

Evolutionary Stability for Large Populations and Backward Induction*

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Abstract

It has been shown (Hart [2002]) that the backward induction (or subgame-perfect) equilibrium of a perfect information game is the unique stable outcome for dynamic models consisting of selection and mutation, when the mutation rate is low and the populations are large, under the assumption that the expected number of mutations per generation is bounded away from zero.

Here it is shown that one can dispense with this last condition. In particular, it follows that the backward induction equilibrium is evolutionarily stable for large populations.

1 Introduction

Conventional game theory relies on the assumption that players are rational. However, in evolutionary models there is no assumption of rationality, and the behavior driving the process by which agents adjust their strategies may be entirely mechanistic. Therefore, it seems surprising that both lead to “rational” stability, i.e., to equilibrium behavior.

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The “rational” outcome of evolutionary models is due to a combination of the two main ingredients of evolution: *selection*, which is a process whereby better strategies prevail, and *mutation*, which is relatively rare and generates random strategies.

The consideration of evolutionary dynamics naturally gives rise to the concept of an *evolutionarily stable strategy* (ESS), introduced by Maynard Smith and Price [1973]. An ESS is a Nash equilibrium in which any small change by mutation will be eliminated by the process of selection. Therefore, an ESS is “stable” with regard to both selection and mutation.

1.1 This Paper

In this paper we follow the work of Hart [2002]. We study the long-run behavior of evolutionary dynamics, we introduce a few notions of stability, and finally, we show the stability properties of the backward induction equilibrium.

As in Hart [2002], the games we consider are finite games in extensive form with perfect information. In these games, an equilibrium point can always be obtained by “backward induction.” The result is also an equilibrium in each subgame, whether that subgame is reached or not. Such a point is called a *subgame-perfect equilibrium*, or a *backward induction equilibrium*.

For each game, there is an associated *population game*; i.e., at each node there is a distinct population of individuals who play the game in the role of the corresponding player. For each population, the proportions of different strategies in the population may be viewed as a mixed strategy.

The dynamic process is a Markov chain on the space of the mixed strategies of the various populations. The model is as follows. At each period, one individual is chosen at random in each population. His current strategy may then change by selection, mutation, or it may not change at all. Selection replaces his strategy by another strategy which, against the other populations

currently playing the game (i.e., “against the field”), yields a higher payoff. Mutation replaces his strategy by an arbitrary strategy, chosen at random. Finally, all the choices at each node are made independently.

Such dynamics yield an irreducible and aperiodic system whose long run behavior is well described by the corresponding unique invariant distribution, which for each state gives the frequency of that state’s occurrence during any large time interval.

As in the case of ESS, we are looking for stable strategies in this model, which are able to withstand the pressure of mutation. Therefore, a strategy profile (i.e., a mixed strategy in each population) is stable if the frequency of its occurrence in the long run depends on selection and not on mutation.

When the populations are fixed, a strategy profile is *evolutionarily stable* if its occurrence is positive independently of the mutation rate, i.e., if its invariant probability is bounded away from zero as the mutation rate goes to zero. When the populations increase, the invariant distribution may change. Therefore, we define a strategy profile to be *evolutionarily stable for large populations* (ESLP) if its invariant probability is bounded away from zero as the mutation rate goes to zero and the populations increase to infinity.

In Hart [2002] it is shown that when the populations are fixed, the backward induction equilibrium is evolutionarily stable. Moreover, it is shown that the backward induction equilibrium becomes in the limit the only stable outcome as the mutation rate decreases to zero and the populations increase to infinity, provided that the expected number of mutations per generation is bounded away from zero.¹ Therefore, only the backward induction equilibrium *may* be ESLP.

In this paper we show that this additional proviso (on the expected number of mutations per generation) is not needed.² Thus, the backward induction equilibrium *must* be ESLP, and we have the stronger result:

¹I.e., the populations go to infinity at least as fast as the mutation rate goes to zero.

²The class of dynamics considered here is less general than Hart [2002]: we assume that the probability for selection is bounded away from zero. See Remark 3 in Subsection 2.3.

Main Result: *The backward induction equilibrium is the unique outcome that is evolutionarily stable for large populations; i.e., it is in the limit the only stable outcome as the mutation rate decreases to zero and the populations increase to infinity.*

The following section presents the model. Section 3 defines stability and presents the Main Theorem as well as some preliminaries results. Section 4 proves the Main Theorem, using a few general propositions on Markov chains. Section 5 concludes with possible extensions of our result.

2 The Model

The model is as in Hart [2002], except for a somewhat less general class of dynamics.³

2.1 The Game

Let Γ be a finite extensive-form game with perfect information. We are thus given a rooted tree; each non-terminal vertex corresponds to a *move*. It may be a chance move, with fixed positive probabilities for its outgoing branches; or it may be a move of one of the players, in which case the vertex is called a *node*. The set of nodes is denoted N . It is convenient to view the game in “agent-normal form”: at each node there is a different agent, and a player consists of a number of agents with identical payoff functions. For each node $i \in N$, the agent there — called “agent i ” — has a set of choices A^i , which is the set of outgoing branches at i . We refer to a^i in A^i as a strategy of i , and we put $A := \prod_{i \in N} A^i$ for the set of N -tuples of strategies. At each terminal vertex (or *leaf*) there are associated payoffs to all agents; let $u^i : A \rightarrow \mathbb{R}$ be the resulting payoff function of agent i (i.e., for each $a = (a^j)_{j \in N} \in A$: if there are no chance moves, then $u^i(a)$ is the payoff of i at the leaf that is

³See Remarks 1 and 3 in Subsection 2.3.

reached when every agent $j \in N$ chooses a^j ; if there are chance moves, it is the appropriate expectation). As usual, the payoff functions are extended multilinearly to *randomized* (or *mixed*) strategies; thus $u^i : X \rightarrow \mathbb{R}$, where $X := \prod_{i \in N} X^i$ and $X^i := \Delta(A^i) = \{x^i \in \mathbb{R}_+^{A^i} \mid \sum_{a^i \in A^i} x_{a^i}^i = 1\}$, the unit simplex on A^i , is the set of probability distributions over A^i .

For each node $i \in N$, let $N(i)$ be the set of nodes that are successors (not necessarily immediate) of i in the tree, and let $\Gamma(i)$ be the subgame starting at the node i .

An N -tuple of randomized strategies $x = (x^i)_{i \in N} \in X$ is a *Nash equilibrium* of Γ if⁴ $u^i(x) \geq u^i(y^i, x^{-i})$ for every $y^i \in X^i$. It is moreover a *subgame-perfect* (or *backward induction*) *equilibrium* of Γ if it is a Nash equilibrium in each subgame $\Gamma(i)$, for all $i \in N$. Such an equilibrium is therefore obtained by *backward induction*, starting from the final nodes (those nodes i with no successors, i.e., with $N(i) = \emptyset$) and going towards the root. We will denote by EQ and BI the set of Nash equilibria and the set of backward induction equilibria, respectively, of the game Γ ; thus $BI \subseteq EQ \subseteq X$.

