φ is linear and symmetric, theorem 3.1 implies φ is a b-weighted Shapley value. Hence, $0 = \psi_1^b(w+\varepsilon v^m) - \psi_1^b(w) = \varepsilon \psi_1^b(v^m)$ [by linearity of ψ^b] $= \varepsilon \frac{1}{n}\left(b_m - b_{\rho(m)}\right)$ [using formula 3.1]. So, $b_m = b_{\rho(m)}$ for all $m \in M - \{k\}$. Since $b_n = 1$, it follows that $b_m = 1$ for all $m \in M$. Therefore, φ is the reduced Shapley value.

Conversely, suppose φ is the reduced Shapley value. By theorem 3.1, φ is linear and symmetric. By formula 2.2, the reduced Shapley value for player i is a positive linear combination of player i's marginal contributions. Hence, φ is margin monotone. ■

In addition to Ω_M being a convex cone and symmetric, the theorem assumes that Ω_M has a nonempty interior. This condition clearly holds for the collections of convex, superadditive, zero-monotonic, and all N-games. Weaker conditions are possible. For example, the conclusion of the theorem holds, with an almost identical proof, if Ω_M is assumed to contain only zero-normalized games and have a nonempty interior in the space of all zero-normalized N-games. (See the Formulation Independence section for definitions of these terms.) The next theorem shows that some condition on Ω_M is required for an allocation method to be uniquely characterized by linearity, symmetry, and margin monotonicity.

Theorem 4.2. *There exists a convex cone and symmetric Ω_M and linear, symmetric, and margin monotone allocation methods on Ω_M which are not the reduced Shapley value.*

Proof. Let $k \in M$ satisfy $\rho(k) = 0$, that is, $k = \min\{m : m \in M\}$. Let Ω be the set of all N-games w satisfying $\sum_{|S|=k} w(S) = 0$. Clearly, Ω_M is a convex cone and symmetric. Note that Ω_M does not have a nonempty interior. Let $\varphi = \psi^b$ where $b_m = 1$ for all $m \in M - \{k\}$ and $b_k > 1$. By theorem 3.1, φ is linear and symmetric. We now show that φ is margin monotone. Let $l \in M$ satisfy $\rho(l) = k$. Suppose player i is marginally favored by $w \in \Omega_M$ over $v \in \Omega_M$. Then $\varphi_i(w) - \varphi_i(v) = \psi_i^b(w) - \psi_i^b(v) = \psi_i^b(w - v)$ [by linearity of ψ^b]

$$= \psi_i(w - v) + \tfrac{1}{n}(b_k - 1)\left(\binom{n-1}{k-1}^{-1} \sum_{\substack{|S|=k \\ i \in S}} (w - v)(S) - \binom{n-1}{k}^{-1} \sum_{\substack{|S|=k \\ i \notin S}} (w - v)(S)\right)$$

[by formulas 2.1 and 3.1]

$$= \psi_i(w - v) + \tfrac{1}{n}(b_k - 1)\left(\binom{n-1}{k-1}^{-1} \sum_{\substack{|S|=k \\ i \in S}} (w - v)(S) + \binom{n-1}{k}^{-1} \sum_{\substack{|S|=k \\ i \in S}} (w - v)(S)\right)$$

[since $v, w \in \Omega_M$ implies $\sum_{|S|=k} (w - v)(S) = 0$]

$= \psi_i(w - v) + \tfrac{1}{n}(b_k - 1)\left(\binom{n-1}{k-1}^{-1} + \binom{n-1}{k}^{-1}\right) \sum_{\substack{|S|=k \\ i \in S}} (w - v)(S)$ [algebra] ≥ 0 [since player i is marginally favored by w over v, and ψ is margin monotone]. Hence, $\varphi_i(w) \geq \varphi_i(v)$. ■

Player i is *coalitionally favored* by M-game w over the M-game v if $w(S) \geq v(S)$ for all coalitions S containing i and $w(S) \leq v(S)$ for all coalitions S not containing i. So, a player i is coalitionally favored by w over v if coalitions containing i are better off in w and coalitions not containing i are worse off in w. In such a circumstance, it is natural to assume that player i receives a higher payoff in w than v. The allocation method φ is *coalition monotone* on Ω_M if $\varphi_i(w) \geq \varphi_i(v)$ whenever a player i is coalitionally favored by an M-game $w \in \Omega_M$ over an M-game $v \in \Omega_M$. Note that if player i is coalitionally favored by w over v, then player i is marginally favored by w over v. So, coalition monotonicity is a weaker condition than margin monotonicity.

Theorem 4.3. *Suppose Ω_M is a convex cone, symmetric, and has a nonempty interior. The allocation method φ is linear, symmetric, and coalition monotone on Ω_M if and only if φ is a b-weighted Shapley value for which $b_m \geq 0$ for all $m \in M$.*

Proof. Suppose the allocation method φ is linear, symmetric, and coalition monotone on Ω_M. For $m \in M$, define the set functions v^m by $v^m(S) = 1$ if $S = \{1, 2, ..., m\}$ and $v^m(S) = 0$ otherwise. Let w be an M-game contained in the nonempty interior of Ω_M. Hence, for sufficiently small $\varepsilon > 0$, the M-games $w + \varepsilon v^m$ are contained in Ω_M for all $m \in M$. Clearly, player 1 is coalitionally favored by $w + \varepsilon v^m$ over w. Because φ is coalition monotone, it follows that $\varphi_1(w + \varepsilon v^m) \geq \varphi_1(w)$. Because φ is linear and symmetric, theorem 3.1 implies φ is a b-weighted Shapley value. Hence, $0 \leq \psi_1^b(w + \varepsilon v^m) - \psi_1^b(w) = \varepsilon \psi_1^b(v^m)$ [by linearity of ψ^b] $= \varepsilon \frac{1}{n} \binom{n-1}{m-1}^{-1} b_m$ [using formula 3.1]. So, $b_m \geq 0$ for all $m \in M$.

Conversely, suppose φ is a b-weighted Shapley value for which $b_m \geq 0$ for all $m \in M$. By theorem 3.1, φ is linear and symmetric. Suppose player i and M-games $w, v \in \Omega_M$ satisfy $w(S) \geq v(S)$ for all coalitions S containing i and $w(S) \leq v(S)$ for all coalitions S not containing i. Since in formula 3.1, the coefficients of $w(S)$ are nonnegative when S contains i and nonpositive when S does not contain i, it follows that $\varphi_i(w) \geq \varphi_i(v)$. Hence, φ is coalition monotone. ■

Player i is favored over player j in the M-game w if $w(S \cup \{i\}) \geq w(S \cup \{j\})$ for all coalitions S satisfying $i, j \notin S$ and $|S| + 1 \in M$. So, player i is favored over player j if substituting i for j can only increase the worth of any coalition. In such a circumstance, it is natural to assume that player i receives a higher payoff than player j. The allocation method φ is *player monotone* on Ω_M if $\varphi_i(w) \geq \varphi_j(w)$ whenever a player i is favored over a player j in an M-game $w \in \Omega_M$. Player monotonicity involves comparisons between players in a single game while margin and coalition monotonicity involve comparisons between games for a single player. It is somewhat remarkable that player and coalition monotonicity characterize the same class of allocation methods.

Theorem 4.4. *Suppose Ω_M is a convex cone, symmetric, and has a nonempty interior. The allocation method φ is linear, symmetric, and player monotone on Ω_M if and only if φ is a b-weighted Shapley value for which $b_m \geq 0$ for all $m \in M$.*

Proof. Suppose the allocation method φ is linear, symmetric, and player monotone on Ω_M. Let w' be an M-game contained in the nonempty interior of Ω_M. Since Ω_M is symmetric, $\sigma w'$ is contained in the interior of Ω_M for all permutations σ of N. Since Ω_M is a convex cone, $w = \sum_\sigma \sigma w'$, where the summation is over all permutations σ of N, is contained in the interior of Ω_M. Note also that all pairs of players are substitutes in w; hence, $\varphi_i(w) = \frac{1}{n} w(N)$ for all $i \in N$. For $m \in M$, define the set functions v^m by $v^m(S) = 1$ if $S = \{1, 2, ..., m\}$ and $v^m(S) = 0$ otherwise. Since w is contained in the interior of Ω_M, there exists a sufficiently small $\varepsilon > 0$ for which the M-games $w + \varepsilon v^m$ are contained in Ω_M for all $m \in M$. Clearly, player 1 is favored over player n in $w + \varepsilon v^m$. Because φ is player monotone, it follows that $\varphi_1(w + \varepsilon v^m) \geq \varphi_n(w + \varepsilon v^m)$. Because φ is linear and symmetric, theorem 3.1 implies φ is a b-weighted Shapley value. Hence, $0 \leq \psi_1^b(w + \varepsilon v^m) - \psi_n^b(w + \varepsilon v^m) = \psi_1^b(w) - \psi_n^b(w) + \varepsilon \psi_1^b(v^m) - \varepsilon \psi_n^b(v^m)$ [by linearity of ψ^b] $= \varepsilon \psi_1^b(v^m) - \varepsilon \psi_n^b(v^m)$ [since 1 and n are substitutes in w] $= \varepsilon \frac{1}{n} \left(\binom{n-1}{m-1}^{-1} + \binom{n-1}{m}^{-1} \right) b_m$ [using formula 3.1]. So, $b_m \geq 0$ for all $m \in M$.

