A Unified Approach to Information, Knowledge, and Stability*

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Abstract

Within the context of strategic interaction, we provide a unified framework for analyzing information, knowledge, and the "stable" pattern of behavior. The major innovations are: (i) unlike the standard ad hoc semantic model of knowledge, the state space is constructed by Harsanyi's types that were explicitly formulated by Epstein and Wang (*Econometrica* **64**, 1996, 1343-1373); (ii) players may be boundedly rational and have non-partitional information structures; and (iii) players may have general preferences, including subjective expected utility and non-expected utility. We first study the interactive epistemology. We then establish an equivalence theorem between a strictly dominated strategy and a never-best reply in terms of epistemic states. Finally, we explore epistemic foundations behind the fascinating idea of stability due to J. von Neumann and O. Morgenstern. *JEL Classification:* C70, C72, D81.

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1 Introduction

In their classics, von Neumann and Morgenstern (1944) enthusiastically advocated the idea of stability by introducing a fascinating solution concept of the vN-M stable set. Ever since then the criterion of stability has been widely applied in economics and other social sciences.[1] Greenberg (1990) took this line of approach one step further by providing an integrated approach to the study of formal models in the social and behavioral sciences, and thereby revitalized this old idea. Chwe (1994), Greenberg *et al.* (1996), Greenberg *et al.* (2002), Luo (2001), Nakanishi (1999), and Xue (1998) are some examples of recent applications in game theory and economic theory.

Recall that von Neumann and Morgenstern (1944, Sections 4.6, 4.7, and 65.1) referred the idea of stability to a wide range of social organizations. More specifically, a *vN-M (abstract) stable set* is defined as a subset $\mathcal{K}$ of ordered outcomes satisfying the following two conditions:

1. [internal stability] no y in $\mathcal{K}$ is dominated by an x in $\mathcal{K}$;

2. [external stability] every y not in $\mathcal{K}$ is dominated by some x in $\mathcal{K}$.

In other words, $\mathcal{K}$ is free of inner contradictions: no outcome in $\mathcal{K}$ can be "upset" by an element in $\mathcal{K}$; $\mathcal{K}$ is free of external inconsistencies: any outcome outside $\mathcal{K}$ can be "overruled" by an outcome in $\mathcal{K}$.

Although the stability criterion appears to be methodologically profound, conceptually sophisticated, theoretically elegant, and applicably fruitful, no formal foundation has been laid for it in the literature. Up until now, most theorists have simply taken this criterion as a normative requirement. To connect with real-life phenomena, von Neumann and Morgenstern literally interpreted a vN-M stable set as a social norm in a society (see, von Neumann and Morgenstern 1947, pp. 40-43). As M. Shubik wrote,

> [A vN-M stable set is] viewed as a *standard of behavior* — or a *tradition, social convention, canon of orthodoxy,* or *ethical norm* (Shubik 1982, p. 161).

[1] See, for instance, Lucas (1994) and Shubik (1982) for surveys.

Following the above interpretation, the idea of stability attributes to the discipline of *homo sociologicus*, which emphasizes that human behavior is primarily driven by a macro-force such as established social norms. Apparently, this line of interpretation deviates from the basic tenet in the discipline of *homo economicus*, which insists that all behavior should be explained in terms of individual rationality.

This paper is therefore motivated mainly by the following fundamental questions about the concept of stability. Which epistemic foundation(s) is behind a social norm? What is the economic rationale of the "stable" pattern of strategic behavior? How does one formally relate the notion of rationality to that of stability?

In an attempt to answer the aforementioned questions, Luo (2002) first studied epistemic foundations behind the criterion of stability within the standard semantic framework, and established the linkage between stability and Bayesian rationality. The purpose of this paper is to further extend this line of research to very general cases of social organizations.

Some salient features in this paper are as follows. Firstly, in recent years, some authors have studied various solution concepts in noncooperative games from a decision-theoretic viewpoint — i.e., in terms of rationality and epistemic states, for example, Aumann (1995, 1987), Aumann and Brandenburger (1995), and Dekel and Gul (1997). However, within the conventional semantic framework used in game theory, the notion of a *state of the world,* or simply a *state,* may be self-referential since it consists of a specification of information, knowledge, and strategy.[2]

In the spirit of Savage's (1954) choice-theoretic approach, Morris (1996) made some progress by deducing information and knowledge from preferences at a state. However, since preferences are *ad hoc* specified at a state, Morris' framework still suffers from the self-referential criticism on the specification of preferences. By employing Epstein and Wang's (1996) general construction of Harsanyi's (1967-1968) types, we provide a unified framework in which the state space represents the exhaustive uncertainty facing each player in a strategic setting — i.e., the primitive uncertainty about the choices of strategy by

[2] See, e.g., Osborne and Rubinstein (1994, p. 77).

all players, as well as the uncertainty about *all* players' types (each type is homeomorphic to an infinite regress of a hierarchy of "beliefs about beliefs"). The complete representation of a state allows for eliciting, as not being *ad hoc,* all aspects of the full description of the world, including information, knowledge, preferences, and the choice of strategy. Among others, we explore the related interactive epistemology and establish an equivalence theorem about games in terms of epistemic states (see Theorem 1). The proposed framework is also conceptually important since it offers a more thorough set-up for thinking about the set-valued solution concept, like the vN-M stable set. To extend Tan and Werlang's (1988) and Brandenburger and Dekel's (1987) results about Bernheim (1984) and Pearce's (1984) rationalizability to general preferences, Epstein (1997) did his analysis in a similar framework.[3]

Secondly, within the conventional semantic framework, the information structure is assumed to be partitional. However, weakening the assumptions on knowledge and on information seems to be appealing since the assumption of a partitional information structure is rather restrictive in many economic applications. See, for example, Bacharach (1985), Dekel and Gul (1997), Geanakoplos (1989, 1994), Rubinstein (1998), Samet (1990), and Shin (1993) for further discussions. In this paper players may have a non-partitional information structure — i.e., players are boundedly rational (see Rubinstein 1998, Chapter 3). In particular, players may be "unaware of awareness," "ignoring ignorance," or even convinced of something objectively incorrect — i.e., they might fail to satisfy basic axioms of knowledge: *the axiom of knowledge, the axiom of transparency*, and *the axiom of wisdom.*

Thirdly, in this paper players may have diverse preferences other than subjective expected utility; for example, probabilistically sophisticated preferences (cf. Machina and Schmeidler 1992), Choquet expected utility (cf. Schmeidler 1989), the ordinal expected utility (cf. Borgers 1993), and so on. Since Epstein and Wang (1996) constructed Harsanyi's types by the hierachy of preferences rather than the hierachy of beliefs (see, e.g., Brandenburger and Dekel 1993, Mertens and Zamir 1985) in the proposed framework, players are therefore allowed to have not only subjective expected utility but also non-expected utility,

[3]In a similar spirit, Zamir and Vassilakis (1993) discussed "common belief and common knowledge" under subjective expected preferences.

such as Choquet expected utility.

The primary reason for pursuing the study of this paper is as follows. Experimental evidence such as the Ellsberg Paradox contradicts some of the tenets in the Savage model; for example, the Sure-Thing Principle. In particular, decision makers usually display an aversion to uncertainty or ambiguity. Consequently, it is a significant research subject to study games where players might have general preferences. See, for example, Dow and Werlang (1994), Epstein (1997), Ghirardato and Le Breton (2000), Klibanoff (1993, 1996), Lo (1996, 1999), Luo and Ma (2001), and Marinacci (2000). It is therefore an intriguing research topic to explore epistemic foundations behind the idea of stability in social organizations where individuals might exhibit general preferences.

More importantly perhaps, this paper suggests a novel interpretation for a "choice set" of strategies. In the proposed framework, the multiplicity of the choice of strategy would be better referred to the uncertainty about epistemic states (see 4.2 in Section 4). Moreover, the proposed framework is methodologically important since it, equipped with a rich state space, is immunized from the intrinsic inconsistency between the non-partitional information structures and Bayesian rationality, as pointed out by Morris (1996).[4]

One of the main results in this paper is to formulate and prove an equivalence theorem between a strictly dominated strategy and a never-best reply in terms of epistemic states (see Theorem 1). This definition of a strict dominance relation improves the one defined by Epstein (1997) and the equivalence theorem is clearly of independent theoretic interests (cf. Appendix IX). This paper also studies the related interactive epistemology. In particular, we extend Morris' (1996) properties of knowledge to the general case of an infinite state space (cf. Subsection 2.2). Finally, this paper explores the epistemic foundation for a stable set within the proposed framework. Under rather mild conditions, rationality and common knowledge of rationality prescribe the "stable" pattern of strategy behavior (see Theorem 2). If, moreover, the set of strategy choices set is publicly known as a "social norm," rationality coincides with stability (see Lemma 7). This paper thus extends some of Luo's (2002) results to the cases of general preferences and as well as to non-partitional information structures.

[4]Cf. 4.4 in Section 4.

The sequel of this paper is as follows. Section 2 offers a framework for analyzing information, knowledge, and the "stable" pattern of behavior. Subsection 2.1 introduces games in terms of epistemic states; Subsection 2.2 investigates the related interactive epistemology; and Subsection 2.3 establishes a fundamental equivalence theorem between a strictly dominated strategy and a never-best reply in terms of epistemic states. Section 3 studies epistemic foundations for stability. Subsection 3.1 introduces the notion of stability; Subsection 3.2 introduces the notion of rationality; and Subsection 3.3 presents the main results to relate information, knowledge, and stability. Section 4 is devoted to discussions. To facilitate reading, the precise definitions of "regular preferences" and "marginal consistency" and as well as some technical proofs are relegated to Appendices I-IX.

2 The Framework

2.1 Games in terms of epistemic states

We first provide a unified framework for analyzing strategic behavior as well as its related interactive epistemology.

Consider an *n-person strategic game* $\mathcal{G} \equiv (N, \{X_i\}, \{\zeta_i\})$, where X_i, for each $i \in N$, is a compact convex metric space of player i's strategies, and $\zeta_i : X \to [0,1]$ (where $X \equiv \times_{i\in N} X_i$) is a continuous payoff function that assigns each strategy profile $x \in X$ to a number in $[0,1]$.

