Aggregation with Many Effective Voters

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Abstract

McGarveys's theorem [6] shows that majority aggregation of a profile of linear orders generates any complete binary relation. Kalai [5] proved that the same is true for any neutral monotone SWF defined by a strong simple game where the power of every voter is sufficiently small, which implies that there are many effective voters. In this paper we study neutral monotone SWFs with many effective voters with no restriction on voter power. We give bounds on the minimal number of relations generated by such SWFs.

1 Introduction

Condorcet's famous paradox [2] shows that majority aggregation of individual preferences between at least 3 alternatives represented by linear orders may lead to an intransitive binary relation. McGarvey [6] extended this result and proved that any complete binary relation can be generated by majority on a sufficient number of voters. In McGarvey's proof at least m^2 voters are required in order to generate all possible binary relations on m alternatives. A counting argument shows that at least $\frac{m}{log(m)}$ are required and Erdős and Moser [3] showed that this number of voters suffices.

Arrow's impossibility theorem [1] shows that any social welfare function (SWF) generates an intransitive relation for some profile of linear orders. Can this result be extended in the same way Condorecet's paradox was extended by McGarvey? Kalai [4] conjectured that the answer is positive, namely for any neutral monotone SWF with a sufficient number of effective voters any complete binary relation can be generated by an appropriate profile of linear orders. Kalai [5] proves this conjecture for a certain class of SWF. He shows that a neutral monotone SWF defined by a strong simple game with small Shapley-Shubik power for every voter generates all binary relations. This results implies that voting systems in which the power of each voter is small leads to *social indeterminacy* – the group preference is a binary relation for which the

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preference on any pair of alternatives is independent of the preference on the other pairs.

In this paper we refute Kalai's conjecture for the general case by constructing a family of SWFs for which there exist binary relations that are not generated by the family for any number of voters. Nevertheless, we show that the set of binary relations generated by a neutral monotone SWF is very large. The set of all binary relations generated by a SWF f is called the *image* and denoted $Im(f)$. Define:

$$X_m = \max_n \min\{|Im(f)| : \ f \text{ neutral monotone with } n \text{ effective voters}\}$$

this number indicates the degree of social determinacy introduced by a voting systems with unbounded numbers of effective voters. Our goal is to prove the following theorem:

Theorem 1.

$$2^{0.25m^2} \leq X_m \leq 2^{0.468m^2}$$

Let us give a brief overview of the proof. There are two main ideas: the first is to analyze the relations generated by a family of SWFs we call *almost dictator* which has many effective voters but nevertheless is quite restrictive. X_m is the smallest number of relations that can be generated by any SWF with many effective voters therefore an upper bound on the number of relations generated by almost dictator passes on to X_m. In section 3 we find both lower (lemma 2) and upper (lemma 3) bounds for the number of relations generated by almost dictator. The second idea is to introduce a notion of embedding such that the image of an embedded SWF is a subset of the image the embedding SWF. In section 4 we show (corollary 1) that for any SWF with a sufficient number of voters there is an embedded almost dictator. This implies that the image of any SWF includes all the relations in the image of almost dictator which shows that the lower bound of lemma 2 is a lower bound for X_m. As we show, this lower bound is high, hence the degree of social determinacy in neutral monotone SWFs with many effective voters is quite low for any power distribution among voters.

2 Preliminaries

We begin by briefly describing the model we will be using. A *strong simple game* G is a tuple $([n], \mathcal{W})$ where $[n] = [1, \ldots, n]$ is a set of voters and $\mathcal{W}$ is a set of coalitions (subsets of $[n]$) such that $\emptyset \notin \mathcal{W}$, $[n] \in \mathcal{W}$ and either $S \in \mathcal{W}$ or $[n] - S \in \mathcal{W}$ for every coalition $S \subset [n]$. The set $\mathcal{W}$ designates the winning coalitions. A game is *monotone* if $S \in \mathcal{W}$ and $S \subset T$ imply $T \in \mathcal{W}$. If $S, S' \in \mathcal{W}$ are two disjoint winning coalitions in a monotone game then $S' \subset [n] - S$ implies $S, [n] - S \in \mathcal{W}$ contradicting monotinicity, hence any two winning coalitions have a nonempty intersection.

