# Utility Equivalence in Auctions *

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**Abstract**

Auctions are considered with a (non-symmetric) independent-private-value model of valuations. It shall be demonstrated that a utility equivalence principle holds for an agent if and only if she has constant absolute risk aversion.

## 1 Introduction

Most of the research in Auction Theory focuses on the seller's perspective. The Optimal Auction Theorem (Myerson (1981)), which characterizes auction mechanisms that maximize the seller's revenue, and the Revenue Equivalence Principle[1], which provides conditions under which a seller is indifferent between various auctions are well-known examples. When following Myerson's proof of the Revenue Equivalence Principle, it can be seen that it follows from a Utility Equivalence Principle for risk-neutral agents, and that these two principles are equivalent. That is, the seller is indifferent between two auction mechanisms if and only if every potential buyer is

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[1]Vickrey (1961), Ortega-Reichert (1968), Holt (1980), Harris and Raviv (1981), Myerson(1981), Riley and Samuelson (1981).

indifferent between them. Matthews (1983, 1987) were the first attempts to compare auction mechanisms from the buyers' point of view, when the buyers were not risk-neutral[2]. Matthews (1987) compared first- and second-price auctions, and showed relationships between the preferences of agents over these two auction mechanisms and monotonicity properties of the Arrow-Pratt measure of risk aversion ($\frac{-u''(x)}{u'(x)}$). In particular, Matthews showed that when an agent has constant absolute risk aversion (CARA), she is indifferent between first- and second-price auctions. This theorem was generalized by Monderer and Tennenholtz (2000a) to all k-price auctions. In this discussion we prove a general utility equivalence principle, that holds for agents with constant absolute risk aversion. Furthermore, we show that this equivalence principle holds if and only if the agents have CARA.

We shall consider a seller that wishes to sell a single[3] item by an auction mechanism to one of $n$ potential buyers. We assume the (non-symmetric) independent-private-value model of valuations[4]. Each potential buyer $a$ is characterized by her utility for money function $u^a$, and by her valuation structure (distribution of types). The set of possible types of $a$ is an interval $[\alpha^a, \beta^a]$. However, we assume the most general structure of distribution functions, and in particular our model treats atoms as well as atomless distributions. The auction mechanism is typically described by sets of messages, one set for each agent, and by functions (of vector of messages) that define the probability of winning the object by each agent, and the payment functions for each agent[5]. The auction mechanism together with the valuation structure define a Bayesian game – the auction game.

For a fixed equilibrium profile in this game, let $Q^a(t)$, $t \in [\alpha^a, \beta^a]$ be the proba-

[2]Maskin and Riley (1984) discussed the revenue of the seller in first-price auctions with risk-averse agents.

[3]This work does not deal with mechanisms for selling several items, i.e; combinatorial auction mechanisms. Krishna and Perry (1998) generalize Myerson's utility equivalence to such auctions, keeping the assumption of risk neutrality.

[4]This assumption is made in all previous works that deal with utility (or revenue) equivalence. However, there are several issues in auction theory that have been analyzed without the independence assumption. The first, and classical, work to remove this assumption was Milgrom and Weber (1982).

[5]There are two such payment functions for each agent. One function describes the payment paid by her, when she wins the item, and the other one gives her payment when she does not win. This splitting of payments is necessary when dealing with agents that are not risk-neutral.

bility that agent $a$ wins the object in equilibrium given that her valuation is $t$, and let $U^a(t)$ be the expected utility of this agent in equilibrium.

Myerson (1981) showed that for a risk-neutral buyer:
For all auction games $A$ and $B$ and equilibrium profiles in those games, such that $Q^a_A(t) = Q^a_B(t)$ for every $t \in [\alpha^a, \beta^a]$,

$$U^a_A(t) - U^a_B(t) = U^a_A(s) - U^a_B(s) \quad \text{for every } t, s \in [\alpha^a, \beta^a].$$

We refer to this result as Myerson's utility equivalence theorem[6], and we show that:

- Myerson's utility equivalence theorem holds only for risk-neutral buyers.

We provide a weaker version of utility equivalence. When a buyer is not necessarily risk-neutral his expected utility function, $U^a$, is naturally written as the sum of his expected win and loss utility functions, $U^a(t) = U^a_w(t) + U^a_l(t)$.

