ASYMPTOTIC ANALYSIS OF LARGE AUCTIONS

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Abstract

We study private-value auctions with a large number of bidders. We first calculate asymptotic approximations of the equilibrium bids and the seller's revenue in first-price auctions, regardless of whether the bidders are symmetric or asymmetric, risk-neutral or risk averse. We then show that with n bidders, the effects of risk aversion and of asymmetry on the equilibrium bids and on the seller's revenue are only $O(1/n^2)$. Furthermore, it is demonstrated that first-price auctions with asymmetric bidders or with risk averse bidders are $O(1/n^2)$ revenue equivalent to large classes of standard auctions.

Keywords: Large auctions, asymmetric auctions, risk-averse bidders, asymptotic methods, revenue equivalence, collusion

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1 Introduction

Many auctions, particularly those which have recently began to appear on the internet, have a large number of bidders. The standard approach to study large auctions has been to consider their limit as the number of bidders n approaches infinity.[1]. Using this approach, it has been shown for quite general conditions that as n goes to infinity, the bid approaches the true value, the seller's expected revenue approaches the maximal possible value, and the auction becomes efficient.[2] Most of the studies that adopted this approach, however, do not provide the rate of convergence to the limit, i.e., a bound on the difference between the limiting value and the value at a finite n. Therefore, it is not clear how large n should be (5, 10, 100?) in order for the auction to be considered "large" (i.e., in order for the limiting results obtained for $n = \infty$ to be applicable). Such rate of convergence results were obtained by Satterthwaite and Williams (1989), who showed that the rate of convergence of the bid to the true value in a double auction is $O(1/m)$, where m is the number of traders on each side of the market, and by Rustichini, Satterthwaite and Williams (1994), who showed that the rate of convergence of the bid to the true value in a k-double auction is $O(1/m)$ and the corresponding inefficiency is $O(1/m^2)$. In this study we go a step further and *calculate explicitly the leading-order deviation of the equilibrium bids and the revenue at a finite n from their limiting values, regardless*

[1]This is the approach, for example, in Wilson (1976), Pesendorfer and Swinkels (1997), Kremer (1999), Swinkels (1999)

[2]See, for example, Swinkels (2001) and Bali and Jackson (2001).

of whether bidders are symmetric or asymmetric, risk-neutral or risk-averse.[3] [4] We do this by employing asymptotic and perturbation analysis tools. These powerful Applied Mathematics techniques have been used with great success in the exact sciences over the last several hundred years to analyze complex mathematical models, but have not been applied to auction theory as well as to most branches of Economics.[5]

Since the pioneering work of Vickrey (1961) who established the revenue equivalence of the classical private-value auctions (first-price, second-price, English, Dutch), a considerable research effort has been devoted to revenue ranking of different auction mechanisms. Vickrey's result was generalized twenty years later by the Revenue Equivalence Theorem (Riley and Samuelson (1981) and Myerson (1981)) according to which the seller's revenue is the same for a wide class of private-value auctions with symmetric and risk-neutral bidders. However, private value auctions are, in general, not revenue equivalent when bidders are asymmetric (Marshall et al. (1994), Maskin (2000)) or risk-averse (Maskin and Riley (1984), Matthews (1987)). As we mentioned, previous studies showed that under quite general conditions, auctions become revenue equivalent as n approaches infinity. In this work we prove a stronger result, namely, that independently of whether bidders are sym-

[3]Since the leading-order deviation is $O(1/n)$, our asymptotic results are $O(1/n^2)$ accurate. Therefore, they become valid at much smaller values of n than the limiting results, which are only $O(1/n)$ accurate.

[4]Caserta and de Vries (2002) used extreme value theory to calculate the leading order correction to the seller's revenue in large auctions. Since the bidders in their model are symmetric and risk-neutral, they were able to utilize the explicit formula for the revenue.

[5]In previous studies 3d perturbation analysis to analyze auctions with weakly asymmetric bidders (Fibich and Gavious (2003), Fibich, Gavious and Sela, (2003)) and with weakly risk-averse bidders (Fibich, Gavious and Sela, (2002)).

metric or asymmetric, risk-neutral or risk-averse, noncooperating or acting collusively, for large classes of standard auctions the $O(1/n)$ deviation of the revenue in a large auction from the limiting revenue is also independent of the auction mechanism. In other words, the revenue difference among large auctions is at most $O(1/n^2)$. This result implies that revenue ranking of large auctions is probably more of academic interest than of practical value.[6]

The paper is organized as follows. In Section 2 we calculate asymptotic approximations to the equilibrium bids and to the seller's revenue in large symmetric first-price auctions, and show that regardless of whether the bidders are risk-neutral or risk-averse the equilibrium bids have differences and generate differences in the seller's revenue of only $O(1/n^2)$. Then we show that all large k price auctions ($k = 1, 2, \dots$) with risk-averse or risk-neutral bidders are $O(1/n^2)$ revenue equivalent. In Section 3 we calculate approximations to the equilibrium bids in large asymmetric first-price auctions. As was pointed out by Swinkels (2001), while in large asymmetric auctions "players' values may come from very different distributions, their environments, and thus their optimal behavior with any given valuation, may be very similar". Since the environment (competition) that players i and j face differs by one out of $n-1$ players (i is facing j but not i and vice versa), one could expect the resulting asymmetry among the bids to be $O(1/n)$. However, our results show that in a first price auction with n asymmetric bidders, the asymmetry among the equilibrium bids is only $O(1/n^2)$. Similarly, one could expect that the revenue differences

[6]In fact, in all the numerical examples that we have tested, we found that for auctions with only six bidders the revenue difference between first- and second-price auctions is in the fourth or fifth digit.

among large asymmetric auctions would be $O(1/n)$. However, we show that large auctions with asymmetric bidders are $O(1/n^2)$ revenue equivalent. Section 4 concludes, and the Appendix contains proofs omitted from the main body of the paper.

An important question is how large n should be for our asymptotic results to be valid. Since our expansions have an $O(1/n^2)$ accuracy, roughly speaking, they have a 1% accuracy for 10 bidders. As the numerical examples in this study show, however, our asymptotic results are already quite accurate even for values as low as $n = 6$. The issue of how small can n be for our asymptotic results to be valid can probably be resolved theoretically by using the same methods to calculate explicitly the next $O(1/n^2)$ term in the asymptotic expansion, but this is beyond the scope of the present study.

2 Symmetric large auctions

Consider a large number ($n \gg 1$) of bidders who are competing for a single object. The bidders are symmetric such that the valuation v_i of bidder i for the object is independently distributed according to a common distribution function $F(v)$ on the interval $[0, 1]$. We denote by $f = F'$ the corresponding density function. We assume that F is twice continuously differentiable and that $f \neq 0$ in $[0, 1]$.

2.1 Symmetric large auctions with risk-neutral bidders

Assume that the bidders are risk-neutral and they compete for a single object in a first-price auction. In this symmetric setup, the equilibrium bid is

$$b(v) = v - \frac{\int_0^v F^{n-1}(x)\,dx}{F^{n-1}(v)}, \tag{1}$$

and the seller's expected revenue is

$$R = \int_0^1 b(v)\,dF^n(v). \tag{2}$$

It is well known that as the number of players increases, the equilibrium bid b approaches the player's value v and the seller's expected revenue R approaches one, i.e., $\lim_{n\to\infty} b(v) = v$ and $\lim_{n\to\infty} R = 1$. Our goal is to achieve a more precise characterization of the asymptotic behavior for large n, i.e., to calculate explicitly the leading-order deviation from these limiting values for a large finite n. To do that, we need the following Lemma:

Lemma 1 *Let $F(v)$ be a monotonically increasing function. Then, for a sufficiently large n,*

$$\int_0^v F^{n-1}(x)\,dx = \frac{1}{n}\frac{F^n(v)}{f(v)} + O\left(\frac{1}{n^2}\right). \tag{3}$$

Proof. See Appendix A.