Each player, as a decision maker, faces uncertainty not only about the primitive uncertainty corresponding to the strategy choices, but also about players' types in Harsanyi's sense. Accordingly, the state space of states of the world is constructed as: $\Omega \equiv X \times T_1 \times T_2 \times \ldots \times T_n$, where T_i is the space of player i's types. We refer to an element $\omega \in \Omega$ as a *state* and to a (Borel measurable) subset $E \subseteq \Omega$ as an *event*. Denote by t_i^{ω} player i's type projected at ω, and denote by x^{ω} the strategy profile at ω. Thus, a state ω can be written as $(x^{\omega}; t_1^{\omega}, t_2^{\omega}, ..., t_n^{\omega})$.

The objects of each player's choice are *acts*; i.e., Borel measurable functions $f : \Omega \to [0,1]$. Denote by $\mathcal{F}(\Omega)$ the set of a player's acts and by $\mathcal{P}(\Omega)$ the set of the *preferences over* $\mathcal{F}(\Omega)$. Throughout this paper, we restrict ourselves to the subclass of regular preferences that admit representation by utility functions — i.e., the subclass of regular preferences that satisfy U.1-6 and U.2$'$ in Appendix I. Based upon Epstein and Wang's (1996) Theorem 6.1, $T_i \sim^{homeomorphic} \mathcal{P}(\Omega)$,[5] and let $\psi : T_i \to \mathcal{P}(\Omega)$ represent such a homeomorphism. Write the utility function associated with t_i^{ω} freely as $\psi \circ t_i^{\omega}$ or u_i^{ω} for convenience.

A strategy $x_i \in X_i$ is referred to as an act $x_i : X \to [0,1]$, satisfying $x_i(x') = \zeta_i\left(x_i, x'_{-i}\right)$ for all $x' \in X$. (The strategy x_i is also referred to as an act from Ω to $[0,1]$, satisfying $x_i(\omega) = \zeta_i\left(x_i, x_{-i}^{\omega}\right)$.) Let $\mathcal{P}_i(X)$ denote the set of the preferences over the set of acts $f : X \to [0,1]$, satisfying $f(x_i, x_{-i}) = f(x'_i, x_{-i})$ for all (x_i, x_{-i}) and (x'_i, x_{-i}) in X. In what follows, we assume that $\mathcal{P}(E)$ and $\mathcal{P}_i(Y)$ are well defined for any $E \subseteq \Omega$ and $Y \subseteq X$. For the sake of brevity, we use $u_i(x_i)$ to represent the utility of the restriction of x_i to E (or Y) if $u_i \in \mathcal{P}(E)$ (or $u_i \in \mathcal{P}_i(Y)$). Let $X^{E} \equiv \left\{x^{\omega} \mid \omega \in E\right\}$. By marginal consistency in Appendix II, $\mathcal{P}(E)$ and $\mathcal{P}_i\left(X^{E}\right)$ can be treated as the same provided that preferences refer only to player i's strategies.

Given an event E, let $\mathcal{P}(\Omega|E)$ denote the set of i's preferences for which the complement of E is null in the sense of Savage; i.e., any two acts that agree on E are ranked as indifferent. Say *i knows E at ω* if there exists a closed subset $\overline{E} \subseteq E$ such that $\psi \circ t_i^{\omega} \in \mathcal{P}(\Omega|\overline{E})$.[6] Let K_iE denote the set of all the states where i knows E; i.e.,

$$K_iE \equiv \left\{\omega \in \Omega |\ \psi \circ t_i^{\omega} \in \mathcal{P}(\Omega|\overline{E}) \text{ for some closed set } \overline{E} \subseteq E\right\}.$$

Thus, for a closed set E, $K_iE = \left\{\omega \in \Omega |\ \psi \circ t_i^{\omega} \in \mathcal{P}(\Omega|E)\right\}$. Player i's information structure generated by the knowledge operator K_i is the correspondence $P_i : \Omega \rightrightarrows \Omega$, such that for all $\omega \in \Omega$,

$$P_i(\omega) = \bigcap_{\{E\subseteq\Omega|\ K_iE\ni\omega\}} E.$$

The set $P_i(\omega)$ represents *all* aspects of uncertainty on the part of player i — including uncertainty about *all* players' strategic behavior, uncertainty about

[5] Within this framework, each player's type space is homogeneous and each player may be ignorant of his own types (cf. 4.3 in Section 4).

[6] Some reader may prefer the term "believes E" rather than "knows E."

the uncertainty of *all* players' strategic behavior, and so on *ad infinitum.* It constitutes the standard model for "differential" information.

Example 1. A state ω^* is said to be a *Nash state* in $\mathcal{G}$ if, for all i,

$$\psi \circ t_i^{\omega^*}\left(x_i^{\omega^*}\right) \geq \psi \circ t_i^{\omega^*}(x_i) \text{ for all } x_i \in X_i,$$

where $x_{-i}^{\omega^*} = x_{-i}^{\omega}$ for all $\omega \in P_i(\omega^*)$. The profile x^{ω^*} is said to be a Nash equilibrium under general preferences.[7]

2.2 Interactive epistemology

We start by presenting two very useful properties for information structures.

Lemma 1 *The information correspondence P_i satisfies the following properties.*

(1.1) $P_i(\omega)$ *is closed.*

(1.2) $P_i(\omega) = P_i(\omega')$ whenever $t_i^{\omega} = t_i^{\omega'}$.

Proof. (1.1) By the definition of K_iE, it is easy to see that $\omega \in K_iE$ if, and only if, $\omega \in K_i\overline{E}$ for some closed subset $\overline{E} \subseteq E$. It therefore follows that

$$\bigcap_{\{E\subseteq\Omega|\ K_iE\ni\omega\}} E = \bigcap_{\{\overline{E}\subseteq\Omega|\ K_i\overline{E}\ni\omega \text{ and } \overline{E} \text{ is closed}\}} \overline{E}.$$

Hence, $P_i(\omega)$ is closed.

(1.2) Since $t_i^{\omega} = t_i^{\omega'}$, $\psi \circ t_i^{\omega} = \psi \circ t_i^{\omega}$. Therefore, for any $E \subseteq \Omega$, $\omega \in K_iE$ iff $\omega' \in K_iE$. Hence, $P_i(\omega) = P_i(\omega')$.■

Lemma 2 *The knowledge operator K_i satisfies the following properties.*

$K1$. $K_i\emptyset = \emptyset$.

[7] By marginal consistency, $\psi \circ t_i^{\omega^*} \in \mathcal{P}_i\left(X^{P_i(\omega^*)}\right)$. Let $\psi \circ t_i^{\omega^*}(x_i) \equiv u_i^*\left(x_i, x_{-i}^{\omega^*}\right)$. Then, $u_i^*\left(x_i^{\omega^*}, x_{-i}^{\omega^*}\right) \geq u_i^*\left(x_i, x_{-i}^{\omega^*}\right)$ for all $x_i \in X_i$.

$K2$. $K_i\Omega = \Omega$.

$K3$. $E \subseteq F \Rightarrow K_iE \subseteq K_iF$.

$K4$. $\bigcap_{\lambda\in\Lambda} K_iE^{\lambda} \subseteq K_i(\bigcap_{\lambda\in\Lambda} E^{\lambda})$ for a family of closed subsets $\{E^{\lambda}\}_{\lambda\in\Lambda}$.

Proof. See Appendix III.■

Remark 1. The knowledge operator K_i may fail to satisfy the other three axioms of knowledge — i.e., *the axiom of knowledge*, *the axiom of transparency*, and *the axiom of wisdom*.[8] In particular, the information structure is possibly non-partitional.

The following lemma provides an alternative definition of knowledge.

Lemma 3 $K_iE = \{\omega \in \Omega |\ P_i(\omega) \subseteq E\}$.

Proof. Let $\omega \in K_iE$. By the definition of $P_i(\omega)$, $P_i(\omega) \subseteq E$. Thus, $K_iE \subseteq \{\omega \in \Omega |\ P_i(\omega) \subseteq E\}$. Conversely, suppose that $P_i(\omega) \subseteq E$. By the proof of (1.1) in Lemma 1, $K3$ and $K4$ jointly imply that

$$\bigcap_{\left\{\overline{E}\subseteq\Omega |\ K_i\overline{E}\ni\omega \text{ and } \overline{E} \text{ is closed}\right\}} K_i\overline{E} \subseteq K_iE.$$

Therefore, $\omega \in K_iE$. Thus, $K_iE \supseteq \{\omega \in \Omega |\ P_i(\omega) \subseteq E\}$.■

An immediate implication of Lemma 3 is the following.

Corollary 1 $\omega \in K_iP_i(\omega)$.

We now introduce the notion of *common knowledge*. Roughly speaking, an event is common knowledge if everyone knows it, and everyone knows that everyone knows it, and everyone knows that everyone knows that everyone knows it, and so on *ad infinitum*. Let $E \subseteq \Omega$. Define $KE \equiv \cap_{i\in N}K_iE$ and $K^lE \equiv K\left(K^{l-1}E\right)$ for all $l \geq 2$. Define

$$CKE \equiv KE \cap K^2E \cap K^3E \cap ....$$

In words, CKE is the event that E is commonly known.

[8] That is, $K_iE \subseteq E$, $K_iE \subseteq K_i(K_iE)$, and $\Omega\backslash K_iE \subseteq K_i(\Omega\backslash E)$.

2.3 Fundamental equivalence theorem

In this subsection we shall formulate and establish a fundamental equivalence theorem between a "strictly dominated strategy" and a "never-best reply" in terms of epistemic states. For any subset $Y \subseteq X$, let $Y_{-i} \equiv \{y_{-i} | \ (x_i, y_{-i}) \in Y$ for some $x_i \in X_i\}$.

Definition 1. A strategy y_i is *strictly dominated given* $Y \subseteq X$ if there exists $x_i \in X_i$ such that

$$\zeta_i(x_i, y_{-i}) > \zeta_i(y_i, y_{-i}) \text{ for all } y_{-i} \in Y_{-i}.$$ [9]

Definition 2. A strategy y_i is a *never-best response given* $E \subseteq \Omega$ if, for every $\omega \in K_i E$,

$$u_i^{\omega}(x_i) > u_i^{\omega}(y_i) \text{ for some } x_i \in X_i.$$

That is, a strategy x_i is a *best response given* $E \subseteq \Omega$ if, for some $\omega \in K_i E$,

$$u_i^{\omega}(x_i) \geq u_i^{\omega}(y_i) \text{ for all } y_i \in X_i.$$

Theorem 1 *Let E be a compact event. Then, a strategy y_i is a never-best response given E if, and only if, it is strictly dominated given X^E.*

To prove Theorem 1, we need the following two lemmas.