The classical result of Kuhn [1953] states that there always exists a *pure* backward induction equilibrium. We assume here that the game Γ has a *unique* backward induction equilibrium, which must therefore be pure; we denote it $b = (b^i)_{i \in N} \in A$, and refer to b^i as the “backward induction strategy of i .” This uniqueness is true generically, i.e., for almost every game. For instance, when there are no chance moves, it suffices for each player to have different payoffs at different leaves.

2.2 The Gene-Normal Form

We now consider a population game associated to Γ : at each node $i \in N$ there is a nonempty finite population $M(i)$ of individuals playing the game in the role of agent i . We assume that the populations at different nodes are

⁴We write x^{-i} for the $(|N| - 1)$ -tuple of strategies of the other agents, i.e., $x^{-i} = (x^j)_{j \in N \setminus \{i\}}$.

distinct:

$$M(i) \cap M(j) = \emptyset \text{ for all } i \neq j. \quad (2.1)$$

Each individual $q \in M(i)$ is characterized by a pure strategy in A^i , which we denote by $\omega_q^i \in A^i$; let $\omega^i = (\omega_q^i)_{q \in M(i)}$ and $\omega = (\omega^i)_{i \in N}$. For each $a^i \in A^i$, let

$$x_{a^i}^i \equiv x_{a^i}^i(\omega^i) := \frac{|\{q \in M(i) \mid \omega_q^i = a^i\}|}{|M(i)|} \quad (2.2)$$

be the proportion of population $M(i)$ that plays the strategy a^i ; then $x^i \equiv x^i(\omega^i) := (x_{a^i}^i(\omega^i))_{a^i \in A^i} \in X^i$ may be viewed as a mixed strategy of i . The payoff of an individual $q \in M(i)$ is defined as his average payoff against the other populations, i.e., $u^i(\omega_q^i, x^{-i})$; we shall slightly abuse notation by writing this as $u^i(\omega_q^i, \omega^{-i})$.

We refer to the above model as the *gene-normal form* of Γ (by (2.1), which is the counterpart, in population games, of the “agent-normal form”).

2.3 The Dynamics

We come now to the dynamic model. Our processes will be *discrete-time stationary Markov chains*:

Let $\Omega = \Omega(\Gamma, (M(i))_{i \in N}) := \prod_{i \in N} (A^i)^{M(i)}$ be the state space. A *state* ω of the system specifies the pure strategy of each individual in each population; i.e., $\omega = (\omega^i)_{i \in N}$, where $\omega^i = (\omega_q^i)_{q \in M(i)}$ and $\omega_q^i \in A^i$ for each $i \in N$ and $q \in M(i)$.

We first present a simple dynamic model, which we call the *basic model*. Assume that all populations are of equal size, say $m = |M(i)|$ for each $i \in N$. Let $\mu > 0$ and $\sigma > 0$ be given, such that $\mu + \sigma \leq 1$. Let $Q = (q_{\omega, \tilde{\omega}})_{\omega, \tilde{\omega} \in \Omega}$ be a one-step transition matrix on Ω , defined by the following process performed independently for each $i \in N$:

- Choose an individual $q(i) \in M(i)$ at random: all m individuals in $M(i)$ have the same probability $1/m$ of being chosen.

- Let $\tilde{\omega}_q^i := \omega_q^i$ for each $q \in M(i)$ such that $q \neq q(i)$; i.e., all individuals in $M(i)$ except $q(i)$ do not change their strategies.
- Choose one of $SE(i)$ (“selection”), $MU(i)$ (“mutation”) and $NC(i)$ (“no change”), with probability σ , μ and $1 - \mu - \sigma$, respectively.
- If selection $SE(i)$ was chosen, then define

$$B^i \equiv B^i(q(i), \omega) := \{a^i \in A^i \mid u^i(a^i, \omega^{-i}) > u^i(\omega_{q(i)}^i, \omega^{-i})\}. \quad (2.3)$$

B^i is the set of “better strategies” — those strategies at node i that are strictly better in Γ , against the populations at the other nodes, than the strategy $\omega_{q(i)}^i$ of the chosen individual $q(i)$. If B^i is not empty, then the new strategy $\tilde{\omega}_{q(i)}^i$ of $q(i)$ is a randomly chosen better strategy: $\tilde{\omega}_{q(i)}^i := a^i$ with probability $1/|B^i|$ for each $a^i \in B^i$. If B^i is empty, then there is no change in $q(i)$ ’s strategy: $\tilde{\omega}_{q(i)}^i := \omega_{q(i)}^i$.

- If mutation $MU(i)$ was chosen, then $\tilde{\omega}_{q(i)}^i$ is a random strategy in A^i ; i.e., $\tilde{\omega}_{q(i)}^i := a^i$ with probability $1/|A^i|$ for each $a^i \in A^i$.
- If no-change $NC(i)$ was chosen, then the strategy of $q(i)$ does not change: $\tilde{\omega}_{q(i)}^i := \omega_{q(i)}^i$.

In general, there is no need for the various probabilities to be equal, just comparable. Therefore the *general model* is as follows. We are given a *mutation rate* parameter $\mu > 0$, and a *population size* m such that $|M(i)| = m$ for all nodes $i \in N$. The process is a stationary Markov chain on $\Omega_m = \Omega(\Gamma, (M(i))_{i \in N})$, whose one-step transition probability $Q = (q_{\omega, \tilde{\omega}})_{\omega, \tilde{\omega} \in \Omega_m}$ satisfies:

- Conditional independence over $i \in N$, i.e.,⁵

$$Q[\tilde{\omega} \mid \omega] = \prod_{i \in N} Q[\tilde{\omega}^i \mid \omega]. \quad (2.4)$$

- For each $i \in N$, one individual $q(i) \in M(i)$ is chosen, such that there exist constants $\gamma_1, \gamma_2 > 0$ with

$$\frac{\gamma_1}{m} \leq Q[q(i) = q \mid \omega] \leq \frac{\gamma_2}{m} \text{ for each } q \in M(i), \text{ and} \quad (2.5)$$

$$Q[\tilde{\omega}_q^i = \omega_q^i \text{ for all } q \in M(i) \setminus \{q(i)\} \mid \omega] = 1. \quad (2.6)$$

- There exists a constant $\sigma > 0$ such that, for each $i \in N$,

$$Q[\tilde{\omega}_{q(i)}^i = a^i \mid \omega] \geq \sigma \text{ for each } a^i \in B^i, \quad (2.7)$$

where $B^i \equiv B^i(q(i), \omega)$ is the set of strictly better strategies, as defined in (2.3).

