Competition for Staff between Two Departments

V.J.Baston
School of Mathematics
University of Southampton
Highfield, Southampton SO17 1BJ, United Kingdom
e-mail: V.J.D.Baston@maths.soton.ac.uk

A.Y.Garnaev *
Department of Computer Modelling and Multiprocessor Systems
Faculty of Applied Mathematics and Control Processes
St Petersburg State University
Universitetskii prospekt 35, Peterhof,
St Petersburg 198504, Russia
e-mail: garnaev@ag2784.spb.edu

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Abstract

In this paper the following scenario is analyzed from a game-theoretical point of view. Two departments in a large organization are each seeking to make an appointment within the same area of expertise, for instance, a computer science specialist. To avoid duplication it has been decided that the heads of the two departments should together interview the applicants in turn and make their decisions on one applicant before interviewing any others. If a candidate is rejected by both departmental heads, the candidate cannot be considered for either post at a later date. If both heads decide to make an offer two cases are considered: (a) the departments are equally attractive so that an applicant has no preference between them (b) one department can offer better prospects to applicants who will always choose that department. The departmental heads know that there are precisely n applicants and that each applicant has an expertise which is random over a known range. If no appointment is made to a department from these n applicants, then the department will suffer from a shortfall of expertise. In the paper it will be shown that that the games (a) and (b) have very different characteristics. The game (b) is straightforward to analyze because it has just one Nash equilibrium. On the other hand, game (a) has many this game has many Nash equilibria and this raises the question of equilibrium selection. We will argue that there are comparatively few natural ones and show that it is reasonable to have several Nash equilibrium solutions as different dynamics within the firm can result in different outcomes. Thus, if one departmental head is aggressive and one passive, we might expect a different outcome to one in which both are of a similar temperament. In the former case we would not necessarily expect a symmetric outcome even though the scenario does not give one player an advantage over the other. Thus, although it may be natural to expect a solution of (a) to be symmetric, we will also investigate non-symmetric solutions. These non-symmetric equilibria have the advantage that the players have pure actions whereas, in our symmetric solution, the players are called upon to employ actions with complicated probabilities.

Key words: Secretary Problem, Nash equilibrium, Stackleberg strategies, Multistage Non-zero Sum Game.

1 Introduction

The Secretary Problem is a well-known problem in the decision sciences. It can be stated as follows. An organization wishes to appoint a secretary from n applicants who appear in random order with all $n!$ permutations equally likely. At any stage the applicants interviewed up to and including that stage can be ranked in a preference order and, at each stage, the organization has to decide whether or not to make the candidate an offer. If an applicant is made an offer, it will be accepted but any applicant who is rejected will not be available for consideration at a later stage. The organization wants to maximize the probability of choosing the best candidate. There has been a lot of interest in this problem and variants of it which attempt to make the problem more realistic; see, for instance Tamaki (1991), who has considered the variation in which there is a fixed probability (independent of rank) of the applicant accepting an offer. Other papers dealing with sequential selection problems include Chun (1992) and Kwan and Yuan (1988). Less attention has been paid to the game theory aspects of hiring an employee although Sakaguchi (1985, 1989) is a notable exception. A selection panel usually involves at least two members so there is the potential for its members to have different interests. In particular Mazalov, Sakaguchi and Zabelin (2002) have modelled a situation in which a labor union and management jointly employ a secretary. We will investigate a variant of their problem.

The main purpose of this paper is to analyze the following scenario. Two departments in a large organization are each seeking to make an appointment within the same area of expertise, for instance, a computer science specialist. To avoid duplication it has been decided that the heads of the two departments should together interview the applicants in turn and make their decisions on one applicant before interviewing any others. The organization has a very good reputation for treating its staff well so it can be assumed that an applicant who is offered an appointment will accept it. If a candidate is rejected by both departmental heads, the candidate cannot be considered for either post at a later date. If both heads decide to make an offer, the candidate will be allowed to choose which department he or she joins; from past experience it is known that applicants do not have a preference for one department over the other so each department is equally likely to be accepted. The departmental heads know that there are precisely n applicants and that each applicant has an expertise which is random over a known range; the expertise of an applicant can only be accurately assessed during the interviewing process and it is assumed that the heads will agree on this value. If no appointment is made to a department from these n applicants, then the department will suffer from a shortfall of expertise.