Lemma 4 *$\mathcal{P}(E)$ is convex.*

Proof. See Appendix IV.■

Lemma 5 *Let Y be a compact set of strategy profiles. Then, a strategy y_i is strictly dominated given Y iff there exists $x_i \in X_i$ such that $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}_i(Y)$.*

[9] Note that this notion of strict dominance in the sense of "payoff dominance" is equivalent to Luce and Raiffa's (1957, p. 286) notion of "strong dominance" in terms of states of the world.

Proof. See Appendix V.■

We now turn to the proof of Theorem 1.

Proof of Theorem 1. *"if part":* Let x_i strictly dominate y_i given X^E. By the proof of Lemma 5, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}_i(Y)$. By marginal consistency, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. By Epstein and Wang's (1996) Theorem 4.3, $\mathcal{P}(E) \sim^{homeomorphic} \mathcal{P}(\Omega|E)$. Let $\varphi : \mathcal{P}(E) \to \mathcal{P}(\Omega|E)$ be such a homeomorphism. By the proof of Epstein and Wang's (1996) Theorem 4.3, $\varphi \circ u_i(x'_i) = u_i(x'_i)$ for all $x'_i \in X_i$. Therefore, $\varphi \circ u_i(x_i) > \varphi \circ u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. Thus, $u_i^{\omega}(x_i) > u_i^{\omega}(y_i)$ for all $u_i^{\omega} \in \mathcal{P}(\Omega|E)$. Since E is compact, it therefore follows that $u_i^{\omega}(x_i) > u_i^{\omega}(y_i)$ for all $\omega \in K_i E$.

"only if part": Consider a zero-sum game $\mathcal{G}' = (N', \{X'_j\}, \{\zeta'_j\})$ such that $N' = \{i, -i\}$, $X'_i = X_i$, and $X'_{-i} = \mathcal{P}(E)$. Define the payoff function in $\mathcal{G}'$ as

$$\zeta'_i(x_i, u_i) = u_i^{\omega}(x_i) - u_i^{\omega}(y_i), \text{ for all } x_i \in X'_i \text{ and } u_i \in \mathcal{P}(E),$$

where $u_i^{\omega} \in \mathcal{P}(\Omega|E)$ and $u_i^{\omega} = \varphi \circ u_i$. By U.6 in Appendix I, $\mathcal{P}(E)$ is compact. By Lemma 4, $\mathcal{P}(E)$ is convex. By Epstein and Wang's (1996) Theorem 3.1, $\mathcal{P}(E)$ is Hausdorff. By continuity of φ and by U.5 in Appendix I, it therefore follows that $\zeta'_i(\cdot,\cdot)$ is continuous (see Appendix VI). Now, by Glicksberg's (1952) Theorem, there exists a Nash equilibrium (x_i^*, u_i^*) in $\mathcal{G}'$. However, since y_i is a never-best response given E, we have

$$\max_{x_i \in X'_i} \zeta'_i(x_i, u_i) = \max_{x_i \in X'_i} \left[u_i^{\omega}(x_i) - u_i^{\omega}(y_i)\right] > 0$$

for all $u_i^{\omega} \in \mathcal{P}(\Omega|E)$. Therefore, for any $u_i \in \mathcal{P}(E)$,

$$\zeta'_i(x_i^*, u_i) \geq \zeta'_i(x_i^*, u_i^*) = \max_{x_i \in X'_i} \zeta'_i(x_i, u_i^*) > 0.$$

Thus, $u_i^{\omega}(x_i^*) > u_i^{\omega}(y_i)$ for all $u_i^{\omega} \in \mathcal{P}(\Omega|E)$. By the proof of Epstein and Wang's (1996) Theorem 4.3, $u_i^{\omega}(x'_i) = \varphi^{-1} \circ u_i^{\omega}(x'_i)$ for all $x'_i \in X_i$. Therefore, $\varphi^{-1} \circ u_i^{\omega}(x_i^*) > \varphi^{-1} \circ u_i^{\omega}(y_i)$ for all $u_i^{\omega} \in \mathcal{P}(\Omega|E)$. Since $\mathcal{P}(E) \sim^{homeomorphic} \mathcal{P}(\Omega|E)$, it follows that $u_i(x_i^*) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. By marginal consistency, $u_i(x_i^*) > u_i(y_i)$ for all $u_i \in \mathcal{P}_i\left(X^E\right)$. By Lemma 5, x_i^* strictly dominate y_i given X^E.■

Similarly, a strategy y_i is said to be a *never-best response given* $Y \subseteq X$ if, for every $u_i \in \mathcal{P}_i(Y)$,

$$u_i(x_i) > u_i(y_i) \text{ for some } x_i \in X_i.$$

An immediate implication of Theorem 1 is the following.

Corollary 2 *Let Y be a compact set of strategy profiles. Then, a strategy y_i is a never-best response given Y iff it is strictly dominated given Y.*

Proof. Consider $E \equiv Y \times T_1 \times T_2 \times \ldots \times T_n$. By Epstein and Wang's (1996) Theorem 6.1, T_i is compact. By the Tychonoff Theorem, E is compact. Therefore, $\left[y_i \text{ is strictly dominated given } Y\right] \Longleftrightarrow^{\text{by Theorem 1}} \left[y_i \text{ is a never-best response given } E\right] \Longleftrightarrow^{\text{by Definition 2}} \left[\text{for every } \omega \in K_iE,\ \overset{\omega}{u_i}(x_i) > \overset{\omega}{u_i}(y_i) \text{ for some } x_i \in X_i\right] \Longleftrightarrow^{\text{by the compactness of } E} \left[\text{for every } u_i \in \mathcal{P}(\Omega|E),\ u_i(x_i) > u_i(y_i) \text{ for some } x_i \in X_i\right] \Longleftrightarrow^{\text{by Epstein and Wang's (1996) Theorem 4.3}} \left[\text{for every } u_i \in \mathcal{P}(E),\ u_i(x_i) > u_i(y_i) \text{ for some } x_i \in X_i\right] \Longleftrightarrow^{\text{by marginal consistency}} \left[\text{for every } u_i \in \mathcal{P}_i(Y),\ u_i(x_i) > u_i(y_i) \text{ for some } x_i \in X_i\right] \Longleftrightarrow \left[y_i \text{ is a never-best response given } Y\right]$.■

Remark 2. In the case of expected utility, Corollary 2 therefore generalizes a result that, in a finite game, a strategy is a never-best response if and only if it is strictly dominated (see, for instance, Luo's (2002) Lemma 1, Osborne and Rubinstein's (1994) Lemma 60.1, and Pearce's (1984) Lemma 3).

3 The Foundation of Stability

3.1 Stability

Within the context of strategic interactions, we shall employ a natural extension of the notion of a vN-M stable set, due to Luo (2001), as follows.

Definition 3. A subset $\mathcal{K} \subseteq X$ is a *(general) stable set* if it is a vN-M stable set with respect to $\succ^{\mathcal{K}}$, where $x \succ^{\mathcal{K}} y$ iff, for some i, x_i strictly dominates y_i given $\mathcal{K}$.

That is, a stable set $\mathcal{K}$ satisfies:

1. [internal stability] $\forall x \in \mathcal{K}$, $y \not\succ^{\mathcal{K}} x$ for all $y \in \mathcal{K}$, and
2. [external stability] $\forall x \notin \mathcal{K}$, $y \succ^{\mathcal{K}} x$ for some $y \in \mathcal{K}$.

In other words, $\mathcal{K}$ is free of inner contradictions — i.e., no element in $\mathcal{K}$ can be dominated by an element in $\mathcal{K}$, with respect to the conditional dominance relation $\succ^{\mathcal{K}}$. Furthermore, $\mathcal{K}$ is free of external inconsistencies — i.e., any element outside $\mathcal{K}$ is dominated by an element in $\mathcal{K}$, with respect to the conditional dominance relation $\succ^{\mathcal{K}}$. Clearly, every stable set must be in Cartesian-product form.

3.2 Rationality

From an epistemic perspective, at a state ω, player i knows only the set $P_i(\omega)$. That is, he considers it possible that the true state could be any state in $P_i(\omega)$, but not any state outside $P_i(\omega)$. In particular, at that state player i can conclude only that all his plausible choices of strategy are within the scope of $X_i^{P_i(\omega)}$.[10] We therefore define the notion of "rationality" by requiring that the choice set $X_i^{P_i(\omega)}$ consists of all the best replies in face of epistemic uncertainty $P_i(\omega)$. Formally, let

$$BR_i(\omega) \equiv \{x_i \in X_i |\ x_i \text{ is a best response given } P_i(\omega)\}.$$

Define i *is rational at* ω if

$$X_i^{P_i(\omega)} = BR_i(\omega).$$

Let

$$R_i \equiv \{\omega \in \Omega |\ i \text{ is rational at } \omega\}.$$

Let $R \equiv \cap_{i \in N} R_i$ denote the event that "everyone is rational."

[10] Recall that $X_i^E \equiv \left\{ \overset{\omega}{x_i} \mid \omega \in E \right\}$ and $X_{-i}^E \equiv \left\{ \overset{\omega}{x_{-i}} | \ \omega \in E \right\}$ for any event E.

3.3 Epistemic foundation of stability

We next start to explore the epistemic condition for the "stable" pattern of behavior. Up until now, we have not imposed any essential condition on regular preferences and hence have allowed for a rather arbitrary information structure. In particular, the knowledge operator might violate the axiom of knowledge. Somewhat surprisingly, without referring to other conditions, the notion of rationality yields no strategic implication since both rationality as well as common knowledge of rationality are consistent with any strategic behavior.

Lemma 6 $X^{R \cap CKR} = X$.

Proof. See Appendix VII.■

Throughout this subsection, we thereby assume the *weak* axiom of knowledge — i.e., we assume that $\omega \in P_i(\omega)$ for all $\omega \in R \cap CKR$. Moreover, we assume that, from an individual's epistemic viewpoint, each player is aware that his choices of strategy are independent of other players' choices of strategy, in the sense that $X^{P_i(\omega)} = X_i^{P_i(\omega)} \times X_{-i}^{P_i(\omega)}$. We are now in a position to present the main results of this section: rationality and common knowledge of rationality jointly imply stability.

Theorem 2 $X^{R \cap CKR}$ *is a stable set.*

To prove Theorem 2, we need the following two lemmas.

Lemma 7 *Suppose* $X^{P_i(\omega)} = \times_{j \in N} X_j^{P_j(\omega)}$ *for all* i. *Then,* $\times_{j \in N} X_j^{P_j(\omega)}$ *is a stable set iff* $\omega \in R$.