- There exist constants $\alpha_1, \alpha_2 > 0$ such that, for each $i \in N$,

$$Q[\tilde{\omega}_{q(i)}^i = a^i \mid \omega] \geq \alpha_1 \mu \text{ for each } a^i \in A^i, \text{ and} \quad (2.8)$$

$$Q[\tilde{\omega}_{q(i)}^i = a^i \mid \omega] \leq \alpha_2 \mu \text{ for each } a^i \notin B^i, a^i \neq \omega_{q(i)}^i. \quad (2.9)$$

Without loss of generality, all parameters $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \sigma$ are taken to be the same for all $i \in N$. It can easily be seen that the basic model is a special case of the general model, where $\gamma_1 = \gamma_2 = 1$, $\alpha_1 = 1/\max_{i \in N} |A^i|$ and $\alpha_2 = 1/\min_{i \in N} |A^i|$.

Remarks:

⁵For each $\omega \in \Omega_m$, we view $Q[\cdot \mid \omega]$ as a probability distribution over Ω_m , such that $Q[\Omega' \mid \omega] = \sum_{\omega' \in \Omega'} q_{\omega, \omega'}$ for all $\Omega' \subseteq \Omega_m$; derived probabilities — like its marginals, etc. — will also be denoted by $Q[\cdot \mid \omega]$.

1. For simplicity we assume that $|M(i)| = m$ for all nodes $i \in N$, but the same proof (up to minor changes) will work even if we assume only that the populations are comparable; i.e., the ratios $|M(i)| / |M(j)|$ for all nodes $i, j \in N$ are bounded away from zero.
2. As in Hart [2002], we assume that the mutation rate μ is fixed, but we only need the mutation rates in the different populations to be comparable.
3. We assume that the probability of switching to a better strategy by selection is bounded away from zero — whether or not that strategy is currently present in the population. Therefore, our class of dynamics is less general than that of Hart [2002] (where the above probability may be proportional to the current proportion of the strategy in the population. See Hart [2002, (2.7)]).
4. A general model with a one-step transition matrix Q satisfying (2.4) – (2.9) yields a Markov chain which is irreducible, since the probability of reaching any state from any other state is positive (as follows from (2.5) and (2.8), by using an appropriate sequence of mutations). Hence there exists a unique invariant distribution π on Ω_m . The Markov chain is moreover aperiodic, since there is positive probability of staying in the same state. Therefore the long-run behavior of the process is well described by π , in the following two senses:

- In any long enough period of time, the relative frequency of visits at a state ω is approximately $\pi[\omega]$; i.e., for every $\omega \in \Omega_m$:

$$\lim_{T_2 - T_1 \rightarrow \infty} \frac{|\{t \mid T_1 < t < T_2, \omega_t = \omega\}|}{T_2 - T_1} = \pi[\omega].$$

- The probability that the state ω occurs at a period t is approxi-

mately $\pi[\omega]$ for large t ; i.e., for every $\omega \in \Omega_m$:

$$\lim_{t \rightarrow \infty} P[\omega_t = \omega] = \pi[\omega].$$

The two properties hold regardless of the initial state; moreover, they hold not only for single states ω but also for any set of states $\Omega' \subseteq \Omega_m$.

3 Stability

In this section we define stability and present the main result of this work.

3.1 Definitions

We are interested in the behavior of the process when the mutation rate is low, i.e., in the limit of the invariant distribution π as $\mu \rightarrow 0$. We will look at the case where the population size m is fixed, and at the case where $m \rightarrow \infty$.

Let the game Γ and the constants $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ and σ be fixed. For every mutation rate $\mu > 0$ and population sizes m , let $\Theta(m, \mu)$ be the set of all one-step transition matrices $Q = (q_{\omega, \tilde{\omega}})_{\omega, \tilde{\omega} \in \Omega_m}$ satisfying (2.4) – (2.9) with mutation rate μ and population sizes m . For every $Q \in \Theta(m, \mu)$, let π_Q be the unique invariant distribution of Q , and let $\pi_{m, \mu}[\omega] = \inf_{Q \in \Theta(m, \mu)} \pi_Q[\omega]$ for all $\omega \in \Omega_m$.

Definition 3.1. A state $\omega \in \Omega_m$ is *m-evolutionarily stable* if

$$\liminf_{\mu \rightarrow 0} \pi_{m, \mu}[\omega] > 0.$$

Recall that each state $\omega \in \Omega_m$ may be viewed as an N -tuple of mixed strategies $x(\omega) = (x^i(\omega^i))_{i \in N} \in X$ (see (2.2)). The invariant distribution π_Q on Ω_m therefore induces a probability distribution $\hat{\pi}_Q := \pi_Q \circ (x)^{-1}$ over X ;

i.e., $\hat{\pi}_Q[Y] := \pi_Q[\{\omega \in \Omega_m \mid x(\omega) \in Y\}]$ for every (measurable) $Y \subseteq X$. Let $\hat{\pi}_{m,\mu}[Y] = \inf_{Q \in \Theta(m,\mu)} \hat{\pi}_Q[Y]$.

Definition 3.2. An N -tuple of mixed strategies $x \in X$ is *m -evolutionarily stable* if

$$\liminf_{\mu \rightarrow 0} \hat{\pi}_{m,\mu}[x] > 0.$$

We now consider the case where the populations increase, i.e., $m \rightarrow \infty$ (while the game and constants remain fixed). As $m \rightarrow \infty$, the state space changes and becomes infinite in the limit; we need therefore to consider the probabilities of neighborhoods rather than the probability of a single point.

For every $\epsilon > 0$ and N -tuple of mixed strategies $x \in X$, let x_ϵ be the ϵ -neighborhood of x , i.e., $x_\epsilon := \{y \in X : \|x - y\| < \epsilon\}$ (we will also use BI_ϵ instead of b_ϵ).

Definition 3.3. An N -tuple of mixed strategies $x \in X$ is *evolutionarily stable for large populations* (ESLP) if

$$\liminf_{\substack{\mu \rightarrow 0 \\ m \rightarrow \infty}} \hat{\pi}_{m,\mu}[x_\epsilon] > 0.$$

We will also use ESLP to denote the set of all N -tuples which are ESLP.

Thus, x is ESLP if, for any neighborhood of x , the relative frequency of visits at that neighborhood is bounded away from zero, for all large population sizes and all small mutation rates.

Remarks:

1. When the populations are fixed, equilibria other than the backward induction equilibrium b may be evolutionarily stable (for example, equilibrium c in the game⁶ Γ_1 of Figure 1; see also Remark 2 in Subsection

⁶This is game Γ_3 of Hart [2002].

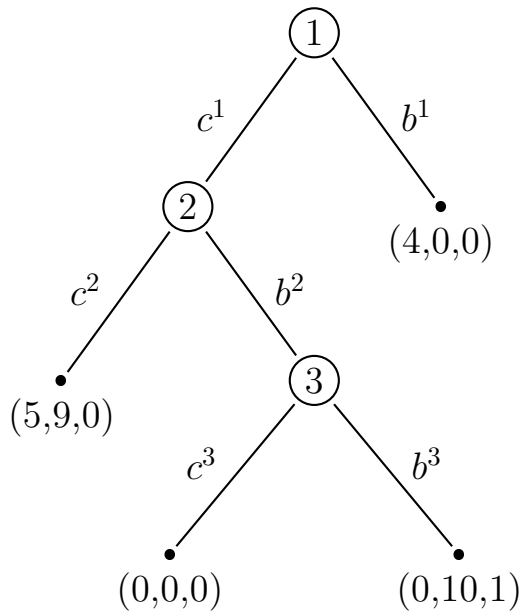


Figure 1: The Game Γ_1

3.1 of Hart [2002]). Therefore, the backward induction equilibrium is not the only equilibrium which is “ m -evolutionarily stable for all m ,” but as seen in Hart [2002], the probabilities of all other equilibria go to zero as $m \rightarrow \infty$.