We say that this buyer (or rather his utility function $U^a$) satisfies the weak utility equivalence principle if:

- There exists a positive function $h$ such that for all auction games $A$ and $B$ and equilibrium profiles in those games,
  for which $Q^a_A(t) = Q^a_B(t)$ and $U^a_{l,A}(t) = U^a_{l,B}(t) \quad$ for every $t \in [\alpha^a, \beta^a]$,
  $$h(t)(U^a_A(t) - U^a_B(t)) = h(s)(U^a_A(s) - U^a_B(s)) \quad \text{for every } t, s \in [\alpha^a, \beta^a].$$

We prove that :

- The weak utility equivalence principle holds for an agent if and only if this agent has CARA.

An important consequence of the utility equivalence principle is:

- If, in addition to the utility equivalence conditions, $U^a_A(\alpha_a) = U^a_B(\alpha_a)$ then
  $$U_A(t) = U_B(t) \quad \text{for every } t \in [\alpha^a, \beta^a].$$

[6]Note that Myerson's theorem implies that for every two auction games $A$ and $B$ such that $Q^a_A(t) = Q^a_B(t)$ for every $t \in [\alpha^a, \beta^a]$ and $U^a_A(\alpha^a) = U^a_B(\alpha^a)$,

$$U^a_A(t) = U^a_B(t) \quad \text{for every } t \in [\alpha^a, \beta^a].$$

Our theorem is particularly useful in dealing with standard auctions (i.e., auctions in which the highest bid wins), in which the loss payment functions are identically zero, and the equilibrium strategies are increasing. A CARA agent in such auctions gets the same expected utility from all auctions that give him the same expected utility at his lowest possible valuation.[7]

In order to prove the utility equivalence theorem we analyze the equilibrium structure in auction games. The properties which are obtained have their own significance and are proved for the *most general utility functions*.
We show that for every equilibrium in an auction game, $U_A^a$ is a Liptchitz non-decreasing function. In addition we state an equation that should be satisfied in equilibrium (the equilibrium equation). All explicit formulas for the bidding functions in equilibrium may be derived from this equation (see, e.g, Maskin and Riley (1984) and Monderer and Tennenholtz (2000a)).

All the results obtained in this paper can be generalized to settings with random numbers of agents, which have been studied in the literature (see, e.g., Monderer and Tennenholtz (2000b), and McAfee and McMillan (1987)).

# 2 Preliminaries

This section presents the basic notations and assumptions that will be used to describe the auction environment throughout this discussion. This environment includes a single owner (a seller), who wishes to sell an item to one of $n \geq 1$ agents (potential buyers) through an auction mechanism.

## 2.1 The agents description

The set of agents is denoted by $N$. We assume $N = \{1, 2, ..., n\}, n \geq 1$.
Every agent $a$ has a von Neumann-Morgenstern utility function for money, $u^a(x), -\infty < x < \infty$, such that

[7] Matthews (1983) also provided a utility equivalence principle for CARA agents. He proved that a CARA buyer is indifferent between any two equilibria of any two auctions, which, in particular, yield the same expected difference, $U_w^a(t) - U_l^a(t)$ for every type $t$. However, this theorem is vacuous for (standard) auctions in which $U_l^a$ is identically zero.

- $u^a(0) = 0$.
- $u^a$ is twice differentiable.
- $(u^a)'(x) > 0$ for every $x \in R$,

where $R$ denotes the set of real numbers. Whenever possible, we will omit the agent superscript.

In this paper we will mainly deal with agents who have *constant absolute risk aversion* ($CARA$). We refer to such an agent as a $CARA$ agent. The set of all utility functions of the $CARA$ agents is denoted by $CARA$. Recall that an agent has $CARA$ if and only if there exists a constant $\lambda$ such that $\frac{u''(x)}{u'(x)} = \lambda$ for all $x$. Note that, for such an agent, the Arrow-Pratt measure of risk aversion is constant and it is $-\lambda$. If $\lambda = 0$ the agent is *risk-neutral* and the utility function has the form $u(x) = cx$ for some $c > 0$. If $\lambda < 0$, the agent is *risk-averse* and $u(x) = c(1 - e^{\lambda x})$, $c > 0$. If $\lambda > 0$, the agent is *risk-seeking* and $u(x) = c(e^{\lambda x} - 1)$, $c > 0$.
The following is a useful characterization of $CARA$.