Proof. Let $X(\omega) \equiv \times_{j \in N} X_j^{P_j(\omega)}$. *"if part":* Let $\omega \in R$. We proceed to verify that $X(\omega)$ is a stable set.

Internal stability. Assume, in negation, that $y \succ^{X(\omega)} x$ for some $x, y \in X(\omega)$. Then, for some i, y_i strictly dominates x_i given $X(\omega)$. By Lemma 5, $u_i(y_i) > u_i(x_i)$ for all $u_i \in \mathcal{P}_i(X(\omega))$. Since $X(\omega) = X^{P_i(\omega)}$, by Theorem 1 and (1.1) of Lemma 1, it therefore follows that $x_i \notin \arg\max_{z_i \in X_i} u_i^{\omega'}(z_i)$ for all $\omega' \in K_i P_i(\omega)$. However, since $\omega \in R_i$, $x_i \notin X_i^{P_i(\omega)}$, which is a contradiction.

External stability. Let $x \in X \backslash X(\omega)$. Since $\omega \in R$, $x_i \notin BR_i(\omega)$ for some i. Since $X(\omega) = X^{P_i(\omega)}$, by Theorem 1 and (1.1) of Lemma 1, there exists $y_i \in X_i$ that dominates x_i given $X(\omega)$. Consider a partial ordered set $(X_i', \succcurlyeq)$, such that

$$X_i' \equiv \{y_i \in X_i |\; y_i \text{ strictly dominates } x_i \text{ given } X(\omega)\},$$

and for all $x_i', y_i' \in X_i'$,

(a) $y_i' \succ x_i'$ iff y_i' strictly dominates x_i' given $X(\omega)$;

(b) $y_i' \sim x_i'$ iff $y_i' = x_i'$.

By U.5 and marginal consistency, u_i is continuous over X_i. By the compactness of X_i, it is easily verified that every totally-ordered subset of X_i' has an upper bound in X_i'. By making use of Zorn's Lemma, there is a maximal strategy $\widehat{y}_i \in X_i'$ that strictly dominates x_i given $X(\omega)$. By Theorem 1, it must be the case that $\widehat{y}_i \in BR_i(\omega)$. But, since $\omega \in R_i$, $\widehat{y}_i \in X_i^{P_i(\omega)}$. Define $y \in X$ be such that, for all i,

$$y_i = \begin{cases} \widehat{y}_i, \text{ if } x_i \notin BR_i(\omega) \\ x_i, \text{ if } x_i \in BR_i(\omega) \end{cases}.$$

Clearly, $y \in X(\omega)$ and $y \succ^{X(\omega)} x$.

"only if part": Suppose that $X(\omega)$ is a stable set. Then, $x \in X \backslash X(\omega)$ if, and only if, $y \succ^{X(\omega)} x$ for some $y \in X(\omega)$. By Lemma 5, $x \in X \backslash X(\omega)$ if, and only if, there exists a player i such that $u_i(y_i) > u_i(x_i)$ for all $u_i \in \mathcal{P}_i(X(\omega))$. Since $X(\omega) = X^{P_i(\omega)}$, by Theorem 1 and (1.1) of Lemma 1, it follows that $x \in X(\omega)$ if, and only if, for all i, $x_i \in \arg\max_{z_i \in X_i} u_i^{\omega'}(z_i)$ for some $\omega' \in K_i P_i(\omega)$. That is, $X_i^{P_i(\omega)} = BR_i(\omega)$ for all i. Therefore, $\omega \in R$.■

Lemma 8 $CKR = K(R \cap CKR)$.

Proof. By $K3$ and $K4$ in Lemma 2, we have

$$\begin{aligned} K(R \cap CKR) &= KR \cap K(CKR) \\ &= KR \cap K^2 R \cap K^3 R \cap \ldots \\ &= CKR. \blacksquare \end{aligned}$$

We now turn to the proof of Theorem 2.

Proof of Theorem 2. Let $\omega \in R \cap CKR$. By Lemma 7, it suffices to verify $X^{P_i(\omega)} = \times_{j \in N} X_j^{R \cap CKR}$. For $\omega' \in R \cap CKR$ define $\widehat{\omega} \equiv \left(x^{\omega'}; t^{\omega}\right)$. For $j = 1, 2, \ldots, n$, by (1.2) of Lemma 1, $P_j(\widehat{\omega}) = P_j(\omega)$. Since $\omega \in CKR$, by Lemma 8, $P_j(\widehat{\omega}) \subseteq R \cap CKR$. Again, by Lemma 8, $\widehat{\omega} \in CKR$. But, since $\omega \in R_j$, $X_j^{P_j(\omega)} = BR_j(\omega)$. Thus, $X_j^{P_j(\widehat{\omega})} = BR_j(\widehat{\omega})$. That is, $\widehat{\omega} \in R_j$ for $j = 1, 2, \ldots, n$. Therefore, $\widehat{\omega} \in R \cap CKR$. By the weak axiom of knowledge, $\widehat{\omega} \in P_i(\widehat{\omega})$. Thus, $\widehat{\omega} \in P_i(\omega)$, hence, $x^{\omega'} \in X^{P_i(\omega)}$ for all $\omega' \in R \cap CKR$, i.e., $X^{P_i(\omega)} \supseteq X^{R \cap CKR}$. Since, by Lemma 8, $P_i(\omega) \subseteq R \cap CKR$, $X^{P_i(\omega)} \subseteq X^{R \cap CKR}$. Therefore, $X^{P_i(\omega)} = X^{R \cap CKR}$. However, since $X^{P_i(\omega)} = X_i^{P_i(\omega)} \times X_{-i}^{P_i(\omega)}$ for all i, it therefore follows that $X^{P_i(\omega)} = \times_{i \in N} X_i^{P_i(\omega)}$. Consequently, $X^{P_i(\omega)} = \times_{j \in N} X_j^{R \cap CKR}$. ■

An immediate implication of Theorem 2 is the following.

Corollary 3 *For any $\omega \in R \cap CKR$, $X^{P_i(\omega)}$ is a stable set and, moreover, $\times_{j \in N} X_j^{P_{i(j)}(\omega)}$ (where $i(j) \in N$) is a stable set.*

Proof. By the proof of Theorem 2, $X^{P_i(\omega)} = X^{R \cap CKR}$ for all i. By Theorem 2, $X^{P_i(\omega)}$ is a stable set. Moreover, $\times_{j \in N} X_j^{P_{i(j)}(\omega)} = X^{R \cap CKR}$ since, by the proof of Theorem 2, $X^{R \cap CKR} = \times_{j \in N} X_j^{R \cap CKR}$. Thus, $\times_{j \in N} X_j^{P_{i(j)}(\omega)}$ is a stable set.■

Remark 3. Following J. von Neumann and O. Morgenstern, a stable set is viewed as a prevailing social norm in a society. Accordingly, a social norm is "well known to the community" (see Shubik 1982, p. 261). Under this sort of assumption of social knowledge (without assuming the axiom of knowledge), Lemma 7 tells that the "stable" pattern of behavior is sustained by rational players and, moreover, the "stable" pattern of behavior is attributed only to rational players. The following example illustrates that the assumption in Lemma 7 plays a crucial role.

Example 2. Consider the following two-person game of "guessing numbers": $\mathcal{G} = (N, \{X_i\}, \{\zeta_i\})$, where $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, $\zeta_i(x_i, x_j) = 1 - (x_i - x_j)^2$ for $i = 1, 2$ and $i \neq j$.

Case 1. Let $\omega \in \Omega$ be such that

- $P_1(\omega) = [2/3, 1] \times [2/3, 1] \times T_1 \times T_2$
- $P_2(\omega) = [0, 1] \times [0, 1] \times T_1 \times T_2$.

Clearly, $\omega \in R$. However, $X_1^{P_1(\omega)} \times X_2^{P_2(\omega)} = [2/3, 1] \times [0, 1]$ is not a stable set since it violates internal stability, i.e., for any $u_2 \in \mathcal{P}_2\left([2/3, 1] \times [0, 1]\right)$,

$$u_2(1) \geq 8/9 > 5/9 \geq u_2(0).$$

Note that these inequalities follow from the certainty equivalence and the weak monotonicity of regular preferences.

Case 2. Let $\omega \in \Omega$ be such that

- $P_1(\omega) = [0, 1] \times [2/3, 1] \times T_1 \times T_2$
- $P_2(\omega) = [0, 1] \times [0, 1] \times T_1 \times T_2$.

Thus, $X_1^{P_1(\omega)} \times X_2^{P_2(\omega)} = [0, 1] \times [0, 1]$ is a stable set. However, at ω, player 1 is not rational since the strategies lying in $[0, 2/3)$ cannot be rationalized by any $u_1^{\omega} \in \mathcal{P}_2(\Omega | P_1(\omega))$.

4 Discussions

4.1 Epistemic games. Note that a strategic game $\mathcal{G} \equiv (N, \{X_i\}, \{\zeta_i\})$ does not specify players' preferences in the face of strategic uncertainty; it specifies only players' payoff functions ζ_i. From an epistemic perspective, a complete outcome of the game $\mathcal{G}$ is summarized by a state. A "transparent" game associated with $\mathcal{G}$ is determined by epistemic types. Formally, a "transparent" game at type profile t is defined as:

$$(\mathcal{G}, \psi \circ t) \equiv \left\{\omega \in \Omega | \ \left(t_1^{\omega}, t_2^{\omega}, \ldots, t_n^{\omega}\right) = t\right\},$$

where $\psi \circ t = (\psi \circ t_1, \psi \circ t_2, \ldots, \psi \circ t_n)$. The state space Ω can be viewed as an "opaque" game in terms of

$$\Omega = \bigcup_{t \in T_1 \times T_2 \times \ldots \times T_n} (\mathcal{G}, \psi \circ t).$$

Moreover, the game associated with a collection of preference models $\mathcal{P}^* (\Omega) \equiv (\mathcal{P}_1^* (\Omega), \mathcal{P}_2^* (\Omega), \ldots, \mathcal{P}_n^* (\Omega))$ in the sense of Epstein (1997) is given by

$$(\mathcal{G}, \mathcal{P}^* (\Omega)) = \bigcup_{u^n \in \mathcal{P}_1^* (\Omega) \times \mathcal{P}_2^* (\Omega) \times \ldots \times \mathcal{P}_n^* (\Omega)} (\mathcal{G}, u^n).$$

Within our framework in this paper, the statement "a game is common knowledge" is a formal statement rather than an informal "meta-sense": A game is common knowledge if, and only if, the game, as a subset of states, is commonly known (cf. also Zamir and Vassilakis 1993, pp. 496-497). In particular, the "opaque" game is commonly known.

