ABILITY AND KNOWLEDGE

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Abstract. In games with incomplete information, more information to a player implies a broader strategy set for this player in the normal form game, hence that more knowledge implies more ability. We prove that, on the other hand, given two normal form games G and G' such that players in a subset J of the set of players possess more strategies in G' than in G, there exist two games with incomplete information with normal forms G and G' such that players in J are more informed in the second than in the first. More ability can then be rationalized by more knowledge, and our result thus establishes the formal equivalence between ability and knowledge.

1. Introduction

"Ability" refers to the possibility of an agent to achieve particular actions. "Knowledge" refers to the information possessed by the agent. For instance, "running 100 m in less than 12 sec." is an ability, whereas "knowing the password required to log into computer account X" refers to some knowledge. Some skills can be described either in terms of knowledge, or as abilities, as for instance "preparation of a particular recipe", or "piloting a plane". In fact, the connections between knowledge and ability are strong, and the aim of this short paper is to clarify these.

Different levels of ability for a player can be represented in normal form games. If an agent possesses more strategies in game G than in G', this expresses more ability for this agent in G than in G'. Knowledge is naturally represented by information structures. Given two information structures $\mathfrak{E}$ and $\mathfrak{E}'$, a player has more knowledge in $\mathfrak{E}$ than in $\mathfrak{E}'$ when his information partition is finer in $\mathfrak{E}$ than in $\mathfrak{E}'$.

An information structure together with a payoff specification with incomplete information define a game with incomplete information, that can be represented in normal form. It is well known that finer information implies larger strategy sets in the associated normal form games. Indeed, agents having more knowledge can use more information in their decision making, which results in more ability. For instance, when a firm discovers the knowledge of some technology, this results in a larger production set.

The aim of this paper is to prove the equivalence of ability and knowledge. Since it is already well known that more knowledge implies more ability, we show a converse to this proposition, namely that more ability can always be rationalized as the consequence of more knowledge. More precisely, given two finite normal form games G and G', and assuming that players in a subset J of the set of players have more ability in G' than in G, we construct two information structures $\mathfrak{E}$ and $\mathfrak{E}'$ and a payoff specification γ, such that:

- $\mathfrak{E}'$ is more informative than $\mathfrak{E}$ for players in J,
- The normal form game associated with $\mathfrak{E}$ and γ is G
- The normal form game associated with $\mathfrak{E}'$ and γ is G'

The proof of this result relies on the following logic. Assume that in game G', player i possesses a strategy a which is not available in G. We try to explain this

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extra strategy by extra knowledge of player i in games with incomplete information. To do this, we construct a game in which player i, in order to play strategy a, must announce a password, which is initially uniformly drawn in the continuum $[0, 1]$. If i is informed of the value of the password, i has the possibility to announce the true value whatever it is, hence to achieve a with probability 1. If i has no information of the password, the announced value will match the password with zero probability, hence a is not an available action to i. In this reasoning, the ability to play a is rationalized as the consequence of the knowledge of the adequate information. Our proof relies on a continuum space of states of nature (the passwords in our previous example). We show in section 4.3 that this assumption is essential, where we provide a counter example when this space is finite or countable.

Our result demonstrates that, without imposing any further structure on the nature of knowledge of the players, the only predictable effect of an increase in information to some player is an increase of the strategy set of this player in the corresponding normal form game.

The equivalence of knowledge and information gives a better understanding of the question of value of information. It is known at least since Hirshleifer's [Hir71] work that the value of information is not always positive in economic situations, neither for the agent for receiving more information, nor for society as a whole. As pointed out by Neyman [Ney91], the reason why information can have a negative value is that other players are aware of this extra information. More information is always beneficial to the agent if other agents are ignoring of it.