Substitution of (3) in (1) provides the equilibrium bids in large first-price auctions:

Proposition 1 *In a symmetric first-price auction with n risk-neutral bidders, the equilibrium bid for a sufficiently large n is*

$$b(v) = v - \frac{1}{n}\frac{F(v)}{f(v)} + O\left(\frac{1}{n^2}\right). \tag{4}$$

Substituting $v = 1$ in eq. (4) shows that the maximal bid $\bar{b} = b(1)$ is given by

$$\bar{b} = 1 - \frac{1}{n}\frac{1}{f(1)} + O\left(\frac{1}{n^2}\right). \tag{5}$$

Expression (4) for the equilibrium bid leads to an asymptotic characterization of the seller's expected revenue:

Proposition 2 *In a symmetric first-price auction with n risk-neutral bidders, the seller's expected revenue for a sufficiently large n is*

$$R[F] = 1 - \frac{2}{n}\frac{1}{f(1)} + O\left(\frac{1}{n^2}\right). \tag{6}$$

Proof. Using (2) and (4) we obtain

$$R[F] = b(1) - \int_0^1 b'(v)F^n(v)\, dv = 1 - \frac{1}{n}\frac{1}{f(v)} + O\left(\frac{1}{n^2}\right) - \int_0^1 [1 + O(1/n)]F^n(v)\, dv.$$

Therefore, by (3), the result follows.

By the Revenue Equivalence Theorem (Riley and Samuelson(1981) and Myerson (1981)), the result of Proposition 2 for symmetric large first-price auctions can be generalized as follows.[7]

Theorem 1 *Consider any auction mechanism with n bidders that satisfies the following conditions:*

1. *All players are risk neutral.*

[7]The result of Theorem 1 is equivalent to the main result (Proposition 6) of Caserta and de Vries (2002), which was derived using extreme value theory.

2. *Player i's valuation is private information to i and is drawn independently by a continuously differentiable distribution function $F(v)$ from a support $[0, 1]$ which is common to all players.*

3. *The object is allocated to the player with the highest valuation.*

4. *Any player with valuation 0 expects zero surplus.*

Then, the seller's expected revenue for a sufficiently large n is given by (6).

Thus, with $O(1/n^2)$ accuracy, the revenue in symmetric auctions depends only on the value of the density function f at the maximal value.

2.2 Symmetric large auctions with risk-averse bidders

Assume now that the bidders are risk-averse, that is, each bidder's utility is given by the function $U(v - b)$, which is twice continuously differentiable, normalized such that $U(0) = 0$, monotonically increasing ($U' > 0$), and concave ($U'' < 0$).[8]

In this setup there is no explicit formula for the equilibrium bids and for the revenue in first-price auctions. Recently, Fibich, Gavious and Sela (2002) obtained explicit approximations of the equilibrium bids for the case of weak risk aversion, by using perturbation methods to expand the solution in the small risk-aversion parameter. Here we take a different approach, where we utilize the existence of the large parameter n (number of

[8]In fact, the results of this section hold with no restriction on U'', i.e., they hold for both risk-averse and risk-loving bidders.

players) to expand the solution in the small parameter $1/n$. Unlike Fibich, Gavious and Sela (2002), we do not assume that risk aversion is weak.

Proposition 3 *In a symmetric first-price auction with n risk-averse bidders, the equilibrium bid for a sufficiently large n is given by (4), and the seller's expected revenue is given by (6).*

Proof. See Appendix D.

Comparison with Propositions 1 and 2 shows that the differences in the equilibrium bids and in the seller's revenue between the case of risk-neutral and risk-averse bidders are $O(1/n^2)$. In other words, risk aversion has only an $O(1/n^2)$ effect on the equilibrium bids and on the revenues in symmetric large first-price auctions.

Propositions 3 can be generalized to any k-price auction:[9]

Proposition 4 *In a k-price auction ($k = 1, 2, 3, \dots$) with n risk-averse bidders, the equilibrium bid for a sufficiently large n is*

$$b(v) = v + \frac{k-2}{n-k}\frac{F(v)}{f(v)} + O\left(\frac{1}{n^2}\right), \tag{7}$$

and the seller's expected revenue is given by (6).

Proof. See Appendix E.

[9] In a k-price auction the bidder with the highest bid wins the auction and pays the k-th highest bid. For more details on k-price auctions, see Monderer and Tennenholtz (2000).

We recall that in the risk-neutral case the equilibrium bids in k-price auctions are given by (Wolfstetter, 1995)

$$b(v) = v + \frac{k-2}{n-k+1}\frac{F(v)}{f(v)}.$$

Comparison with equation (7) shows that in large symmetric k-price auctions, risk aversion only has an $O(1/n^2)$ effect on the equilibrium bids. Similarly, comparison of Proposition 4 with Theorem 1 shows that risk aversion only has an $O(1/n^2)$ effect on revenues in large symmetric k-price auctions. This implies, in particular, that all large symmetric k-price auctions, with risk-neutral or with risk-averse bidders, are $O(1/n^2)$ revenue equivalent.

3 Asymmetric large auctions

3.1 First-price auctions

Consider a large number ($n \gg 1$) of risk-neutral bidders who are competing for a single object in a first-price auction. The bidders are asymmetric such that the valuation of bidder i for the object v_i is independently distributed according to a distribution function $F_i(v)$ on the common interval $[0, 1]$. We denote by $f_i = F_i'$ the corresponding density functions.

Let $b_i = b_i(v_i)$ be the equilibrium bid function of bidder i. Since the equilibrium bids are strictly monotonic (Maskin and Riley (2000)), we can define the inverse bid functions as $v_i = v_i(b_i)$. The equations for the inverse bid functions are (Lebrun (1999), Fibich and

Gavious (2003))

$$v_i'(b) = \frac{F_i(v_i(b))}{f_i(v_i(b))} \left[\left(\frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{(v_j(b) - b)} \right) - \frac{1}{(v_i(b) - b)} \right], \qquad i = 1, \ldots, n. \tag{8}$$

The initial conditions for the system (8) are given by

$$v_i(b = 0) = 0, \qquad i = 1, \ldots, n. \tag{9}$$

The equilibrium bids also satisfy the condition that all bidders with the highest valuation $v = 1$ place the same (unknown) maximal bid, denoted by $\bar{b}$. Hence,

$$v_i(\bar{b}) = 1, \qquad i = 1, \ldots, n. \tag{10}$$

In this asymmetric setup there are no explicit solutions of the equilibrium bids. Recently, Fibich and Gavious (2003) obtained explicit approximations of the solutions of eqs. (8)–(10) for the case of weak asymmetry by using perturbation methods to expand the solution in the small asymmetry parameter. In this study we take a different approach and utilize the existence of the large parameter n to explicitly calculate an asymptotic approximation of the equilibrium bids. Unlike Fibich and Gavious (2003), we do not assume that the asymmetry among the players is weak.