4.2 The rationale behind a choice set. Within the conventional semantic framework, Luo (2002) studied epistemic foundations behind the criterion of stability. In particular, at a state of the world each player is *exogenously* associated with a nonempty subset of strategies that is interpreted as a choice set. In our framework in this paper, the choice set $X_i^{P_i(\omega)}$ should be viewed as *endogenous* since it is deduced from the information structure $P_i(\omega)$ (cf. also Appendix VIII for further discussion).

While in Savage's framework of a single-person's decision making, the decision maker would be well aware of his choice that affects no states, this is not appropriate here. In the context of strategic interaction, each player's choice of strategy should be included in the description of a state since each player must take into account the choices of the other players. For example, the choice of strategy by i should depend on the choice of strategy by j that, in turn, should depend on the choice of strategy by i.[11] Of course, a player can do whatever he

[11] J. von Neumann and O. Morgenstern offered a *defensive and concealment* rationale for mixing play in zero-sum games:

wants, but he might *not* know what it is he wants, because what a player wants to do often depends on what others want to do (see also 4.3). Consequently, if a player unconsciously makes a choice, then he certainly does not know his own choice; if a player consciously makes a choice, then he perhaps does not know his own choice, because the player might not know what it is he wants. Although a state of the world does specify a strategy for a player, the player simply may not know his own strategy in the face of uncertainty. What he knows is only the scope of strategies. The correlation of strategy allowed in our framework could be another origin for the ignorance of strategy. In addition, a player may not know his own choice of strategy in games with imperfect recall (cf. Rubinstein 1998, Chapter 4).[12]

It is easy to see that i knows his strategy x_i^ω at ω if, and only if, $X_i^{P_i(\omega)} = \{x_i^\omega\}$. From an epistemic viewpoint, the requirement that a player knows his using strategy seems to be rather a restrictive assumption in strategic settings. The following example demonstrates this point.

Example 3. Consider a two-person game. For simplicity, we consider only the probabilistic notion of knowledge — i.e., "belief with probability 1." Consider four states as follows:

> Thus one important consideration for a player in such a game is to protect himself against having his intentions found out by his opponent. Playing several such strategies at random, so that only their probabilities are determined is a very effective way to achieve a degree of such protection: By this device the opponent cannot possibly find out what the player's strategy is going to be, since the player does not know it himself (von Neumann and Morgenstern 1944, p. 146).

Therefore, this classical rationale posits that a player may show a tendency to *consciously choose not to know his choice.* Walker and Wooders (2001) found that the serve-and-return play of John McEnroe, Bjorn Borg, Boris Becker, and Pete Sampras at Wimbledon and other professional tennis players is largely consistent with the minimax hypothesis. For non-zero-sum games, see Reny and Robson (2002). See also Lo (1996) and Eichberger and Kelsey (1996) for discussions on mixing behavior in the non-expected utility model.

[12] In practical decision-making, people screw up, break hearts, and get annoyed. They often suffer from the difficulty to make a decision.

$$\begin{cases} \omega_1 = (x_1, x_2; t_1, t_2) \\ \omega_2 = (x_1', x_2; t_1, t_2) \\ \omega_3 = (x_1, x_2; t_1', t_2) \\ \omega_4 = (x_1', x_2; t_1', t_2) \end{cases}.$$

Since $T_i \sim^{homeomorphic} \Delta(T_i \times T_j \times X)$, we let $\mu_{t_1} = \psi \circ t_1$ and $\mu_{t_1'} = \psi \circ t_1'$ such that

$$\mu_{t_1}(\omega_i) = \begin{cases} 1/2, & \text{if } i = 1, 2 \\ 0, & \text{if } i = 3, 4 \end{cases} \text{ and } \mu_{t_1'}(\omega_i) = 1/4 \text{ for } i = 1, 2, 3, 4.$$

Thus, we have

$$P_1(\omega) = \begin{cases} \{\omega_1, \omega_2\}, & \text{if } \omega = \omega_1, \omega_2 \\ \{\omega_1, \omega_2, \omega_3, \omega_4\}, & \text{if } \omega = \omega_3, \omega_4 \end{cases}.$$

While player 1 knows his own type at ω_1, he does not know his own strategy at that state. In particular, $X_1^{P_i(\omega)} = \{x_1, x_1'\}$.

4.3 Ignorance of own type. Note that $T_i \sim^{homeomorphic} \mathcal{P}(\Omega)$. A player with an epistemic type is uncertain not only about the strategy profiles, but also about the type profiles.[13] In particular, the player is uncertain about his own types or own preferences (see also Heifetz and Samet's (1998, p. 330) Remark). In Example 3, at ω_3 player 1 does not know whether his type is t_1 or t_1'. As Epstein and Wang (1996, p. 1352) wrote, "... it seems natural given an agent who does not perfectly understand the nature of the primitive state space ... and who reflects on the nature and degree of his misunderstanding. ... uncertainty about own preferences has been shown to be useful also in modeling preference for flexibility (Kreps (1979)) and behavior given unforeseen contingencies (Kreps (1992))."

[13] To expound his theory of games with incomplete information, Harsanyi (1967, p.171) articulated that "Each player is assumed to know his own actual type" (cf. also Harsanyi 1995, p.296). To make sense of the notion of a Bayesian equilibrium, each player should be aware of his own using strategy, of course. As Binmore (1992, p. 502) pointed out, "Harsanyi's theory ... leaves a great deal to the judgment of those who use it. It points a figure at what is missing in an information structure, but does not say where the missing information is to be found." See also Myerson (1985, pp. 238-239).

In the case of a single-person decision making, this viewpoint relates to the decision maker's introspection — i.e., he is uncertain not only about the true *state of nature*, but also about his preferences about this uncertainty, his preferences about his preferences about this uncertainty, and so on. The viewpoint of the ignorance of own type subsequently puts forward a novel interpretation for using the notion of choice sets in orthodox choice theory.

4.4 The definition of rationality. To study the set-valued solution concept of a stable set, the notion of rationality used in this paper is a bit different from the conventional one used in the literature — e.g., Aumann and Brandenburger (1995) and Epstein (1997) defined "player i is rational at ω" as: $u_i^{\omega}\left(x_i^{\omega}\right) \geq u_i^{\omega}(y_i)$ for all $y_i \in X_i$.[14] (See Appendix IX for the relationship between this *ex post* rationality and stability.) However, as Aumann and Brandenburger (1995, Section 7a) pointed out, this sort of definition of rationality is purely *descriptive*; it purports to describe what *do* players do and what *do* they believe; not *why* players do what they do, not what *should* they do. To make sense of it from an individual's viewpoint, a player should be aware of (and hence know) his own *true* type and of his own *using* action. By (1.2) of Lemma 1, the information structure is therefore partitional. Subsequently, this definition of rationality arises the question about its applicability in general cases where players are boundedly rational.

The notion of rationality used in this paper is based upon the epistemic aspects and, hence, it is also *prescriptive*. Furthermore, the notion of rationality should be referred to a player's type since preferences are determined by the player's type. For example, let t_i^* be i's *rational* type, and let t_i' and t_i'' be two plausible types that t_i^* cannot exclude. Suppose that x_i' and x_i'' are best responses with respect to $\psi \circ t_i'$ and $\psi \circ t_i''$, respectively.[15] It seems natural that the rational type t_i^* would not preclude the choices of x_i' and x_i'' from i's disposal and, moreover, the rational type t_i^* should preclude all the strategies that are not a best response to any of his types that he cannot exclude. That is exactly the definition used in this paper.

[14] Aumann (1995) also defined a weak version of rationality — roughly, a player is rational if, and only if, he does not know that he would be able to do better.

[15] The true preferences are irrelevant to evaluating optimal choices. Only the perceivable and conscious preferences matter for this evaluation. See also Harsanyi's (1997) discussion on "actual" vs. "informed" preferences.

4.5 The completeness of a state space. In this paper we view a state as a *full* description of the world. It is best to think of a state as an *endogenous* variable since a state is constructed by strategies and Harsanyi's types.[16] A state specifies what every player does, and what every player thinks about what every player does, and so on; it specifies every player's preferences, and every player's preferences about every player's preferences, and so on; it specifies what every player knows, and what every player knows about what every player knows, and so on. The state space includes all possible states and is intrinsically infinite. The completeness of a state space is crucial for our main results in this paper. In particular, we cannot expect a similar result as Theorem 1 within an *exogenous* finite model of state space.

4.6. Applications in other models of preferences. Throughout this paper we restrict attention to the class of regular preferences, and all results here are not confined with the restriction. As pointed out in Epstein (1997), our analysis can be applied to other specific models of preferences; for example, the subjective expected utility model, the ordinal expected utility model, the probabilistic sophistication model, the Choquet expected utility model, and so on.

4.7 The significance of the "stable" pattern of behavior. To connect with real-life phenomena, von Neumann and Morgenstern (1947) literally interpreted a vN-M stable set as a social norm in a society. As von Neumann and Morgenstern put it so bluntly, "This is clearly how things are in actual social organizations ..." (see von Neumann and Morgenstern 1947, p. 42). The rationales behind the idea of stability are deeply profound.

[16] In Kripke's model $\mathcal{M}$, a state is "endogenously" defined as a closed, coherent, and complete list of all formulae ϕ that are true at that state, i.e., $\omega = \{\phi| \ (\mathcal{M}, \omega) \models \phi\}$ (cf. Rubinstein's (1998) Chapter 3). The notion of a state is also given another interpretation in the literature. A state can be viewed as an *exogenous* variable in economic models of uncertainty — i.e., a description of the contingencies that the decision-maker perceives to be relevant in the context of a certain decision problem. In accordance with this sort of interpretation, it seems fairly natural to assume that the decision-maker knows his own type and his action; see also Aumann and Brandenburger (1995, pp. 1175-1176).

APPENDIX I: REGULAR PREFERENCE

Let $\mathcal{F}^u(\Omega) = \{f \in \mathcal{F}(\Omega) |\ f(\Omega) \text{ is finite; } f^{-1}([r,1]) \text{ is closed for any } r \in [0,1]\}$. Let $\mathcal{F}^l(\Omega) = \{f \in \mathcal{F}(\Omega) |\ f(\Omega) \text{ is finite; } f^{-1}((r,1]) \text{ is open for any } r \in [0,1]\}$. A preference is said to be *regular* if it has a numerical representation $u : \mathcal{F}(\Omega) \to [0,1]$ satisfying:

U.1. Certainty Equivalence: $u(r) = r$, $\forall r \in [0,1]$.