2. We use x_ϵ and not x , because $\liminf_{\mu \rightarrow 0} \hat{\pi}_{m,\mu}[b]$ may be zero: in the simple case of a one-person game with the basic model, we have⁷ $\hat{\pi}[b] \doteq e^{-m\mu}$, which goes to zero as $m\mu \rightarrow \infty$.

⁷We write $f(m, \mu) \doteq g(m, \mu)$ if $f(m, \mu) = O(g(m, \mu))$ and $g(m, \mu) = O(f(m, \mu))$, where we use the “big-O” notation: $f(m, \mu) = O(g(m, \mu))$ if there exists a constant $c = c(\Gamma, \epsilon)$ which does not depend on m or μ such that $|f(m, \mu)| \leq c|g(m, \mu)|$ for all $m > m_0$ and $\mu < \mu_0$ for some m_0 and $\mu_0 > 0$.

3.2 The Main Result

In Hart [2002] it is shown that the backward induction equilibrium is m -evolutionarily stable for all m , and that only the backward induction equilibrium may be evolutionarily stable for large populations, i.e., $ESLP \subseteq BI$. Here we show that the backward induction equilibrium is always ESLP. Moreover, any neighborhood of the backward induction occurs in the limit with probability 1.

Theorem 3.4 (Main Theorem). *ESLP = BI. Moreover, for every $\epsilon > 0$,*

$$\lim_{\substack{\mu \rightarrow 0 \\ m \rightarrow \infty}} \hat{\pi}_{\mu, m}[BI_\epsilon] = 1. \quad (3.1)$$

Thus, in the long run, as the mutation rate is low and the populations are large, the dynamic system is most of the time in states where almost every individual plays his backward induction strategy; i.e., for all $\rho > 0$, when $\mu < \mu_0 \equiv \mu_0(\epsilon, \rho)$ and $m > m_0 \equiv m_0(\epsilon, \rho)$, at least $1 - \rho$ of the time, the proportion of individuals at all nodes who do not play their backward induction strategy is $< \epsilon$, for all dynamics $Q \in \Theta(m, \mu)$.

The backward induction equilibrium is m -evolutionarily stable for all m , and therefore any neighborhood of it is also m -evolutionarily stable. Moreover, from the double limit (3.1) it follows that the iterative limit $\lim_m \lim_\mu$ is 1 — and thus positive, which implies that BI is m -evolutionarily stable *uniformly* in m .

Corollary 3.5. *For every $\epsilon > 0$,*

$$\lim_{m \rightarrow \infty} \lim_{\mu \rightarrow 0} \hat{\pi}_{\mu, m}[BI_\epsilon] = 1.$$

Therefore, any neighborhood of backward induction equilibrium is uniformly evolutionarily stable, i.e., the probability, as $\mu \rightarrow 0$, to be in BI_ϵ is bounded away from zero for all m (and goes to 1 as $m \rightarrow \infty$).

3.3 Preliminary Results

For each m , let ES_m be the set of all m -evolutionarily stable N -tuples of mixed strategies.

The following result describes the relations between the different sets of equilibria.

Proposition 3.6. *$ESLP \subseteq BI \subseteq ES_m \subseteq EQ$, and each one of the last two inclusions may be strict.⁸*

Proof. The inclusions are all results of Hart [2002]. By Theorem 3.1 of Hart [2002], the backward induction equilibrium is m -evolutionarily stable for all m ; therefore, $BI \subseteq ES_m$. The dynamic here satisfies (3.4) of Hart [2002]; therefore, by Remark 1 in Subsection 3.1 of Hart [2002], we have $ES_m \subseteq EQ$. And finally, let $x \in X$ such that $x \neq b$, and take $\epsilon > 0$ small enough, such that $x_\epsilon \cap BI_\epsilon = \phi$. By Theorem 3.2 of Hart [2002], for any sequence $\{Q_{m,\mu}\}$ such that $Q_{m,\mu} \in \Theta_{m,\mu}$ and $m\mu > \delta$ for some $\delta > 0$, we have $\lim_{m \rightarrow \infty \mu \rightarrow 0} \hat{\pi}_{Q_{m,\mu}}[BI_\epsilon] = 1$. Hence, $\lim_{m \rightarrow \infty \mu \rightarrow 0} \hat{\pi}_{Q_{m,\mu}}[x_\epsilon] = 0$ and $\liminf_{m \rightarrow \infty \mu \rightarrow 0} \hat{\pi}_{m,\mu}[x_\epsilon] = 0$. Therefore, $ESLP \subseteq BI$.

As we have seen in Remark 1 in Subsection 3.1, equilibria other than the backward induction equilibrium may be m -evolutionarily stable. Therefore, we have $BI \subsetneq ES_m$.

In order to prove that ES_m may be a proper subset of EQ , consider the game Γ_2 of Figure 2. Assume that $m > 2$ is even and let $x \in X$ be the mixed profile $(1, 1/2)$ — i.e., the proportion of c^1 is 1 and the proportion of c^2 is $1/2$ — then x is a Nash equilibrium. The backward induction equilibrium $(0, 0)$

⁸Of course, the result of our Main Theorem is that the first inclusion is actually an equality.

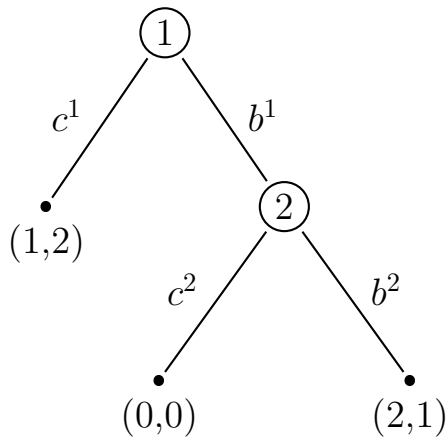


Figure 2: The Game Γ_2

(i.e., $b = (b^1, b^2)$) is *one-mutation-reachable* (see the proof of Theorem 3.1. of Hart [2002]) from⁹ x ; i.e., b can be reached from x by one mutation step in node 2 (from c^2 to b^2) followed by selection steps in the two nodes. Therefore, there is a constant $C = C(m) > 0$, such that the probability of leaving x and reaching b before returning to x is at least $C\mu$ (as m is fixed and for all $\mu > 0$). In order to reach x from b , we must have two consecutive moves by mutation in any of the nodes (either a mutation from b^1 to c^1 or a mutation from b^2 to c^2) while there is selection in both nodes towards b . Therefore, the mean first entrance time from b to x , denoted $u_{b,x}$, is of the order of¹⁰ $(\sigma/m\mu)^2 \doteq 1/\mu^2$. The mean recurrence time of x satisfies $u_x \geq C\mu \cdot u_{b,x} \doteq 1/\mu$. The invariant probability of x is¹¹ $1/u_x \doteq \mu$, and therefore x is a Nash equilibrium which is not m -evolutionarily stable.¹² \square

⁹To streamline the argument, we are ignoring the distinction between a state ω and its corresponding pair of mixed strategies $x = x(\omega)$ (given by (2.2)).