Although the problem is a simply stated and natural one, we will see that it is not at all straightforward to analyze it using the methods of game theory. As stated earlier, the competitive hiring of secretaries has already been analyzed in the literature but we shall see that our analysis differs markedly from that in previous papers. Because these papers have concentrated on the situation in which one secretary has to be hired, the players are essentially playing the same game at each stage; from one stage to the next the only essential difference is that the number of applicants decreases by one. However, in our scenario, this is not true; as soon as the decision to appoint is made, the problem becomes a decision problem rather than a game. Thus our problem is a bridge between the classical sequential selection problems and the game theoretic ones. A similarity between our paper and previous ones is that, if there are a comparatively large number of applicants, the interviewers can hold out for an applicant of comparatively high quality when the interviewing process is in its early stages; this is because there is a high probability that a good, or two good, applicants can be found from the remaining ones. A further similarity is that arguments of a backward induction nature are used. However a major difference is that our problem gives rise to many Nash equilibria and this raises the question of equilibrium selection. We will argue that there are comparatively few natural ones in the sense that they could be explained and justified to non-mathematical personnel. We will also argue that it is reasonable to have several Nash equilibrium solutions as different dynamics within the firm can result in different outcomes. Thus, if one departmental head is aggressive and one passive, we might expect a different outcome to one in which both are of a similar temperament. In the former case we would not necessarily expect a symmetric outcome even though the scenario does not give one player an advantage over the other. Thus, although it may be natural to expect a solution to be symmetric, we will also investigate non-symmetric solutions These non-symmetric equilibria have the advantage that the players have pure actions whereas, in our symmetric solution, the players are called upon to employ actions with complicated probabilities.

Before analyzing the above scenario, we will consider the modification in which one department has better prospects than the other; these prospects could take several forms such as being able to offer a higher salary, better training or faster promotion than the other. In this case it can be assumed that, if a candidate is accepted by both departmental heads, then the applicant will choose the one with the better prospects. Generally speaking the analysis of this problem follows the pattern of previous papers and provides a good lead in to the main problem.

2 Preliminary Notions

By choosing the units appropriately we can suppose that each applicant has an expertise drawn at random from the interval $[0, 1]$ and that the shortfall in expertise of a department not employing an applicant is c. We can think of the situation as a multi-stage game with the r-th stage being one in which rejection of the current applicant would leave $r-1$ applicants available for interview. However we will adopt a naive approach and not use the general theory of such games. Consider first the decision problem in which a player knows that there are r applicants remaining and the other player has filled the post in his department. We will denote the expectation of the player in this case by u_r. Clearly, if $r = 1$, the player will accept the applicant and $u_r = 1/2$. Further, when there are $r-1$ applicants available if the current one is rejected, he will accept the applicant he is interviewing if the applicant has expertise x where $x \geq u_{r-1}$ and reject otherwise. Thus

$$u_r = \int_0^{u_{r-1}} u_{r-1}\,dx + \int_{u_{r-1}}^1 x\,dx = u_{r-1}^2 + (1 - u_{r-1}^2)/2 = (1 + u_{r-1}^2)/2. \qquad (2.1)$$

The recurrence relation $u_r = (1+u_{r-1}^2)/2$ is known in the literature as Moser's relation (see, for example, Petrosjan and Zenkevich (1996)).

3 The Dominant Firm Problem

Suppose one department is dominant so that an applicant will always choose it in preference to the other. We will assume the dominant department is represented by player 1.

The one applicant game is trivial because player 1 will always accept and receive an expectation of $1/2$ so that player 2 will always get $-c$.