U.2. Weak Monotonicity: $f' \geq f \Rightarrow u(f') \geq u(f)$, $\forall f, f' \in \mathcal{F}(\Omega)$.

U.3. Inner Regularity: $u(f) = \sup \{u(g) : g \leq f, g \in \mathcal{F}^u(\Omega)\}$, $\forall f \in \mathcal{F}(\Omega)$.

U.4. Outer Regularity: $u(g) = \inf \{u(h) : h \geq g, h \in \mathcal{F}^l(\Omega)\}$, $\forall g \in \mathcal{F}^u(\Omega)$.

A regular preference u is said to be "strongly monotonic" if it satisfies:

U.2′. Strong Monotonicity: $f' > f \Rightarrow u(f') > u(f)$, $\forall f, f' \in \mathcal{F}(\Omega)$.

For the purpose of this paper, we also need the following two additional conditions:

U.5. Uniform Equicontinuity:[17] $\forall \varepsilon > 0$, $\exists \delta$ such that for every $u \in \mathcal{P}(\Omega)$

$$|u(f) - u(f')| < \varepsilon, \text{ whenever } sup_{\omega \in \Omega} |f(\omega) - f'(\omega)| < \delta.$$

U.6. Preference-model Closedness:[18] For any closed set $E \subseteq \Omega$, $\mathcal{P}(E)$ is closed.

APPENDIX II: MARGINAL CONSISTENCY

[17] We assume that $\mathcal{F}(\Omega)$ is endowed with sup-norm topology.

[18] See Epstein and Wang (1996) for the definition of the topology on $\mathcal{P}(E)$.

Marginal consistency is introduced as a primitive requirement in a case where a player is endowed with an arbitrary set of preferences. For the special case of regular preferences, the "marginal consistency" can be defined as follows. Let $\mathcal{F}_i(X)$ denote the set of acts $f : X \to [0,1]$, satisfying $f(x_i, x_{-i}) = f(x_i', x_{-i})$ for all (x_i, x_{-i}) and (x_i', x_{-i}) in X. For any $E \subseteq \Omega$ and $u \in \mathcal{P}(E)$, the "restriction of u to $\mathcal{F}_i(X)$" is referred as a preference in $\mathcal{P}_i(X^E)$, denoted by $mrg_{\mathcal{F}_i(X)}u$. Say u satisfies the *marginal consistency* if, $\forall g \in \mathcal{F}_i(X)$, $\forall f \in \mathcal{F}(E)$, $mrg_{\mathcal{F}_i(X)}u(g) = u(f)$ whenever $g(x^\omega) = f(\omega)$ (in particular, $mrg_{\mathcal{F}_i(X)}u(x_i) = u(x_i)$ $\forall x_i \in X_i$). By Epstein and Wang's (1996) Theorem D.2, u must satisfy marginal consistency and, hence, $\left\{ mrg_{\mathcal{F}_i(X)}u \mid u \in \mathcal{P}(E) \right\} = \mathcal{P}_i(X^E)$.

APPENDIX III: PROOF OF LEMMA 2

Proof of Lemma 2. Clearly, $K1$ holds by U.1; $K2$ and $K3$ hold by the definition of knowledge. To prove $K4$, note that X satisfies the *second axiom of countability* — i.e., the topology on X has a countable basis — since X is a compact metric space (see, e.g., Aliprantis and Border 1999, Chapter 3). We divide this proof into the following three steps.

Step 1. Ω satisfies the second axiom of countability.

By the construction of a type space, $\Omega \subseteq \Omega_0 \times (\times_{k=0}^{\infty} \mathcal{P}^n(\Omega_k))$, where $\Omega_0 = X$ and $\Omega_k = \Omega_{k-1} \times \mathcal{P}^n(\Omega_{k-1})$ for $k \geq 1$. By the fact that the countable Cartesian product of the second countable spaces is the second countable, it suffices to show that $\mathcal{P}(X)$ satisfies the second axiom of countability.

Following Epstein and Wang (1996), consider the topology on $\mathcal{P}(X)$ generated by the subbasis consisting of:

$$\{u : \ u(g) < r,\, g \in \mathcal{F}^u(X),\, r \in [0,1]\} \text{ and } \{u : \ u(h) > r,\, h \in \mathcal{F}^l(X),\, r \in [0,1]\}.$$

Let $\mathcal{B}_\tau$ be a countable basis of this topology on X, and let

$$\mathcal{B} \equiv \left\{ B \mid B = \cup_{k=1}^{K} B_k, B_k \in \mathcal{B}_\tau \right\},$$

where K is a positive integer, and let

$$\mathcal{C} \equiv \{C \mid C = X \backslash B, B \in \mathcal{B}\}.$$

Consider the following two classes of functions:

$$\begin{aligned}\widehat{\mathcal{F}}^u(X) &\equiv \left\{ f \in \mathcal{F}^u(X) |\ f = \sum_{k=1}^{K} q_k 1_{C_k};\ q_k \in Q \text{ and } C_k \in \mathcal{C} \right\} \text{ and} \\ \widehat{\mathcal{F}}^l(X) &\equiv \left\{ f \in \mathcal{F}^l(X) |\ \ f = \sum_{k=1}^{K} q_k 1_{B_k};\ q_k\ \in Q \text{ and } B_k \in \mathcal{B} \right\},\end{aligned}$$

where Q is the set of all rational numbers in $[0, 1]$. Clearly, $\widehat{\mathcal{F}}^u(X)$ and $\widehat{\mathcal{F}}^l(X)$ are both countable sets. Now consider the following class of sets:

$$\{u : u(g) < q,\ g \in \widehat{\mathcal{F}}^u(X),\ q \in Q\} \ \text{ and } \{u : u(h) > q,\ h \in \widehat{\mathcal{F}}^l(X),\ q \in Q\}.$$

Note that $h \in \mathcal{F}^l(X)$ can be expressed as $h = \Sigma_{k=1}^{K} r_k 1_{G_k}$, where $r_k \in [0, 1]$ and G_k is open in X (cf. Epstein and Wang 1996, p. 1366). Since $\mathcal{B}_\tau$ is a countable basis, $G_k = \cup_{l=1}^{\infty} B_{l,k}$ (where $B_{l,k} \in \mathcal{B}_\tau$). For r, $r_k \in [0, 1]$, we can find $q_{m,k} \uparrow r_k$ and $q_k \downarrow r$, where $q_{m,k}$, $q_k \in Q$. Define $h_m \equiv \Sigma_{k=1}^{K} q_{m,k} 1_{\cup_{l=1}^{m} B_{l,k}}$. Clearly, $h_m \leq h$ and $h_m(x) \uparrow h(x)$ for each $x \in X$. Now by U.3, for any $\varepsilon > 0$, there exists $g \leq h$, $g \in \mathcal{F}^u(\Omega)$ such that

$$u(h) - \varepsilon < u(g) \leq u(h).$$

By U.5, without loss of generality we may assume $g < h$. Since $g \in \mathcal{F}^u(X)$ can be expressed as $g = \Sigma_{k=1}^{K'} r'_k 1_{F_k}$, where $r'_k \in [0, 1]$ and F_k is closed in X (cf. Epstein and Wang 1996, p. 1366). Therefore, $h_m \geq g$ for sufficiently large m. Thus, $u(h_m) \uparrow u(h)$. Hence,

$$\{u : u(h) > r\} = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \{u : u(h_m) > q_k\}.$$

Similarly, we have

$$\{u : u(g) < r\} = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \{u : u(g_m) < q_k\}.$$

Thus, $\mathcal{E}$ generates the topology on $\mathcal{P}(X)$. Since $\mathcal{E}$ is countable, $\mathcal{P}(X)$ satisfies the second axiom of countability.

Step 2. $K_iE \cap K_iF \subseteq K_i(E \cap F)$ *for any closed sets* $E, F \subseteq \Omega$.

Let $\omega \in K_iE \cap K_iF$. Since E and F are closed, $u_i^{\omega} \in \mathcal{P}(\Omega|E)$ and $u_i^{\omega} \in \mathcal{P}(\Omega|F)$. Therefore,

$$u_i^{\omega}(f) = u_i^{\omega}(1_E f) \text{ and } u_i^{\omega}(f) = u_i^{\omega}(1_F f), \forall f \in \mathcal{F}(\Omega).$$

Thus,

$$u_i^{\omega}(f) = u_i^{\omega}(1_E f) = u_i^{\omega}(1_{E\cap F} f), \forall f \in \mathcal{F}(\Omega).$$

That is, $\omega \in K_i(E \cap F)$.

Step 3. $\bigcap_{\lambda\in\Lambda} K_iE^{\lambda} \subseteq K_i(\bigcap_{\lambda\in\Lambda} E^{\lambda})$.

Let $\{E^{\lambda}\}_{\lambda\in\Lambda}$ be a family of closed subsets of Ω . Since, by Step 1, Ω satisfies the second axiom of countability, it follows that there exists a countable sequence of open sets $\left\{\Omega \backslash E^k\right\}_{k=1}^{\infty}$ such that

$$\bigcup_{\lambda\in\Lambda} \left[\Omega \backslash E^{\lambda}\right] = \bigcup_{k=1}^{\infty} \left[\Omega \backslash E^k\right]$$

or equivalently

$$\bigcap_{\lambda\in\Lambda} E^{\lambda} = \bigcap_{k=1}^{\infty} E^k.$$

Without loss of generality, $\forall k \geqslant 1$, $E^k \supseteq E^{\lambda}$ for some $\lambda \in \Lambda$. Let $\omega \in \bigcap_{\lambda\in\Lambda} K_iE^{\lambda}$. Then, $\omega \in K_iE^k$, $\forall k \geqslant 1$. Now consider the sequence $\left\{\overline{E}^k\right\}_{k=1}^{\infty}$ such that $\overline{E}^1 = E^1, \overline{E}^2 = \overline{E}^1 \cap E^2, ..., \overline{E}^k = \overline{E}^{k-1} \cap E^k, ...$ Clearly, $\overline{E}^k \downarrow \cap_{\lambda\in\Lambda} E^{\lambda}$. By Step 2, $\omega \in K_i^k\overline{E}$, $\forall k \geqslant 1$. The result therefore follows from Epstein and Wang's (1996) Theorem 4.4. ■