A voter is *influential* or *effective* if his or her vote may have some impact on the outcome. In a strong simple game G this means that the voter is a *pivot* for at least one coalition, namely $S \notin \mathcal{W}$ and $S \cup \{i\} \in \mathcal{W}$ for some coalition $S \subset [n] - \{i\}$. A coalition $S \subset [n]$ is *minimal* if $S \in \mathcal{W}$ and $S' \notin \mathcal{W}$ for any $S' \subsetneq S$. If G is monotone then a coalition is minimal if every voter in the coalition is a pivot. If $S \in \mathcal{W}$ then there exists a minimal winning coalition $S' \subset S$. Notice that a minimal winning coalition may not be the smallest winning coalition, indeed in the almost dictator game we introduce in the next section there is a minimal winning coalition consisting on $n-1$ out of n voters while there are other winning coalitions consisting of two voters.

Let Pr_p be a distribution on the set of all coalitions such that every voter independently belongs to a random coalition with probability p. The *Banzhaf Power index* of voter $j \in [n]$ in G, denoted $I_j^p(G)$, is the probability that j is a pivot, thus $I_j^p(G) = Pr_p(\{S \subset [n] - \{j\} : S \notin \mathcal{W} \wedge S \cup \{j\} \in \mathcal{W}\})$. Voter j is called a *dummy* in G if he or she has no influence on the outcome of G in such a case $I_j^p(G) = 0$ for all p. Another common power index is the Shapley-Shubik power index which also measures the influence of a voter. The relation between these two indices is given by Owen's identity [7]

$$\phi_i(G) = \int_0^1 I_k^p(G)dp$$

A key to the proof of the upper and lower bounds is the analysis of neutral monotone SWFs defined by a family of strong simple games called *almost dictator*. For n voters almost dictator is the game $AD_n = ([n], \mathcal{W})$ defined by $\mathcal{W} = \{S \cup \{1\} : \emptyset \neq S \subset [n]\} \cup \{[n] - \{1\}\}$. Thus voter 1 – the 'almost dictator' – imposes his choice unless all the other voters are lined up against him. This gives a monotone game with no dummies. Voter 1 is a pivot unless all the others are unanimous one way or another hence $I_1^p = 1 - p^{n-1} - (1-p)^{n-1}$. Any other voters is a pivot if all the voters apart from 1 agree hence $I_k^p = p^{n-2}(1-p) + p(1-p)^{n-2}$. It follows from Owen's identity that the Shapley-Shubik power indices are given by:

$$\begin{aligned}
\phi_1(G) &= \int_0^1 I_1^p(G)dp = \int_0^1 [1 - p^{n-1} - (1-p)^{n-1}]dp = 1 - \frac{2}{n} > 0 \\
\phi_k(G) &= \int_0^1 I_k^p(G)dp = \int_0^1 [p^{n-2}(1-p) + p(1-p)^{n-2}]dp = \frac{2}{n(n-1)} > 0 \\
&\text{for } k > 1
\end{aligned}$$

Let $[m]$ be a set of $m > 2$ alternatives. Designate the set of all complete antisymmetric binary relations on $[m]$ by Δ and the set of all linear orders $\Omega \subset \Delta$. In our model a preference is a linear order (we disregard indifference). An n voter *social welfare function* (SWF) is a function $f : \Omega^n \to \Delta$ such that any $R = f(R_1, \ldots, R_n)$ satisfies *independence of irrelevant alternatives* (IIA) : the preference of P on alternatives $a, b \in [m]$ depends only on the individual preferences of each voter between these two alternatives, and the *Pareto* condition: if all voters prefer a to b then so does R. It is implied by these conditions that a function f is a SWF iff there exists a collection of strong simple games $\{G_{ab}\}_{a,b\in[m]}$ such that aRb iff $\{j \in [n] : aR_jb\}$ is a winning coalition in G_{ab}. In this paper we assume neutrality and monotinicity namely $G_{ab} = G$ for all $a, b \in [m]$, we shall occasionally identify a SWF with the game defining it.