Some classes of games are known to show either social or private positive value of information. In decision problems (one player games), the value of information is positive if the agent is a Bayesian and expected utility maximizer. Indeed, more strategies are always beneficial, as the only choice to be made is the choice of the utility maximizing strategy. Works by Wakker [Wak88] and Chassagnon and Vergnaud [CV99] show that value of information can be negative for a non expected utility maximizer. For more than one player, the logic of socially positive value of information extends to games of common interests. Bassan, Gossner, Scarsini and Zamir [BSGZ03] show that the common interest condition is necessary and sufficient for a property of socially positive value of information to hold. The private value of information is positive in purely antagonistic zero-sum games, where finer information, or a larger strategy set, can only be beneficial to the player receiving it, and harmful for the other player. Gossner and Mertens [GM01] and Lehrer and Rosenberg [LR03a] study the value of information in these games. For general games, examples of situations with negative value of information can be found e.g. in [BSZ97] or in [KTZ90]. Lehrer and Rosenberg [LR03b] study the maps from partitional information structures to values of games that arise as values of games with incomplete information.

Blackwell [Bla51], [Bla53] shows that a statistical experiment yields a better payoff than another in every decision problem if and only if it is more informative. Gossner [Gos00] characterizes information structures that induce more correlated equilibrium distributions than others in every game. This order between information structures is compatible with the social value of information in all games.

Our result allows to view the value of more information as the value of a larger strategy set. Of course, such a value cannot be positive in general. For instance, by deleting the "defect" strategy for both players in the prisoner's dilemma, one transforms a game with defection as unique Nash equilibrium into a game with cooperation as unique Nash outcome. Hence, more strategies for both players is harmful for them both. In other words, committing not to use some information is formally equivalent to committing not to use certain strategies, and such a commitment may have positive effects. We introduce the comparison concepts between normal form games in section 2, and between information structures in section 3. We establish the connexion between the two in section 4, and briefly discuss applications to the value of information in section 5.

2. Normal form games

An arbitrary set I of players is fixed. If $(X_i)_i$ is a family of sets, X and X_{-i} denote $\Pi_i X_i$ and $\Pi_{j\neq i} X_j$ respectively. For a family of maps $\alpha_i\colon X_i \to Y_i$, $\alpha\colon X \to Y$ is defined by $\alpha(x) = (\alpha_i(x_i))_i$.

A game in normal form $G = ((S_i), g)$ is given by a strategy space S_i for each player i and by a payoff function $g\colon S \to \mathbb{R}^I$. A game $G = ((S_i)_i, g)$ is a finite game in mixed strategies when each S_i is the space of probabilities over a finite (pure) actions space A_i, and when g is multilinear on S. Strategies $s_i, s'_i \in S_i$ are payoff equivalent in G whenever for every $s_{-i} \in S_{-i}$, $g(s_i, s_{-i}) = g(s'_i, s_{-i})$.

2.1. **Equivalent games.** We now define equivalence between games.

Definition 1. *Given two games G and G' in normal form, G is equivalent to G', and we note $G \sim G'$, when there exists a family of mappings $\psi = (\psi_i)_i$, $\psi_i\colon S_i \to S'_i$ such that:*

(1) $g = g' \circ \psi$,
(2) Every element of S'_i is payoff equivalent to an element of $\mathtt{Im}\psi_i$

Proposition 1. *$\sim$ is an equivalence relation.*

Proof. The relation $\sim$ is clearly reflexive since the identity from G to itself does the job. We prove $\sim$ is symmetric. Assume $G \sim G'$, and let ψ be the corresponding family of mappings. For $s'_i \in S'_i$, select $t'_i \in \mathtt{Im}\psi_i$ payoff equivalent to s'_i and select $\psi'_i(s'_i) \in \psi^{-1}(t'_i)$. Then, for every $s' \in S'$, $g(\psi'(s')) = g'(\psi(\psi'(s'))) = g'(s')$ so $g \circ \psi' = g'$. Now, for $s_i \in S_i$ let $t_i = \psi'_i(\psi_i(s_i)) \in \mathtt{Im}\psi'_i$. Then, $\psi_i(t_i) = \psi_i(s_i)$ and