Conversely, suppose φ is a b-weighted Shapley value for which $b_m \geq 0$ for all $m \in M$. By theorem 3.1, φ is linear and symmetric. Using formula 3.1, we obtain that $\psi_i^b(w) - \psi_j^b(w) =$

$$\frac{1}{n} \left(\binom{n-1}{m-1}^{-1} + \binom{n-1}{m}^{-1} \right) \sum_{m \in M} b_m \left(\sum_{\substack{|S|=m \\ i \in S \\ j \notin S}} w(S) - \sum_{\substack{|S|=m \\ i \notin S \\ j \in S}} w(S) \right).$$

If player i is favored over player j in the M-game $w \in \Omega_M$, then the last bracketed term in the above expression for $\psi_i^b(w) - \psi_j^b(w)$ is nonnegative. Since the b_m are also nonnegative, it follows that $\psi_i^b(w) - \psi_j^b(w) \geq 0$. Hence, φ is player monotone. ■

5. Formulation Independence

Consider the following joint cost allocation problem. Suppose each player must make use of some shared resource, and the most economical method of obtaining a sufficient amount of the shared resource for each player in a coalition S results in a cost of $c(S)$. If all players cooperate, what is a fair way to allocate the total cost $c(N)$? There are at least two reasonable ways to make use of a cooperative game allocation method φ to solve this joint cost allocation problem. First, define the worth of coalition S to be the negative of its cost, $w^1(S) = -c(S)$, and then allocate the total cost via the negative of the allocation method: $-\varphi_i(w^1)$ would be the cost allocated to player i. Second, define the worth of coalition S to be the savings resulting via cooperation, $w^2(S) = \sum_{i \in S} c(\{i\}) - c(S)$, and then allocate the total savings via the allocation method: $c(\{i\}) - \varphi_i(w^2)$ would be the cost allocated to player i. It would be desirable for the cost allocation not to depend upon the solution approach used, that is, $-\varphi_i(w^1) = c(\{i\}) - \varphi_i(w^2)$. Our next condition ensures this.

The *zero-normalization* of the M-game w is the M-game $\bar{w}$ defined by the formula $\bar{w}(S) = w(S) - \sum_{i \in S} w(\{i\})$. an allocation method φ is *formulation independent* on the collection Ω_M if $\varphi_i(w) = \varphi_i(\bar{w}) + w(\{i\})$ for all $w \in \Omega_M$ and $i \in M$. Note that the equality of this definition is equivalent to the last equality of the previous paragraph if we set $w = -c$. Young (1994) calls formulation independence, "invariance in direct costs." Formulation independence and *proportionality* ($\varphi(aw) = a\varphi(w)$ for all positive real numbers a and M-games w, which is implied by linearity) is equivalent to another property often cited in the literature: *covariance with respect to strategic equivalence* ($\varphi_i(v) = a\varphi_i(w) + b_i$ for all players $i \in N$, positive real numbers a, real numbers b_i, and M-games v and w satisfying $v(S) = aw(S) + \sum_{j \in S} b_j$ for all coalitions $S \subseteq N$). The following theorem characterizes linear, symmetric, and formulation independent allocation methods.

Theorem 5.1. *Suppose the set of known coalition sizes M contains 1. Suppose the collection of M-games Ω_M is a convex cone, is symmetric, and contains its zero normalizations. The allocation method φ on Ω_M is linear, symmetric, and formulation independent if and only if φ is a b-weighted Shapley value satisfying $\sum_{m \in M} b_m = n$.*

Proof. We begin by determining a relationship between the b-weighted Shapley values of an M-game and its zero-normalization:

$$\psi_i^b(w) - \psi_i^b(\bar{w})$$

$$= \frac{1}{n} \sum_{m \in M} b_m \left(\binom{n-1}{m-1}^{-1} \sum_{\substack{|S|=m \\ i \in S}} \left(w(S) - \bar{w}(S) \right) - \binom{n-1}{m}^{-1} \sum_{\substack{|S|=m \\ i \notin S}} \left(w(S) - \bar{w}(S) \right) \right)$$

[using formula 3.1]

$$= \frac{1}{n} \sum_{m \in M} b_m \left(\binom{n-1}{m-1}^{-1} \sum_{\substack{|S|=m \\ i \in S}} \left(\sum_{j \in S} w(\{j\}) \right) - \binom{n-1}{m}^{-1} \sum_{\substack{|S|=m \\ i \notin S}} \left(\sum_{j \in S} w(\{j\}) \right) \right)$$

[using the definition of zero-normalization]

$$= \frac{1}{n} \sum_{j \in N} w(\{j\}) + \frac{1}{n} \sum_{m \in M - \{n\}} b_m \left(w(\{i\}) + \binom{n-1}{m-1}^{-1} \binom{n-2}{m-2} \sum_{j \in N - \{i\}} w(\{j\}) \right.$$
$$\left. - \binom{n-1}{m}^{-1} \binom{n-2}{m-1} \sum_{j \in N - \{i\}} w(\{j\}) \right)$$

[separating out the $m = n$ term of the outside summation and reversing the double summations over coalitions and players]

$$= \frac{1}{n} \sum_{j \in N} w(\{j\}) + \frac{1}{n} \sum_{m \in M - \{n\}} b_m \left(w(\{i\}) - \frac{1}{n-1} \sum_{j \in N - \{i\}} w(\{j\}) \right)$$

$$= \frac{1}{n} w(\{i\}) \sum_{m \in M} b_m + \frac{1}{n} \left(1 - \frac{1}{n-1} \sum_{m \in M - \{n\}} b_m \right) \sum_{j \in N - \{i\}} w(\{j\})$$

$$= w(\{i\}) + \frac{1}{n} \left(\sum_{m \in M} b_m - n \right) \left(w(\{i\}) - \frac{1}{n-1} \sum_{j \in N - \{i\}} w(\{j\}) \right).$$

Thus, $\psi_i^b(w) = \psi_i^b(\bar{w}) + w(i)$ if and only if

$$\left(\sum_{m \in M} b_m - n \right) \left((n-1) w(\{i\}) - \sum_{j \in N - \{i\}} w(\{j\}) \right) = 0.$$

Suppose $\varphi = \psi^b$ where $\sum_{m \in M} b_m = n$. By theorem 3.1, φ is linear and symmetric. By the result of the previous paragraph, $\varphi_i^b(w) = \varphi_i^b(\bar{w}) + w(\{i\})$ for all M-games w, and so φ is formulation independent.

Conversely, suppose φ is an allocation method on Ω_M which is linear, symmetric, and formulation independent. By theorem 3.1, $\varphi = \psi^b$ for some b satisfying $b_n = 1$. We need to show that b can be chosen so that $\sum_{m \in M} b_m = n$. We first consider the case when $w \in \Omega_M$ implies $w(1) = w(2) = ... = w(n)$. In this case, the expression b_1 multiplies in equation 3.1 always equals zero. So, changing b_1 does not change ψ^b. Hence, we can choose $b_1 = n - \sum_{m \in M - \{1\}} b_m$.

We now consider the case when there is a $w \in \Omega_M$ and $i, j \notin N$ satisfying $w(\{i\}) \neq w(\{j\})$. Without loss of generality, we can assume that i is chosen so that $w(\{i\}) \geq w(\{j\})$ for all $j \in N$ and $w(\{i\}) > w(\{j\})$ for some $j \in N$. Since φ is formulation independent, the result of the proof's first paragraph yields $\left(\sum_{m \in M} b_m - n\right) a = 0$ where $a > 0$. Hence, $\sum_{m \in M} b_m = n$. ■

Formulation independence and margin monotonicity are largely incompatible properties as shown by the following theorem.