**Lemma 1** *$u \in CARA$ if and only if there exists $\Gamma \in R$ such that*

$$u(a+b) = u(a) + u(b) + \Gamma u(a)u(b) \quad \forall a, b. \tag{2.1}$$

**Proof:** If $u \in CARA$, then (2.1) is satisfied by $\Gamma = \lambda/u'(0)$.

Suppose there exists $\Gamma$ such that (2.1) is satisfied. Differentiating both sides of (2.1) with respect to $a$ yields:

$$u'(a+b) = u'(a)[1 + \Gamma u(b)] \quad \forall b.$$

Differentiating both sides again according to $a$ yields:

$$u''(a+b) = u''(a)[1 + \Gamma u(b)] \quad \forall b.$$

By dividing the two equations (note that $(u)'(x) > 0$ for every $x \in R$) we get that $u''(x)/u'(x)$ is constant. Hence, $u \in CARA$. ∎
We will use the following equality derived from the proof of Lemma 1:

$$u'(a) = u'(0)[1 + \Gamma u(a)] \quad \forall a. \tag{2.2}$$

We proceed to discuss the agents' valuations. We use the (non-symmetric) independent-private-value model. In this model, every agent $a \in N$ knows her own valuation (type, willingness to pay), $t^a \in T^a$, where $T^a = [\alpha^a, \beta^a]$, $0 \leq \alpha^a \leq \beta^a$. This valuation is a realization of a random variable $\tilde{Z}^a$ which takes values in $T^a$ and has a distribution function $F^a$ [8]. Let $F(t) = \prod_{a=1}^{n} F^a(t^a)$ be the common distribution function on $T = \times_{a=1}^{n} T^a$. The triplet $(N, T, F)$ is called a *valuation structure*.

## 2.2 The auction mechanism

The auction mechanism comprises sets of messages, one for each agent, as well as rules that determine the winner and the payments.
An agent $a \in N$ has a message set $M^a$ that contains a message $e^a$ that is called a *null message*. Such a message is never actually sent, but if $a$ does not send any actual message, the seller relates to it as if $a$ sent $e^a$.

Let $M = \times_{a \in N} M^a$ be the set of vectors of messages, and let $e \in M$ be the vector of null messages. We assume that

- $M^a \setminus \{e^a\}$ is a subset of some Euclidean space for every $a \in N$.[9]

Note that $M^a$ is a metric space with the natural metric of Euclidean spaces, and with agreement that the distance between $e^a$ and a real message $m$ is 1. Hence, $M^a$ and $M$ have a natural Borel structure.
A subset $B^a$ of $M^a$ is *bounded* if $B^a \setminus \{e^a\}$ is bounded.

The rest of the auction mechanism is defined by three functions

$$\tau : M \to [0,1]^n, \quad x : M \to \Re^n, \quad y : M \to \Re^n.$$

If the agents send the vector of messages $m \in M$, the seller conducts a lottery to determine the winner. The probability that $a$ is the winner is $\tau^a(m)$. The seller may keep the item to himself. Hence,

$$\sum_{a \in N} \tau^a(m) \leq 1 \quad and \quad \tau^a(m) \geq 0, \quad \forall a \in N, \quad \forall m \in M.$$

[8]That is, $F^a(t^a) = Prob(\tilde{z}^a \leq t^a)$. Our model covers both a continuous and a discrete distribution of types.

[9]We do not exclude finite sets of messages.

$x^a(m)$ is the amount of money that agent $a$ has to pay if she gets the object and, $y^a(m)$ is the amount of money that agent $a$ has to pay if she does not get the object.

We assume that

- $\tau, x, y$ are Borel measurable, and $x$ and $y$ are bounded on bounded subsets of $M$.

Naturally a non-participating agent neither wins nor pays. Hence, we assume

- $\tau^a(m) = x^a(m) = y^a(m) = 0$, whenever $m^a = e^a$.

Every auction mechanism $C = C(N, M, \tau, x, y)$, alongside a valuation structure $I = (N, T, F)$, defines a Bayesian game, $A = A(C, I)$, which we call an *auction game.*

A *strategy*[10] of agent $a$ is a bounded Borel measurable function $b^a : T^a \longrightarrow M^a$.