$$g(t_i, s_{-i}) = g'(\psi_i(s_i), \psi_{-i}(s_{-i})) = g(s_i, s_{-i})$$

so that s_i and t_i are payoff-equivalent. To prove that $\sim$ is transitive, assume $G \sim G'$ and $G' \sim G''$, and $\psi = (\psi_i)_i$ from G to G' and $\psi' = (\psi'_i)_i$ from G' to G'' be the corresponding mappings. It is immediate that $\psi'' = (\psi''_i)_i$ with $\psi''_i = \psi'_i \circ \psi_i$ from G to G'' verifies the requested properties. □

The notion of equivalence between games is closely related to that of reduced normal forms, indeed we have:

Proposition 2. *Any game G is equivalent to its reduced normal form.*

Proof. The family $(\psi_i)_i$ that maps any strategy s_i to its equivalence class verifies the condition of definition 1. □

Example 1. *G and G' are two finite games in mixed strategies given by the payoff matrices:*

	l	m	r
t	$1,0$	$5,2$	$3,1$
b	$5,0$	$3,6$	$4,3$

G

	L	R
T	$1,0$	$5,2$
M	$5,0$	$3,6$
B	$3,0$	$4,4$

G'

Define ψ_1 and ψ_2 on pure strategies by $\psi_1(t) = T$, $\psi_1(b) = M$, $\psi_2(l) = L$, $\psi_2(m) = R$, $\psi_2(r) = \frac{1}{2}L + \frac{1}{2}R$, and extend them linearly to the mixed strategy spaces. Then, $g = g' \circ \psi$, and to see that every strategy in G' is payoff equivalent to a strategy in the image of ψ, note that B is payoff equivalent to $\frac{1}{2}T + \frac{1}{2}M$.

2.2. **Restrictions of a games.** Deleting elements of the strategy space for player i transforms a game G into a game G' in which allows less strategic choices for player i. This idea for any subset of the set of players is captured by the definition of a restriction below.

Definition 2. *G is a restriction for players in J of G', and we note $G \subseteq_J G'$, when there exists a family of mappings $\varphi = (\varphi_i)_i$, $\varphi_i \colon S_i \to S'_i$ such that:*

- *$g = g' \circ \varphi$,*
- *For $i \notin J$, every element of S'_i is payoff equivalent to an element of $\mathtt{Im}\varphi_i$.*

Example 2. *Consider the finite games in mixed strategies G and G' given by the payoff matrices:*

	l	R
t	$1,-1$	$-1,1$
b	$0,0$	$0,0$

G

	L	R
T	$1,-1$	$-1,1$
B	$-1,1$	$1,-1$

G'

Define φ_1 and φ_2 on pure strategies by $\varphi(t) = T$, $\varphi(b) = \frac{1}{2}T + \frac{1}{2}B$, $\varphi(l) = L$, $\varphi(r) = R$ and extend them linearly to the mixed strategy spaces. Then, φ, verifies the properties of the definition with $J = \{1\}$, and so $G \subseteq_{\{1\}} G'$. In fact, G is a version of G' in which player 1 is restricted to play mixed strategies that put weight no more than $\frac{1}{2}$ on B.

2.3. **Restrictions and equivalences.** The aim of this section is to address the following question: Assume that G is a restriction (for any subset J of the players, or more generally for $J = I$) of G', and that G' is a restriction of G. Can we infer that G and G' are equivalent? Answering this question is not necessary for establishing the formal equivalence of Ability and Knowledge, but we think it may help understanding the concepts of equivalences and restrictions.

We first provide a counter-example to this conjecture for general games.