Since $\lim_{n\to\infty} b_i(v) = v$, we can look for equilibrium bids of the form

$$b_i(v) = v + \frac{1}{n} u_i(v) + O(\frac{1}{n^2}), \qquad i = 1, \ldots, n.$$

Substitution in (8)–(10) leads to the following result:

Proposition 5 *In an asymmetric first-price auction with n bidders, the equilibrium bids for a sufficiently large n are*

$$b_i(v) = v - \frac{1}{n} u(v) + O(\frac{1}{n^2}), \qquad i = 1, \ldots, n \tag{11}$$

where

$$u(v) = \frac{1}{\frac{1}{n}\sum\limits_{j=1}^{n} \frac{f_j(v)}{F_j(v)}}. \tag{12}$$

Proof. See Appendix B.

Note that u is the harmonic average of $\{F_i(v)/f_i(v)\}_{j=1}^{n}$. Clearly, in the symmetric case, (11)-(12) reduces to (4).

A-priori, one could expect that the differences among the players' equilibrium bids in large asymmetric first-price auctions would be $O(1/n)$. However, since u_i turns out to be independent of i, we conclude that the differences among the players' equilibrium bids in large asymmetric first-price auctions are only $O(1/n^2)$. Moreover, the equilibrium bids in an asymmetric first-price auction can be approximated, with $O(1/n^2)$ accuracy, with those of a symmetric auction as follows:

Corollary 1 *The equilibrium bids in large asymmetric first-price auctions are $O(1/n^2)$ equivalent to those in symmetric first-price auctions with the same number of players whose (symmetric) distribution function F_{sym} is the geometric average of the asymmetric distribution functions, i.e., $F_{\mathrm{sym}} = (\prod_{j=1}^{n} F_j)^{1/n}$.*

Proof. This follows from Proposition 5, since $u(v)$ is the same in both cases.

Substituting $v = 1$ in Proposition 5 gives:

Corollary 2 *In an asymmetric large first-price auction with n players, the maximal equilibrium bid for a sufficiently large n is*

$$\bar{b} = \bar{b}_{\mathrm{approx}} + O\left(\frac{1}{n^2}\right),$$

where

$$\bar{b}_{\text{approx}} = 1 - \frac{1}{n}u(1), \qquad u(1) = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} f_i(1)}. \tag{13}$$

Note that $u(1)$ is the harmonic average of $\{1/f_i(1)\}_{j=1}^{n}$. In the symmetric case $u(1) = 1/f(1)$, and Corollary 2 reduces to eq. (5).

We can also use Proposition 5 to calculate an asymptotic approximation of the seller's expected revenue in asymmetric first-price auctions:

Proposition 6 *In an asymmetric large first-price auction with n bidders, the seller's expected revenue for a sufficiently large n is*

$$R^{1\text{st}}[F_1, \ldots, F_n] = 1 - \frac{2}{n}u(1) + O\left(\frac{1}{n^2}\right), \tag{14}$$

where $u(1)$ is given by (13).

Proof. See Appendix C.

The following example shows that the results of our asymptotic analysis are valid even for a relatively small number of asymmetric bidders (e.g., six players).[10]

[10] In other numerical experiments we also observed that the predictions of our asymptotic analysis are valid already for $n = 6$ (see, e.g., Section 3.3). This, of course, does not constitute a proof that our results are always valid for such small values of n. However, there are numerous examples in asymptotic analysis where results which are formally derived for $n \to \infty$, become in fact valid already for n not much larger than one. Therefore, it is reasonable to expect that, generically, the asymptotic results already become valid for n less than 10.

Example 1 *Consider a large first-price auction where bidders are equally split to* $n/2$ *bidders with a distribution function* $F_1(v) = v^{1/2}$ *and* $n/2$ *bidders with a distribution function* $F_2(v) = v^2$.

In this case, an application of Proposition 5 yields

$$u(v) = \frac{2}{f_1/F_1 + f_2/F_2} = \frac{4}{5}v.$$

Hence, the equilibrium bids are $b_i(v) \approx \tilde{b}(v) = v - (4/5)v/n$. To compare these approximations with the exact bids, we solve numerically the following system of equations

$$\begin{aligned} v_1'(b) &= \frac{v_1(b)}{1/2}\left[\frac{1}{2(n-1)}\left(\frac{2-n}{v_1(b)-b} + \frac{n}{v_2(b)-b}\right)\right], \\ v_2'(b) &= \frac{v_2(b)}{2}\left[\frac{1}{2(n-1)}\left(\frac{n}{v_1(b)-b} + \frac{2-n}{v_2(b)-b}\right)\right], \end{aligned}$$

subject to (9,10). The exact and approximate equilibrium bids are shown in Figure 1 for $n = 2$, 4, and 6 bidders. Since the asymmetry among the distribution functions is not small, in the case of two players the equilibrium bids are clearly asymmetric. However, the asymmetry in the equilibrium bids vanishes quickly as n increases. Indeed, already for six bidders the equilibrium bids are nearly indistinguishable from each other, as well as from the asymptotic approximation.

We can also use Corollary 2 to obtain an asymptotic approximation for the maximal bid. Since $u(1) = \frac{4}{5}$, the approximate maximal bid is given by $\bar{b}_{\text{approx}} = 1 - u(1)/n = 1 - 4/5n$. In Table 1 we compare the exact maximal bid $\bar{b}$ with its asymptotic approximation $\bar{b}_{\text{approx}}$. This comparison shows that the accuracy of the asymptotic approximation is good when n is relatively small or not very large. For example, for $n = 6$ bidders, the accuracy is 2%.

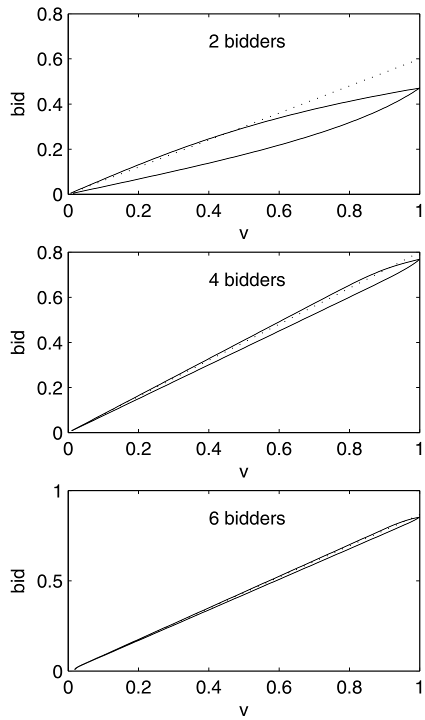

Figure 1: Exact (solid) and approximate (dots) equilibrium bids for 2,4, and 6 bidders.

In order to see whether the $1/n^2$ convergence rate of the asymptotic approximation for the equilibrium bids is already valid at these small values of n, we plot in Figure 2 the approximation error $\bar{b} - \bar{b}_{\text{approx}}$ as a function of n on a loglog scale. The slope of the best-fitting least-square line going through these points is -2.05. Clearly, if the approximation error would have been exactly a $constant/n^2$, the resulting loglog curve would have been a straight line with slope -2. Thus, we conclude that the $1/n^2$ convergence rate is already reached at these small values of n.