For $a \in N$ we denote $T^{-a} = \times_{i \in N, i \neq a} T^i$. For $t \in T$ we denote by $t^{-a}$ the projection of $t$ on $T^{-a}$.

Let $b = (b^a)_{a \in N}$ be a fixed strategy profile in the auction game A. Consider a fixed agent $a \in N$.

Let $Q^a_A(m^a|t^a)$ and $U^a_A(m^a|t^a)$ be the probability that $a$ is the winner and the expected utility of $a$ respectively, when $a$ sends the message $m^a$, given that her type is $t^a$ and all the other players use their strategies in $b$.

More precisely,

$$Q^a_A(m^a|t^a) = E_{T^{-a}} \left\{ \tau^a(m^a, b^{-a}) \right\}$$

and

$$U^a_A(m^a|t^a) = E_{T^{-a}} \left\{ u^a \left( t^a - x^a(m^a, b^{-a}) \right) \tau^a(m^a, b^{-a}) + u^a \left( -y^a(m^a, b^{-a}) \right) \left[ 1 - \tau^a(m^a, b^{-a}) \right] \right\},$$

where $b^{-a}$ is the vector $(b^j(t^j), j \neq a)$.

Recall that $b$ is an equilibrium strategy profile if for every agent $a$,

$$U^a_A(b^a(t^a)|t^a) \geq U^a_A(m^a|t^a)$$

[10]In this paper we deal with pure strategies. However, a simple application of the revelation principle yields that for a discrete distribution of types (i.e., when there are potential problems with the existence of equilibrium), our result can be extended to a setting with mixed strategies.

for every $t^a \in T^a \quad and \quad m^a \in M^a$.
The expected utility function $U^a_A$ is decomposed to an expected utility-when-win function $U^a_{A,w}$ and an expected utility-when-loss function $U^a_{A,l}$, where

$$U^a_{A,w}(m^a|t^a) = E_{T^{-a}}\left\{u^a\left(t^a - x^a(m^a, b^{-a})\right)\tau^a(m^a, b^{-a})\right\}$$

and

$$U^a_{A,l}(m^a|t^a) = E_{T^{-a}}\left\{u^a\left(-y^a(m^a, b^{-a})\right)\left[1 - \tau^a(m^a, b^{-a})\right]\right\}.$$

When $b$ is a fixed equilibrium strategy profile in $A$, we denote $U^a_A(b^a(t^a))|t^a)$ by $U(t)$, and $Q^a_A(b^a(t^a)|t^a)$ by $Q(t)$.

# 3 Myerson's utility equivalence theorem

Myerson (1981) proved that a risk-neutral agent is indifferent up to a constant between any two auction mechanisms which have the same probability-of-winning function, $Q$. We will prove that such a result holds only for risk-neutral agents.

**Theorem (Myerson 1981)** Let $a$ be a fixed risk-neutral agent. Let $T^a = [\alpha^a, \beta^a]$. Then the following holds: let $A$ and $B$ be two auction games in which the set of types of $a$ is $T^a$, and let $b$ and $d$ be fixed equilibrium profiles in $A$ and $B$ respectively. If

$$Q^a_A(t) = Q^a_B(t) \quad \text{for Borel almost every } t \in T^a,$$

then

$$U^a_A(t) - U^a_B(t) = U^a_A(s) - U^a_B(s), \tag{3.1}$$

for every $t, s \in T^a$.
We proceed to show that Myerson's equivalence principle holds only for risk-neutral agents.

**Theorem 1** *Let $a$ be an agent with utility function $u$, and let $T^a = [\alpha^a, \beta^a]$, $\alpha^a < \beta^a$. If the the following condition holds, $a$ is risk-neutral:*

*Let $A$ and $B$ be two auction games, in which the set of types of $a$ is $T^a$, and let $b$ and $d$ be fixed equilibrium profiles in $A$ and $B$ respectively. If*

$$Q^a_A(t) = Q^a_B(t) \quad \textit{for Borel almost every } t \in T^a,$$

*then (3.1) holds.*

**Proof:** We will consider auctions in which the set of agents is $N = \{a\}$. Let $z$ be a real number. Let $k$ be a positive integer satisfying

$$u(-z) + (k-1)u(1) \geq 0. \qquad (3.2)$$

Consider the following direct auction mechanisms in which the probability to win and the payments do not depend on the vector of messages. In particular, let $A_{z,k}$ be such a mechanism when $\frac{1}{k}$ is the probability to win and the payment-when-win is z and when-lose is -1. That is, for every $m \in M^a \setminus \{e^a\}$,

$\tau^a(m) = \frac{1}{k}$, $x^a(m) = z$, and $y^a(m) = -1$.