Example 3. *We consider a version of an "iron arm" fight in which player's strengths may vary. There are 2 players, 1 and 2. In G, player i's chooses some energy put in the fight, $a_i \in [0,1]$. The payoff to player i is 1 is $a_i > a_{3-i}$ (i wins the fight), 0 if $a_i = a_{3-i}$ (draw), and -1 is $a_i < a_{3-i}$ (i loses the fight). The game G' is the same as G except that player 1's strategy set is $[0,2]$. The game G'' is the same as G except that both player's strategy sets are $[0,2]$. Considering the maps $\psi_i \colon a_i \mapsto 2a_i$ from $[0,1]$ to $[0,2]$ show that G and G'' are equivalent. By the definition of the games, $G \subseteq_{\{1\}} G'$ and $G' \subseteq_{\{2\}} G''$, hence $G \subseteq_{\{1,2\}} G' \subseteq_{\{1,2\}} G$. But G and G' are not equivalent: indeed, player 1 has a strategy that guarantees a win in G', but not in G.*

The previous counter example relies on infinite pure strategy spaces. We now state a positive answer for finite games in mixed strategies.

Theorem 1. *Assume that G and G' are finite games in mixed strategies such that $G \subseteq_I G'$ and $G' \subseteq_I G$, then $G \sim G'$.*

We start with two lemmata.

Lemma 1. *If $G \sim G' \subseteq_J G''$ or $G \subseteq_J G' \sim G''$ then $G \subseteq_J G''$.*

Proof. It is immediate to check that the composition of an inclusion map and an equivalence map is an inclusion map. □

Lemma 2. *If G is finite game with no equivalent pair of pure strategies and $\phi = (\phi_i)_i$ a family of maps such that $g \circ \phi = g$, then each ϕ_i is a permutation of A^i.*

Proof. Proof : Let M be the $A^i \times A^{-i}$ matrix with elements in $\mathbb{R}^I$ defined by $m_{a_i,a_{-i}} = g(a_i, a_{-i})$. Let S and T be the transition matrices over A^i and A^{-i} respectively given by $s_{a_i,b_i} = \phi(a_i)(b_i)$ and $t_{a_{-i},b_{-i}} = \phi(a_{-i})(b_{-i})$. The relation $g \circ \phi = g$ rewrites $M = SM\,{}^tT$. Let $k \in \mathbb{N}$ be such that both S^k and T^k are transitions of aperiodic Markov chains, and let S^∞ and T^∞ denote the limits of the sequences $(S^{nk})_n$ and $(T^{nk})_n$. We deduce from the above that $M = S^\infty M\,{}^tT^\infty$. If S^∞ had two equal rows, then so would M, and player i would have two equivalent pure strategies. Hence all rows of S^∞ are distinct, which implies that S^∞ and S^k are the identity matrix and that S is a permutation matrix. □

Proof of theorem 1. From proposition 2 and lemma 1, it suffices to prove the theorem when G and G' are reduced normal forms. Let φ and φ' be the maps from G to G' and from G' to G as in definition 2, and let $\phi = \varphi' \circ \varphi$. Then $g = g \circ \phi$, and by lemma 2 each ϕ_i is a permutation. For $a'_i \in A'_i$, let $a_i = \phi_i^{-1}(\varphi'_i(a'_i))$, and $\tilde{a}'_i = \varphi_i(a_i)$. Then $\varphi'_i(\tilde{a}'_i) = \varphi'_i(a_i)$, and for all $a'_{-i} \in A'_{-i}$,

$$g'(a'_i, a'_{-i}) = g(\varphi'_i(a'_i), \varphi'_{-i}(a'_{-i})) = g(\varphi'_i(\tilde{a}'_i), \varphi'_{-i}(a'_{-i})) = g'(\tilde{a}'_i, a'_{-i})$$

Hence, $a'_i \in A'_i$ is equivalent to $\tilde{a}'_i = \varphi_i(a_i)$, which implies $a'_i = \varphi_i(a_i)$ since G' is a reduced normal form. Furthermore, $a'_i = \varphi_i(a_i)$ implies $a_i = \phi_i^{-1}(\varphi'_i(a_i))$, so that $a_i \in A_i$ such that $a'_i = \varphi_i(a_i)$ is unique. Each map φ'_i is thus one-to-one from A_i to A'_i, and the inverse map is $\phi_i^{-1} \circ \varphi'_i$.