We say that a set of voters vote in blocks if there is a division of $[m]$ into blocks such that voters within each block behave identically; voting according to party lines is an example. Formally, let $G = ([n], \mathcal{W})$ and $G' = ([n'], \mathcal{W}')$ be strong simple games. We say that G *embeds* G' by *block voting* if there exists a function $\varphi : [n] \to [n']$ such that $S \in \mathcal{W}$ iff $\varphi(S) \in \mathcal{W}'$ for every $S \subset [n]$, i.e. all the voters in 'block' $\varphi^{-1}(j)$ vote identically for every $j \in [n']$. We denote this relation $G' = G \circ \varphi^{-1}$. If $G_2 = G_1 \circ \varphi^{-1}$ and $G_3 = G_2 \circ \psi^{-1}$ then $G_3 = G_1 \circ (\varphi\psi)^{-1}$ hence embedding is a transitive relation between strong simple games. If a game defining a SWF f embeds a game defining another SWF f' by φ then we say that f embeds f' by φ and denote $f' = f \circ \varphi^{-1}$. It is easy to see that embedding is also transitive relation on SWFs.

The *image* $Im(f)$ is the set of all binary relations generated by profiles of linear orders. The Pareto principle implies $\Omega \subset Im(f)$. If f embeds f' by φ then for any $R = f'(R_1, \ldots, R_k)$ let $(Q_1, \ldots, Q_n)$ be the corresponding block profile namely $Q_i = R_j$ if $\varphi(i) = j$. By definition $R = f(Q_1, \ldots, Q_n)$ hence $Im(f') \subset Im(f)$.

3 Weak Social Determinism

In this section we find lower and upper bounds of the number of relations in the image of almost dictator. A relation R is *weakly determined* by a linear order R_0 if for any three alternatives $a, b, c \in [m]$ such that aR_0bR_0c there is no cycle $aRcRbRa$. Thus if we think of R_0 as determining a clockwise direction for any triple then we say that a relation is weakly determined if it has no counterclockwise cycles. Denote by $\mathfrak{C}$ the set of all weakly determined relations (where R_0 runs over all linear orders). An SWF satisfies *weak social determinacy* if its image is a subset of $\mathfrak{C}$.

Lemma 1. *$Im(AD_n) \subset \mathfrak{C}$ and if $n > \binom{m}{2}$ then $Im(AD_n) = \mathfrak{C}$*

proof: Let R be a relation generated by AD_n with R_0 as the preference of voter 1. For any three alternatives $a, b, c \in [m]$ such that aR_0bR_0c it follows from cRb and bRa that all voters apart from 1 prefer b to c and a to b. Since the voter preferences are transitive it follows that all these voters prefer a to c hence cRa. Thus no relation with a counterclockwise cycle $aRcRbRa$ can be generated. This implies that any relation generated with R_0 as the preference of the almost dictator is weakly determined by R_0 hence $Im(AD_n) \subset \mathfrak{C}$.

The idea in the second part of the proof is to construct a set of linear orders that agree with R whenever R disagrees with R_0 and agrees with R_0 on at least one pair, for any R weakly determined by R_0. For a voter profile with R_0 as the almost dictator preference and the constructed linear orders as the preferences of the other voters, it follows that all voters line up against the almost dictator if he or she disagrees with R and at least one voter joins the almost dictator of he or she agrees with R, hence the image of this profile is R.

Let $R \in \mathfrak{C}$ be weakly determined by R_0. If R disagrees with R_0 on all pairs then R is the inverse of R_0 and therefore a linear order which implies that it belongs to the image of any SWF. Otherwise, for any pair (a', b') such that $a'Rb'$ and $a'R_0b'$ there is a partial relation $P_{a'b'}$ such that $a'P_{a'b'}b'$, and on any other pair $aP_{a'b'}b$ if aRb and bR_0a and is undefined otherwise. If $aP_{a'b'}b$, $bP_{a'b'}c$ and $\{a', b'\} \not\subset \{a, b, c\}$ then by definition cR_0bR_0a and $aRbRc$, if we had cRa then this would contradict weak determinism thus aRc and consequently $aP_{a'b'}c$. This shows that if $a_1P_{a'b'}a_2 \ldots a_lP_{a'b'}a_1$ is a cycle then there exists $c \neq a', b'$ such that $a'P_{a'b'}b'P_{a'b'}cP_{a'b'}a'$. This implies $a'R_0cR_0b'R_0$ and $a'Rb'RcRa'$ again contradicting weak determinism. Consequently $P_{a'b'}$ is transitive and by Szpilrajn theorem (see for instance [8]) can be extended to a linear order $R_{a'b'}$.