Finally, from Proposition 6, the seller's expected revenue $R^{1\text{st}}$ can be approximated with $1 - \frac{2}{n}u(1) = 1 - \frac{8}{5n}$. Comparison of the exact value[11] with its asymptotic approximation

[11] The exact value of $R^{1\text{st}}$ was calculated numerically. See Fibich and Gavious (2003) for details on the

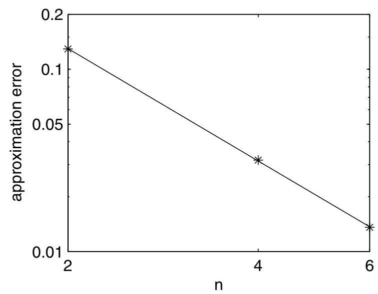

Figure 2: Difference between the approximate and exact equilibrium bids as a function of n (*), plotted on a loglog scale. Slope of best fitting line (solid) is -2.05.

number of bidders (n)	exact maximal bid ($\bar{b}$)	approximate maximal bid ($\bar{b}_{\text{approx}}$)
2	0.47	0.60
4	0.77	0.80
6	0.85	0.87

Table 1: Exact and approximate values of the maximal bid in a first-price auction.

(Table 2) shows that the approximation error is 2% for $n = 6$ players, and that the approximation error also scales as a $constant/n^2$ already at these small values of n.

3.2 Second-price auctions

Consider a large number ($n \gg 1$) of asymmetric bidders with distribution functions $F_1, \ldots, F_n$ who are competing for a single object in a second-price auction.

numerical method.

number of bidders (n)	exact revenue ($R^{1\text{st}}$)	approximate revenue $\left(1 - \frac{2u(1)}{n}\right)$
2	0.3117	0.2000
4	0.6369	0.6000
6	0.7518	0.7333

Table 2: Exact and approximate values of the seller's expected revenue in a first-price auction.

In a second price auction with either symmetric or asymmetric bidders, the equilibrium bid functions are $b(v) = v$. The seller's expected revenue is given by (see, e.g., Fibich and Gavious (2003))

$$R^{2\text{nd}}[F_1, \ldots, F_n] = 1 - \int_0^1 \prod_{i=1}^{n} F_i(v)\, dv - \sum_{i=1}^{n} \int_0^1 (1 - F_i(v)) \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(v)\, dv. \qquad (15)$$

Expansion of $R^{2\text{nd}}$ in $1/n$ leads to the following result:

Proposition 7 *In an asymmetric large second-price auction with n bidders, the seller's expected revenue for a sufficiently large n is*

$$R^{2\text{nd}}[F_1, \ldots, F_n] = 1 - \frac{2}{n} u(1) + O\left(\frac{1}{n^2}\right), \qquad (16)$$

where $u(1)$ is given by (13).

Proof. See Appendix F.

Cantillon (2003) showed that the revenue in asymmetric second-price auctions is always smaller than in a symmetric second-price auctions with the same number of players whose (symmetric) distribution function F_{sym} is the geometric average of the asymmetric distribution functions, i.e., $F_{\text{sym}} = (\prod_{j=1}^{n} F_j)^{1/n}$. Since $u(1)$ is the same in both cases, Proposition 7 shows that the revenue difference between the two is only $O(1/n^2)$.

Example 2 *We consider a large asymmetric second-price auction with the same distribution functions as in Example 1.*

Since $u(1) = 4/5$ (see Example 1), the asymptotic approximation (16) yields

$$R^{\text{2nd}} = 1 - \frac{8}{5n} + O\left(\frac{1}{n^2}\right). \tag{17}$$

By (15), the exact value of the seller's expected revenue is given by

$$R^{2nd} = 1 + \frac{n-1}{1.25n+1} - \frac{n/2}{1.25n-1} - \frac{n/2}{1.25n+0.5}.$$

Expanding this in $1/n$ gives

$$R^{2nd} = 1 - \frac{8}{5n} + \frac{104}{125n^2} + \ldots,$$

which is in agreement with (17).

3.3 Asymptotic revenue equivalence

A comparison of the seller's expected revenue in an asymmetric first-price auction (14) and in an asymmetric second-price auction (16) shows that these auctions are $O(1/n^2)$

revenue equivalent. In the following we show that this asymptotic revenue equivalence can be generalized to a wide class of asymmetric auctions.[12]

Theorem 2 *Consider any auction mechanism that satisfies the following assumptions:*

1. *All n players are risk neutral.*

2. *Player i's valuation is private information to i and is drawn independently by a continuously differentiable distribution function $F_i(v)$ from a support $[0,1]$ which is common to all players.*

3. *The object is allocated to the player with the highest bid.*[13]

4. *Any player with valuation 0 expects zero surplus.*

5. *In equilibrium, any player with valuation 1 has the same maximal bid.*

Then, the seller's expected revenue for a sufficiently n is given by

$$R[F_1, \ldots, F_n] = 1 - \frac{2}{n} u(1) + O\left(\frac{1}{n^2}\right), \tag{18}$$

where $u(1)$ is given by (13).

[12]Fibich, Gavious and Sela (2003) showed that weakly-asymmetric auctions are asymptotically revenue equivalent, by expanding the revenue in the small asymmetry parameter. Unlike that study, here we do not assume that the asymmetry among players is weak.

[13]In the symmetric case, assumption 3 is equivalent to the assumption that in equilibrium the object is allocated to the player with the highest valuation. This equivalence, however, does not hold in the asymmetric case since asymmetric auctions are not necessarily efficient.

Proof. See Appendix G.

Notice that in the case of symmetric auctions, Theorem 2 reduces to Theorem 1.

In order to get a feeling for the level of revenue equivalence when the number of players is not really large, in Table 3 we compare the expected revenue in asymmetric first-price and second-price auction for 6 bidders with various distributions. Since in all cases the differences between R^{1st} and R^{2nd} are only in the fourth or fifth digit, this suggests (again) that the results of the asymptotic analysis are already valid for $n \approx 6$. In fact, these results suggest that, for all practical purposes, first-price and second-price auction with $n \geq 6$ bidders are revenue equivalent.

distributions	R^{1st}	R^{2nd}	$R^{1st} - R^{2nd}$
$F_1 = F_2 = F_3 = F_4 = v^4$, $F_5 = F_6 = v^{1/2}$	0.877782	0.877778	0.000004
$F_1 = F_2 = F_3 = v^4$, $F_4 = F_5 = F_6 = v^{1/2}$	0.84487	0.84483	0.00004
$F_1 = F_2 = F_3 = v^2$, $F_4 = F_5 = F_6 = v^{1/2}$	0.751773	0.751697	0.00008
$F_i = v^i, \ i = 1, \ldots, 6$	0.900224	0.900143	0.000081

Table 3: Expected revenue $R^{1st}[F_1, \ldots, F_6]$ and $R^{2nd}[F_1, \ldots, F_6]$.

3.4 Collusion in large auctions

Our results for asymmetric auctions can be applied to analyze the effect of collusion in large auctions. Consider a large symmetric first-price auction with $n \gg 1$ bidders, where the value of the object for each bidder is independently drawn according to a distribution

$F(v)$ defined on the interval $[0,1]$. Assume that m bidders compete as single players, whereas the rest k, $k = n - m$ bidders set up a coalition and act as a single bidder whose value is the maximum over the values of the bidders participating in the coalition. As such, the coalition's value is distributed according to $F^k(v)$ over the support $[0,1]$.