¹⁰Note that m is fixed and therefore $m \doteq 1$.

¹¹See (7.4) of Feller [1950].

¹²See also Subsection 2.3 of Nöldeke and Samuelson [1993]. There it is shown (Proposition 1) that if a component of states is not one-mutation-reachable from any other component, but is one mutation away from some other component, then that component is not in the support of the limiting distribution.

4 The Outcome of the Dynamics

In this section we prove a few general propositions on Markov chains, and we use these propositions to describe the long-run behavior of our dynamics, and to prove the Main Theorem.

4.1 An Outline of the Proof

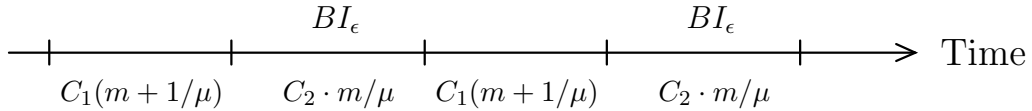
We begin by providing an outline of the proof. By Theorem 3.2 of Hart [2002], in order to show (3.1), it is enough to assume that $m\mu \leq D$ for some $D < \infty$.

The proof is based on the following two properties of our dynamic system, regardless of what specific matrix $Q \in \Theta_{m,\mu}$ we choose:

P1 The expected time to reach BI_ϵ is at most $C_1(m + 1/\mu)$ for some constant C_1 .

P2 The expected time to leave BI_ϵ is at least $C_2 m/\mu$ for some constant C_2 .

By **P1** and **P2**, over time, the Markov chain looks like this:



Therefore (Proposition 4.4), on average, the system will be in BI_ϵ a proportion of at least $\frac{C_2 m/\mu}{C_2 m/\mu + C_1(m+1/\mu)} \stackrel{\circ}{=} \frac{1}{1+\mu+1/m}$ of the time, or $\pi[BI_\epsilon] \geq \frac{1}{1+\mu+1/m} \rightarrow 1$ as $m \rightarrow \infty$ and $\mu \rightarrow 0$.

In order to show **P2**, note that if the populations at all nodes play the backward induction strategy, then the unique local best reply for each i is b^i (recall that b is the unique backward induction equilibrium). Therefore,

there exists a $\lambda > 0$ such that b^i is the unique local best reply of i when the proportion of the individuals at each node who do not play the backward induction strategy is less than λ , and therefore a change from b^i towards any other strategy can be done only by mutation.

If $\epsilon < \lambda$, the proportion of b^i for each state in BI_ϵ will decrease only by mutation, and if we start in $BI_{\epsilon/2} \subseteq BI_\epsilon$ we will need $\geq m\epsilon/2$ mutations. The expected time of each mutation is $1/\mu$; therefore, the expected time to leave BI_ϵ (starting from $BI_{\epsilon/2}$, we use **P1** with $\epsilon/2$) is at least C_2m/μ (Proposition 4.8).

In order to show **P1**, we need the following proposition (Proposition 4.3):

Let Ω be a finite state space, which can be partitioned into $K+1$ different subspaces $\Omega_0, \dots, \Omega_K$. Our aim is to calculate the expected time it takes to get to Ω_0 , starting from Ω_K (denote that average by U). Let u_k , for $k = 1, \dots, K$, be the expected time it takes to go from Ω_k to Ω_{k-1} . Then it is easy to see that $U \leq u_K + \dots + u_1$.

Now assume that we have the one-step probabilities. Assume that for every $k = 1, \dots, K$, there is a probability of f_k of going (in one step) “forward” from Ω_k to Ω_{k-1} , and there is also a probability of g_k of going “back” from Ω_k to Ω_K (and a probability of $(1 - f_k - g_k)$ of staying in Ω_k). If $g_k = 0$ for all k , then the expected time to go from Ω_k to Ω_{k-1} is $1/f_k$, and we get $U \leq 1/f_K + \dots + 1/f_1$.

If $g_k \geq 0$ (for the case $K = 2$, see Figure 3), the expected time to go from Ω_K to Ω_{K-1} is $1/f_K$. However, in Ω_{K-1} , for each “successful” move to Ω_{K-2} , there are, on average, g_{K-1}/f_{K-1} moves back to Ω_K . Therefore, the expected time to go from Ω_{K-1} to Ω_{K-2} is $1/f_{K-1} + g_{K-1}/f_{K-1} \cdot 1/f_K$. Adding for all k and rearranging yield

$$U \leq \sum_{k=1}^K \left[\frac{1}{f_k} \prod_{l=1}^{k-1} \left(1 + \frac{g_l}{f_l} \right) \right] \leq \left(\sum_{k=1}^K \frac{1}{f_k} \right) \left[\prod_{l=1}^{K-1} \left(1 + \frac{g_l}{f_l} \right) \right]. \quad (4.1)$$

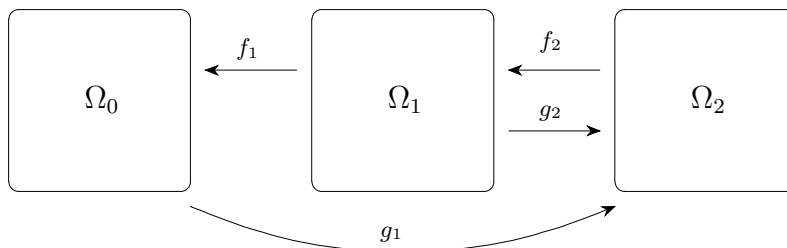


Figure 3: The State Space for $K = 2$

The proof of **P1** is by backward induction. Let us show it in the game Γ_1 of Figure 1 with the basic dynamics, for node $i = 2$.

Let Ω' be the set of states with $x^3(b^3) \geq 1 - \epsilon$ (for some $\epsilon < 1/10$, which is an appropriate λ for this game). A move at 3 from b^3 to c^3 can only occur by mutation; therefore the proportion of b^3 will stay above $1 - \epsilon$ with probability greater than^{13,14} $(1 - \mu)$.

Assume by induction that it takes $O(m + 1/\mu)$ periods to reach Ω' . Let U be the expected time until both $x^2(b^2) \geq 1 - \epsilon$ and $x^3(b^3) \geq 1 - \epsilon$. Then we have $U \leq O(m + 1/\mu) + U_1 + U_2$, where U_1 is the expected time until node 2 becomes a *reached node*¹⁵ (while in Ω' and not leaving it) and U_2 is the expected time until $x^2(b^2) \geq 1 - \epsilon$ (while in Ω' and node 2 is reached).