Since the probability to win and the payments do not depend on the the messages and (3.2) holds, then $A_{z,k}$ generates a truth telling auction game. Moreover, $Q^a_{A_{z,k}}(t^a) = \frac{1}{k}$ for every $t^a \in T^a$.

In addition, note that (3.2) is satisfied for $z = 0$. Hence, the auction game $A_{0,k}$ has the same properties as $A_{z,k}$. Hence, $A_{z,k}$ and $A_{0,k}$ satisfy $Q^a_{A_{z,k}} = Q^a_{A_{0,k}}$.

Therefore, by (3.1),

$$u(t-z) - u(t) = u(s-z) - u(s), \quad \text{for all } t, s \in T^a. \qquad (3.3)$$

Recall that (3.3) holds for every real number, $z$, and differentiate (3.3) with respect to $z$ to get: $u'(t-z) = u'(s-z)$ for every $t, s \in [\alpha^a, \beta^a]$, and for every $-\infty < z < \infty$. Hence $u'$ is a constant function. Therefore $u(x) = cx$, $c > 0$, for every $x$. ▮

A slight modification in the proof of Theorem 1 shows that this theorem holds also for a fixed set of agents. That is, given a set of agents, if an agent is indifferent up to a constant between any two auctions which have the same probability-to-win function, then she must be a risk-neutral agent.

# 4 The utility equivalence theorem

We introduce a generalized utility equivalence theorem which holds if and only if the agent is a $CARA$ agent.

To prove this, we present important properties of the expected utility and probability-to-win in-equilibrium functions. Although those properties will be used for $CARA$ agents only, we state and prove the results for an agent with an arbitrary attitude to risk.

**Lemma 2** *Let $A$ be an auction game, $b$ be a fixed equilibrium profile in $A$ and let $a$ be a fixed agent with utility function $u$.*
*Then $U_A^a$ is a Liptchitz non-decreasing function.*
*Moreover for Borel almost every $t$ in $[\alpha^a, \beta^a]$,*

$$[U_A^a(t)]' = E_{T^{-a}} \left\{ \left[ u\left(t - x^a(b^a(t), b^{-a})\right)\right]' \tau^a(b^a(t), b^{-a}) \right\}. \tag{4.1}$$

**Proof:** Consider an auction game $A$.
First we prove that $U^a(.)$ is a non-decreasing function.
Let $s, t$ be in $T^a$. The difference between the expected utility given $t$ and the expected utility given $s$ is,

$$U^a(s) - U^a(t) =$$

$$E_{T^{-a}} \left\{ u\left(s - x^a(b^a(s), b^{-a})\right) \tau^a(b^a(s), b^{-a}) + u\left(-y^a(b^a(s), b^{-a})\right) \left[1 - \tau^a(b^a(s), b^{-a})\right] \right\} -$$
$$E_{T^{-a}} \left\{ u\left(t - x^a(b^a(t), b^{-a})\right) \tau^a(b^a(t), b^{-a}) + u\left(-y^a(b^a(t), b^{-a})\right) \left[1 - \tau^a(b^a(t), b^{-a})\right] \right\}. \tag{4.2}$$

Given s, to bid $b^a(s)$ is at least as good as $b^a(t)$.
Therefore, by plugging in (4.2) $b^a(t)$ instead of $b^a(s)$ we get:

$$U^a(s) - U^a(t) \geq$$
$$E_{T^{-a}} \left\{ \left(u(s - x^a(b^a(t), b^{-a})\right) \tau^a(b^a(t), b^{-a}) + u\left(-y^a(b^a(t), b^{-a})\right) \left[1 - \tau^a(b^a(t), b^{-a})\right] \right\} -$$
$$E_{T^{-a}} \left\{ \left(u(t - x^a(b^a(t), b^{-a})\right) \tau^a(b^a(t), b^{-a}) + u\left(-y^a(b^a(t), b^{-a})\right) \left[1 - \tau^a(b^a(t), b^{-a})\right] \right\}.$$