When the coalition is small, i.e., $k = O(1)$ and $n \gg k$, we can apply Proposition 5 to calculate the equilibrium bids. Since there are effectively m players with the distribution function F and a single player with the distribution function F^k, the equilibrium bids, both for the individual bidders and for the coalition, are

$$b_i(v) = v - \frac{1}{m+1}\frac{m+1}{m+k}\frac{F(v)}{f(v)} + O\left(\frac{1}{(m+1)^2}\right) = v - \frac{1}{n}\frac{F(v)}{f(v)} + O\left(\frac{1}{n^2}\right).$$

Note that the $O(1/n)$ correction term turns out to be independent of the coalition size k. In addition, by Proposition 6, the expected revenue is

$$R[F] = 1 - \frac{2}{n}\frac{1}{f(1)} + O\left(\frac{1}{n^2}\right).$$

We have thus shown that collusion of players in large first-price auctions has a negligible effect on the equilibrium bids as well as on the seller's revenue. In particular, the coalition size is almost meaningless. Furthermore, by Theorem (2) we can conclude that in all large auctions, collusion has a negligible effect on revenue.

4 Concluding remarks

Fibich, Gavious and Sela (2003) showed that for weakly-asymmetric auction, if ϵ is the small asymmetry parameter, all standard auctions are $O(\epsilon^2)$ revenue equivalent. Hence, the revenue difference among weakly-asymmetric auctions is negligible. In this study we do not assume that the asymmetry is weak, yet we find that the revenue difference among large asymmetric auctions is also negligible. Together, these two studies suggest that in most cases asymmetry among bidders valuations plays a minor role in revenue ranking of auctions. A similar conclusion, however, cannot be applied to risk-aversion. Indeed, Fibich, Gavious and Sela (2002) showed that for symmetric auction with weakly-risk averse bidders, if ϵ is a small risk-aversion parameter, risk aversion has an $O(\epsilon)$ effect on the equilibrium bids, and then the revenue difference among auctions is $O(\epsilon)$. Therefore, risk-aversion can play an important role in revenue ranking of small auctions, but not in large ones.

If we try to generalize the results of this research, we can identify several unifying themes: 1) Auctions with a large number of bidders are considerably simpler to analyze than auctions with a small number of bidders, since the effects of various "complications" such as asymmetry, risk-aversion, or collusion are negligible. 2) The leading-order deviation of the revenue from the limiting value depends only on the value of the density function(s) at the maximal value. 3) Auctions with as few as six bidders can be considered as large auctions. Naturally, further research is required to test the validity of these general statements for cases which are not considered in this study.

A Proof of Lemma 1

In the following we calculate an asymptotic expansion of the integral $\int_0^v F^n(x)\,dx$ using integration by parts (for an introduction to asymptotic calculation of integrals using integration by parts, see, e.g., Murray (1984)).

$$\int_0^v F^n(x)\,dx = \int_0^v [F^n(x)f(x)]\frac{1}{f(x)}\,dx = \frac{1}{n+1}\frac{F^{n+1}(v)}{f(v)} + \frac{1}{n+1}\int_0^v F^{n+1}(x)f(x)\frac{f'(x)}{f^3(x)}\,dx.$$

A similar integration by parts shows that $\int_0^v F^{n+1}(x)f(x)\frac{f'(x)}{f^3(x)}\,dx = O(1/n)$. Therefore,

$$\int_0^v F^n(x)\,dx = \frac{1}{n+1}\frac{F^{n+1}(v)}{f(v)} + O\left(\frac{1}{n^2}\right).$$

B Proof of Proposition 5

Since $\lim_{n\to\infty} v_i(b) = b$, the equilibrium bids for $n \gg 1$ can be expanded as

$$v_i(b) = b + \frac{1}{n}u_i(b) + \frac{1}{n^2}w_i(b) + O(\frac{1}{n^3}), \quad i = 1, \dots, n. \tag{19}$$

Substitution of (19) in (8) gives

$$1 + O\left(\frac{1}{n}\right) = \tag{20}$$
$$\frac{F_i(b + \frac{1}{n}u_i(b) + \frac{1}{n^2}w_i(b))}{f_i(b + \frac{1}{n}u_i(b) + \frac{1}{n^2}w_i(b))}\left[\frac{n}{n-1}\sum_{j=1}^{n}\frac{1}{u_j(b) + \frac{1}{n}w_j(b)} - \frac{n}{u_i(b) + \frac{1}{n}w_i(b)}\right] + O\left(\frac{1}{n}\right).$$

A-priori, to leading order the left-hand-side is equal to 1 whereas the right-hand-side is $O(n)$. Therefore, we first impose the condition that to leading order of the right side is also equal to 1. This implies that

$$\frac{n}{n-1}\sum_{j=1}^{n}\frac{1}{u_j(b)} - \frac{n}{u_i(b)} = O(1), \qquad i = 1, \dots, n,$$

or

$$\frac{1}{u_i(b)} = \frac{1}{\tilde{U}(b)} + O(\frac{1}{n}), \qquad \tilde{U}(b) = \frac{n-1}{\sum\limits_{j=1}^{n} \frac{1}{u_j(b)}}.$$

Thus, $u_i(b) = \tilde{U}(b) + O(1/n)$ for all i. Since the $O(1/n)$ difference among $\{u_i\}_{i=1}^n$ translates into $O(1/n^2)$ difference among $\{v_i\}_{i=1}^n$, we can set

$$u_i(b) = u(b), \quad i = 1, \ldots, n,$$

i.e., absorb the $O(1/n)$ difference among the $\{u_i\}_{i=1}^n$ into the $\{w_i\}_{i=1}^n$ terms.

Substituting $u_i(b) = u(b)$ and equating the $O(1)$ terms gives

$$1 = \frac{F_i(b)}{f_i(b)} \left[\frac{n}{n-1} \sum_{j=1}^{n} \frac{1}{u(b) + \frac{1}{n} w_j(b)} - \frac{n}{u(b) + \frac{1}{n} w_i(b)} \right] + O(1/n), \qquad i = 1, \ldots, n.$$

Therefore,

$$\frac{f_i(b)}{F_i(b)} = \frac{n}{n-1} \frac{1}{u} + \frac{w_i(b)}{u^2} - \frac{1}{n-1} \sum_{j=1}^{n} \frac{w_j(b)}{u^2} + O(1/n), \qquad i = 1, \ldots, n. \tag{21}$$

Summing (21) over i gives

$$\sum_{i=1}^{n} \frac{f_i(b)}{F_i(b)} = \frac{n^2}{(n-1)u} + \frac{1}{u^2} \sum_{i=1}^{n} w_i(b) - \frac{n}{n-1} \frac{1}{u^2} \sum_{j=1}^{n} w_j(b) = \frac{n^2}{(n-1)u} - \frac{1}{n-1} \frac{1}{u^2} \sum_{j=1}^{n} w_j(b) + O(1).$$

Hence,

$$\frac{1}{n} \sum_{i=1}^{n} \frac{f_i(b)}{F_i(b)} = \frac{1}{u} + O(1/n).$$

We thus proved that the inverse equilibrium bids are given by (19), where u is defined in (12). To invert this relation, we substitute in the identity $v_i \equiv v_i(b_i(v_i))$ the relations $b_i(v) = v + (1/n)B_i(v) + O(1/n^2)$ and $v_i(b) = b + (1/n)u(b) + O(1/n^2)$, to get that $B_i(v) = B(v) = -u(v)$.