APPENDIX IV: PROOF OF LEMMA 4

Proof of Lemma 4. For any $u_1, u_2 \in \mathcal{P}(E)$ and $\alpha \in [0,1]$, we proceed to verify that $\alpha u_1 + (1-\alpha)u_2 \in \mathcal{P}(E)$. Obviously, U.1, U.2, U.2′, U.5, and U.6 hold. Let $f \in \mathcal{F}(E)$. Then,

$$\begin{aligned}
& [\alpha u_1 + (1-\alpha)u_2](f) \\
= \;& \alpha u_1(f) + (1-\alpha)u_2(f) \\
= \;& \sup\{\alpha u_1(g) : g \leq f, g \in \mathcal{F}^u(E)\} + \sup\{(1-\alpha)\,u_2(g) : g \leq f, g \in \mathcal{F}^u(E)\} \\
\geq \;& \sup\{\alpha u_1(g) + (1-\alpha)u_2(g) : g \leq f, g \in \mathcal{F}^u(E)\} \\
= \;& \sup\{[\alpha u_1 + (1-\alpha)u_2](g) : g \leq f, g \in \mathcal{F}^u(E)\}.
\end{aligned}$$

Moreover, for sufficiently small $\varepsilon > 0$, there exist $g_1, g_2 \in \mathcal{F}^u(E)$ such that $g_1 \leq f$, $g_2 \leq f$, $u_1(g_1) > u_1(f) - \varepsilon$, and $u_2(g_2) > u_2(f) - \varepsilon$. Define $g'(\omega) \equiv \max[g_1(\omega), g_2(\omega)]$. Clearly, $g' \in \mathcal{F}^u(E)$ and $g' \leq f$. By U.2, it follows that

$$\begin{aligned}
& \sup\{[\alpha u_1 + (1-\alpha)u_2](g) : g \leq f, g \in \mathcal{F}^u(E)\} \\
\geq \;& \alpha u_1(g') + (1-\alpha)u_2(g') \\
\geq \;& \alpha u_1(g_1) + (1-\alpha)u_2(g_2) \\
\geq \;& \alpha u_1(f) + (1-\alpha)u_2(f) - \varepsilon \\
= \;& [\alpha u_1 + (1-\alpha)u_2](f) - \varepsilon.
\end{aligned}$$

Thus, U.3 holds. Similarly, U.4 holds. ■

APPENDIX V: PROOF OF LEMMA 5

Proof of Lemma 5. Let E be a compact event satisfying $X^E = Y$. Clearly, $X_{-i}^E = Y_{-i}$.

"if part": Suppose that $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}_i(Y)$. By marginal consistency, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. For any $\omega \in E$, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E|\{\omega\})$. Since $\mathcal{P}(E|\{\omega\}) \sim^{homeomorphic} \mathcal{P}(\{\omega\})$, by U.1

$$x_i(\omega) = u_i(x_i) > u_i(y_i) = y_i(\omega)$$

for all $u_i \in \mathcal{P}(\{\omega\})$. Thus, $\zeta_i(x_i, x_{-i}^\omega) = x_i(\omega) > y_i(\omega) = \zeta_i(y_i, x_{-i}^\omega)$ for all $\omega \in E$. Since $X_{-i}^E = Y_{-i}$, it therefore follows that $\zeta_i(x_i, y_{-i}) > \zeta_i(y_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$.

"only if part": If $\zeta_i(x_i, y_{-i}) > \zeta_i(y_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$, we have

$$x_i(\omega) = \zeta_i(x_i, \overset{\omega}{x}_{-i}) > \zeta_i(y_i, \overset{\omega}{x}_{-i}) = y_i(\omega), \text{ for all } \omega \in E,$$

since $X^{E}_{-i} = Y_{-i}$. Let $\overline{x}_i, \overline{y}_i \in \mathcal{F}(\Omega)$ satisfying

$$\overline{x}_i(\omega) = \begin{cases} x_i(\omega), & \text{if } \omega \in E \\ 1, & \text{if } \omega \in \Omega/E \end{cases} \text{ and } \overline{y}_i(\omega) = \begin{cases} y_i(\omega), & \text{if } \omega \in E \\ 0, & \text{if } \omega \in \Omega/E \end{cases}.$$

By Epstein and Wang's (1996) Theorem 4.3, $\mathcal{P}(E) \sim^{homeomorphic} \mathcal{P}(\Omega|E)$. Let $\varphi : \mathcal{P}(E) \to \mathcal{P}(\Omega|E)$ be such a homeomorphism. By strong monotonicity, $\varphi \circ u_i(\overline{x}_i) > \varphi \circ u_i(\overline{y}_i)$ for all $u_i \in \mathcal{P}(E)$. Thus, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. By marginal consistency, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}_i(Y)$. ■

APPENDIX VI: CONTINUITY OF $\zeta'(.,.)$

Proof of Continuity of $\zeta'(.,.)$. The proof is split into the following three steps.

Step 1. If $x_i^m \to x_i$, $sup_{\omega\in\Omega}|x_i^m(\omega) - x_i(\omega)| \to 0$.

Since $\zeta_i(.)$ is continuous and X is compact, $\zeta_i(.)$ is uniformly continuous on X.[19] Hence, for any $\varepsilon > 0$, there exists δ such that whenever $d_i(x_i^m, x_i) < \delta$, we have $|x_i^m(\omega) - x_i(\omega)| = |\zeta_i(x_i^m, \overset{\omega}{x}_{-i}) - \zeta_i(x_i, \overset{\omega}{x}_{-i})| < \varepsilon$ for all ω.

Step 2. For any continuous function $f \in \mathcal{F}(\Omega)$, $u^m(f) \to u(f)$ as $u^m \to u$.

To prove this, it suffices to show that, for all real numbers r, $\{u : u(f) > r\}$ and $\{u : u(f) < r\}$ are open. Since f is continuous, we can find $f_n \in \mathcal{F}^l(\Omega)$ that

$$f_n = \frac{1}{2^n}\sum_{j=1}^{2^n} 1_{G_{nj}}, \text{ where } G_{nj} = \{\omega : f(\omega) > j2^{-n}\}.$$

Clearly, $f_n \uparrow f$ uniformly. By U.5,

$$\{u : u(f) > r\} = \bigcup_{n=1}^{\infty}\{u : u(f_n) > r\}.$$

Thus, $\{u : u(f) > r\}$ is open. Similarly, $\{u : u(f) < r\}$ is open.

[19] We denote the metric for X_i by d_i and denote the metric for X by $d(x, x') = \left(\sum_{i=1}^{n} d_i(x_i, x_i')^2\right)^{1/2}$ for all $x, x' \in X$.

Step 3. $\zeta_i'(x_i, u_i)$ *is jointly continuous.*

Let $(x_i^m, u_i^m) \to (x_i, u_i)$ be a sequence in $X_i \times \mathcal{P}(E)$. Let $\varepsilon > 0$ be sufficiently small. Then, by Step1 and U.5, for sufficiently large m, $|u_i'(x_i^m) - u_i'(x_i)| < \varepsilon/3$ for all $u_i' \in \mathcal{P}(E)$. Since the payoff function $\zeta_i(\cdot)$ is continuous, it therefore follows that x_i is a continuous act. By Step 2, for sufficiently large m, $|u_i^m(x_i) - u_i(x_i)| < \varepsilon/3$ and $|u_i^m(y_i) - u_i(y_i)| < \varepsilon/3$. Hence, we have

$$\begin{aligned}
|\zeta_i'(x_i^m, u_i^m) - \zeta_i'(x_i, u_i)| &\leq |\overset{\omega m}{u_i}(x_i^m) - \overset{\omega}{u_i}(x_i)| + |\overset{\omega m}{u_i}(y_i) - \overset{\omega}{u_i}(y_i)| \\
&\leq |\overset{\omega m}{u_i}(x_i^m) - \overset{\omega m}{u_i}(x_i)| + |\overset{\omega m}{u_i}(x_i) - \overset{\omega}{u_i}(x_i)| \\
&\quad + |\overset{\omega m}{u_i}(y_i) - \overset{\omega}{u_i}(y_i)| \\
&< \varepsilon. \blacksquare
\end{aligned}$$

APPENDIX VII: PROOF OF LEMMA 6

Proof of Lemma 6. The proof is split into two steps. For any event E, let $T_i^E \equiv \left\{ \overset{\omega}{t_i} \mid \omega \in E \right\}$.

Step 1. $R = X \times T_1^{R_1} \times T_2^{R_2} \times \ldots \times T_n^{R_n}$.

It suffices to show that $R_i \supseteq X \times T_i^{R_i} \times T_{-i}$ for all i. Let $\omega' \equiv (x; \overset{\omega}{t_i}, t_{-i})$ such that $(x; t_{-i}) \in X \times T_{-i}$ and $\omega \in R_i$. By (1.2) of Lemma 1, $P_i(\omega') = P_i(\omega)$. Thus, $BR_i(\omega') = BR_i(\omega)$ and $X_i^{P_i(\omega')} = X_i^{P_i(\omega)}$. Since $\omega \in R_i$, it therefore follows that $X_i^{P_i(\omega')} = BR_i(\omega')$. That is, $\omega' \in R_i$.

Step 2. $X^{CKR} = X$.