Let $(Q_1, Q_2, \ldots, Q_n)$ be a profile of linear orders representing voter preferences such that $Q_1 = R_0$ and for any $a, b \in [m]$ on which R and R_0 agree there exists $Q_j = R_{ab}$ (for R_{ab} constructed as above). If $n > \binom{m}{2}$ then such a profile exists. For any $c, d \in [m]$, if cRd and dR_0c then cQ_jd for any $j \neq 1$. If cRd and cR_0d then cQ_jd for at least one voter in addition to voter 1. By definition of almost dictator $R = AD_n(Q_1, \ldots, Q_n)$ implying $R \in Im(AD_n)$ □

The next lemma gives a lower bound on the image of almost dictator, which is obtained by an explicit construction of linear orders that generate a set of relations that includes all the possible preferences on a large set of pairs.

Lemma 2. *If $n > m$ then* $2^{0.25m^2} \leq |Im(AD_n)| \leq |\mathfrak{C}|$

proof: Let A, B be a division of $[m]$ ($A \cup B = [m]$ and $A \cap B = \emptyset$) and let R_0 be a linear order such that aR_0b for every $a \in A$ and $b \in B$. Let R be a relation that agrees with R_0 on any pair of alternatives that are both in A or both in B. We prove that such an R can be generated by almost dictator by showing that for any $a \in A$ there exists a linear order R_a that agrees with R on any pair (a, b) for $b \in B$ and disagrees with R_0 on any pair (a', b) for $a' \in A - \{a\}$ and $b \in B$. For a profile with R_0 as the preference of the almost dictator and R_a as preferences of the other voters, it follows that on all pairs (a, b) $a \in A$ and $b \in B$ all voters oppose the almost dictator if he or she disagrees with R, and at least one voter $j \neq 1$ agrees with R_0 otherwise. The relation R_a is constructed by dividing $[m]$ into four blocks and rearranging the block order while preserving the order of R_0 within the blocks.

If $|B| \leq 1$ then R is a linear order thus $R \in Im(AD_n)$. If $|B| \geq 2$ then for every $a \in A$ let $B_a = \{b \in B : bRa\}$ and let R_a be a linear order that agrees with R_0 on the blocks B_a, $B - B_a$ and $A - \{a\}$ and orders the blocks by $[B_a, a, B - B_a, A - \{a\}]$, in other words R_a prefers a to every alternative in B_a, prefers every alternative in $B - B_a$ to a and prefers every alternative in $A - \{a\}$ to every alternative in $B - B_a$. Let R_1 be a linear order that prefers every alternative in A to every alternative in B and agrees with R_0 on A and B. Let $(Q_1, \ldots, Q_n)$ be a profile of $n > m \geq |A| + 2$ voters such that $Q_1 = R_0$, $Q_2 = R_1$ and for every $a \in A$ there exists at least one j such that $Q_j = R_a$. If a, b are both in A or both in B then voters 1 and 2 agree with R_0. If $a \in A$ then

$R_a = Q_j$ for some j. If aRb then voters 1 and j agree with R and all voters apart from 1 agree with R otherwise. Again by definition $R = AD_n(Q_1, \ldots Q_n)$.

There exist $|A| \cdot |B|$ pairs of alternatives on which R may agree or disagree with R_0 therefore $|Im(AD_n)| \geq 2^{|A||B|}$ and the best bound we get this way is $2^{\frac{m^2}{4}}$ $\square$

Lemma 3. $|\mathfrak{C}| \leq 2^{0.468m^2}$

proof: We apply a probabilistic argument to find an upper bound on the set $\mathfrak{C}$. The idea is to upper bound the number of relations by upper bounding the probability of a corresponding event. We begin with upper bounding the number of relations that are weakly determined by to a linear order R_0. We do this by bounding the probability that a uniformly random relation induces no counterclockwise cycles on a large set of edge disjoint triangles (triples of alternatives that do not intersect on more than one alternative). Clearly this event contains the event that a random relation is weakly determined. The disjointness of the triangles implies that the event that a triangle is a counterclockwise cycle is independent of the relation induced on any one of the other triangles thus we can easily compute the event that no triangle is induced a counterclockwise cycle.