C Proof of Proposition 6

The seller's expected revenue in asymmetric first-price auctions can be written as (see, e.g., Fibich and Gavious (2003))

$$R^{1\text{st}}[F_1, \ldots, F_n] = \bar{b} - \int_0^{\bar{b}} \prod_{j=1}^{n} F_j(v_j(b))\, db.$$

In Corollary 2 we showed that $\bar{b} = 1 - u(1)/n + O(1/n^2)$. In addition,

$$\int_0^{\bar{b}} \prod_{j=1}^{n} F_j(v_j(b))\, db = \int_0^{\bar{b}} H^n(b)\, db, \qquad H(b) = \left(\prod_{j=1}^{n} F_j(v_j(b)) \right)^{1/n}.$$

Therefore, by Lemma 1,

$$\int_0^{\bar{b}} \prod_{j=1}^{n} F_j(v_j(b))\, db = \frac{1}{n+1} \frac{H^{n+1}(\bar{b})}{H'(\bar{b})} + O\left(\frac{1}{n^2}\right).$$

It is easy to verify that $H(\bar{b}) = 1$ and $H'(\bar{b}) = \frac{1}{n} \sum_{j=1}^{n} f_j(1)$. Summarizing the above gives

$$\int_0^{\bar{b}} \prod_{j=1}^{n} F_j(v_j(b))\, db = \frac{n}{n+1} \frac{1}{\sum_{j=1}^{n} f_j(1)} + O\left(\frac{1}{n^2}\right) = \frac{1}{\sum_{j=1}^{n} f_j(1)} + O\left(\frac{1}{n^2}\right).$$

Therefore, the result follows.

D Proof of Proposition 3

The method of this proof is similar to that of Proposition 5. The inverse equilibrium bids satisfy

$$v'(b) = \frac{1}{n-1} \frac{F(v(b))}{f(v(b))} \frac{U'(v(b) - b)}{U(v(b) - b)}. \tag{22}$$

Since $\lim_{n \to \infty} v(b) = b$, we can look for a solution of the form

$$v(b) = b + \frac{1}{n-1} v_1(b) + O\left(\frac{1}{n^2}\right).$$

Substitution in (22) gives

$$\begin{aligned}1 &+ O\left(\frac{1}{n}\right)\\ &= \frac{1}{n-1}\frac{F(b)+(v_1/(n-1))f(b)+O(n^{-2})}{f(b)+(v_1/(n-1))f'(b)+O(n^{-2})}\cdot\frac{U'(0)+(v_1/(n-1))U''(0)+O(n^{-2})}{U(0)+(v_1/(n-1))U'(0)+O(n^{-2})}.\end{aligned}$$

Since $U(0)=0$ and $U'(0)>0$, the balance of the leading order terms gives

$$1=\frac{F(b)}{f(b)}\cdot\frac{U'(0)}{v_1U'(0)}.$$

Therefore, $v_1(b)=F(b)/f(b)$ and the inverse equilibrium bids are given by

$$v(b)=b+\frac{1}{n-1}\frac{F(b)}{f(b)}+O\left(\frac{1}{n^2}\right).$$

Inverting this relation (see end of Appendix B) shows that the equilibrium bids are given by (4). The calculation of the expected revenue is identical to the one in the proof of Proposition 2.

E Proof of Proposition 4

The case $k=1$ was proved in Proposition 3. When $k=2$ the result follows since $b(v)=v$. Therefore, we only need to prove for $k\geq 3$. In that case, the equilibrium strategies in k-price auctions are the solutions of (see Monderer and Tennenholtz (2000))

$$\int_0^v U(v-b(t))F^{n-k}(t)(F(v)-F(t))^{k-3}f(t)\,dt=0. \tag{23}$$

Defining $m = n - k$ and $t = v - s$, we can rewrite eq. (23) as

$$\begin{aligned} 0 &= \int_0^v U(v-b(t))F^m(t)(F(v)-F(t))^{k-3} f(t)\,dt \qquad (24) \\ &= \int_0^v e^{m\ln(F(t))} U(v-b(t))(F(v)-F(t))^{k-3} f(t)\,dt \\ &= e^{m\ln F(v)} \int_0^v e^{-m[\ln F(v)-\ln F(v-s)]}\, U(v-b(v-s))(F(v)-F(v-s))^{k-3} f(v-s)\,ds. \end{aligned}$$

Since the maximum of $\ln(F(v-s))$ is attained at $s = 0$, we can calculate an asymptotic approximation of this integral using Laplace method (see, e.g., Murray (1984)). To do that, we make the change of variables $x(s) = [\ln F(v) - \ln F(v-s)]$ and expand all the terms in the last integral in a Taylor series in s near $s = 0$.

Expansion of $x(s)$ near $s = 0$ gives $x = sf(v)/F(v) + O(s^2)$. Therefore,

$$\frac{dx}{ds} = f(v)/F(v) + O(s), \qquad s = x\frac{F(v)}{f(v)} + O(s^2), \qquad ds = \frac{dx}{f(v)/F(v)}\,[1 + O(x)]$$

Let us expand the solution $b(v)$ in a power series in m, i.e.,

$$b(v) = b_0(v) + \frac{1}{m} b_1(v) + O\left(\frac{1}{m^2}\right).$$

Therefore, near $s = 0$,

$$b(v-s) = b_0(v) - sb_0'(v) + \frac{1}{m} b_1(v) - \frac{1}{m} sb_1'(v) + O(s^2) + O\left(\frac{1}{m^2}\right).$$

In addition,

$$(F(v) - F(v-s))^{k-3} = (sf(v) + O(s^2))^{k-3} = s^{k-3} f^{k-3}(v)\,[1 + O(s)],$$

and

$$f(v-s) = f(v) + O(s).$$

Substitution all the above in (24) gives

$$\begin{aligned} 0 &= \int_0^v \left\{ e^{-mx}\, U\left[v - \left(b_0(v) - s b_0'(v) + \frac{b_1(v)}{m} - \frac{s b_1'(v)}{m} + O(s^2) + O\left(\frac{1}{m^2}\right)\right)\right] \times \right. \\ &\qquad \left. s^{k-3} f^{k-3}(v)\,[1+O(s)]\,[f(v)+O(s)] \right\} ds \\ &\sim \int_0^\infty \left\{ e^{-mx}\, U\left[v - \left(b_0(v) - x\frac{F(v)}{f(v)} b_0'(v) + \frac{1}{m} b_1(v) - \frac{x}{m}\frac{F(v)}{f(v)} b_1'(v) + O(x^2) + O\left(\frac{1}{m^2}\right)\right)\right] \times \right. \\ &\qquad \left. x^{k-3} F^{k-3}(v)\,[1+O(x)]\,[f(v)+O(x)]\,\frac{dx}{f(v)/F(v)}\,[1+O(x)] \right\} \\ &= F^{k-2}(v) \int_0^\infty \left\{ e^{-mx} \left[U(v-b_0(v)) + U'(v-b_0(v)) \left(x\frac{F(v)}{f(v)} b_0'(v) - \frac{b_1(v)}{m} + \frac{x}{m}\frac{F(v)}{f(v)} b_1'(v)\right) \right.\right. \\ &\qquad \left.\left. + O(x^2) + O\left(\frac{1}{m^2}\right)\right] x^{k-3}\,[1+O(x)] \right\} dx. \end{aligned} \tag{25}$$