We proceed to show that, for any event E,

$$KE = X \times T_1^{K_1 E} \times T_2^{K_2 E} \times \ldots \times T_n^{K_n E}.$$

To show this, it suffices to prove $K_i E \supseteq X \times T_i^{K_i E} \times T_{-i}$ for all i. Let $\omega' \equiv (x; \overset{\omega}{t_i}, t_{-i})$ such that $(x; t_{-i}) \in X \times T_{-i}$ and $\omega \in K_i E$. By (1.2) of Lemma 1,

$P_i(\omega') = P_i(\omega)$. Since $\omega \in K_i E$, it therefore follows that $\omega' \in K_i E$. Thus, for all $l \geq 1$

$$K^l R = X \times T_1^{K_1\left(K^{l-1}R\right)} \times T_2^{K_2\left(K^{l-1}R\right)} \times \ldots \times T_n^{K_n\left(K^{l-1}R\right)}.$$

Hence, $X^{R \cap CKR} = X$.■

APPENDIX VIII: CONSISTENCY

In this appendix, we assume the axiom of knowledge and the axiom of transparency, rather than the axiom of wisdom.[20]

Proposition 1. *For all i, $X^{P_i(\omega)} = \cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right]$ iff $\times_{j \in N} X_j^{P_j(\omega)} \subseteq X^{P_i(\omega)}$. In particular, for all i and j, $X_j^{P_i(\omega)} = \cup_{\omega' \in P_i(\omega)} X_j^{P_j(\omega')}$ iff $X_j^{P_j(\omega)} \subseteq X_j^{P_i(\omega)}$.*

Proof. Suppose $\times_{j \in N} X_j^{P_j(\omega)} \subseteq X^{P_i(\omega)}$ for all ω. Then, $\cup_{\omega' \in P_i(\omega)} \times_{j \in N} X_j^{P_j(\omega')} \subseteq \cup_{\omega' \in P_i(\omega)} X^{P_i(\omega')}$. By the axiom of transparency, $\cup_{\omega' \in P_i(\omega)} X^{P_i(\omega')} \subseteq X^{P_i(\omega)}$. Therefore, $\cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right] \subseteq X^{P_i(\omega)}$. However, by the axiom of knowledge, $x^{\omega'} \in \times_{j \in N} X_j^{P_j(\omega')}$ for all $\omega' \in P_i(\omega)$. Thus, $X^{P_i(\omega)} \subseteq \cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right]$. Hence, $X^{P_i(\omega)} = \cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right]$. Conversely, suppose $X^{P_i(\omega)} = \cup_{\omega' \in P_i(\omega)} \times_{j \in N} X_j^{P_j(\omega')}$. By the axiom of knowledge, $\times_{j \in N} X_j^{P_j(\omega)} \subseteq \cup_{\omega' \in P_i(\omega)} \times_{j \in N} X_j^{P_j(\omega')}$. Hence, $\times_{j \in N} X_j^{P_j(\omega)} \subseteq X^{P_i(\omega)}$. Similarly, it is easy to verify that, for all i and j, $X_j^{P_i(\omega)} = \cup_{\omega' \in P_i(\omega)} X_j^{P_j(\omega')}$ iff $X_j^{P_j(\omega)} \subseteq X_j^{P_i(\omega)}$. ■

To view $X^{P_i(\omega)}$ as the choice set of i, it seems natural to require the "consistency" — i.e., $\mathcal{P}_i(X^{P_i(\omega)}) = \mathcal{P}_i(\cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right])$, since all information structures $P_i(\cdot)$ are commonly known. Consider the following two conditions on each player i's information structure.

[20] Under these two axioms, $P_i(\omega) = K_i P_i(\omega)$. Thus, "$i$ is rational at ω" iff $X_i^{P_i(\omega)} = \left\{x_i \in X_i \mid u_i^{\omega'}(x_i) \geq u_i^{\omega'}(y_i) \text{ for some } \omega' \in P_i(\omega) \text{ and all } y_i \in X_i\right\}$.

A1. $X_j^{P_j(\omega)} \subseteq X_j^{P_i(\omega)}$ for all j.
A2. $\times_{j \in N} X_j^{P_i(\omega)} \subseteq X^{P_i(\omega)}$.

That is, A1 states that each player has better information regarding his own choice(s) than an opponent does; A2 states that each player is aware of the independence of his opponents' choices. The following proposition provides epistemic conditions that guarantee this sort of "consistency."

Proposition 2. *Under A1 and A2,* $P_i(X^{P_i(\omega)}) = P_i(\cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right])$ *for all* i.

Proof. It suffices to show $X^{P_i(\omega)} = \cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right]$. Clearly, $\cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right] \subseteq \times_{j \in N} \left[\cup_{\omega' \in P_i(\omega)} X_j^{P_j(\omega')}\right]$. By Proposition 1, $\times_{j \in N} \left[\cup_{\omega' \in P_i(\omega)} X_j^{P_j(\omega')}\right] = \times_{j \in N} X_j^{P_i(\omega)}$. By A2, it follows that $\cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right] \subseteq X^{P_i(\omega)}$. However, by the axiom of knowledge, $x^{\omega'} \in \times_{j \in N} X_j^{P_j(\omega')}$ for all $\omega' \in P_i(\omega)$. Therefore, $X^{P_i(\omega)} \subseteq \cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right]$. Hence, $X^{P_i(\omega)} = \cup_{\omega' \in P_i(\omega)} \left[\times_{j \in N} X_j^{P_j(\omega')}\right]$. ■

APPENDIX IX: EX POST RATIONALITY VS. STABILITY

Let $\widehat{R}_i \equiv \{\omega | \; u_i^{\omega}(x_i^{\omega}) \geq u_i^{\omega}(x_i), \forall x_i \in X_i\}$, and let $\widehat{R} \equiv \cap_{i \in N} \widehat{R}_i$.

Proposition 3. $X^{CK\widehat{R} \cap \widehat{R}}$ *is a stable set.*

To prove Proposition 3, we need the following four lemmas.

Lemma 9 $X^{CK\widehat{R} \cap \widehat{R}} = \times_{i \in N} \{x_i | \; u_i^{\omega}(x_i) \geq u_i^{\omega}(y_i)$ *for all* $y_i \in X_i$ *and for some* $\omega \in CK\widehat{R} \cap \widehat{R}\}$.

Proof. For $i = 1, 2, \ldots, n$, let x_i be such that $u_i^{\omega_i}(x_i) \geq u_i^{\omega_i}(y_i)$ for all $y_i \in X_i$ and for some $\omega_i \in CK\widehat{R} \cap \widehat{R}$. Define $\widehat{\omega} \equiv (x_1, \ldots, x_n; t_1^{\omega_1}, \ldots, t_n^{\omega_n})$. By (1.2) of Lemma 1, $\widehat{\omega} \in CK\widehat{R}$. Moreover, $\widehat{\omega} \in \widehat{R}$ since $\varphi \circ t_i^{\widehat{\omega}} = u_i^{\omega_i}$ for all i. Thus, $(x_1, \ldots, x_n) \in X^{CK\widehat{R} \cap \widehat{R}}$. Conversely, let $\omega \in CK\widehat{R} \cap \widehat{R}$. Since $\omega \in \widehat{R}_i$, $u_i^{\omega}(x_i^{\omega}) \geq u_i^{\omega}(x_i)$, $\forall x_i \in X_i$. ■

Lemma 10 $T_i^{CK\widehat{R} \cap \widehat{R}} = T_i^{K_i(CK\widehat{R} \cap \widehat{R})}$.

Proof. The proof is split into the following three steps.

Step 1. $CK\widehat{R} = K\left(CK\widehat{R} \cap \widehat{R}\right)$.

The proof of this equality is totally similar to that of Lemma 8.

Step 2. $T_i^{CK\widehat{R}\cap\widehat{R}} = T_i^{CK\widehat{R}}$.

It suffices to prove $T_i^{\widehat{R}} = T_i$. For $j = 1, 2, \ldots, n$, let $t_j \in T_j$. Since X_j is compact and $\psi \circ t_j$ is continuous, there exists x_j^* in X_j that is a best reply with respect to $\psi \circ t_j$. Define $\widehat{\omega} \equiv (x_i^*, x_{-i}^*; t_i, t_{-i})$. By (1.2) of Lemma 1, $\widehat{\omega} \in \widehat{R}$. Hence, $t_i \in T_i^{\widehat{R}}$.

Step 3. $T_i^{K_iE} = T_i^{KE}$ *for every event* E.

Clearly, $KE \subseteq K_iE$. Note that $KE = \emptyset$ implies $K_iE = \emptyset$ since each player's type space is homogeneous (see footnote 4). It therefore suffices to prove that $T_i^{K_iE} \subseteq T_i^{KE}$ if $KE \neq \emptyset$. Let $\omega_i \in K_iE$ and let $\omega \in KE$. Define $\widehat{\omega} \equiv (x^{\omega}; t_i^{\omega_i}, t_{-i}^{\omega})$. By (1.2) of Lemma 1, $\widehat{\omega} \in KE$. Hence, $t_i^{\omega_i} \in T_i^{KE}$. ■

Lemma 11 $x_i \in X_i^{CK\widehat{R}\cap\widehat{R}}$ *iff it is a best response given* $CK\widehat{R} \cap \widehat{R}$.

Proof. By Lemmas 9-10, $x_i \in X_i^{CK\widehat{R}\cap\widehat{R}}$ if, and only if, for some $\omega \in K_i\left(CK\widehat{R} \cap \widehat{R}\right)$, $u_i^{\omega}(x_i) \geq u_i^{\omega}(y_i)$ for all $y_i \in X_i$. ■

Lemma 12 $CK\widehat{R} \cap \widehat{R}$ *is compact.*

Proof. The proof is split into the following two steps.

Step 1. $\widehat{R}$ *is compact.*

Consider a sequence $\{\omega_m\}$ in $\widehat{R}_i$ such that $\omega_m \to \omega$. It follows that for $m = 1, 2, \ldots$, $u_i^{\omega_m}(x_i^{\omega_m}) \geq u_i^{\omega_m}(x_i)$, $\forall x_i \in X_i$. Similarly to Steps 2 and 3 in Appendix VI, $u_i^{\omega_m}(x_i) \to u_i^{\omega}(x_i)$ and $u_i^{\omega_m}(x_i^{\omega_m}) \to u_i^{\omega}(x_i^{\omega})$. Hence, $u_i^{\omega}(x_i^{\omega}) \geq u_i^{\omega}(x_i)$, $\forall x_i \in X_i$. Thus, $\widehat{R}_i$ is compact for all i. Hence, $\widehat{R}$ is compact.

Step 2. $CK\widehat{R}$ *is compact.*

Let E be an arbitrary compact event. It suffices to show that KE is compact. Since $\mathcal{P}(E) \sim^{homeomorphic} \mathcal{P}(\Omega|E)$, by U.6 $\mathcal{P}(\Omega|E)$ is compact. Since $K_iE = \left\{\omega | \varphi \circ \overset{\omega}{t_i} \in \mathcal{P}(\Omega|E)\right\}$, $T_i^{K_iE} = \{\varphi^{-1} \circ u_i | u_i \in \mathcal{P}(\Omega|E)\}$ is compact. By the proof of Lemma 6, $K_iE = X \times T_i^{K_iE} \times T_{-i}$. By Epstein and Wang's (1996) Theorem 6.1, T_{-i} is compact. Hence, K_iE is compact for all i. ■

We now turn to the proof of Proposition 3.

Proof of Proposition 3. Let $P_i(\omega) = CK\widehat{R} \cap \widehat{R}$. Since, by Lemma 11, $X^{P_i(\omega)} = BR_i(\omega)$, $\omega \in R_i$. By Lemma 12, $P_i(\omega)$ is compact. The result of Proposition 3 is therefore followed directly from Lemma 7. ■

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