Theorem 5.2. *Suppose the set of known coalition sizes M contains 1. Suppose the collection of games Ω_M is a convex cone, is symmetric, contains its zero normalizations, and has a nonempty interior. The reduced Shapley value is formulation independent on Ω_M if and only if $M = N$.*

Proof. By theorem 3.1, the reduced Shapley value is linear and symmetric. So, theorem 5.1 implies that the reduced Shapley value is formulation independent if and only if it equals a b-weighted Shapley value satisfying $\sum_{m \in M} b_m = n$. Since Ω_M has a nonempty interior, theorem 3.2 implies that the reduced Shapley value is a b-weighted Shapley value if and only if $b_m = 1$ for all $m \in M$. Therefore, the reduced Shapley value is formulation independent if and only if $n = \sum_{m \in M} 1 = |M|$ which is true if and only if $M = N$. ■

The preceeding two theorems suggest that the reduced Shapley value need not be the best value to use in many circumstances. A possible alternative is the *normalized Shapley value* $\bar{\psi} = \psi^b$ where $b_m = 1$ for all $m \in M - \{1\}$ and $b_1 = n - \sum_{m \in M - \{1\}} b_m$. By theorem 5.1, the normalized Shapley value is formulation independent, and so $\bar{\psi}_i(w) = w(\{i\}) + \bar{\psi}_i(\bar{w}) = w(\{i\}) + \psi_i(\bar{w})$ [since b_1 does not affect $\psi^b(\bar{w})$ because $\bar{w}(\{i\}) = 0$ for all $i \in N$], that is, $\bar{\psi}_i(w)$ allocates to player i the player's individual worth and the player's reduced Shapley value in the zero-normalized game. Of course, by theorem 4.1, the normalized Shapley value must not be margin monotone. The following example illustrates the differences between the formulation independent normalized Shapley value and the margin monotone reduced Shapley value.

Example 5.3. *Suppose $M = \{1, n\}$. The reduced Shapley value is given by the formula $\psi_i(w) = \frac{1}{n-1} w(\{i\}) + \frac{1}{n}\left(w(N) - \frac{1}{n-1}\sum_{j\in N} w(\{j\})\right)$. The normalized Shapley value is given by the formula $\bar{\psi}_i(w) = w(\{i\}) + \frac{1}{n}\left(w(N) - \sum_{j\in N} w(\{j\})\right)$. An interpretation of the normalized Shapley value formula is that each player is first allocated his or her individual worth, and then the remaining benefits due to cooperation are divided evenly. Let the M-game w be defined by $w(N) = 2$ and $w(\{i\}) = 1$ for all $j \in N$. Clearly, all pairs of players are substitutes, and so $\psi_1(w) = \bar{\psi}_1(w) = \frac{2}{n}$. Let the M-game v be defined by $v(N) = v(\{1\}) = 1$ and $v(\{j\}) = 0$ for all $j \in N - \{1\}$. Player 1 is marginally favored by both w over v and v over w. Since the reduced Shapley value is margin monotone, it follows that $\psi_1(v) = \psi_1(w) = \frac{2}{n}$. Since the normalized Shapley value is formulation independent, $\bar{\psi}_1(v) = v(\{i\}) + \bar{\psi}_i(\bar{v}) = 1$. One interpretation of v is that player 1 generates all of the potential benefits of cooperation, and so player 1 should be allocated the entire amount of benefits. It is also difficult to interpret the two games as being equivalent from player 1's perspective. These interpretations support the normalized over the reduced Shapley value.*

The results of this section suggest that the reduced Shapley value should not be used as a value for partially defined games because it is not formulation independent and margin monotonicity is not intuitively appealing when worths of singleton coalitions are changed. Although the normalized Shapley value has been suggested as an alternative, there are many other linear, symmetric, and formulation independent allocation methods according to theorem 5.1. Finally, it should be noted that the reduced and normalized Shapley values are identical whenever $1 \notin M$, $M = N$, or Ω contains only zero normalized games.

6. Subsidy Freedom

Player i is *null* in the N-game w if $w(S) = w(S - \{i\})$ for all coalitions S containing i. We are again using the convention that $w(\phi) = 0$. Player i is *null* in the M-game w with respect to Ω if i is null in every Ω-extension of w. An allocation method φ is *subsidy free* if $\varphi_i(w) = 0$ whenever $w \in \Omega_M$ and i is a null player.

The subsidy freedom property has also been called the null player axioms. An interpretation of subsidy freedom is that zero is the fair share for a player that contributes zero worth to any coalition joined. In the literature, subsidy free has also been called the null player axiom. Shapley's (1953) efficiency axiom, called the carrier axiom in the subsequent literature, is logically equivalent to subsidy freedom, given our definition of an allocation. In Example 5.3, $M = \{1, n\}$ and v defined by $v(N) = v(\{1\}) = 1$ and $v(\{j\}) = 0$ for all $j \in N - \{1\}$, players $2, 3, ..., n$ are null with respect to zero-monotonic games, but $\psi_1(v) \neq 0$. Hence, the reduced Shapley value need not be subsidy free.

Notice that subsidy freedom is the first condition we have stated which crucially depends on the underlying space Ω of N-games rather than the space Ω_M of M-games. For example, subsidy freedom will not impose any restrictions on our choice of an allocation method if Ω contains no N-game with a null player. This occurs when Ω is the collection of *strictly superadditive* games $\{w : w(S \cup T) > w(S) + w(T)$ for all disjoint coalitions S and $T\}$. So, subsidy freedom is useful as a restrictive condition only if Ω contains games with null players. Furthermore, Ω cannot be "too large": if Ω is the collection of all N-games and $M \neq N$, then no M-game $w \in \Omega_M$ has a null player even though many N-games have null players. So, our first goal is to develop appropriate conditions on Ω in order to generalize Shapley's characterization theorem for linear, symmetric, and subsidy free allocation methods on N-games to characterization results on M-games.

The *unanimity game* u^R on the *unanimity coalition* R is defined by $u^R(S) = 1$ if $R \subseteq S$ and $u^R(S) = 0$ otherwise. We will write $u^{R,M}$ instead of u^R when we wish to make it clear that a unanimity game is an M-game, and so defined only on coalitions S satisfying $|S| \in M$. Unanimity games play a crucial role in Shapley's (1953) characterization of the Shapley value on cooperative games because symmetry and subsidy freedom uniquely determine the allocation for a unanimity game: players not in the unanimity coalition must be null and so are allocated zero by subsidy freedom, and the players in the unanimity coaltion are substitutes and so are allocated equal amounts by symmetry. Since the unanimity games form a basis for the space of all games, linearity can then be used to extend the definition of the allocation method to all games. Unfortunately, the allocation for a partially defined unanimity game may not be determined uniquely by symmetry and subsidy freedom because players outside of the unanimity coalition need not be null. Proposition 6.1 shows that the crucial issue, under a broad set of circumstances, is whether the unanimity M-game has a unique Ω-extension.

Recall that the N-game w is *zero-monotonic* if $w(S) + w(\{i\}) \leq w(S \cup \{i\})$ for all coalitions S and players $i \notin S$. In a possible allocation problem, adding a player to an already formed coalition should not be detrimental because one way for the expanded coalition to "cooperate" is for the original coalition and new player to continue to act separately. Note that w is zero-monotonic if and only if its zero normalization $\bar{w}$ is *monotone*, that is, $\bar{w}(S) \leq \bar{w}(T)$ for all coalitions S and T satisfying $S \subseteq T$.

Proposition 6.1. *If the unanimity M-game $u^{R,M}$ has the unique Ω-extension $u^{R,N}$, then each player $i \in N - R$ is null in $u^{R,M}$. Conversely, if each player $i \in N - R$ is null in the unanimity M-game $u^{R,M}$, M contains 1, and Ω contains only zero-monotonic games, then $u^{R,M}$ has the unique Ω-extension $u^{R,N}$.*

Proof. Clearly, each player $i \in N - R$ is null in $u^{R,N}$. Since $u^{R,N}$ is the unique Ω-extension of $u^{R,M}$, each player $i \in N - R$ is null in $u^{R,M}$.

Conversely, suppose each player $i \in N - R$ is null in $u^{R,M}$, M contains 1, and Ω contains only zero-monotonic games. Suppose $\hat{u}$ is an Ω-extension of $u^{R,M}$. We will show that $\hat{u} = u^{R,N}$. Suppose S is a coalition, and consider the following three cases. First, suppose $R \subset S$. Let $\{i_1, i_2, ..., i_k\} = N - S$. Since $(N - S) \cap R = \phi$, the players in $N - S$ are null. Hence, $\hat{u}(S) = \hat{u}(S \cup \{i_1\}) = \hat{u}(S \cup \{i_1, i_2\}) = ... = \hat{u}(N) = u^{R,M}(N) = 1$. Second, suppose $|S| > |R|$ and $R \nsubseteq S$. Then $k = |S| - |R| > 0$, and there exist $i_1, i_2, ..., i_k, i_{k+1} \in S - R$. Since $i_1, i_2, ..., i_k$ are null, it follows that $\hat{u}(S) = \hat{u}(S - \{i_1\}) = \hat{u}(S - \{i_1, i_2\}) = ... = \hat{u}(S - \{i_1, i_2, ..., i_k\}) = u^{R,M}(S - \{i_1, i_2, ..., i_k\}) = 0$ [since $|S - \{i_1, i_2, ..., i_k\}| = |R| \in M$, and $i_{k+1} \in S - R$ implies that $S - \{i_1, i_2, ..., i_k\} \neq R$]. Third, suppose $|S| < |R|$. Then there exist a player $i \in S$ and a coalition $T \neq R$ satisfying $|T| = |R|$ and $S \subset T$. Since Ω contains only zero-monotonic games, $\hat{u}$ must be zero-monotonic. Hence, $0 = u^{R,M}(\{i\}) = \hat{u}(\{i\}) \leq \hat{u}(S) \leq \hat{u}(T) = u^{R,M}(T) = 0$, and so $\hat{u}(S) = 0$. Thus, $u^{R,M}$ has the unique Ω-extension $u^{R,N}$. ■

The theorem suggests the following condition would be useful for characterizing allocation methods satisfying subsidy freedom. The collection of M-games Ω_M is *unanimity proper* if $u^{R,M} \in \Omega_M$ and has the unique Ω-extension $u^{R,N}$ for each coalition R satisfying $|R| \in M - \{n\}$.