Now consider the $n \geq 2$ applicant game and the stage where an applicant is being interviewed and there will be $r-1$ applicants available if the current one is rejected by both players. If one player accepts and the other player rejects, or both players accept, the current applicant, the players clearly know what their expectations are. However, if both reject, it is not so clear that the players know what their expectations are because these expectations depend on actions in subsequent stages. Indeed, in the other scenario we consider, this is in fact the case and assumptions have to be made for an analysis of it to be undertaken. Thus let us assume that the players do know what their expectations are if they both reject the current applicant, say v_{r-1} and w_{r-1}, for player 1 and player 2 respectively. The situation can then be represented by a bimatrix given by

$$M_r(x) = \begin{array}{c} \\ \text{Accept} \\ \text{Reject} \end{array} \begin{array}{c} \begin{array}{cc} \text{Accept} & \text{Reject} \end{array} \\ \begin{pmatrix} (x, u_{r-1}) & (x, u_{r-1}) \\ (u_{r-1}, x) & (v_{r-1}, w_{r-1}) \end{pmatrix} \end{array}.$$

We now show by induction that $v_r = u_r$ for all r. The result holds for $r = 1$ so suppose it holds for $r = k$. Thus, at the $k+1$-stage player 1 gets x if he accepts the current applicant and u_k if he doesn't which leads to expression for u_r in (2.1) and the result is established. Not surprisingly the dominant player can behave as though he has no rival. Player 2 will accept a candidate with expertise at least w_{r-1} so, when $w_k \geq 0$, his expectation w_{k+1} is given by

$$w_{k+1} = \int_{u_k}^{1} u_k \, dx \; + \; \int_{w_k}^{u_k} x \, dx \; + \; \int_{0}^{w_k} w_k \, dx = u_k - u_k^2/2 + w_k^2/2$$

and, when $w_k < 0$, by

$$w_{k+1} = u_k - u_k^2/2.$$

It is easy to see that the latter case arises only when $k = 1$. This reflects the fact that player 2 can ensure a non-negative expectation by always accepting the final two applicants. The following table shows that, even when there are 14 applicants, the dominant player does considerably better.

Stage	*Dominant*	*NotDominant*
1	0.5	0
2	0.625	0.375
3	0.6953	0.5
4	0.7417	0.5786
5	0.7750	0.6340
6	0.8004	0.6757
7	0.8203	0.7084
8	0.8364	0.7347
9	0.8498	0.7565
10	0.8611	0.7749
11	0.8707	0.7906
12	0.8791	0.8042
13	0.8864	0.8160
14	0.8929	0.8265

4 The One and Two Applicant Games with No Candidate Preference

We now investigate the scenario described in Section 1 in which an applicant is indifferent between the two departments. The case when there is only one applicant presents no difficulties because both players will want to appoint as they will not want a shortfall of expertise in their department. Thus there is a unique Nash equilibrium and the expected payoff for each player is $(1/2 - c)/2 = (1 - 2c)/4$.

The position is very different when there are two applicants. The game is now a two-stage one rather than single shot and what is meant by a Nash equilibrium has to be considered. We will take a naive view and assume the players will play a Nash equilibrium when interviewing an individual applicant. In essence we are restricting attention to subgame perfect Nash equilibria. When the first applicant is interviewed, each player knows that he can expect $1/2$ if he rejects and the other player accepts and $(1 - 2c)/4$ if both reject. Hence both players will accept an applicant with expertise $x \geq 1/2$ and reject an applicant with expertise $x < (1 - 2c)/4$. For $x \in [(1 - 2c)/4, 1/2]$, each player would prefer the other player to accept the applicant so that he could reject and obtain an expectation of $1/2$ at the second stage. We can represent the situation at the first stage by a game bimatrix $M_2(x)$ given by

$$M_2(x) = \begin{array}{c} \\ \text{Accept} \\ \text{Reject} \end{array} \begin{array}{c} \begin{array}{cc} \quad\quad\quad\text{Accept} & \quad\quad\quad\quad\quad\text{Reject} \end{array} \\ \begin{pmatrix} ((x + 1/2)/2, (x + 1/2)/2) & (x, 1/2) \\ (1/2, x) & ((1 - 2c)/4, (1 - 2c)/4) \end{pmatrix} \end{array}$$

Note that, for $x \geq 1/2$ Accept dominates Reject for both players whereas for $x \leq (1 - 2c)/4$, Reject dominates Accept for both players. For $x \in [\max\{0, (1 - 2c)/4\}, 1/2]$ there are two pure Nash equilibria, namely (Accept, Reject) and (Reject, Accept) and one mixed in which each player accepts an applicant with expertise x with probability $(4x-1+2c)/(2x+2c)$. The payoff vector for the mixed strategy pair is given by (v,v) where

$$v = \frac{4cx+2c+6x-1}{8(x+c)} = \frac{2c+3}{4} - \frac{(2c+1)^2}{8(x+c)}. \tag{3.1}$$

Thus v is an increasing function of x in $[(1-2c)/4, 1/2]$ for fixed $c \in [0,1/2]$.