That is,

$$U^a(s) - U^a(t) \geq E_{T^{-a}} \left\{ \left[ u\left(s - x^a(b^a(t), b^{-a})\right) - u\left(t - x^a(b^a(t), b^{-a})\right) \right] \tau^a(b^a(t), b^{-a}) \right\}. \tag{4.3}$$

For every $s > t$, we get $U^a(s) - U^a(t) \geq 0$ and therefore $U^a$ is non-decreasing.
We show that $U^a$ is a Liptchitz function. Given t, to bid $b^a(t)$ is at least as good as $b^a(s)$. Therefore, by plugging in (4.2) $b^a(s)$ instead of $b^a(t)$ we get, analogously to the way we got (4.3):

$$U^a(s) - U^a(t) \leq E_{T^{-a}} \left\{ \left[ u\left(s - x^a(b^a(s), b^{-a})\right) - u\left(t - x^a(b^a(s), b^{-a})\right) \right] \tau^a(b^a(s), b^{-a}) \right\}. \tag{4.4}$$

As $u$ itself is a Liptchitz function on bounded intervals, and $Q^a$ is bounded, there exists a constant $C > 0$ such that,

$$U^a(s) - U^a(t) \leq C(s - t),$$

and hence

$$|U^a(s) - U^a(t)| \leq C|s-t|, \quad \text{for all } s,t \in T^a.$$

We proceed to prove (4.1). As $U^a$ is a Liptchitz function $U'(t)$ exists Borel almost everywhere in $T^a$. Let $U'(t)$ exist at $t$, $\alpha^a < t < \beta^a$.
Let $s > t$, by (4.3)

$$\frac{U^a(s) - U^a(t)}{s-t} \geq \frac{E_{T^{-a}}\left\{\left[u\left(s - x^a(b^a(t), b^{-a})\right) - u\left(t - x^a(b^a(t), b^{-a})\right)\right]\tau^a(b^a(t), b^{-a})\right\}}{s-t}. \tag{4.5}$$

The limit of the left-hand side of (4.5) when $s \to t$ is $U'(t)$.
On the other hand by the Lebesqe Converges Theorem the right-hand side of (4.5) converges to $E_{T^{-a}}\left\{u'\left(t - x^a(b^a(t), b^{-a})\right)\tau^a(b^a(t), b^{-a})\right\}$. That is,

$$U'(t) \geq E_{T^{-a}}\left\{u'\left(t - x^a(b^a(t), b^{-a})\right)\tau^a(b^a(t), b^{-a})\right\}.$$

Similarly, by (4.4) we get:

$$U'(t) \leq E_{T^{-a}}\left\{u'\left(t - x^a(b^a(t), b^{-a})\right)\tau(b^a(t), b^{-a})\right\}.$$

Therefore

$$U'(t) = E_{T^{-a}}\left\{u'\left(t - x^a(b^a(t), b^{-a})\right)\tau^a(b^a(t), b^{-a})\right\}.$$

∎

In order to prove the utility equivalence, Myerson (1981) proved that for a risk-neutral agent, for every $t \in [\alpha^a, \beta^a]$,

$$U_A^a(t) = \int_{z=\alpha^a}^{t} u'(0)Q_A^a(z)dz + U_A^a(\alpha^a).$$

We generalize this to $CARA$ agents.
Explicitly, we show that the expected utility depends also on the expected utility-when-lose, $U_{A,l}^a(.)$ function. Therefore, the difference between what you may pay if you win and what if you lose affects the expected utility. Recall that for a risk-neutral agent, we get the same result as Myerson (1981) since $U_{A,l}^a(.)$ multiple by $\Gamma = 0$.