Theorem 6.2. *Suppose the collection of M-games Ω_M is a convex cone, symmetric, and unanimity proper. If a linear, symmetric, and subsidy free allocation method exists on Ω_M, then it is the b-weighted Shapley value satisfying $b_r = \binom{n}{r} - \sum_{\substack{m \in M \\ m > r}} \binom{m-1}{r-1} b_m$ for all $r \in M$.*

Proof. By theorem 3.1, φ is a b-weighted Shapley value (satisfying $b_n = 1$). Note that the formula in the statement of the theorem also yields $b_n = 1$. Suppose now that $r \in M - \{n\}$, and R is a coalition satisfying $|R| = r$. By Proposition 6.1, each player $i \in N - R$ is null in the unanimity game $u^{R,M}$. Since φ is subsidy free, $\varphi_i\left(u^{R,M}\right) = 0$ for all $i \in N - R$. Using formula 3.1, we obtain $0 = \psi_i^b\left(u^{R,M}\right)$

$$= \frac{1}{n}\sum_{m\in M} b_m \binom{n-1}{m-1}^{-1} |\{S : |S| = m, i \in S, R \subseteq S\}|$$

$$- \frac{1}{n}\sum_{m\in M} b_m \binom{n-1}{m}^{-1} |\{S : |S| = m, i \notin S, R \subseteq S\}|$$

$$= -\frac{1}{n} b_r \binom{n-1}{r}^{-1} + \frac{1}{n}\sum_{\substack{m\in M\\ r<m<n}} b_m \left(\binom{n-1}{m-1}^{-1}\binom{n-r-1}{m-r-1} - \binom{n-1}{m}^{-1}\binom{n-r-1}{m-r}\right) + \frac{1}{n} b_n.$$

Rearranging and simplifying this equality, we obtain

$$\begin{aligned} b_r &= \sum_{\substack{m\in M\\ r<m<n}} b_m \binom{n-1}{r} \left(\binom{n-1}{m-1}^{-1}\binom{n-r-1}{m-r-1} - \binom{n-1}{m}^{-1}\binom{n-r-1}{m-r}\right) + \binom{n-1}{r} b_n \\ &= \left(\binom{n}{r} - \binom{n-1}{r-1}\right) b_n - \sum_{\substack{m\in M\\ r<m<n}} \binom{m-1}{r-1} b_m = \binom{n}{r} - \sum_{\substack{m\in M\\ m>r}} \binom{m-1}{r-1} b_m. \ \blacksquare \end{aligned}$$

We can now reprove Shapley's (1953) theorem as a corollary.

Corollary 6.3. *Suppose the collection of N-games Ω is a convex cone, symmetric, and contains the unanimity games. The unique linear, symmetric, and subsidy free allocation method on $\Omega = \Omega_N$ is the Shapley value.*

Proof. Clearly, Ω is unanimity proper. By theorem 6.2, if the allocation method φ is linear, symmetric, and subsidy free on Ω_M, then φ is the b-weighted Shapley value satisfying $b_r = \binom{n}{r} - \sum_{m=r+1}^{n} \binom{m-1}{r-1} b_m$ for all $r \in N$. We now prove, by induction, that $b_r = 1$ for all $r \in N$. Clearly, $b_n = 1$. If $b_{r+1} = b_{r+2} = ... = b_n = 1$, then $b_r = \binom{n}{r} - \sum_{m=r+1}^{n} \binom{m-1}{r-1} = \binom{n}{r} - \sum_{m=r+1}^{n} \left(\binom{m}{r} - \binom{m-1}{r}\right)$ [by Pascal's Triangle equality] $= \binom{r}{r}$ [since the sum is telescoping] $= 1$. Thus, $b_r = 1$ for all $r \in N$. Finally, it is clear from formula 2.2 that when $M = N$, the Shapley value is subsidy free. $\blacksquare$

The allocation method described by theorem 6.2 is the normalized Shapley value if and only if $M = \{k, k+1, ..., n\}$ or $M = \{1, k, k+1, ..., n\}$ for some integer k satisfying $2 \leq k \leq n$. The allocation method described by theorem 6.2 is the reduced Shapley value if and only if $M = N - \{1\}$ or $M = N$. The proofs of these two remarks is similar to the induction part of the proof of Corollary 6.3.

While the allocation method described by theorem 6.2 is linear and symmetric, it is important to emphasize that theorem 6.2 does not state whether the allocation method it describes is actually subsidy free. Depending on the circumstances, either the allocation method described by theorem 6.2 will be the unique linear, symmetric and subsidy free allocation method, or there will exist *no* linear, symmetric, and subsidy free allocation method. Of course, if Ω_M is not unanimity proper, then there can be linear, symmetric, and subsidy free allocation methods not described by theorem 6.2. Describing some of the possibilities is the primary goal of this section.

Before we procede to the main characterization results of this section, we state and prove two useful lemmas. Given the importance of the unanimity proper condition, the first lemma states some sufficient conditions for unanimity M-games to have unique Ω-extensions. The second lemma shows that subsidy freedom is usually a stronger condition than formulation independence.

Lemma 6.4. *Suppose the set of known coalition sizes M contains 1, the coalition R satisfies $|R| \in M - \{n\}$, and the collection of N-games Ω contains the unanimity game $u^{R,N}$. The unanimity M-game $u^{R,M}$ has the unique Ω-extension $u^{R,N}$ if any of the following conditions hold: (1) $|R| = 1$ and Ω contains only zero-monotonic games; (2) $n - 1 \in M$ and Ω contains only zero-monotonic games; and (3) Ω contains only convex games.*

Proof. By supposition, $u^{R,N}$ is an Ω-extension of the unanimity M-game $u^{R,M}$. Suppose $\hat{u}$ is an Ω-extension of $u^{R,M}$. For each condition, we will show that $\hat{u} = u^{R,N}$

Suppose condition (1) holds. Then the zero normalization $v = \overline{u^{R,M}}$ satisfies $v(S) = 0$ for all coalitions S satisfying $|S| \in M$. If $\hat{v}$ is a zero-monotonic extension of v and S is a coalition containing a player i, then $0 = v(\{i\}) = \hat{v}(\{i\}) \leq \hat{v}(S) \leq \hat{v}(N) = v(N) = 0$ which implies $\hat{v}(S) = 0$. Now $\hat{v}$ must be the zero normalization of $\hat{u}$ which implies that $\hat{u} = u^{R,N}$.

Suppose condition (2) holds. If $R \subseteq S$, then by zero-monotonicity $1 = u^{R,M}(R) = \hat{u}(R) \leq \hat{u}(S) \leq \hat{u}(N) = u^{R,M}(N) = 1$ which implies that $\hat{u}(S) = 1$. Given condition (1), we may assume $|R| > 1$. If $R \not\subseteq S$, then there exist players

$i \in S$ and $j \in R - S$. By zero-monotonicity and $|N - \{j\}| \in M$, it follows that $0 = u^{R,M}(\{i\}) = \hat{u}(\{i\}) \leq \hat{u}(S) \leq \hat{u}(N - \{j\}) = u^{R,M}(N - \{j\}) = 0$ which implies that $\hat{u}(S) = 0$. Hence, $\hat{u} = u^{R,N}$.