Of course neither of the pure Nash equilibria are symmetric and this could be seen as a drawback because the scenario does not present any differences between the players. However we should not dismiss these equilibria out of hand as they could arise from different dynamics in the organization. For instance, if one departmental head is somewhat aggressive and the other more easy-going, it is quite likely that the more aggressive head would state what he was going to do; in essence the aggressive head assumes the role of a Stackleberg leader. The mixed Nash equilibrium has the advantage that each player adopts the same strategy so that it can be argued that it is fairer than either of the pure equilibria but it has the drawback that the players use complicated probabilities to decide their actions.

So far we have looked at the situation where the players are actually confronted with the applicant's expertise but, from a game point of view, the strategies should be announced before the interview takes place; in other words we should be providing a function $s : [0,1] \to [0,1]$ for each player where $s(x)$ represents the probability of accepting an applicant with expertise x. However this means that, unless restrictions are imposed, the set of Nash equilibria becomes unmanageable. Given any subset S of $I = [(1-2c)/4, 1/2]$ suppose player 1 accepts $x \in S \cap I$ and rejects $x \in I \setminus S$ while player 2 accepts $x \in I \setminus S$ and rejects $x \in S \cap I$; neither player benefits from unilaterally deviating so this strategy pair is a Nash equilibrium. Clearly most such Nash equilibria are unsatisfactory. From the purely mathematical standpoint, one would not be able to calculate an expectation for our problem for non-measurable S. However most measurable S would have undesirable properties from the modelling viewpoint. If a player accepts an applicant with expertise x, then it would appear unreasonable to expect the player to reject an applicant of expertise greater than x; indeed, in many countries, in doing so, an organization might find itself in conflict with the law. If the condition is imposed that acceptance of an applicant with expertise x implies acceptance of every applicant with expertise greater than x,, there are just two pure Nash equilibria for the players; one player accepts the first applicant if and only if the applicant has expertise at least $1/2$ and the other player accepts if and only if the applicant has expertise at least $(1-2c)/4$. Note that, with these Nash equilibria, both applicants will be accepted if $c \geq 1/2$ but that, for $c < 1/2$, there is a probability that only one applicant will be accepted.

We now calculate the expectations for the pure Nash equilibrium in which player 1 accepts an applicant if and only if his expertise x is at least $1/2$ and player 2 accepts if and only if $x \geq (1-2c)/4$. Assuming $c \leq 1/2$, for player 1 we have

$$\begin{aligned} v_2 &= \int_0^{(1-2c)/4} (1-2c)/4\,dx \;+ \int_{(1-2c)/4}^{1/2} 1/2\,dx \;+ \int_{1/2}^{1} (x+1/2)/2\,dx \\ &= \frac{1}{2} + \frac{c^2}{4}. \end{aligned}$$

For player 2 we have

$$\begin{aligned} w_2 &= \int_0^{(1-2c)/4} (1-2c)/4\,dx \;+ \int_{(1-2c)/4}^{1/2} x\,dx \;+ \int_{1/2}^{1} (x+1/2)/2\,dx \\ &= \frac{(1-2c)^2}{32} + \frac{7}{16} = \frac{15}{32} - \frac{c}{8} + \frac{c^2}{8}. \end{aligned}$$

Clearly, for $c \geq 1/2$, the integrals and therefore the expectations are the same as for $c = 1/2$. Thus they are independent of c and $v_2 = 9/16$ and $w_2 = 7/16$.

It is interesting to note that, as a function of c, v_2 is increasing whereas w_2 is decreasing so the Stackleberg leader gains more the larger the value of c. As the leader expects more than the follower when $c = 0$, the most unfair value of c is $1/2$. Furthermore as c goes from 0 to $1/2$, the leader gains more than the follower loses because the leader's expectation rises from $1/2$ to $9/16$ while the follower's expectation declines from $15/32$ to $7/16$. In a sense, the situation can be regarded as a game of Chicken. For $x \leq 1/2$, the leader rejects the applicant and dares the follower to also reject the applicant. The larger the value of c, the more pressure there is on the follower to accept. We shall see in the next section that a similar position arises in the general game but that the effects become insignificant when there are still a large number of applicants remaining.