Step 1: We compute U_1 (Proposition 4.5).

Divide Ω' into two sets: in Ω_1 put all the states where all the population at 1 plays b^1 (and therefore node 2 is not a reached node), and in Ω_0 put all the states where node 2 is a reached node. There is a probability of at least μ of going from Ω_1 to Ω_0 (a mutation at node 1 from b^1 to c^1), and the probability of going back — outside of Ω' , i.e., the probability that the

¹³We are ignoring all the constants.

¹⁴This is also true when $|N(i)| > 1$.

¹⁵A node i is a *reached node* in state ω , if in all the nodes between i and the root, at least one individual plays the strategy towards node i .

proportion of b^3 will decrease below $1 - \epsilon$ — is only μ . Using (4.1) we have¹⁶

$$U_1 \leq (O(m + 1/\mu) + 1/\mu)(1 + \frac{\mu}{\mu}) = O(m + 1/\mu).$$

Step 2: We compute U_2 (Proposition 4.6).

Node 2 will remain a reached node, unless there is only one individual in node 1 who plays c^1 , and that individual is chosen (with probability $1/m$). Therefore the node will remain a reached node with probability $(1 - 1/m)$.

If the proportion of c^2 is greater than ϵ , then the probability of choosing such an individual is $\geq \epsilon$, and the probability that the individual will change his strategy to b^2 by selection is $\geq \sigma$. Therefore, the probability that the proportion of c^2 will decrease is $\geq \epsilon\sigma$.

A move from b^2 to c^2 can be done only by mutation. Therefore the probability that the proportion of c^2 will increase is $\leq \mu$.

Using (4.1), we have (we assume that $m\mu \leq D$)

$$U_2 \leq \left(O(m + 1/\mu) + \sum_{k=1}^m \frac{1}{\epsilon\sigma} \right) \left(\prod_{k=1}^m \left(1 + \frac{\mu}{\epsilon\sigma} \right) \right) \leq O(m + 1/\mu) e^{m\mu} = O(m + 1/\mu).$$

Therefore, $U \leq O(m + 1/\mu)$, and by induction we have **P1**.

4.2 General Propositions

Let Ω be a finite state space with a transition matrix $Q = (q_{\omega, \omega'})_{\omega, \omega' \in \Omega}$. For each $\Omega' \subseteq \Omega$ and $\omega \in \Omega$, let $\{X_t\}_{t=0}^{\infty}$ be a Markov chain on Ω with initial state ω . Define $T(\omega, \Omega') = \min\{t \geq 0 \mid X_t \in \Omega'\}$; i.e., $T(\omega, \Omega')$ is the number of periods it takes to get to Ω' starting from ω .

Let $U(\omega, \Omega') = E[T(\omega, \Omega')]$.

For each $\Omega_1, \Omega_2 \subseteq \Omega$ let $U(\Omega_1, \Omega_2) = \max_{\omega \in \Omega_1} U(\omega, \Omega_2)$.

¹⁶For a node at distance k from the root, Ω' is partitioned into $(k - 1)$ sets, which yields an expected time of the order of $(k - 1) \cdot (1/\mu + m)$ and not $(1/\mu + m)^{k-1}$.

Proposition 4.1. *Let Ω be a finite state space with a transition matrix Q and let $\Omega_0, \Omega_1, \Omega_2 \subseteq \Omega$. Then $U(\Omega_0, \Omega_2) \leq U(\Omega_0, \Omega_1) + U(\Omega_1, \Omega_2)$.*

Proof. Let $\omega_0 \in \Omega_0$ such that $U(\omega_0, \Omega_2) = U(\Omega_0, \Omega_2)$, and let $\{X_t\}_{t=0}^\infty$ be a Markov chain on Ω with initial state ω_0 . Define $T_1 = T(\omega_0, \Omega_2)$ and $T_2 = \min\{t \geq T_1 \mid X_t \in \Omega_2\}$. Therefore $E[T_1] = U(\omega_0, \Omega_2)$, and as $T_2 \geq \min\{t \geq 0 \mid X_t \in \Omega_2\} = T(\omega_0, \Omega_2)$ we get $E[T_2] \geq U(\omega_0, \Omega_2)$.

$$\begin{aligned} E[T_2 - T_1] &= \sum_{\omega \in \Omega} P[X_{T_1} = \omega] E[T_2 - T_1 \mid X_{T_1} = \omega] \\ &= \sum_{\omega \in \Omega_1} P[X_{T_1} = \omega] E[T_2 - T_1 \mid X_{T_1} = \omega] \end{aligned}$$

and

$$\begin{aligned} E[T_2 - T_1 \mid X_{T_1} = \omega] &= \sum_{k=1}^{\infty} P[T_2 - T_1 \geq k \mid X_{T_1} = \omega] \\ &= \sum_{k=1}^{\infty} \sum_{\omega_1, \dots, \omega_{k-1} \notin \Omega_2} q_{\omega, \omega_1} \cdots q_{\omega_{k-2}, \omega_{k-1}} \\ &= \sum_{k=1}^{\infty} P[T(\omega, \Omega_2) \geq k] = U(\omega, \Omega_2). \end{aligned}$$

Therefore

$$U(\Omega_0, \Omega_2) \leq E[T_2] = E[T_1] + E[T_2 - T_1] \leq U(\Omega_0, \Omega_1) + U(\Omega_1, \Omega_2). \quad \square$$

Proposition 4.2. *Let $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be two real sequences such that $f_n > 0$ and $g_n \geq 0$ for all n . Let $\{a_n\}_{n=0}^\infty$ satisfy $a_0 = 0$ and for all $n > 0$,*

$$a_n \leq \frac{1}{f_n} + \left(1 + \frac{g_n}{f_n}\right) a_{n-1}.$$

Then for all $n > 0$,

$$a_n \leq \sum_{k=1}^n \left[\frac{1}{f_k} \prod_{l=k+1}^n \left(1 + \frac{g_l}{f_l} \right) \right].$$

Proof. We will use induction on n . For $n = 1$ we have $a_1 \leq 1/f_1 + (1 + e_1/f_1)a_0 = 1/f_1$. Assume that

$$a_{n-1} \leq \sum_{k=1}^{n-1} \left[\frac{1}{f_k} \prod_{l=k+1}^{n-1} \left(1 + \frac{g_l}{f_l} \right) \right].$$

Then

$$\begin{aligned} a_n &\leq \frac{1}{f_n} + \left(1 + \frac{e_n}{f_n} \right) a_{n-1} \leq \frac{1}{f_n} + \left(1 + \frac{e_n}{f_n} \right) \sum_{k=1}^{n-1} \left[\frac{1}{f_k} \prod_{l=k+1}^{n-1} \left(1 + \frac{g_l}{f_l} \right) \right] \\ &= \frac{1}{f_n} + \sum_{k=1}^{n-1} \left[\frac{1}{f_k} \prod_{l=k+1}^n \left(1 + \frac{g_l}{f_l} \right) \right] = \sum_{k=1}^n \left[\frac{1}{f_k} \prod_{l=k+1}^n \left(1 + \frac{g_l}{f_l} \right) \right]. \quad \square \end{aligned}$$

Let Ω be a state space with a transition matrix Q . For all $\omega \in \Omega$ and $A \subseteq \Omega$, let $Q(\omega, A) = \sum_{\omega' \in A} q_{\omega, \omega'}$ — the probability of going from ω to A in one step.