Suppose condition (3) holds. Given condition (1), we may assume $|R| > 1$. Suppose S is a coalition, and consider the following three cases. First, suppose $|S| < r$. Then there exist a player $i \in S$ and a coalition $T \not\supseteq R$ satisfying $S \subset T$ and $|T| = r$. Hence, $0 = u^{R,M}(\{i\}) = \hat{u}(\{i\}) \leq \hat{u}(S) \leq \hat{u}(T) = u^{R,M}(T) = 0$, and so $\hat{u}(S) = 0$. Second, suppose $R \subseteq S$. Hence, $1 = u^{R,M}(R) = \hat{u}(R) \leq \hat{u}(S) \leq \hat{u}(N) = u^{R,M}(N) = 1$, and so $\hat{u}(S) = 1$. Third, suppose $|S| > r$ and $R \not\subset S$. Choose $i \in S$. Let $T = (N - S) \cup R$. Since $|S \cap T| < r$, the first case implies $\hat{u}(S \cap T) = 0$. Since $R \subset T$, the second case implies $\hat{u}(T) = 1$. Hence, $0 = u^{R,M}(\{i\}) = \hat{u}(\{i\}) \leq \hat{u}(S) \leq \hat{u}(S \cup T) + \hat{u}(S \cap T) - \hat{u}(T) = \hat{u}(N) + 0 - 1$ $= u^{R,M}(N) - 1 = 0$, and so $\hat{u}(S) = 0$. Taking the three cases together, we obtain that $\hat{u} = u^{R,N}$. ∎

Lemma 6.5. *Suppose M is a set of known sizes containing 1. Suppose the collection of games Ω is a convex cone, contains its zero normalizations, contains the singleton unanimity games, and contains only zero-monotonic games. If the allocation method φ is linear and subsidy free, then φ is formulation independent.*

Proof. Suppose $w \in \Omega_M$. Then $w = \bar{w} + \sum_{j \in N} w(\{j\}) u^{\{j\},M}$. Since Ω contains its zero normalizations and the singleton unanimity games, $\bar{w} \in \Omega_M$ and $u^{\{j\},M} \in \Omega_M$ for all $j \in N$. Since φ is linear, $\varphi(w) = \varphi(\bar{w}) + \sum_{j \in N} w(\{j\}) \varphi\left(u^{\{j\},M}\right)$. Since Ω contains only zero-monotonic games, condition (1) of Lemma 6.4 implies that $u^{\{j\},N}$ is the unique Ω-extension of $u^{\{j\},M}$. Clearly, each $k \in N - \{j\}$ is null in $u^{\{j\},N}$ and so in $u^{\{j\},M}$. Since φ is subsidy free, $\varphi_k\left(u^{\{j\},M}\right) = 0$ for all $k \in N - \{j\}$. Since φ is an allocation method, $\varphi_j\left(u^{\{j\},M}\right) = u^{\{j\},M}(N) - \sum_{k \in N - \{j\}} \varphi_k\left(u^{\{j\},M}\right) = 1$. Hence, $\varphi_i(w) = \varphi_i(\bar{w}) + \sum_{j \in N} w(\{j\}) \varphi_i\left(u^{\{j\},M}\right) = \varphi_i(\bar{w}) + w(\{i\})$. Thus, φ is formulation independent. ∎

We now turn to characterization theorems for linear, symmetric, and subsidy free allocation methods on four special classes of cooperative games. We start with the largest class of games, zero monotonic, and work our way to the smallest class of games, convex. The next theorem provides a complete characterization for zero monotonic games. Notice that when Ω is the class of zero monotonic games, Ω_M is unanimity proper if and only if $n - 1 \in M$. The statement and proof of the theorem do not notice whether Ω_M is unanimity proper.

Theorem 6.6. *Suppose the set of known coalition sizes M contains 1, and Ω is the collection of zero-monotonic games. If $M = \{1, k, k+1, ..., l, n\}$ where $1 \leq k \leq l \leq n$, then the unique linear, symmetric and subsidy free allocation method on Ω_M is the normalized Shapley value. Otherwise, there is no linear, symmetric, and subsidy free allocation method on Ω_M.*

Proof. Suppose φ is a linear, symmetric, and subsidy free allocation method on Ω_M. We first show that if φ exists, then φ must be the normalized Shapley value. By Lemma 6.5 and Theorem 5.1, φ is a b-weighted Shapley value where b satisfies $\sum_{m \in M} b_m = n$. Now for $m \in M - \{1, n\}$, define the M-game v^m by $v^m(S) = 1$ if $|S| > m$, $v^m(S) = 1$ if $|S| = m$ and $1 \notin S$, and $v^m(S) = 0$ otherwise. Suppose $\hat{v}$ is a zero-monotonic extension of v^m. Given any coalition S, there is a player $i \in S$, and so $\{i\} \subseteq S \subseteq N$. Since $\hat{v}$ is zero-monotonic, $0 = v^m(\{i\}) \leq \hat{v}(S) \leq v^m(N) = 1$. If $|S| < m$, then by adding some players, including player 1, to S, we can construct a coalition T satisfying $S \cup \{1\} \subseteq T$ and $|T| = m$. So, $0 \leq \hat{v}(S) \leq v^m(T) = 0$ which implies $\hat{v}(S) = 0$. If $|S| > m$, then by removing some players, including player 1, from S, we can construct a coalition R satisfying $R \subseteq S - \{1\}$ and $|R| = m$. So, $1 \geq \hat{v}(S) \geq v^m(R) = 1$ which implies $\hat{v}(S) = 1$. Thus, $\hat{v}$ is uniquely determined to be $\hat{v}(S) = 1$ if $|S| > m$, $\hat{v}(S) = 1$ if $|S| = m$ and $1 \notin S$, and $\hat{v}(S) = 0$ otherwise. Clearly, $\hat{v}$ is zero-monotonic, and so $v^m \in \Omega_M$. Player 1 is null in $\hat{v}$ and so is null in v^m. Since $\varphi = \psi^b$ is subsidy free, $0 = \psi_1^b(v^m) = \frac{1}{n}(1 - b_m)$ [by formula 3.1]. Hence, $b_m = 1$ for all $m \in M - \{1, n\}$. Since we already have that $b_n = 1$ and $\sum_{m \in M} b_m = n$, it follows that φ is the normalized Shapley value.

Suppose now that $M = \{1, k, k+1, ..., l, n\}$ where $1 \leq k \leq l \leq n$. By theorem 3.1, the normalized Shapley value is linear and symmetric. We now show that the normalized Shapley value $\bar{\psi}$ is subsidy free. Suppose player i is null in $w \in \Omega_M$. We must show that $\bar{\psi}_i(w) = 0$. Note that for zero-monotonic M-games, player i is null in w implies player i is null in the zero normalization $\bar{w}$. Since $\bar{\psi}$ is formulation independent, $\bar{\psi}_i(w) = 0$ if $\bar{\psi}_i(\bar{w}) = 0$. Hence, we may assume that w is zero-normalized. Consider the N-game $\hat{w}$ defined by $\hat{w}(S) = w(N)$ if $n \geq |S| > l$, $\hat{w}(S) = w(S)$ if $l \geq |S| \geq k$, and $\hat{w}(S) = 0$ if $k > |S| \geq 1$. Since $w \in \Omega_M$, it follows that $\hat{w}$ is a zero-monotonic extension of w. Since player i is null in w, player i is null in $\hat{w}$. We now consider the marginal contributions $w(S) - w(R)$ where $i \in S$, $|S| \in M$, $R \subseteq S - \{i\}$, and $|R| = \rho(|S|)$. If $|S| = 1$, then $w(S) = w(\{i\}) = 0$ [since w is zero-normalized] $= w(\phi) = w(R)$. If $|S| = k > 1$, then $w(S) = \hat{w}(S) = \hat{w}(S - \{i\})$ [since i is null in $\hat{w}$] $= 0 = w(R)$. If $|S| \in \{k+1, k+2, ..., l\}$, then $w(S) = \hat{w}(S) = \hat{w}(S - \{i\})$ [since i is null in $\hat{w}$]

$= w(S-\{i\}) = w(R)$. If $|S| = n > l$, then $w(S) = \hat{w}(N) = \hat{w}(N-\{i\})$ [since i is null in $\hat{w}$] $= w(N)$. Thus, the marginal contributions all equal zero. Since $\bar{\psi}(w) = \psi(w)$ when w is zero-normalized, formula 2.2 implies that $\bar{\psi}_i(w) = 0$. Therefore, the normalized Shapley value $\bar{\psi}$ is subsidy free.

Suppose now that $M \neq \{1, k, k+1, ..., l, n\}$ for all integers k and l satisfying $1 \leq k \leq l \leq n$. Then there exist $k, l \in M$ satisfying $1 < k < l-1 < n-1$ and $k < m < l$ implies $m \notin M$. Define the M-game w by

$$w(S) = \begin{cases} 2, \text{ if } |S| > l \text{ and } |S| \in M \\ 2, \text{ if } |S| = l \text{ and } 1 \notin S \\ 2, \text{ if } |S| = l, \ 1 \in S, \text{ and } n \in S \\ 1, \text{ if } |S| = l, \ 1 \in S, \text{ and } n \notin S \\ 2, \text{ if } |S| = k, \ 1 \notin S, \text{ and } n \in S \\ 1, \text{ if } |S| = k, \ 1 \notin S, \text{ and } n \notin S \\ 0, \text{ if } |S| = k \text{ and } 1 \in S \\ 0, \text{ if } |S| < k \text{ and } |S| \in M \end{cases}$$