So far we have looked at the situation from the heads of department viewpoint but their interests do not necessarily coincide with that of the organization as a whole. If we think of the organization's interest as the totality of expertise employed, this interest is effectively the sum of the expectations of the two players. Thus it would clearly be better for the organization if both applicants are accepted. We have seen that both applicants are accepted if $c \geq 1/2$ so there is no conflict of interest in this case. Whether the interests of the organization are represented by an increasing function of c as c ranges over the interval [0,1/2] is not so intuitively obvious. We have seen that, as c goes from 0 to $1/2$, the Stackleberg leader gains more than the follower loses over the interval so, as a whole, the function increases. However, the smaller the value of c, the more likely it is that the first applicant will be rejected and the penalty c brought into operation. The interest of the organization E is given by

$$E = v_2 + w_2 = 1 + \frac{(1-2c)(3-6c-4)}{32} = 1 - \frac{(1-2c)(1+6c)}{32}$$

which is a convex quadratic with a minimum at $c = 1/6$ giving $E = 23/24$ at the minimum. Thus the worst value of c for the organization is $1/6$. Further the most unfair value of c, namely $1/2$, is actually the best value for the organization.

There is also a mixed Nash equilibrium for the players. Let $p(x) = (1-2x)/(2x+2c)$ then $p(x)$ is the probability that a player will reject $x \in [(1-2c)/4, 1/2]$ so, using (3.1), the payoff Q for each player in the symmetric Nash equilibrium is given by

$$
\begin{aligned}
Q &= \int_{1/2}^{1} \frac{(x+1/2)}{2}\,dx \;+\; \int_{(1-2c)/4}^{1/2} \Big(\frac{2c+3}{4} - \frac{(2c+1)^2}{8(x+c)}\Big)\,dx \\
&+ \int_{0}^{(1-2c)/4} \frac{1-2c}{4}\,dx \\
&= \frac{5}{16} + \frac{(2c+3)(2c+1)}{16} - \frac{(2c+1)^2 \ln 2}{8} + \frac{(1-2c)^2}{16} \\
&= \frac{9+4c+8c^2}{16} - \frac{(2c+1)^2 \ln 2}{8}
\end{aligned}
$$

Note that this is also a convex quadratic with a minimum in $(0,\ 1/2)$.

Intuition tells us that a player in the role of Stackleberg leader in a pure Nash equilibrium does better than a player in the symmetric Nash equilibrium because his position is essentially unchanged if the first applicant has expertise outside of $[(1-2c)/4, 1/2]$ and he does better if the applicant has expertise within that range because he is guaranteed $1/2$ as the Stackleberg leader. To check this, consider, for $c \le 1/2$,

$$
\begin{aligned}
D_1 = v_2 - Q &= \frac{1}{2} + \frac{c^2}{4} - \frac{9+4c+8c^2}{16} + \frac{(2c+1)^2 \ln 2}{8} \\
&= \frac{2\ln 2 - 1}{16}(1+2c)^2 > 0
\end{aligned}
$$

However it is not clear intuitively whether a player in the role of Stackleberg follower does better or worse than a player in the symmetric Nash equilibrium. To see whether he does so, for $c \le 1/2$ we consider

$$
\begin{aligned}
D_2 = w_2 - Q &= \frac{15}{32} - \frac{c}{8} - \frac{9+4c+8c^2}{16} + \frac{(2c+1)^2 \ln 2}{8} \\
&= \frac{4\ln(2) - 3}{32}(1+2c^2) < 0.
\end{aligned}
$$

Note that

$$
\frac{v_2 + w_2}{2} - Q = \frac{8\ln(2) - 5}{64}(1+2c)^2 > 0. \tag{4.1}
$$

When $c \ge 1/2$, we have

$$
Q = \frac{11+4c}{16} + \frac{(1+2c)^2}{8} \ln \frac{2c}{1+2c}
$$

and

$$
D_1 = 9/16 - Q > 0 \qquad D_2 = 7/16 - Q < 0
$$

so we have the same properties as the case $c = 1/2$. Thus a player in the symmetric game does better than a Stackleberg follower in a pure Nash equilibrium whatever the value of c.