Let now $s'_i = \sum_{a'_i} \lambda_{a'_i} a'_i$, and consider $s_i = \sum_{a'_i} \lambda_{a'_i} \varphi_i^{-1}(a'_i)$. Then

$$\begin{aligned}
g'(\varphi_i(s_i), a'_{-i}) &= g'(\varphi_i(s_i), \varphi_{-i} \circ \varphi_{-i}^{-1}(a'_{-i})) \\
&= g(s_i, \varphi_{-i}^{-1}(a'_{-i})) \\
&= \sum\nolimits_{a'_i} \lambda_{a'_i} g(\varphi_i^{-1}(a'_i), \varphi_{-i}^{-1}(a'_{-i})) \\
&= \sum\nolimits_{a'_i} \lambda_{a'_i} g'(a'_i, a'_{-i}) \\
&= g'(s'_i, a'_{-i})
\end{aligned}$$

Hence, any s'_i is equivalent to an element of the image of φ_i, so that φ verifies the conditions of definition 1. G and G' are then equivalent. □

3. Knowledge: Comparison of information structures

3.1. **Description of information.** K is a measurable space of states of nature. An information structure is given by $\mathfrak{E} = (\Omega, \mathcal{E}, P, (\mathcal{E}_i)_i, \kappa)$, where $(\Omega, \mathcal{E}, P)$ is a probability space of states of the world, $\mathcal{E}_i$ is a sub σ-algebra of $\mathcal{E}$ that describes the information of player i, and κ is a $\mathcal{E}$-measurable application to K that describes the state of the nature.

Definition 3. *We say that $\mathfrak{E}$ is less informative for players in J than $\mathfrak{E}'$, and we note $\mathfrak{E} \subseteq_J \mathfrak{E}'$ when $\mathfrak{E}$ can be obtained from $\mathfrak{E}$ by replacing the σ-algebras $\mathcal{E}'_j$ by sub σ-algebras $\mathcal{E}_j$ for $j \in J$.*

Example 4. *Choose $\Omega = K = \{k_1, k_2\}$ endowed with the discrete σ-algebra and the uniform probability, and κ is the identity. Set $\mathcal{E}'_1 = \mathcal{E}_1 = \mathcal{E}_2 = \{\emptyset, \Omega\}$, and $\mathcal{E}'_2$ the discrete σ-algebra. Then, player 1 is never informed of k, whereas player 2 knows k in $\mathfrak{E}'$ but not in $\mathfrak{E}$. We have $\mathfrak{E} \subseteq_{\{2\}} \mathfrak{E}'$.*

3.2. **Games of incomplete information.** For a given space of states of nature K, a payoff specification is given by measurable spaces X_i and by a measurable and bounded from above or below[1] map $\gamma\colon \Pi_i X_i \times K \to \mathbb{R}^I$.

An information structure $\mathfrak{E}$ and a payoff specification γ on the same space K define a normal form game $G(\mathfrak{E},\gamma)$ in which a strategy for player i is a measurable map from $\mathcal{E}_i$ to X_i and payoffs are given by the relation $g_{\mathfrak{E},\gamma}(f) = \mathbf{E}_P \gamma((f_i)(\omega), \kappa(\omega))$.