Suppose $\hat{w}$ is a zero-monotonic extension of w. If $l < |S| < n$, then players can be removed from S to construct a coalition $R \subseteq S - \{1\}$ satisfying $|R| = l$. Since $\hat{w}$ is a zero monotonic extension of w, it follows that $2 = \hat{w}(R) \leq \hat{w}(S) \leq \hat{w}(N) = 2$. If $k < |S| < l$ and $n \in S$, then players can be removed from S to construct a coalition $R \subseteq S - \{1\}$ satisfying $|R| = k$ and $n \in R$, and players can be added to S to construct a coalition T satisfying $|T| = l$ and $S \cup \{1, n\} \subseteq T$. Since $\hat{w}$ is zero-monotone and an extension of w, it follows that $2 = \hat{w}(R) \leq \hat{w}(S) \leq \hat{w}(T) = 2$. If $k < |S| < l$ and $n \notin S$, then players can be removed from S to construct a coalition $R \subseteq S - \{1, n\}$ satisfying $|R| = k$, and players can be added to S to construct a coalition $T \supseteq S \cup \{1\}$ satisfying $|T| = l$ and $n \notin T$. Since $\hat{w}$ is a zero monotonic extension of w, it follows that $1 = \hat{w}(R) \leq \hat{w}(S) \leq \hat{w}(T) = 1$. If $|S| < k$, choose a player $i \in S$ and a coalition T satisfying $S \cup \{1\} \subseteq T$ and $|T| = k$. Since $\hat{w}$ is a zero monotonic extension of w, it follows that $0 = \hat{w}(\{i\}) \leq \hat{w}(S) \leq \hat{w}(T) = 0$. Thus, $\hat{w}$ is uniquely defined by

$$\hat{w}(S) = \begin{cases} 2, \text{ if } |S| > l \\ 2, \text{ if } k < |S| < l \text{ and } n \in S \\ 1, \text{ if } k < |S| < l \text{ and } n \notin S \\ 0, \text{ if } |S| < k \\ w(S), \text{ if } |S| \in M \end{cases}$$

Clearly, $\hat{w}$ is zero-monotone, so $w \in \Omega_M$. Clearly, player 1 is null in $\hat{w}$ and so in w. Since φ is subsidy free, $0 = \varphi_1(w) = \bar{\psi}_1(w)$ [by first paragraph of proof]

$= \psi_1(w)$ [since $w(\{i\}) = 0$ for all $i \in N$] $= \frac{2}{n} + \frac{1}{n}\binom{n-1}{l-1}^{-1}\left(2\binom{n-2}{l-2} + \binom{n-2}{l-1}\right) - \frac{1}{n}\binom{n-1}{l}^{-1} 2\binom{n-1}{l} - \frac{1}{n}\binom{n-1}{k}^{-1}\left(2\binom{n-2}{k-1} + \binom{n-2}{k}\right)$ [using formula 2.1] $= \frac{l-1-k}{n(n-1)}$ [by straightforward but tedious algebra] $\neq 0$. This contradiction ($0 \neq 0$) implies that there is no linear, symmetric, and subsidy free allocation method on Ω_M. ■

We now consider a class of games in which the size of the coalition is at least as important as the composition of the coalition in determining its worth. The N-game w is *size monotonic* if its zero normalization $\bar{w}$ satisfies $\bar{w}(R) \leq \bar{w}(S)$ for all coalitions R and S satisfying $|R| < |S|$. Note that $u^{R,N}$ is size monotonic only if $|R| \in \{n-1, n\}$, and so the class of size monotonic M-games is *not* unanimity proper unless $M = \{n-1, n\}$. Yet, we obtain the best possible result: existence of a unique linear, symmetric, and subsidy free allocation method for all possible M containing 1.

Theorem 6.7. *Suppose the set of known coalition sizes M contains 1, and Ω is the collection of size monotonic games. The unique linear, symmetric and subsidy free allocation method on Ω_M is the reduced Shapley value.*

Proof. Suppose φ is a linear, symmetric, and subsidy free on Ω_M. By Theorem 3.1, φ is a b-weighted Shapley value. Now for $m \in M - \{n\}$, define the M-game v^m by $v^m(S) = 1$ if $|S| > m$, $v^m(S) = 1$ if $|S| = m$ and $1 \notin S$, and $v^m(S) = 0$ otherwise. It is easily seen that v^m has a unique size monotonic extension, and player 1 is null in the extension and v^m. Since $\varphi = \psi^b$ is subsidy free, $0 = \psi_1^b(v^m) = \frac{1}{n}(1 - b_m)$ [by formula 3.1]. Hence, $b_m = 1$ for all $m \in M - \{n\}$. Since we already have that $b_n = 1$, it follows that φ is the reduced Shapley value.

By Theorem 3.1, the reduced Shapley value ψ is linear and symmetric. We now show that ψ is subsidy free. Suppose player i is null in $w \in \Omega_M$. Let $c_m = \max\{w(R) : |R| = m\}$ for all $m \in M$. Recall $\rho(s) = \max\{0, m \in M : m < s\}$. Consider the N-game $\hat{w}$ defined by $\hat{w}(S) = w(S)$ if $|S| \in M$, and $\hat{w}(S) = c_{\rho(|S|)}$ if $|S| \notin M$. Clearly, $\hat{w}$ is a size monotonic extension of w. Since player i is null in w, player i must be null in $\hat{w}$. Suppose R and S are coalitions satisfying $i \in S$, $|S| \in M$, $R \subseteq S - \{i\}$, and $|R| = \rho(|S|)$. If $|R| = |S| - 1$, then $w(S) = \hat{w}(S) = \hat{w}(S - \{i\}) = \hat{w}(R) = w(R)$. If $|R| < |S| - 1$, then $w(S) = \hat{w}(S) = \hat{w}(S - \{i\}) = c_{\rho(|S|)} = \hat{w}(R \cup \{i\}) = \hat{w}(R) = w(R)$. In either case, the marginal $w(S) - w(R) = 0$. Now by formula 2.2, it follows that $\psi_i(w) = 0$. Thus, ψ is subsidy free. ■

For an unspecified cost allocation problem, the most reasonable class of games to consider are the superadditive ones. Recall that w is superadditive if $w(R) +$

$w(S) \leq w(R \cup S)$ for all disjoint coalitions R and S. One way for disjoint coalitions to "cooperate" would be for each to work independently, and so the savings obtained by the union of the two disjoint coalitions should be at least the sum of the savings each coalition obtains separately. The next theorem characterizes the linear, symmetric, and subsidy free allocation methods for unanimity proper classes of superadditive games.

Theorem 6.8. *Suppose the set of known coalition sizes M contains 1 and $n-1$, and Ω is the collection of superadditive games. If $M = \{1, l, l+1, ..., n\}$ where $2 \leq l \leq n-1$, then the unique linear, symmetric and subsidy free allocation method on Ω_M is the normalized Shapley value. Otherwise, there is no linear, symmetric, and subsidy free allocation method on Ω_M.*

Proof. Suppose M contains 1 and $n-1$. Let k and l satisfy $1 \leq k$, $k+2 \leq l \leq n-1$, $k \in M$, $k < m < l$ implies $m \notin M$, and $l \leq m \leq n$ implies $m \in M$. Suppose φ is a linear, symmetric, and subsidy free allocation method on Ω_M. By condition (2) of Lemma 6.4 and Theorem 6.2, φ is a b-weighted Shapley value where b satisfies $b_r = \binom{n}{r} - \sum_{\substack{m \in M \\ m > r}} \binom{m-1}{r-1} b_m$ for all $r \in M$. By the same argument as used in the proof of corollary 6.3, it follows that $b_n = b_{n-1} = ... = b_l = 1$ and $b_k = \binom{l-1}{k}$. If $k = 1$, then φ is the normalized Shapley value, and we must show that $\varphi = \bar{\psi}$ is subsidy free. If $k > 1$, then we must show that φ is *not* subsidy free, which we will do in two cases: $k \geq \frac{n}{2}$ and $\frac{n}{2} > k > 1$.

Suppose $k = 1$. We will show that $\varphi = \bar{\psi}$ is subsidy free. Suppose player i is null in $w \in \Omega_M$. We must show that $\bar{\psi}_i(w) = 0$. Note that for superadditive M-games, player i is null in w implies player i is null in the zero normalization $\bar{w}$. Since $\bar{\psi}$ is formulation independent, $\bar{\psi}_i(w) = 0$ if $\bar{\psi}_i(\bar{w}) = 0$. Hence, we may assume that w is zero normalized. Consider the N-game $\hat{w}$ defined by $\hat{w}(S) = w(S)$ if $|S| \geq l$, and $\hat{w}(S) = 0$ if $|S| < l$. Clearly, $\hat{w}$ is a superadditive extension of w, and so player i is null in $\hat{w}$. Hence, $w(S) = w(S - \{i\})$ for all S satisfying $|S| > l$, and $w(S) = \hat{w}(S - \{i\}) = 0$ if $|S| = l$ and $i \in S$. Now using formula 3.1 with the b_m as defined above, we obtain $\bar{\psi}_i(w) = 0$. Thus, $\bar{\psi}$ is subsidy free.