Let Pr be the uniform distribution on the set of all binary relations. The probability that a triangle $a, b, c \in [m]$ is not a counterclockwise cycle relative to R_0 is $\frac{7}{8}$. The probability of the event that there are no counterclockwise cycles is bounded by the probability that on a set of edge disjoint triangles no triangle is induced a counterclockwise cycle. Due to disjointness the latter probability is $(\frac{7}{8})^K$ where K is the size of a set of edge disjoint triangles. Consequently the best bound we can hope to obtain this way would be for a maximal set of edge disjoint triangles.

A maximal set of edge disjoint triangles on m vertices is a *Steiner triple system*. It is not too difficult to show that such a set can consist of at most $K = \frac{m(m-1)}{6}$ triangles. It is a well known theorem in combinatorial design that a Steiner triple system of this size exists for $m = 1, 3 \ mod\ 6$ (see van Lint [9] for more on combinatorial design), this implies the existence of a set of edge disjoint triangles of size $\frac{m^2}{6} - \theta(m)$[1] for all m.

It follows that the probability of a relation having a counterclockwise cycle is bounded by $(\frac{7}{8})^{\frac{m^2}{6} - \theta(m)}$. Consequently the number of relations without these cycles is bounded by

$$(\frac{7}{8})^{\frac{m^2}{6} - \theta(m)} 2^{\frac{m(m-1)}{2}} = 2^{(\frac{log_2 \frac{7}{8}}{3} + 1)\frac{m^2}{2} - \theta(m)}$$

There are $m! = 2^{\theta(mlog(m))}$ choices for the order R_0 hence for large m

$$|\mathfrak{C}| \leq 2^{(\frac{log_2 \frac{7}{8}}{3} + 1)m^2 + \theta(mlog(m)) - \theta(m)} < 2^{0.468m^2} \ll 2^{\frac{m(m-1)}{2}} \quad \square$$

[1] We replace a function $\varphi(m)$ with $\theta(m)$ when there exist constants $c, c' \in \mathbb{R}^+$ such that $cm \leq \varphi(m) \leq c'm$ for m large.

The lemma shows that not all relations can be generated by SWFs in the almost dictator family. Since almost dictator defines a SWF it follows that this bound passes on to X_m.

4 An Embedding Theorem

In this section we show that any SWF defined by a game with $n > k2^{k^2+1}$ voters embeds an almost dictator with k voters, hence the lower bound on the number of relations generated by almost dictator applies to X_m as well.

Lemma 4. *For a strong simple game with $n > 3$ effective voters there exists a minimal winning coalition with more than two voters.*

proof: If a singleton is a winning coalition then the game is dictatorial and there is only one effective voter. Suppose every minimal coalition in $\mathcal{W}$ is a two voter coalition and let $T = \{a_1, a_2\} \in \mathcal{W}$ be such a coalition. Any two effective voters $b_1 \neq b_2$ belong to minimal winning coalitions that intersect T. If $\{a_1, b_1\} \in \mathcal{W}$ and $\{a_2, b_2\} \in W$ then we have two disjoint winning coalitions which contradicts simplicity. Consequently all effective voters belong to a minimal winning coalition that intersects T on the same voter thus w.l.g $\{a_1, b\} \in \mathcal{W}$ for any effective voter b.