Example 5. *Take up the information structures $\mathfrak{E}$ and $\mathfrak{E}'$ of example 4, and let $X_1 = \{T, B\}$, $X_2 = \{L, R\}$, and γ be given by the two payoff matrices:*

$k = 1$

	L	R
T	$0,0$	$1,2$
B	$2,0$	$0,2$

$k = 2$

	L	R
T	$0,1$	$1,0$
B	$2,1$	$0,0$

In $G_{\mathfrak{E},\gamma}$ the only strategies for $i \in \{1,2\}$ are the constant ones in X_i, and the payoff matrix of this game is:

	L	R
T	$0, \frac{1}{2}$	$1,1$
B	$2, \frac{1}{2}$	$0,1$

$G_{\mathfrak{E},\gamma}$

In $G_{\mathfrak{E}',\gamma}$ the strategies for player 1 are the constant strategies, and player 2 has 4 strategies. For instance LR is the strategy of player 2 that plays L if $k = k_1$ and R if $k = k_2$. The payoff matrix of this game is

	LL	LR	RL	RR
T	$0, \frac{1}{2}$	$\frac{1}{2}, 0$	$\frac{1}{2}, \frac{3}{2}$	$1,1$
B	$2, \frac{1}{2}$	$1,0$	$1, \frac{1}{2}$	$0,1$

$G_{\mathfrak{E}',\gamma}$

4. Relations between knowledge and ability

4.1. **More knowledge implies more ability.** We recall the well known fact that more knowledge implies more ability.

Theorem 2. *$\mathfrak{E} \subseteq_J \mathfrak{E}'$ implies $G_{\mathfrak{E},\gamma} \subseteq_J G_{\mathfrak{E}',\gamma}$.*

Proof. Let Σ_i and Σ'_i be the sets of measurable maps from $(\Omega, \mathcal{E}_i)$ and $(\Omega, \mathcal{E}'_i)$ respectively to X_i. It is straightforward that the family of inclusion maps ψ_i from Σ_i to Σ'_i verifies the conditions of definition 2. □

Example 6. *It is seen in the previous example that $G_{\mathfrak{E},\gamma} \subseteq_J G_{\mathfrak{E}',\gamma}$.*

4.2. **Question about a converse theorem.** Given K, and two games in mixed strategies such that $G \subseteq_J G'$, we address the existence of $\mathfrak{E}$, $\mathfrak{E}'$, and γ, such that

- $G_{\mathfrak{E},\gamma} \sim G$;
- $G_{\mathfrak{E}',\gamma} \sim G'$;
- $\mathfrak{E} \subseteq_J \mathfrak{E}'$.

4.3. **A counter example if K is finite or countable.** Let G and G' be the one-player finite games in mixed strategies:

B	0

G

T	1
B	0

G'

[1]These assumptions are to ensure that expected payoffs are well defined.

Proposition 3. *Consider the above games G and G', and assume K is finite or countable. There does not exist $\mathfrak{E}$, $\mathfrak{E}'$, and γ, such that*

- $G_{\mathfrak{E},\gamma} \sim G$;
- $G_{\mathfrak{E}',\gamma} \sim G'$;
- $\mathfrak{E} \subseteq_{\{1\}} \mathfrak{E}'$.

Proof. We go by contradiction, and assume wlog. that $(\Omega, \mathcal{E}')$ is K with the discrete σ-algebra, $\mathcal{E} = \{\emptyset, K\}$ and $P(k) > 0$ for all k. From the equivalence between G' and $G_{\mathfrak{E}',\gamma}$, we deduce that $\min_{(x_k)_k} \sum P(k)g(x_k, k)$ is well defined and has value 0. Hence, for each $k \in K$ let x_k that minimizes $g(x_k, k)$. For every $x \in X$:

$$\sum\nolimits_k P(k)g(x,k) \geq \sum\nolimits_k P(k)g(x_k,k)$$

with strict inequality if there exists k such that $g(x,k) > g(x_k,k)$. By equivalence of G and $G_{\mathfrak{E},\gamma}$, $\sum_k P(k)g(x,k) = 0$ for every x. Hence, for every x, k $g(x,k) = g(x_k,k)$. Then $\sum P(k)g(x'_k,k)$ is independent of $(x'_k)_k$, so that the payoff function of G' must be identically 0. A contradiction. □

4.4. **A positive result.**

Theorem 3. *Given two finite games in mixed strategies such that $G \subseteq_J G'$, there exists K, $\mathfrak{E}$, $\mathfrak{E}'$, and γ, such that:*

(1) $G_{\mathfrak{E},\gamma} \sim G$;
(2) $G_{\mathfrak{E}',\gamma} \sim G'$;
(3) $\mathfrak{E} \subseteq_J \mathfrak{E}'$.