**Lemma 3** *Let $A$ be an auction game, $b$ be a fixed equilibrium profile in $A$, and let $a$ be a fixed $CARA$ agent with utility function $u$.*
*Then*

$$U_A^a(t) = u'(0)\int_{z=\alpha^a}^{t} e^{u'(0)\Gamma(t-z)}[Q_A^a(z) - \Gamma U_{A,l}^a(z)]dz + U_A^a(\alpha^a)e^{u'(0)\Gamma(t-\alpha^a)}. \tag{4.6}$$

**Proof:** Recall that $U$ is a Liptchitz function, and let $t \in T^a$, such that $U'(t)$ exists. By (4.1),

$$U'(t) = E_{T^{-a}} \left\{ u' \left( t - x^a(b^a(t), b^{-a}) \right) \tau^a(b^a(t), b^{-a}) \right\}.$$

Moreover, for any $CARA$ agent, by (2.2), $u'(a) = u'(0)[1 + \Gamma u(a)]$.
Therefore

$$U'(t) = u'(0)[E_{T^{-a}} \left\{ \tau^a(b^a(t), b^{-a}) \right\} + \Gamma E_{T^{-a}} \left\{ u \left( t - x^a(b^a(t), b^{-a}) \right) \tau^a(b^a(t), b^{-a}) \right\}].$$

Hence,

$$U'(t) = u'(0)[Q^a(t) + \Gamma(U^a(t) - U_l^a(t))]. \tag{4.7}$$

Multiplying both sides of (4.7) by $e^{-u'(0)\Gamma t}$ and rewriting, yields

$$(U^a(t)e^{-u'(0)\Gamma t})' = u'(0)e^{-u'(0)\Gamma t}(Q^a(t) - \Gamma U_l^a(t)).$$

As $U^a(t)$ is a Liptchitz function, $U^a(t)e^{-u'(0)\Gamma t}$ is absolutely continuous in $T^a$, and it is the integral of its derivative.
Therefore,

$$U^a(t) = u'(0)e^{u'(0)\Gamma t} \int_{z=\alpha}^{t} e^{-u'(0)\Gamma z}(Q^a(z) - \Gamma U_l^a(z))dz + U^a(\alpha)e^{u'(0)\Gamma(t-\alpha)}.$$

▮

Note that, $U^a(.)$ depends only on $u$,$Q^a$ , $U_l^a(.)$ and $U^a(\alpha)$.
The following theorem generalizes Myerson utility equivalence theorem to $CARA$ agents:

**Theorem 2** *Let $a$ be a $CARA$ agent with the utility function $u$, and let $T^a = [\alpha^a, \beta^a]$. Then there exists a positive function $h(t)$, $t \in T^a$ such that the following holds: let $A$ and $B$ be two auction games, in which the set of types of $a$ is $T^a$, and let $b$ and $d$ be fixed equilibrium profiles in $A$ and $B$ respectively. If*

$$Q_A^a(t) = Q_B^a(t) \quad \textit{for Borel almost every } t \in T^a, \tag{4.8}$$

*and*

$$U_{A,l}^a(t) = U_{B,l}^a(t) \quad \textit{for Borel almost every } t \in T^a, \tag{4.9}$$

*then*

$$h(t)\left(U_A^a(t) - U_B^a(t)\right) = h(s)\left(U_A^a(s) - U_B^a(s)\right), \tag{4.10}$$

*for every* $t, s \in T^a$.
*Moreover, when* $u(x) = c(1 - e^{\lambda x})$, $\lambda < 0$, *or* $u(x) = c(e^{\lambda x} - 1)$, $\lambda > 0$, *one can choose* $h(t) = e^{\lambda(\alpha - t)}$ *for every* $t \in T^a$.
*In addition, if* $a$ *is risk-neutral, then (4.8) without (4.9) implies (4.10) with* $h(t) = 1$ *for every* $t \in T^a$.

**Proof:** Let $A$ and $B$ be two auction games.
By Lemma 3:

$$U_A^a(t) = u'(0) \int_{z=\alpha^a}^{t} e^{u'(0)\Gamma(t-z)} [Q_A^a(z) - \Gamma U_{A,l}^a(z)] dz + U_A^a(\alpha^a) e^{u'(0)\Gamma(t-\alpha^a)}.$$

And

$$U_B^a(t) = u'(0) \int_{z=\alpha^a}^{t} e^{u'(0)\Gamma(t-z)} [Q_B^a(z) - \Gamma U_{B,l}^a(z)] dz + U_B^a(\alpha^a) e^{u'(0)\Gamma(t-\alpha^a)}.$$

Therefore, if $Q_A^a(t) = Q_B^a(t)$ and $U_{A,l}^a(t) = U_{B,l}^a(t)$ for Borel almost every $t$, then

$$U_A(t) - U_B(t) = e^{u'(0)\Gamma(t-\alpha)} (U_A^a(\alpha) - U_B^a(\alpha)) \quad \text{for every } t \in T^a.$$