We recall that for p integer, $\int_0^\infty e^{-mx} x^p\, dx = p!/m^{p+1}$. Therefore, balancing the leading $O(m^{-(k-2)})$ terms gives

$$U(v-b_0(v))F^{k-2}(v) \int_0^\infty e^{-mx}\, x^{k-3}\, dx = 0.$$

Since $U(z) = 0$ only at $z = 0$, this implies that $b_0(v) \equiv v$. Using this and $U'(0) = 0$, eq. (25) reduces to

$$0 = \int_0^\infty \left\{ e^{-mx} \left(x\frac{F(v)}{f(v)} - \frac{1}{m} b_1(v) + \frac{x}{m}\frac{F(v)}{f(v)} b_1'(v) \right) \left[x^{k-3} + O(x^{k-2}) \right] \right\} dx$$

Therefore, balance of the next-order $O(m^{-(k-1)})$ terms gives

$$\frac{F(v)}{f(v)} \int_0^\infty e^{-mx} x^{k-2}\, dx - \frac{1}{m} b_1(v) \int_0^\infty e^{-mx} x^{k-3}\, dx = 0,$$

or

$$\frac{F(v)}{f(v)} \frac{(k-2)!}{m^{k-1}} - \frac{(k-3)!}{m^{k-1}} b_1(v) = 0.$$

Therefore,

$$b_1(v) = (k-2)\frac{F(v)}{f(v)}.$$

Hence, we proved (7).

The seller's expected revenue in a k-price auction is given by

$$R_k = \int_0^1 b(v) dF_k(v),$$

where $b(v)$ is the equilibrium bid in the k price auction and $F_k(v)$ is the distribution of the k-th valuation in order (i.e., k-order statistic of the bidders private valuations). Substituting the asymptotic expansion for the equilibrium bids gives

$$R_k = \int_0^1 \left[v + \frac{k-2}{n-k}\frac{F(v)}{f(v)} \right] dF_k(v) + O(\frac{1}{n^2}).$$

Since the asymptotic expansion for the equilibrium bid is independent of the utility function U until order $O(\frac{1}{n^2})$ and since $dF_k(v) = O(1)$, the revenue in the risk-averse case is the same as in the risk-neutral case, with $O(\frac{1}{n^2})$ accuracy. By Theorem 1, the latter is given by (6).

F Proof of Proposition 7

From (15) it follows that

$$R^{\text{2nd}}[F_1, \ldots, F_n] = 1 - \int_0^1 \prod_{i=1}^n F_i(v)\, dv - \sum_{i=1}^n I_i,$$

where

$$I_i = \int_0^1 (1 - F_i(v)) H_i^{n-1}(v)\, dv, \qquad H_i = \left(\prod_{\substack{j=1 \\ j \neq i}}^n F_j \right)^{1/(n-1)}.$$

Calculations similar to those in Appendix C show that

$$\int_0^1 \prod_{i=1}^{n} F_i(v)\,dv = \frac{1}{\sum_{j=1}^{n} f_j(1)} + O\left(\frac{1}{n^2}\right).$$

Similarly, since $H_i'(0) = H_i''(0) = 0$, two integration by parts give that

$$I_i = -\frac{1}{n}\int_0^1 H_i^n \left(\frac{1-F_i}{H_i'}\right)' dv = \frac{1}{n(n+1)}\frac{f_i(1)}{[H_i'(1)]^2} + O\left(\frac{1}{n^3}\right).$$

Since

$$H_i'(1) = \frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} f_j(1) = \frac{1}{n}\sum_{j=1}^{n} f_j(1) + O\left(\frac{1}{n}\right),$$

it follows that

$$I_i = \frac{f_i(1)}{[\sum_{j=1}^{n} f_j(1)]^2} + O\left(\frac{1}{n^3}\right),$$

and

$$\sum_{i=1}^{n} I_i = \frac{1}{\sum_{j=1}^{n} f_j(1)} + O\left(\frac{1}{n^2}\right).$$

Therefore, we obtain that

$$R^{\text{2nd}}[F_1,\ldots,F_n] = 1 - \frac{2}{\sum_{j=1}^{n} f_j(1)} + O\left(\frac{1}{n^2}\right),$$

which concludes the proof.

G Proof of Theorem 2

Let $E_i(v)$, $S_i(v)$ and $P_i(v)$ be the expected payment, the expected surplus, and the probability of winning for bidder i with type v at equilibrium, respectively. Therefore,

$$S_i = vP_i(v) - E_i(v)\ . \tag{26}$$

It is well known (see, e.g., Krishna (2002)) that

$$\frac{dS_i}{dv} = P_i(v) \ . \tag{27}$$

From (26),(27) it follows that

$$E_i'(v) = vP_i'(v) \ . \tag{28}$$

Let R_i be the expected payments of player i averaged across her types. Then,

$$\begin{aligned} R_i &= \int_{\underline{v}}^{\bar{v}} E_i(v)F_i'(v)\,dv = E_i(v)F_i\Big|_{\underline{v}}^{\bar{v}} - \int_{\underline{v}}^{\bar{v}} E_i'(v)F_i(v)\,dv \\ &= E_i(\bar{v}) - \int_{\underline{v}}^{\bar{v}} vP_i'(v)F_i(v)\,dv \ . \end{aligned}$$

Since $S_i(\underline{v}) = 0$ (Condition 4) we have from (28) that $E_i(\bar{v}) = \int_{\underline{v}}^{\bar{v}} vP_i'(v)\,dv$. Hence,

$$R_i = \int_{\underline{v}}^{\bar{v}} vP_i'(v)(1 - F_i(v))\,dv = -\int_{\underline{v}}^{\bar{v}} P_i(v)[v(1 - F_i(v))]'\,dv \ .$$

The seller's expected revenue is thus given by

$$R = \sum_{i=1}^{n} R_i = -\sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} P_i(v)[v(1 - F_i(v))]'\,dv \ . \tag{29}$$

Since in equilibrium

$$P_i(v) = P\left(b_i(v) > \max_{j\neq i} b_j\right) = \prod_{\substack{j=1 \\ j\neq i}}^{n} F_j(b_j^{-1}(b_i(v)) \ = \frac{1}{F_i(v)} \prod_{j=1}^{n} F_j(b_j^{-1}(b_i(v)),$$

we have

$$R = -\sum_{i=1}^{n} \int_0^1 \left(\prod_{j=1}^{n} F_j(b_j^{-1}(b_i(v))\right) \frac{1 - F_i(v) - vf_i(v)}{F_i(v)}\,dv.$$

In order to approximate R for large n, we use an asymptotic method known as Laplace method for integrals (see, e.g., Murray (1984)). We first define

$$G_i(v) = \left(\prod_{j=1}^{n} F_j(b_j^{-1}(b_i(v)) \right)^{1/n}, \quad g_i(v) = G_i'(v),$$

and observe that $G_i(1) = 1$ (Assumption 5). Then, we have

$$\begin{aligned} R &= -\sum_{i=1}^{n} \int_0^1 e^{n \ln G_i(v)} \cdot \left(\frac{1 - F_i(v) - v f_i(v)}{F_i(v)} \right) dv = \\ &= -\sum_{i=1}^{n} \int_0^1 e^{n \ln G_i(1-s)} \cdot \left(\frac{1 - F_i(1-s) - (1-s) f_i(1-s)}{F_i(1-s)} \right) ds, \end{aligned}$$

where the last equality is obtained by the change of variables $s = 1 - v$. Defining $u_i(s) = -\ln G_i(1-s)$, we have that

$$\begin{aligned} u_i &= s g_i(1) + 1/2 s^2 \left(\frac{g_i(1-s)}{G_i(1-s)} \right)'_{s=0} + O(s^3), \\ \frac{du_i}{ds} &= g_i(1) + s \left(\frac{g_i(1-s)}{G_i(1-s)} \right)'_{s=0} + O(s^2). \end{aligned}$$