Since any singleton is a loosing coalition $[n]-\{a_1\} \in \mathcal{W}$ and $[n]-\{a_1, b\} \notin \mathcal{W}$ for any effective voter b. It follows that any minimal winning coalition which is a subset of $[n] - \{a_1\}$ must include all effective voters. Since there are more than three effective voters this minimal coalition must have more than two voters contradicting the initial assumption □

Theorem 2. *A strong simple game with $n > k2^{k^2+1}$ effective voters has a minimal winning coalition with more than k voters.*

proof: Let $G = ([n], \mathcal{W})$ be a strong simple game and assume any minimal winning coalition has at most k voters. Let T_0 be a minimal winning coalition of maximal cardinality, lemma 4 implies $2 < |T_0| \leq k$. Any minimal winning coalition $S \neq T_0$ must intersect T_0 hence induces a partition of T_0. Associate the voters in $S - T_0$ to this partition. Since every effective voter belongs to at least one minimal winning coalition the pigeonhole principle implies that there exists a winning coalition S that induces a partition $T_0' = T_0 \cap S$ and $T_0'' = T_0 - T_0'$ with at least $\frac{k2^{k^2+1}-k}{2^k}$ associated effective voters.

For a partition (T_0', T_0'') as above let $n_1 = n - |T_0| + 2$ and take $\varphi_0 : [n] \to [n_1]$ such that $\varphi_0^{-1}(1) = T_0'$, $\varphi_0^{-1}(2) = T_0''$ and $|\varphi_0^{-1}(j)| = 1$ for $j \in [n_1] - \{1, 2\}$. Then we define a new embedded game $G_1 = G \circ \varphi_0^{-1} = ([n_1], \mathcal{W}_1)$. If $l \in [n] - T_0$ is an effective voter associated with the partition (T_0', T_0'') then there exists a minimal winning coalition such that $S \cap T_0 = T_0'$. Thus $S \in \mathcal{W}$ but $S - \{l\} \notin \mathcal{W}$ so by definition of embedding $\varphi(S) \in \mathcal{W}_1$ and $\varphi(S) - \{\varphi(l)\} \notin \mathcal{W}_1$. This implies that the image of a G-effective voter associated with the partition is G_1-effective hence there are at least $\frac{k2^{k^2+1}-k}{2^k}$ effective voters in G_1.

Suppose there exists a minimal winning coalition $S \subset [n_1]$ in G_1 with more than k voters. Minimality implies that $S - \{j\} \notin \mathcal{W}_1$ for every $j \in S$, hence by definition of embedded game $\varphi^{-1}(S) \in \mathcal{W}$ but $\varphi^{-1}(S - \{j\}) \notin \mathcal{W}$. Consequently any minimal winning coalition $S' \subset [n]$ such that $S' \subset \varphi^{-1}(S)$ must intersect $\varphi^{-1}(j)$ for every $j \in S$. Since inverses are disjoint it follows that $|S'| > k$ contrary to the assumption on G. If S is a minimal winning coalition with more than two voters then $|S \cap \{1,2\}| = 1$ since $\{1,2\} \in \mathcal{W}_1$. If $2 \in S$ then for $j \in S - \{2\}$ we know $([n] - S) \cup \{j\} \in \mathcal{W}_1$ and therefore there exists a minimal winning coalition $S' \subset ([n] - S) \cup \{j\}$, from $[n] - S \notin \mathcal{W}_1$ follows that $1, j \in S'$. This shows that for every effective voter $j \neq 1, 2$ there exists a minimal coalition that includes 1 and j. If every minimal winning coalition that includes 1 is a two voter coalition then the set of all effective voters apart from 1 is a minimal winning coalition with more than k voters which as we have seen is a contradiction. Thus G_1 is a strong simple game with $\frac{k2^{k^2+1}-k}{2^k}$ effective voters such that any minimal winning coalition is bounded by k, with a minimal winning coalition T_1 such that $1 \in T_1$ and $|T_1| > 2$ and a 'special' voter $a_1 = 2$ such that $\{1, a_1\} \in \mathcal{W}_1$.