Therefore, for $h(t) = e^{u'(0)\Gamma(\alpha - t)} = e^{\lambda(\alpha - t)}$, $t \in T^a$,

$$h(t) \left( U_A(t) - U_B(t) \right) = h(s) \left( U_A(s) - U_B(s) \right) \quad \text{for every } t, s \in T^a.$$

Finally note that if $a$ is risk-neutral, $\Gamma = 0$, and therefore $h(t) = 1$ and the right side of (4.6) depends on $Q$ and $U(\alpha^a)$ only. ∎
The proof of the following corollary follows easily from Theorem 2, and therefore it is omitted.

**Corollary 1** *Let* $a$ *be a fixed CARA agent with the utility function* $u$. *Let* $A$ *and* $B$ *be two auction games with the same set of types for* $a$, *and let* $b$ *and* $d$ *be fixed equilibrium profiles in* $A$ *and* $B$ *respectively. Assume that* $Q_A^a(t) = Q_B^a(t)$, $U_{A,l}^a(t) = U_{B,l}^a(t)$ *for Borel almost every* $t \in T^a$ , *and* $U_A^a(\alpha^a) = U_B^a(\alpha^a)$.
*Then*

$$U_A^a(t^a) = U_B^a(t^a) \quad \textit{for every } t^a \in T^a. \tag{4.11}$$

Corollary 1 enables us to characterize utility-equivalent auctions for a CARA agent. In particular, all symmetric (standard) auctions in which the highest bid wins, losers do not pay and equilibrium bid function is necessarily increasing are equivalent for the participant. One class of such standard auctions are $k$-price auctions, $k \geq 1$ ( if a symmetric equilibrium is assumed). That is, a CARA agent is indifferent to all $k$-price auctions with the same number of participants. For this class it is possible to show more: A CARA agent is indifferent to all $k$-price auctions with the same number of participants, and with the same constant entree fee, provided that the distribution of types is atomless.

We proceed to prove a converse to Theorem 2.

**Theorem 3** *Let $a$ be a fixed agent with the utility function $u$ and a set of types $T^a$ such that $\alpha^a < \beta^a$. Assume there exists a positive function $h(t^a)$, $t^a \in T^a$ such that the following holds: for every two auction games $A$ and $B$, in which the set of types of $a$ is $T^a$, and for all equilibrium profiles $b$ and $d$ in $A$ and $B$ respectively such that $Q^a_A(.) = Q^a_B(.)$ and $U^a_{A,l}(.) = U^a_{B,l}(.)$,*

$$h(t^a)\,(U^a_A(t^a) - U^a_B(t^a)) = h(s^a)\,(U_A(s^a) - U_B(s^a)) \quad \textit{for every } t, s \in T^a. \tag{4.12}$$

*Then $a$ is a CARA agent.*

**Proof:** We consider the auctions $A = A_{z,k}$ and $B = A_{0,k}$ defined in the proof of Theorem 1. Recall that $Q^a_{A_{z,k}}(.) = Q^a_{A_{0,k}}(.) = \frac{1}{k}$, $U^a_{A_{z,k},l}(.) = U^a_{A_{0,k},l}(.) = [1 - \frac{1}{k}]u(1)$, $U_{A_{z,k}}(t) = \frac{1}{k}u(t-z) + [1 - \frac{1}{k}]u(1)$ and $U_{A_{0,k}}(t) = \frac{1}{k}u(t) + [1 - \frac{1}{k}]u(1)$.
By (4.12),

$$h(t)\,((u(t-z) - u(t)) = h(s)\,(u(s-z) - u(s)) \tag{4.13}$$

for every $s, t \in T^a = [\alpha^a, \beta^a]$, and $-\infty < z < \infty$. Twice differentiating both sides of (4.13) with respect to $z$ yields

$$h(t)u'(t-z) = h(s)u'(s-z)$$

and

$$h(t)u''(t-z) = h(s)u''(s-z).$$

Therefore, $\frac{u''}{u'}$ is a constant function, and therefore $u \in CARA$. ▮
A slight modification in the proof of Theorem 3, as in Theorem 1, shows that this theorem holds also for a fixed set of agents.

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