From (4.1) the sum of the expectations for a pure Nash equilibrium is greater than twice that for the symmetric one. This accords with intuition in that $x \in ((1-2c)/4, 1/2)$ is always accepted by the pure strategy pair but only with a probability less than 1 by the symmetric strategy pair so there is a positive probability that such values will lead to a shortfall of expertise in one of the departments when the symmetric strategy pair is used.

The players' actions for x not in the range are the same for both strategy pairs. Thus the players would have better expectations by choosing one of the pure Nash equilibria at random than using the symmetric Nash equilibrium. From a practical standpoint randomizing the role of Stackleberg leader seems a more attractive option than using the symmetric Nash equilibrium. It is a lot easier to understand and to put into practice; a simple toss of a coin rather than a complicated randomization if $x \in (1-2c)/4, 1/2]$.

5 The General Game with No Candidate Preference

We saw in the last section that the two applicant game presented difficulties so we must expect similar ones to occur in the general game. However, when there are three or more applicants, further complications arise. In the two applicant game, the players know their expectations if both players reject the first applicant but, when there are three applicants and both reject the first applicant, their expectations will depend on which Nash equilibrium is adopted at the second stage. We will assume that there is some kind of consistency in the actions of the players at the various stages and investigate a number of scenarios to see how the payoffs differ and how they are affected by the value of c.. In the two applicant case, we have suggested that, it might benefit the players to agree to randomize who takes the role of Stackleberg leader so we will consider the situation in which, at each stage, the players agree to randomize who takes that role. By doing this the players create a symmetric situation where, at each stage, the players have the same expectation if both players reject the current applicant. It is therefore very different from the situation in which the players agree to take it in turns to assume the role of Stackleberg leader; in this case, at any given stage, the players will in general have different expectations. If there is no agreement to randomize concerning who takes the role of Stacklberg leader, it seems natural to assume that a player who adopts the role of a Stackleberg leader for one stage would take that role at every stage; we investigate how unfair this is on the Stackleberg follower. Similarly, if a symmetric (mixed) strategy is used at one stage then one might expect it to be used at every stage. Intuitively we would expect that, when the number of applicants is large, there would not be very much difference in the expectations of the players from any of these scenarios.

Assuming players 1 and 2 believe their expectations are respectively v_{r-1} and w_{r-1} if they both reject the applicant interviewed at stage r, the r-stage matrix is

$$M_r(x) = \begin{array}{c} \\ \text{Accept} \\ \text{Reject} \end{array} \begin{array}{c} \begin{array}{cc} \quad\quad\text{Accept} & \quad\quad\quad\text{Reject} \end{array} \\ \begin{pmatrix} ((x+u_{r-1})/2, (x+u_{r-1})/2) & (x, u_{r-1}) \\ (u_{r-1}, x) & (v_{r-1}, w_{r-1}) \end{pmatrix} \end{array}$$

The result of the following lemma enables us to carry out a similar analysis to that of the two applicant case.

Lemma. For all r, $u_r \geq \max\{v_r, w_r\}$.

Proof. For any strategy pair in a two-person non-zero sum game, the expectations in a Nash equilibrium are non-decreasing functions of the elements in the matrix. Because $u_1 = 1/2 \geq (1-2c)/4$, the result follows by induction.

If player 1 assumes the role of Stackleberg leader and Player 2 that of Stackleberg follower the respective expectations are

$$\begin{aligned} v_r &= \int_{u_{r-1}}^{1} (x+u_{r-1})/2\,dx \;+\; \int_{w_{r-1}}^{u_{r-1}} u_{r-1}\,dx \;+\; \int_{0}^{w_{r-1}} v_{r-1}\,dx \\ &= \frac{(1+u_{r-1})^2}{4} - w_{r-1}(u_{r-1}-v_{r-1}) \end{aligned}$$

and

$$\begin{aligned} w_r &= \int_{u_{r-1}}^{1} (x+u_{r-1})/2\,dx \;+\; \int_{w_{r-1}}^{u_{r-1}} x\,dx \;+\; \int_{0}^{w_{r-1}} w_{r-1}\,dx \\ &= \frac{1-u_{r-1}^2+2u_{r-1}+2w_{r-1}^2}{4}. \end{aligned}$$