Hence,

$$\begin{aligned} ds &= \frac{du_i}{g_i(1) + s \left(\frac{g_i(1-s)}{G_i(1-s)} \right)'_{s=0} + O(s^2)} = \frac{du_i}{g_i(1)} \left[1 - s \frac{1}{g_i(1)} \left(\frac{g_i(1-s)}{G_i(1-s)} \right)'_{s=0} + O(s^2) \right] \\ &= \frac{du_i}{g_i(1)} \left[1 - u_i \frac{1}{g_i^2(1)} \left(\frac{g_i(1-s)}{G_i(1-s)} \right)'_{s=0} + O(u_i^2) \right]. \end{aligned}$$

In addition,

$$\begin{aligned} \frac{1 - F_i(1-s) - (1-s) f_i(1-s)}{F_i(1-s)} &= -f_i(1) - s(f_i^2(1) - f_i'(1) - 2 f_i(1)) + O(s^2) \\ &= -f_i(1) - \frac{u_i}{g_i(1)} (f_i^2(1) - f_i'(1) - 2 f_i(1)) + O(u_i^2). \end{aligned}$$

Summarizing the above gives that

$$
\begin{aligned}
R &= \int_0^1 \sum_{i=1}^n \left\{ e^{-nu_i} \cdot \left(f_i(1) + \frac{u_i}{g_i(1)} (f_i^2(1) - f_i'(1) - 2f_i(1)) \right) \times \right. \\
&\qquad \left. \frac{1}{g_i(1)} \left[1 - u_i \frac{1}{g_i^2(1)} \left(\frac{g_i(1-s)}{G_i(1-s)} \right)'_{s=0} \right] + O(u_i^2) \right\} du_i
\end{aligned}
$$

Since u_i is a dummy variable, we can rename $u_i = u$. Therefore,

$$
\begin{aligned}
R &= \sum_{i=1}^n \int_0^1 \left\{ e^{-nu} \cdot \left(f_i(1) + \frac{u}{g_i(1)} (f_i^2(1) - f_i'(1) - 2f_i(1)) \right) \times \right. \\
&\qquad \left. \frac{1}{g_i(1)} \left[1 - u \frac{1}{g_i^2(1)} \left(\frac{g_i(1-s)}{G_i(1-s)} \right)'_{s=0} \right] + O(u^2) \right\} du \\
&\sim \sum_{i=1}^n \frac{1}{g_i(1)} \int_0^\infty e^{-nu} \cdot \\
&\quad \left(f_i(1) + u \left[\frac{1}{g_i(1)} (f_i^2(1) - f_i'(1) - 2f_i(1)) - \frac{f_i(1)}{g_i^2(1)} \left(\frac{g_i(1-s)}{G_i(1-s)} \right)'_{s=0} \right] + O(u^2) \right) du \\
&= \sum_{i=1}^n \frac{f_i(1)}{g_i(1)} \int_0^\infty e^{-nu} du + \\
&\quad \sum_{i=1}^n \frac{1}{g_i(1)} \left[\frac{1}{g_i(1)} (f_i^2(1) - f_i'(1) - 2f_i(1)) - \frac{f_i(1)}{g_i^2(1)} \left(\frac{g_i(1-s)}{G_i(1-s)} \right)'_{s=0} \right] \int_0^\infty e^{-nu} u \, du \\
&\quad + O(n^{-2}).
\end{aligned}
$$

We thus see that

$$
\begin{aligned}
R &= \frac{1}{n} \sum_{i=1}^n \frac{f_i(1)}{g_i(1)} \\
&\quad + \frac{1}{n^2} \sum_{i=1}^n \frac{1}{g_i^2(1)} \left[(f_i^2(1) - f_i'(1) - 2f_i(1)) + \frac{f_i(1)}{g_i(1)} \left(\frac{g_i(v)}{G_i(v)} \right)'_{v=1} \right] + O(n^{-2}).
\end{aligned}
\tag{30}
$$

We now calculate $g_i(1)$. Since $G_i(1) = 1$,

$$\begin{aligned} g_i(1) &= (\ln G_i(v))'_{v=1} = \frac{1}{n}\frac{d}{dv}\left(\sum_{j=1}^{n} \ln\left[F_j(b_j^{-1}(b_i(v))\,\right]\right)_{v=1} \\ &= \frac{1}{n}\sum_{j=1}^{n} \frac{f_j(b_j^{-1}(b_i(1))}{F_j(b_j^{-1}(b_i(1))}\left(b_j^{-1}(b_i(v)\right)'_{v=1} \\ &= \frac{1}{n}\sum_{j=1}^{n} f_j(1)\left(b_j^{-1}(b_i(v)\right)'_{v=1} = \frac{b_i'(1)}{n}\sum_{j=1}^{n}\frac{f_j(1)}{b_j'(1)}. \end{aligned}$$

Therefore,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{f_i(1)}{g_i(1)} = 1.$$

Substituting the above in (30) yields

$$R = 1 + \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{g_i^2(1)}\left[(f_i^2(1) - f_i'(1) - 2f_i(1)) + f_i(1)\frac{1}{g_i(1)}\left(\frac{g_i(v)}{G_i(v)}\right)'_{v=1}\right] + O\left(\frac{1}{n^2}\right). \tag{31}$$

Let us expand $b_i(v) = v + O(1/n)$. Inverting this relation gives

$$b_i^{-1}(b) = v_i(b) = b + O(1/n).$$

Hence, substituting $b_i(v) = v + O(1/n)$ in $b_j^{-1}(b) = b + O(1/n)$ gives

$$b_j^{-1}(b_i(v)) = v + O(1/n),$$

$$F_j(b_j^{-1}(b_i(v))) = F_j(v) + O(1/n),$$

$$G_i = \prod_{j=1}^{n}\left[F_j(v) + O(1/n)\right]^{1/n},$$

$$\ln G_i = \frac{1}{n}\sum_{j=1}^{n}\ln\Big[F_j(v) + O(1/n)\Big] = \frac{1}{n}\sum_{j=1}^{n}\ln F_j(v) + O(1/n),$$

$$(\ln G_i)' = \frac{1}{n}\sum_{j=1}^{n} \frac{f_j(v)}{F_j(v)} + O(1/n),$$

and

$$(\ln G_i)'' = \frac{1}{n}\sum_{j=1}^{n} \frac{f_j'(v)}{F_j(v)} - \frac{1}{n}\sum_{j=1}^{n} \frac{f_j^2(v)}{F_j^2(v)} + O(1/n).$$

Therefore,

$$g_i(1) = (\ln G_i)'_{v=1} = \frac{1}{n}\sum_{j=1}^{n} f_j(1) + O(1/n),$$

and

$$\left(\frac{g_i}{G_i}\right)'_{v=1} = (\ln G_i)''_{v=1} = \frac{1}{n}\sum_{j=1}^{n} f_j'(1) - \frac{1}{n}\sum_{j=1}^{n} f_j^2(1) + O(1/n).$$

Substitution of the last two relations in (31) gives (18).

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