Assume G_j is a game on voter set $[n_j]$ with at least $\frac{k}{2^{jk}}[2^{k^2+1} - 2^{jk} + 1]$ effective voters such that any minimal winning coalition has less than k voters with a minimal winning coalition T_j such that $1 \in T_j$ and $|T_j| > 2$ and with special voters $a_1, \ldots, a_j$ such that $\{1, a_l\} \in \mathcal{W}_j$ for any $l = 1, \ldots, j$. As before there is a partition T_j' and T_j'' with $\frac{k}{2^{(j+1)k}}[2^{k^2+1} - 2^{(j+1)k} + 1]$ associated G_j-effective voters, we choose the sets so that $1 \in T_j'$. Let $n_{j+1} = n_j - |T_j| + 2$ and take $\varphi_j : [n_j] \to [n_{j+1}]$ such that $\varphi_j^{-1}(1) = T_j'$, $\varphi_j^{-1}(2) = T_j''$ and $|\varphi_j^{-1}(l)| = 1$ for $l \in [n_j] - \{1,2\}$. We define for this partition the game $G_{j+1} = G_j \circ \varphi_j^{-1} = ([n_{j+1}], \mathcal{W}_{j+1})$.

As before the image of any G_j-effective voter associated with the partition is G_{j+1}-effective hence G_{j+1} has at least $\frac{k}{2^{(j+1)k}}[2^{k^2+1} - 2^{(j+1)k} + 1]$ effective voters. If S is a minimal winning coalition with more than k voters then $S - \{l\} \notin \mathcal{W}_{j+1}$ for every $l \in S$ so $\varphi_j^{-1}(S) \in \mathcal{W}_j$ but $\varphi_j^{-1}(S - \{l\}) \in \mathcal{W}_j$ therefore any minimal coalition $S' \subset \varphi_j^{-1}(S)$ must intersect $\varphi_j^{-1}(l)$ implying $|S| > k$ which contradicts the assumption on G_j. If l is effective there exists a minimal winning coalition that includes voters 1 and l, if every such coalition is a two voter coalition then the set of all effective voters is a minimal winning coalition with more than k voters again producing a contradiction. This shows that there exists a minimal winning coalition T_{j+1} with more than two voters and $1 \in T_{j+1}$. Finally, $\{1, a_l\} \in \mathcal{W}_j$ for any special voter therefore $\varphi(\{1, a_l\}) = \{1, \varphi(a_l)\} \in \mathcal{W}_{j+1}$ $l = 1, \ldots, j$ thus the image of every G_j-special voter is G_{j+1}-special. Since T_j is minimal with more than two voters $a_1, \ldots, a_j \notin T_j$ and consequently $\varphi(a_1), \ldots, \varphi(a_j) \neq 2$. By defintion $\{1,2\} \in \mathcal{W}_{j+1}$ therefore $2, \varphi(a_1), \ldots, \varphi(a_j)$ are $j+1$ special voters in G_{j+1}.

Repeated use of this construction gives a sequence of games $G_1, \ldots, G_k$. Let $\varphi = \varphi_k \circ \varphi_{k-1} \circ \ldots \circ \varphi_1 : [n] \to [n_k]$; φ defines an embedding of G_k in G. Let $a_1, \ldots, a_k$ be the special voters of G_k, thus $\{1, a_l\} \in \mathcal{W}_k$ for every $l = 1, \ldots, k$

therefore by definition of embedding $\varphi^{-1}(\{1, a_l\}) = \varphi^{-1}(1) \cup \varphi^{-1}(a_l) \in \mathcal{W}$. Since G_k is not dictatorial $[n] - \varphi^{-1}(1) \in \mathcal{W}$. Thus any minimal winning coalition $S \subset [n] - \varphi^{-1}(1)$ must intersect each one of $\varphi^{-1}(a_1), \ldots, \varphi^{-1}(a_k)$ implying $|S| > k$ contrary to the assumption on G $\square$

Corollary 1. *A game with $k2^{k^2+1}$ effective voters or more embeds AD_{k+1}*

proof: It follows from the theorem that for any game with $n > 2^{k^2+1}$ there exists a minimal winning coalition S with more than k voters. Minimality implies that $([n] - S) \cup \{j\}$ is a winning coalition for every $j \in S$ therefore G embeds AD_{k+1} with the block $[n]-S$ as the almost dictator and the k singletons of voters in S $\square$

Corollary 2. *If a SWF f is defined by a game with more than $m2^{m^2+1}$ effective voters then $2^{0.25m^2} \leq |Im(f)|$.*

proof: This is an immediate conclusion of corollary 1 and lemma 2.

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