We now consider the case in which the players use the symmetric Nash equilibrium at each stage. At stage r the players will have a common expectation which will be denoted by q_r. The probability of a player accepting $x \in [q_{r-1}, u_{r-1}]$ at stage r is given by $2(x - v_{r-1})/(x+u_{r-1}-2v_{r-1})$ and the corresponding payoff is

$$2u_{r-1} - q_{r-1} - \frac{2(u_{r-1}-q_{r-1})^2}{x+u_{r-1}-2q_{r-1}}$$

Hence the expectation of the symmetric Nash equilibrium is

$$\begin{aligned} q_r &= \int_{u_{r-1}}^{1} \frac{x+u_{r-1}}{2}\,dx \\ &+ \int_{q_{r-1}}^{u_{r-1}} (2u_{r-1}-q_{r-1}-\frac{2(u_{r-1}-q_{r-1})^2}{x+u_{r-1}-2q_{r-1}})\,dx \\ &+ \int_{0}^{q_{r-1}} q_{r-1}\,dx = \frac{(1+u_{r-1})^2}{4} \\ &+ (u_{r-1}-q_{r-1})(u_{r-1}-2q_{r-1}) + 2(u_{r-1}-q_{r-1})^2 \ln 2. \end{aligned}$$

When $c = 0$, the values are

Stage	*U value*	*Leader*	*Follower*	*Sum/2*	*Symmetric*
1	0.5	0.25	0.25	0.25	0.25
2	0.625	0.5	0.4688	0.4844	0.4759
3	0.6953	0.6016	0.5747	0.5881	0.5806
4	0.7417	0.6646	0.6419	0.6533	0.6468
5	0.7750	0.7089	0.6894	0.6991	0.6935
6	0.8004	0.7421	0.7250	0.7335	0.7286
7	0.8203	0.7681	0.7528	0.7605	0.7561
8	0.8364	0.7891	0.7753	0.7822	0.7782
9	0.8498	0.8064	0.7939	0.8001	0.7965
10	0.8611	0.8210	0.8095	0.8152	0.8119
11	0.8707	0.8335	0.8228	0.8281	0.8250
12	0.8791	0.8442	0.8343	0.8393	0.8364
13	0.8864	0.8537	0.8444	0.8490	0.8463
14	0.8929	0.8620	0.8533	0.8576	0.8551

When $c = 1/2$, the values are

Stage	*U value*	*Leader*	*Follower*	*Sum/2*	*Symmetric*
1	0.5	0	0	0	0
2	0.625	0.5625	0.4375	0.5	0.4659
3	0.6953	0.6328	0.5605	0.5967	0.5763
4	0.7417	0.6835	0.6339	0.6587	0.6444
5	0.7750	0.7215	0.6842	0.7028	0.6921
6	0.8004	0.7511	0.7214	0.7362	0.7276
7	0.8203	0.7748	0.7503	0.7625	0.7553
8	0.8364	0.7942	0.7734	0.7838	0.7777
9	0.8498	0.8104	0.7924	0.8014	0.7961
10	0.8611	0.8243	0.8083	0.8163	0.8116
11	0.8707	0.8362	0.8218	0.8290	0.8248
12	0.8791	0.8465	0.8335	0.8400	0.8362
13	0.8864	0.8558	0.8437	0.8497	0.8462
14	0.8929	0.8636	0.8527	0.8582	0.8550

These tables demonstrate that the general case follows a similar pattern to that of the two applicant case. The Stackleberg leader benefits from a higher value of c whereas the follower benefits from a lower value of c. However, by the time there are fourteen applicants, the value of c makes very little difference to either of the expectations. The tables also show that, for fourteen applicants, the expectations of the players are very close to each other. Thus aggression manifested by taking the role of Stackleberg leader is of little benefit when there are a comparatively large number of applicants.