Proposition 4.3. *Let $\Omega = \bigcup_{k=0}^K H_k$ be a finite state space, where $H_k \neq \emptyset$ for all $k=0, \dots, K$ and $H_k \cap H_l = \emptyset$ for all $0 \leq l < k \leq K$. Let Q be a transition matrix on Ω such that for all $1 \leq k \leq K$ there are constants $f_k > 0$ and g_k , such that $Q(\omega, \bigcup_{l=0}^{k-1} H_l) \geq f_k$ and $Q(\omega, \bigcup_{l=k+1}^K H_l) \leq g_k$ for all $\omega \in \Omega_k$. Then*

$$U(\Omega, H_0) \leq \sum_{k=1}^K \left[\frac{1}{f_k} \prod_{l=1}^{k-1} \left(1 + \frac{g_l}{f_l} \right) \right] \leq \left(\sum_{k=1}^K \frac{1}{f_k} \right) \left[\prod_{l=1}^{K-1} \left(1 + \frac{g_l}{f_l} \right) \right].$$

Proof. For every $1 \leq k \leq K$, let $A_k = \bigcup_{l=0}^{k-1} H_l$, $B_k = \bigcup_{l=k}^K H_l$, $u_k = U(H_k, A_k)$ and $v_k = u_K + \dots + u_k$, and let $v_{K+1} = 0$.

For every $1 \leq k \leq l \leq K$ we will prove by induction on $s = l, l-1, \dots, k$ that $U(H_l, A_s) \leq u_l + u_{l-1} + \dots + u_s$. For $s = l$, we have $U(H_l, A_s) =$

$$U(H_l, A_l) = u_l.$$

Assume by induction that $U(H_l, A_{s+1}) \leq u_l + \dots + u_{s+1}$. Since $U(A_s, A_s) = 0$ and $U(H_s, A_s) \geq 0$, we get $U(A_s \cup H_s, A_s) = \max\{U(A_s, A_s), U(H_s, A_s)\} = U(H_s, A_s)$. Therefore, using Proposition 4.1,

$$\begin{aligned} U(H_l, A_s) &\leq U(H_l, A_s \cup H_s) + U(A_s \cup H_s, A_s) = U(H_l, A_{s+1}) + U(H_s, A_s) \\ &\leq u_l + \dots + u_{s+1} + u_s. \end{aligned}$$

Therefore $U(B_l, A_k) = \max_{s=l, \dots, K} U(H_s, A_k) \leq v_k$ for every $1 \leq k \leq l \leq K$.

Let $\omega_k \in H_k$ such that $U(H_k, A_k) = u_k$ is attained at ω_k , i.e., $u_k = U(\omega_k, A_k)$. Then $\omega_k \notin A_k$ and we have

$$\begin{aligned} u_k &= U(\omega_k, A_k) = 1 + \sum_{\omega \in \Omega} q_{\omega_k, \omega} U(\omega, A_k) \\ &= 1 + \sum_{\omega \in B_{k+1}} q_{\omega_k, \omega} U(\omega, A_k) + \sum_{\omega \in H_k} q_{\omega_k, \omega} U(\omega, A_k) + \sum_{\omega \in A_k} q_{\omega_k, \omega} U(\omega, A_k) \\ &\leq 1 + U(B_{k+1}, A_k) Q(\omega_k, B_{k+1}) + U(\Omega_k, A_k) Q(\omega_k, H_k) + 0 \\ &\leq 1 + v_{k+1} g_k + u_k (1 - f_k) + 0. \end{aligned}$$

After rearrangement we have

$$f_k u_k + f_k v_{k+1} \leq 1 + g_k v_{k+1} + f_k v_{k+1}, \text{ and}$$

$$v_k \leq \frac{1}{f_k} + \left(1 + \frac{g_k}{f_k}\right) v_{k+1}.$$

Therefore, by Proposition 4.2,

$$U(\Omega, H_0) = U(B_1, A_1) \leq v_1 \leq \sum_{k=1}^K \left[\frac{1}{f_k} \prod_{l=1}^{k-1} \left(1 + \frac{g_l}{f_l}\right) \right]. \quad \square$$

Remark 4.1. In Proposition 4.3 it is sufficient to assume that $U(\Omega, \Omega \setminus H_K) = U(H_K, A_K) = v_K \leq 1/f_K$, without any other assumptions on the one-step

transition probabilities on H_K .

Proposition 4.4. *Let a finite state space Ω and a transition matrix Q be an irreducible and aperiodic Markov chain with an invariant distribution π . Let $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$ and let C_1 and C_2 be constants which satisfy:*

1. $U(\Omega, \Omega_1) \leq C_1$.
2. $0 < C_2 \leq U(\omega, \Omega \setminus \Omega_2)$ for every $\omega \in \Omega_1$.

Then

$$\frac{\pi[\Omega \setminus \Omega_2]}{\pi[\Omega_2]} \leq \frac{C_1}{C_2}.$$

Proof. Let $X = (X_t)_{t=0}^\infty \in \Omega^\mathbb{N}$. Let $T_0 = T_0(X) = 0$ and define successively for $n \geq 1$:

$$T_{2n-1} = T_{2n-1}(X) = \min\{t > T_{2n-2} \mid X_t \in \Omega_1\}$$

$$T_{2n} = T_{2n}(X) = \min\{t > T_{2n-1} \mid X_t \in \Omega \setminus \Omega_2\}.$$

For every $t \geq 1$ define:

$$H_t = H_t(X) = \max\{n \mid T_{2n} < t\}$$

$$P_t = P_t(X) = \frac{1}{t} \sum_{n=0}^{t-1} 1_{\{X_n \in \Omega \setminus \Omega_2\}}$$

$$Q_t = Q_t(X) = \frac{1}{t} \sum_{n=0}^{t-1} 1_{\{X_n \in \Omega_2\}}.$$

Then for every t

$$\begin{aligned}
P_t &= \frac{1}{t} \sum_{n=0}^{t-1} 1_{\{X_n \in \Omega \setminus \Omega_2\}} = \frac{1}{t} \left[\sum_{n=0}^{T_1-1} 1_{\{X_n \in \Omega \setminus \Omega_2\}} + \cdots + \sum_{n=T_{2H_t}}^{t-1} 1_{\{X_n \in \Omega \setminus \Omega_2\}} \right] \\
&\leq \frac{1}{t} [T_1 + \cdots + (T_{2H_t-1} - T_{2H_t-2}) + (T_{2H_t+1} - T_{2H_t})] \\
&= \frac{\sum_{n=1}^{H_t+1} (T_{2n-1} - T_{2n-2})}{t} \\
Q_t &= \frac{1}{t} \sum_{n=0}^{t-1} 1_{\{X_n \in \Omega_2\}} = \frac{1}{t} \left[\sum_{n=0}^{T_1-1} 1_{\{X_n \in \Omega_2\}} + \cdots + \sum_{n=T_{2H_t}}^{t-1} 1_{\{X_n \in \Omega_2\}} \right] \\
&\geq \frac{1}{t} [(T_2 - T_1) + \cdots + (T_{2H_t} - T_{2H_t-1})] = \frac{\sum_{n=1}^{H_t} (T_{2n} - T_{2n-1})}{t}.
\end{aligned}$$