Proof. We construct the information structures and the payoff specification, and later verify the equivalences of games.

The information structures Let $(M_i, \mathcal{M}_i, m_i)$ for $i \in I$ and $(K_j, \mathcal{K}_j, \beta_j)$ for $j \in J$ be independent copies of $[0,1]$ endowed with the Borel sets and the Lebesgue measure. Let $(\Omega, \mathcal{E}, P)$ be the product of these spaces M_i and K_j. The space K is the product of the K_j's, and κ is the projection from Ω to K.

For every $i \in I$, $\mathcal{E}_i$ is generated by $\mathcal{M}_i$ on Ω. For $j \in J$, $\mathcal{E}'_j$ is generated by $\mathcal{M}_i$ and $\mathcal{K}_i$. For $j \notin J$ $\mathcal{E}'_j = \mathcal{E}_j$. It is thus verified that $\mathfrak{E} \subseteq_J \mathfrak{E}'$.

Game and payoff specification Assume wlog. the A_i's and A'_i's are disjoint. For $i \in I$, let $X_i = (A_i \cup A'_i) \times K$. We define an outcome function from $X \times K \to X$. Let $x = (x_i)_{i\in I} = ((a_i)_{i\in I}, (b_i)_{i\in J})$ and $k = (k_j)_j$. For $i \notin J$, $o_i(x,k) = a_i$. For $j \in J$, select $a^0_j \in A_j$, and let

$$o_j(x,k) = \begin{cases} a_j & \text{if } a_j \in A_i \text{ or } b_j = k_j \\ a^0_j & \text{otherwise} \end{cases}$$

This defines $o = (o_i)_i \colon X \times K \to X$. We now define $\tilde{\varphi} \colon X \to S'$ by its coordinates $(\tilde{\varphi}_i)_i$. If $a_i \in A'_i$, $\tilde{\varphi}_i(a_i)$ is the unit mass at a_i. It $a_i \in A_i$, $\tilde{\varphi}_i(a_i) = \varphi(a_i)$. The (measurable) payoff function with incomplete information is $\gamma = g' \circ \tilde{\varphi} \circ o$.

Verification of (2) To any strategy $f_i \colon (\Omega, \mathcal{E}'_i) \to X_i$ we associate the probability $\psi'_i(f_i)$ induced by P, f_i, o_i and $\tilde{\varphi}_i$ on A'_i. A profile $f = (f_i)_i$ of strategies induces with P, o and $\tilde{\varphi}$ the product measure $\psi'(f)$ on A' that has marginals $\psi'_i(f_i)$ on A'_i. Hence the corresponding payoff is

$$g_{\gamma,\mathfrak{E}'}(f) = \mathbf{E}_{\psi'(f)} g' = g'((\psi'_i(f_i))_i)$$

which is point (1) of definition 1 for G'. For point (2) of this definition applied to G', for every $s'_i \in S'_i$, let $a_i(\omega)$ be a $\mathcal{M}_i$-measurable random variable with values in A'_i and law s'_i. Set $b_i(\omega) = k_i$, and $f_i = (a_i, b_i)$ ($\mathcal{E}'_i$ measurable). Then $o_i(a_i(\omega), b_i(\omega)) = a_i(\omega)$, and $\psi'_i(f_i) = s'_i$. So, $\mathtt{Im}\psi'_i = S'_i$.

Verification of (1) We now define $\psi_i(f_i) \in S_i$ for $\mathcal{E}_i$-measurable f_i.