When $c = 1/6$, the values are

Stage	*U value*	*Leader*	*Follower*	*Sum/2*	*Symmetric*
1	0.5	0.1667	0.1667	0.1667	0.1667
2	0.625	0.5069	0.4514	0.4792	0.4640
3	0.6953	0.6069	0.5667	0.5868	0.5754
4	0.7417	0.6684	0.6374	0.6529	0.6440
5	0.7750	0.7117	0.6864	0.6991	0.6918
6	0.8004	0.7442	0.7230	0.7376	0.7274
7	0.8203	0.7697	0.7514	0.7605	0.7552
8	0.8364	0.7904	0.7742	0.7823	0.7776
9	0.8498	0.8075	0.7930	0.8002	0.7960
10	0.8611	0.8219	0.8088	0.8153	0.8115
11	0.8707	0.8342	0.8223	0.8282	0.8247
12	0.8791	0.8449	0.8339	0.8394	0.8362
13	0.8864	0.8542	0.8440	0.8491	0.8461
14	0.8929	0.86250	0.8530	0.8577	0.8549

This table confirms the analysis of Section 4 that, when there are just two applicants the organization can expect less expertise for $c = 1/6$ than for $c = 0$. We will have further comments on this table in the next section.

Now consider the case in which, at each stage, the players randomize for who should play the role of leader, then $v_{r-1} = w_{r-1}$ in the above matrix and the expectation v_r^* is

$$v_r^* = \frac{1 + 2u_{r-1} + v_{r-1}^*(3v_{r-1}^* - 2u_{r-1})}{4}.$$

Stage	*U value* $c = 0$	*value* $c = 1/2$
1	0.25	0
2	0.4844	0.5
3	0.5938	0.6240
4	0.6599	0.6780
5	0.7057	0.7177
6	0.7397	0.7483
7	0.7662	0.7727
8	0.7876	0.7927
9	0.8052	0.8091
10	0.8200	0.8232
11	0.8326	0.8352
12	0.8435	0.8457
13	0.8530	0.8549
14	0.8614	0.8630

6 Discussion of the Results

The tables in the previous section suggest some interesting conclusions. Perhaps the most illuminating is that it appears that, whatever the value of c in [0,1/2], the players have a better expectation by randomizing who should adopt the role of Stackleberg leader at each stage other than the first than by using any of the other Nash equilibria considered. In particular, for the two ïair ösolutions in the sense that they give the same expectation to each player at each stage, the simpler method which can be implemented by a simple toss of a fair coin at each stage appears preferable to the one in which a complicated randomization is necessary at each stage. Furthermore the simple fair method is best from the organization's standpoint in that a higher level of expertise is appointed than in any of the others, certainly in the cases when $c = 0$ and $c = 1/2$ and there are more than two applicants. There is a striking difference between the symmetric Nash equilibrium and the others. In the former the expectation for a given stage is higher when $c = 0$ than it is when $c = 1/2$ whereas, for the others, the expectation is higher when $c = 1/2$ than when $c = 0$.

In Section 4 we saw that, when there are just two applicants, the average expectation of the players has a minimum w.r.t c in $(0, 1/2)$ for both the symmetric and the pure Nash equilibrium. The tables for $c = 0$ and $c = 1/6$ illustrate a difference between these Nash equilibria in the general case. The symmetric Nash equilibrium has a lower expectation for $c = 1/6$ than for $c = 0$ for all but the one applicant case whereas the pure Nash equilibrium has this property for only a small number of applicants.

References

[1] Young Hak Chun, *On the Sequential Selection Problem*, Decision Sciences **23**(1992) 1251-1262.

[2] C.C.Y. Kwan, Y. Yuan, *A Sequential Selection Problem*, Decision Sciences **19** (1988) 762-770.

[3] V.V. Mazalov, M. Sakaguchi and A.A. Zabelin, *Multistage Arbitration Game with Random Offers*, International Journal of Mathematics, Game Theory and Algebra **12**(2002) 409-417

[4] L.A. Petrosjan and N.A. Zenkevich, *Game Theory*. World Scientific, London, 1996.

[5] M. Sakaguchi, *The Bilateral Secretary Problem with a Random Population Size*, Mathematica Japonica, **30** (1985) 457-469.

[6] M. Sakaguchi, *Multiperson Secretary Problems*, Mathematica Japonica, **34** (1989) 459-473.

[7] M. Tamaki, *A Secretary Problem with Uncertain Employment and Best Choice of Available Candidates*, Operations Research **39** (1991) 274-284.