Suppose $k \geq \frac{n}{2}$. We now show that φ is not subsidy free. Define the M-game v by $v(S) = 1$ if $|S| = l, l+1, ..., n$, $v(S) = 1$ if $|S| = k$ and $n \notin S$, and $v(S) = 0$ otherwise. Suppose $\hat{v}$ is a superadditive extension of v. If $|S| < k$, then there is a coalition T satisfying $S \cup \{n\} \subseteq T$ and $|T| = k$. Because $\hat{v}$ is superadditive and zero normalized, it follows that $0 = \sum_{i \in S} \hat{v}(\{i\}) \leq \hat{v}(S) = \hat{v}(S) + \sum_{i \in T-S} \hat{v}(\{i\}) \leq \hat{v}(T) = 0$, and so $\hat{v}(S) = 0$. If $|S| > k$, then there is a coalition R satisfying

$R \subset S - \{n\}$ and $|R| = k$. Because $\hat{v}$ is superadditive and zero normalized, it follows that $1 = \hat{v}(R) + \sum_{i \in S-R} \hat{v}(\{i\}) \leq \hat{v}(S) = \hat{v}(S) + \sum_{i \in N-S} \hat{v}(\{i\}) \leq \hat{v}(N) = 1$, and so $\hat{v}(S) = 1$. It is easy to see that $\hat{v}$ is superadditive, and so v has a unique superadditive extension. Player n is null in $\hat{v}$, and so player n is null in v. Nonetheless, using formula 3.1 with the b_m as defined above, we obtain $\varphi_n(v) = \frac{1}{n}\left(1 - \binom{l-1}{k}\right) \neq 0$ [since $l - 1 > k$]. Hence, φ is not subsidy free.

Suppose $\frac{n}{2} > k > 1$. We will show that $\varphi = \bar{\psi}$ is not subsidy free. Define the M-game v by $v(S) = 1$ if $|S \cap \{1, 2, ..., 2k-1\}| \geq k$, and $v(S) = 0$ otherwise. Suppose $\hat{v}$ is a superadditive extension of v. If $|S \cap \{1, 2, ..., 2k-1\}| \geq k$, then there is a coalition R satisfying $R \subset S \cap \{1, 2, ..., 2k-1\}$ and $|R| = k$. Because $\hat{v}$ is superadditive and zero normalized, it follows that $1 = \hat{v}(R) + \sum_{i \in S-R} \hat{v}(\{i\}) \leq \hat{v}(S) = \hat{v}(S) + \sum_{i \in N-S} \hat{v}(\{i\}) \leq \hat{v}(N) = 1$, and so $\hat{v}(S) = 1$. If $|S \cap \{1, 2, ..., 2k-1\}| < k$, then $|(N - S) \cap \{1, 2, ..., 2k-1\}| \geq k$ which implies $\hat{v}(N - S) = 1$ by previous work. Because $\hat{v}$ is superadditive and zero normalized, it follows that $0 = \hat{v}(N) - \hat{v}(N - S) \geq \hat{v}(S) \geq \sum_{i \in S} \hat{v}(\{i\}) = 0$, and so $\hat{v}(S) = 0$. It is easy to see that $\hat{v}$ is superadditive, and so v has a unique superadditive extension. Player n is null in $\hat{v}$, and so player n is null in v. Nonetheless, using formula 3.1 with the b_m as defined above, we obtain $\varphi_n(v) = \frac{1}{n}\left(\binom{n-1}{l-1}^{-1} q - \binom{l-1}{k}\binom{n-1}{k}^{-1}\binom{2k-1}{k}\right)$ where q is the number of coalitions of size l containing player n and at least k players from $\{1, 2, ..., 2k-1\}$. After some algebra, we obtain $\varphi_n(v) = \frac{1}{n}\binom{n-1}{l-1}^{-1}\left(q - \binom{2k-1}{k}\binom{n-1-k}{l-1-k}\right)$. Now $\binom{2k-1}{k}\binom{n-1-k}{l-1-k}$ is the number of ways of first coloring k players from $\{1, 2, ..., 2k-1\}$ blue and then coloring player n and $l - k - 1$ of the other $n - k$ players red. Each such coloring (by combining the blue and red colored players) results in a coalition of size l containing player n and at least k players from $\{1, 2, ..., 2k-1\}$. Because there are more distinct colorings than there are resulting coalitions, $\binom{2k-1}{k}\binom{n-1-k}{l-1-k} > q$, and so $\varphi_n(v) < 0$. Hence, φ is not subsidy free. ■

The only general result for superadditive games when $1 \in M$ but $n - 1 \notin M$ known to the author is that if a linear, symmetric, and subsidy free allocation method exists, then it is unique. The proof uses the same games and arguments as in the last two cases of the proof of Theorem 6.8. Sometimes the defined allocation method is subsidy free, and other times it is not. Although a general result is not yet available, it is useful to consider one example.

Example 6.9. *Let the set of known coalition sizes $M = \{1, 2, 5\}$, and Ω be the collection of superadditive games. Then the unique linear, symmetric and subsidy free allocation method on Ω_M is the b-weighted Shapley value satisfying $b_5 = 1$*

and $b_1 = b_2 = 2$. The somewhat tedious proof is left to the reader. Notice that this allocation method is neither the reduced or normalized Shapley value.

For our last characterization, we consider the class of convex games. The N-game w is convex if $w(R) + w(S) \leq w(R \cup S) + w(R \cap S)$ for all coalitions R and S. It can be shown that w is convex if and only if $w(R) - w(R - \{i\}) \leq w(S) - w(S - \{i\})$ for all coalitions $R \subseteq S$. Hence, convex games are useful for modeling situations in which there are increasing returns to scale. Unfortunately, linear, symmetric, and subsidy free allocation methods do not exist for useful cases.

Theorem 6.10. *Suppose the set of known coalition sizes M contains 1, and Ω is the collection of convex games. If $M = \{1, n\}$ or $M = N$, then the unique linear, symmetric, and subsidy free allocation method on Ω_M is the normalized Shapley value. Otherwise, there is no linear, symmetric, and subsidy free allocation method on Ω_M.*

Proof. The conclusion for when $M = N$ follows from Corollary 6.3. Suppose $M = \{1, n\}$ and φ is a linear, symmetric, and subsidy free allocation method on Ω_M. By condition (3) of Lemma 6.4 and Theorem 6.2, φ is the normalized Shapley value: $\varphi_i(w) = w(\{i\}) + \frac{1}{n}\left(w(N) - \sum_{j \in N} w(\{j\})\right)$. By Theorem 3.1, the normalized Shapley value is linear and symmetric. We now show that it is subsidy free. Suppose player i is null in w. Define $\hat{w}$ by $\hat{w}(S) = \sum_{j \in S} w(\{j\})$ for all $S \neq N$. Clearly, $\hat{w}$ is a convex extension of w, and so player i is null in $\hat{w}$. Hence, $w(\{i\}) = \hat{w}(\{i\}) = 0$ and $w(N) = \hat{w}(N) = \hat{w}(N - \{i\}) = \sum_{j \in N - \{i\}} \hat{w}(\{j\}) = \sum_{j \in N} \hat{w}(\{j\}) = \sum_{j \in N} w(\{j\})$. Substitution of these results back into the formula for φ yields $\varphi_i(w) = 0$.

For the remainder of the proof, suppose $M \neq \{1, n\}$ and $M \neq N$. We will show that there is no linear, symmetric, and subsidy free allocation method on Ω_M. On the contrary, suppose φ is a linear, symmetric, and subsidy free allocation method on Ω_M. We will derive a contradiction by defining an M-game w in which player n is null but $\varphi_n(w) \neq 0$.

By condition (3) of Lemma 6.4 and Theorem 6.2, φ is the b-weighted Shapley value satisfying $b_r = \binom{n}{r} - \sum_{\substack{m \in M \\ m > r}} \binom{m-1}{r-1} b_m$ for all $r \in M$. In particular, if $k \geq 1$ and $k + 1, k + 2, ..., n \in M$, then by the same argument used in the proof of Corollary 6.3, it follows that $1 = b_n = b_{n-1} = ... = b_{k+1}$. So, $\varphi(w) = \psi(w)$ if

$w(S) = 0$ for all S satisfying $|S| \in M$ and $|S| \leq k$. Alternatively, if $k \in M$ and $k+1, k+2, ..., n-1 \notin M$, then $b_n = 1$ and $b_k = \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}$.