For $i \notin J$ select for each $s'_i \in S'_i$ an element $\varphi_i^{-1}(s'_i)$ such that s'_i is payoff equivalent to $\varphi_i(\varphi_i^{-1}(s'_i))$ in G'. Let then $\psi_i(f_i) = \varphi_i^{-1}(\psi'_i(f_i))$.

For $i \in J$, note that f_i being $\mathcal{E}_i$ measurable implies $o_i(f_i(\omega)) \in A_i$ P almost surely. Let then $\psi_i(f_i) \in S_i$ be the image of P by f_i and o. Then $\varphi_i(\psi_i(f_i)) = \psi'_i(f_i)$.

We deduce,

$$g(\psi(f)) = g'(\varphi(\psi(f))) = g'(\psi'(f)) = g_{\gamma,\mathfrak{E}'}(f) = g_{\gamma,\mathfrak{E}}(f)$$

where the first equality comes from $g = g' \circ \phi$, the second from the equivalences in G' of $\varphi_i(\psi_i(f))$ with $\psi'_i(f_i)$ for all i, next from the equivalences of G' and $G_{\gamma,\mathfrak{E}'}$, and the last one from $g_{\gamma,\mathfrak{E}'} = g_{\gamma,\mathfrak{E}}$ on the domain of $g_{\gamma,\mathfrak{E}}$. Hence point (1) of definition 1 for G'.

For point (2) of this definition, for every $s_i \in S_i$, let $a_i(\omega)$ be a $\mathcal{M}_i$-measurable random variable with values in A_i and law s_i. Set $b_i(\omega) = 0$ constant, and let f_i $\mathcal{E}_i$ measurable be given by $f_i = (a_i, b_i)$. Then $o_i(a_i(\omega), b_i(\omega)) = a_i(\omega)$, and $\psi_i(f_i) = s_i$. So, $\mathtt{Im}\psi_i = S_i$. □

Remark that the constructed information structures $\mathfrak{E}$ and $\mathfrak{E}'$ depend on J, but not on the games G and G'. Note also that the payoff specification γ has the same image as g'. In particular, γ is zero-sum game whenever g' is, and a group of players have common interests in γ whenever they do in g'. This leads us the the following statement that strengthens theorem 3.

Theorem 4. *For every subset J of players, there exist information structures $\mathfrak{E} \subseteq_J \mathfrak{E}'$ such that for any two finite games in mixed strategies $G \subseteq_J G'$, there exists a payoff specification γ that verifies:*
(1) $G_{\mathfrak{E},\gamma} \sim G$;
(2) $G_{\mathfrak{E}',\gamma} \sim G'$;
(3) $\mathtt{Im}\gamma = \mathtt{Im}g'$.

5. On the value of information

More information is beneficial in one player games, socially beneficial in games with common interest, and privately beneficial for the player receiving it in zero-sum games. These results can be seen as a consequence that a broader strategy set is beneficial in these classes of games. On the other hand, many situations are known in which more information to some player may hurt this player, or the group of players. Theorem 3 can be used to construct such games with negative value of information.

Example 7. *Consider the games G and G' given by the payoff matrices:*

	L
T	$3,3$
M	$2,0$

G

	l	r
t	$3,3$	$0,4$
b	$2,0$	$1,1$

G'

Both games are dominance solvable, with $(3,3)$ as unique Nash payoff in G, and $(1,1)$ as unique Nash payoff in G'. Since G is a restriction for player 2 of G', G is equivalent to some $G_{\mathfrak{E},\gamma}$, and G' to some $G_{\mathfrak{E},\gamma}$, with $\mathfrak{E} \subseteq_{\{2\}} \mathfrak{E}'$. We are then facing a situation where the value of information is negative, since the better information of player 2 in $\mathfrak{E}'$ has a negative effect on the Nash payoff for both players.

Along the same lines, it is possible to construct examples in which the value of more information for player 1 is for instance positive for player 2, but negative for player 1.

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