Consider the special case $n - 1 \in M$. Since $M \neq N$, there exists an integer k satisfying $1 < k < n-1$, $k \notin M$, and $k < m \leq n$ implies $m \in M$. Define the M-game w by $w(S) = 1$ if $\{1, 2, ..., k\} \subset S$, and $w(S) = 0$ otherwise. Suppose $\hat{w}$ is a convex extension of w, and S is any coalition. We will show what $\hat{w}(S)$ must be. Choose $i \in S$. Then $0 = w(\{i\}) = \hat{w}(\{i\}) \leq \hat{w}(S) \leq \hat{w}(N) = w(N) = 1$ which implies $0 \leq \hat{w}(S) \leq 1$. If $|S| < k$, then by adding players to S, we can construct a coalition T satisfying $S \subset T$, $|T| = k+1$, and $\{1, 2, ..., k\} \nsubseteq T$; hence, $\hat{w}(S) \leq \hat{w}(T) = w(T) = 0$, and so $\hat{w}(S) = 0$. Now $\hat{w}(\{1, 2, ..., k\}) \geq \hat{w}(\{1, 2, ..., k, k+1\}) + \hat{w}(\{1, 2, ..., k, k+2\}) - \hat{w}(\{1, 2, ..., k+2\}) = w(\{1, 2, ..., k, k+1\}) + w(\{1, 2, ..., k, k+2\}) - w(\{1, 2, ..., k+2\}) = 1 + 1 - 1 = 1$, and so $\hat{w}(\{1, 2, ..., k\}) = 1$. Finally, if $|S| = k$ and $S \neq \{1, 2, ..., k\}$, then $\hat{w}(S) \leq \hat{w}(N) + \hat{w}(S \cap \{1, 2, ..., k\}) - \hat{w}((N - S) \cup \{1, 2, ..., k\}) = 1 + 0 - 1 = 0$, and so $\hat{w}(S) = 0$. Therefore, $\hat{w}$ must be the unanimity game on $\{1, 2, ..., k\}$. Clearly, player n is null in $\hat{w}$ and so in w. Since φ is subsidy free, $\varphi_n(w) = 0$. By a result in the previous paragraph, $0 = \varphi_n(w) = \psi_n(w) = \psi_n\left(u^{R,M} - v\right)$ [where $R = \{1, 2, ..., k\}$ and v is defined by $v(R) = 1$ and $v(S) = 0$ otherwise] $= \psi_n\left(u^{R,N} - v\right)$ [since $\left(u^{R,N} - v\right)(S) = 0$ if $|S| \notin M$] $= \psi_n\left(u^{R,N}\right) - \psi_n(v)$ [since ψ is linear] $= 0 - \frac{1}{n}\binom{n-1}{k}^{-1}$ [since n is null in $u^{R,N}$ and using formula 2.1] $\neq 0$. This contradiction implies that there is no linear, symmetric, and subsidy free allocation method.

Consider the remaining special case $n - 1 \notin M$. Since $M \neq \{1, n\}$, there exists an integer $k \in M$ such that $2 \leq k \leq n-2$ and $k < m < n$ implies $m \notin M$. Define the M-game w by

$$w(S) = \begin{cases} n-k, \text{ if } S = N \\ 1, \text{ if } |S| = k \text{ and } n \notin S \\ 0, \text{ if } |S| = k \text{ and } n \in S \\ 0, \text{ if } |S| < k \text{ and } |S| \in M \end{cases}$$

Suppose $\hat{w}$ is a convex extension of w, and S is any coalition. We will show what $\hat{w}(S)$ must be by considering two cases. First, suppose $|S| < k$. Choose a player $i \in S$ and a coalition T satisfying $S \cup \{n\} \subset T$ and $|T| = k$. Hence, $0 = w(\{i\}) = \hat{w}(\{i\}) \leq \hat{w}(S) \leq \hat{w}(T) = w(T) = 0$, and so $\hat{w}(S) = 0$. Second, suppose $|S| > k$. Choose a player $i \neq n$ and coalition R satisfying $i \in R \subseteq S - \{n\}$ and $|R| = k$. So, $\hat{w}(R) = w(R) = 1$, and $\hat{w}((S - \{i\}) \cap R) = \hat{w}(R - \{i\}) = 0$ since $|R - \{i\}| < k$. Since $\hat{w}$ is convex, $\hat{w}(S) = \hat{w}((S - \{i\}) \cup R) \geq \hat{w}(S - \{i\}) +$

$\hat{w}(R) - \hat{w}((S-\{i\}) \cap R) = \hat{w}(S-\{i\})+1$. Repeating this argument $l = |S|-k$ times, we obtain $\hat{w}(S) \geq \hat{w}(R)+l$ where $|R| = k$ and $n \in R$ if and only if $n \in S$. Hence, $\hat{w}(S) \geq 1 + l = |S| - k + 1$ if $n \notin S$, and $\hat{w}(S) \geq 0 + l = |S| - k$ if $n \in S$. Applying this argument to $S = N$, we obtain that $\hat{w}(N) \geq |N| - k$. Since $\hat{w}(N) = w(N) = n - k$, it follows that all the nonstrict inequalities must be equalities. Thus, $\hat{w}(S) = |S| - k + 1$ if $n \notin S$, and $\hat{w}(S) = |S| - k$ if $n \in S$. Combining the results of the two cases, we obtain

$$\hat{w}(S) = \begin{cases} |S| - k + 1, & \text{if } |S| \geq k \text{ and } n \notin S \\ |S| - k, & \text{if } |S| \geq k \text{ and } n \in S \\ 0, & \text{if } |S| < k \end{cases}$$

Clearly, player n is null in $\hat{w}$ and so in w. Since φ is subsidy free, $\varphi_n(w) = 0$. By a result in the second paragraph of the proof, $\varphi_n(w) = \psi_n^b(w)$ where $b_n = 1$ and $b_k = \binom{n-1}{k}$. Hence, $0 = \frac{n-k}{n} - \frac{1}{n}\binom{n-1}{k}\binom{n-1}{k}^{-1} |\{S : |S| = k \text{ and } n \notin S\}| = \frac{n-k}{n} - \frac{1}{n}\binom{n-1}{k} \leq \frac{n-k}{n} - \frac{n-1}{n} = -\frac{k-1}{n} < 0$. This contradiction implies that there is no linear, symmetric, and subsidy free allocation method. ■

7. Conclusion

The practioner must exercise caution in choosing an allocation method for partially defined cooperative games. The axioms of linearity, symmetry, and subsidy freedom, which uniquely determine the Shapley value on fully defined cooperative games, characterize different allocation methods (sometimes nonuniquely) or no allocation method depending upon the class of partially defined cooperative games under consideration. The axioms of symmetry and margin monotonicity, which uniquely determine the Shapley value again on fully defined cooperative games, characterize an allocation method that usually does not satisfy subsidy freedom. The most positive results suggest the use of the normalized Shapley value for zero monotonic or superadditive games and to determine coalitional worths of the singletons and a block of the largest coalitions.

In the example examined in Section 2, we found that no zero monotonic or superadditive extension had a Shapley value equal to the reduced (or normalized) Shapley value. Notice that no linear, symmetric, and subsidy free allocation method exists for the example's set of known coalition sizes. Conversely, for all classes of partially defined cooperative games for which we have found a unique linear, symmetric, and subsidy free allocation method, extensions always exist for which the Shapley value allocates the same way as the characterized allocation method. Whether this relationship holds in general is an open question. The nonexistence of linear, symmetric, and subsidy free allocation methods for certain classes of games suggests that the linearity condition is too strong.

8. References

Curiel I, Potters J, Prasad R, Tijs S, and Veltman B (1993) Cooperation in One Machine Scheduling. Z. Oper. Res. 38: 113-129.

Driessen T (1994) Generalized Concavity in Cooperative Game Theory: Characterizations in Terms of the Core. In Pecs (ed) Generalized Convexity, Lecture Notes in Econom. and Math. Systems, 405, Springer, Berlin, pp. 40-52.

Dubey P, Neyman A, and Weber RJ (1981) Value Theory Without Efficiency. Math Oper Res 6: 122-128.

Housman D (1992) Values for Partially Defined Games. International Conference on Game Theory, SUNY Stony Brook.

Gilboa I and Monderer D (1991) Quasi-Values on Subspaces. Int J Game Theory 19: 353-363.

Kalai E and Samet D (1988) Weighted Shapley Values. In Roth A (ed) The Shapley Value, Cambridge University Press, Cambridge, pp. 83-100.

Letscher D (1990) The Shapley Value on Partially Defined Games. Research Experiences for Undergraduates Final Report.

Monderer D (1988) Values and Semivalues on Subspaces of Finite Games. Int J Game Theory 17: 301-310.

Nowak AS and Radzik T (1995) On Axiomatizations of the Weighted Shapley Values. Games and Economic Behavior 8: 389-405.

Shapley LS (1953) A Value for n-Person Games. In Kuhn HW and Tucker AW (eds.) Contributions to the Theory of Games II, Annals of Mathematics Studies No. 28, Princeton University Press, Princeton, New Jersey, pp. 307-317.

Skorin-Kapov D (1993) On a Cost Allocation Problem Arising from a Capacitated Concentrator Covering Problem. Oper Res Lett 13: 315-323.

Willson SJ (1993) A Value for Partially Defined Cooperative Games. Int J Game Theory 21: 371-384.

Young HP (1985) Monotonic Solutions of Cooperative Games. Int J Game Theory 14: 65-72.

Young HP (1994) Cost Allocation. In Handbook of Game Theory with Economic Applications, Vol. II, Handbooks in Economics, Vol. 11, North-Holland, Amsterdam, pp. 1193-1235.