Let $\{X_t\}_{t=0}^\infty$ now be a Markov chain on Ω with some initial distribution $(q_\omega)_{\omega \in \Omega}$, and define as above T_n , H_t , P_t and Q_t for all $n \geq 0$, $t \geq 1$. Note that $\lim_{t \rightarrow \infty} E[P_t] = \pi[\Omega \setminus \Omega_2]$ and $\lim_{t \rightarrow \infty} E[Q_t] = \pi[\Omega_2]$. As in the proof of Proposition 4.1, for every $n \geq 1$, we have $E[T_{2n-1} - T_{2n-2}] \leq U(\Omega, \Omega_1) \leq C_1$ and $C_2 \leq E[T_{2n} - T_{2n-1}]$.

Therefore

$$\begin{aligned}
\pi[\Omega \setminus \Omega_2] &= \lim_{t \rightarrow \infty} E[P_t] \leq \limsup_{t \rightarrow \infty} E \left[\frac{\sum_{n=1}^{H_t+1} (T_{2n-1} - T_{2n-2})}{t} \right] \\
&\leq \limsup_{t \rightarrow \infty} \frac{E[H_t] + 1}{t} \cdot C_1 = C_1 \limsup_{t \rightarrow \infty} \frac{E[H_t]}{t} = \frac{C_1}{C_2} \limsup_{t \rightarrow \infty} \frac{E[H_t] C_2}{t} \\
&\leq \frac{C_1}{C_2} \limsup_{t \rightarrow \infty} E \left[\frac{\sum_{n=1}^{H_t} (T_{2n} - T_{2n-1})}{t} \right] \leq \frac{C_1}{C_2} \limsup_{t \rightarrow \infty} E[Q_t] \\
&= \frac{C_1}{C_2} \pi[\Omega_2]. \quad \square
\end{aligned}$$

4.3 The Proof of the Main Theorem

We now use the previous propositions to prove the Main Theorem. Let the game Γ be fixed. We start with a number of useful notations.

- For each node $i \in N$, let $Y^i(\omega) = 1 - x_{b^i}^i(\omega)$. This is the proportion of population i that does not play the backward induction strategy in state ω .
- Given two nodes $i, j \in N$ such that i is a descendant of j (i.e., $i \in N(j)$), let $R^{j,i}(\omega)$ be an indicator random variable, defined as 1 if node i is reached from node j in state ω , and 0 otherwise; i.e., $R^{j,i}(\omega) = 1$ if and only if for every node $k \in N$ on the path from j to i there is at least one individual $q \in M(k)$ whose choice ω_q^k is the strategy that leads toward i . Let $a^{k,i} \in A^k$ be that strategy. When j is the root we will write $R^i(\omega)$ for the indicator that i is reached.
- For each node $i \in N$, when every $j \in N(i)$ plays the backward induction strategy — i.e., when $Y^j(\omega) = 0$ — the unique local best reply for i is b^i (recall that b is the unique backward induction equilibrium). Therefore there exists a $\lambda_i > 0$ (appropriately small) such that b^i is the unique local best reply of i when $Y^j(\omega) < \lambda_i$ for all $j \in N(i)$. Let $\lambda = \min_{i \in N} \lambda_i$. This λ depends on the game only, and will be fixed from now on.
- Let $L^i(\omega)$ be an indicator random variable, defined as 1 if $Y^j(\omega) < \lambda$ for all $j \in N(i)$, and 0 otherwise. Thus, when $L^i(\omega) = 1$ the backward induction strategy b^i is the unique local best reply of i .
- Assume that the nodes are numbered $\{1, \dots, n\}$, where $n = |N|$, such that $j \in N(i) \Rightarrow j > i$. Let $\epsilon > 0$. Define $G_i(\epsilon) = \{\omega \mid Y^j(\omega) \leq \epsilon \forall j \geq i\}$ for all $i \in N$, and $G_{n+1}(\epsilon) = \Omega$. Note that if $\epsilon < \lambda$, then for every $\omega \in G_i(\epsilon)$ and every $j_1 \geq i$, we have $Y^{j_2}(\omega) \leq \epsilon < \lambda$ for all $j_2 > j_1$. Therefore, $L^{j_1}(\omega) = 1$ for all $j_1 \geq i$.
- Let $\epsilon > 0$. For all $i, j \in N$, define $G_{j,i}(\epsilon) = \{\omega \in G_j(\epsilon) \mid R^i(\omega) = 1\}$. Note that if $\epsilon < \lambda$ and $N(i) \subseteq \{j, j+1, \dots, n\}$, then $R^i(\omega) = 1$ and $L^i(\omega) = 1$ for every $\omega \in G_{j,i}(\epsilon)$.

For every mutation rate $\mu > 0$ and population size m , let $Q \in \Theta(m, \mu)$. For every $\omega \in \Omega_m$ let $\tilde{\omega}$ be the next state, i.e., for all $\omega' \in \Omega_m$, $P[\tilde{\omega} = \omega' | \omega] = P[X_{t+1} = \omega' | X_t = \omega] = q_{\omega, \omega'}$.

We use the notations above to state the following implications of the dynamics given by (2.4) – (2.9):

1. All strategies have a positive probability of being chosen by mutation:

$$P[\tilde{\omega}_{q(i)}^i = a^i | \omega] \geq \alpha_1 \mu \text{ for every } a^i \in A^i. \quad (4.2)$$

2. If node i is not reached, i.e., $R^i(\omega) = 0$, then all strategies of i yield the same payoff and only mutation affects ω^i . Therefore:

$$\text{If } R^i = 0 \text{ then } P[\tilde{\omega}_{q(i)}^i \neq \omega_{q(i)}^i | \omega] \leq \alpha_2 \mu. \quad (4.3)$$

3. If $R^i(\omega) = 1$ and $L^i(\omega) = 1$ then b^i is the global best reply of i and thus certainly a “better strategy” for a “non- b^i individual” (i.e., $b^i \in B^i(q(i), \omega)$ when $\omega_{q(i)}^i \neq b^i$), and there isn’t any “better strategy” for a “ b^i individual” (i.e., $B(q(i), \omega) = \phi$ when $\omega_{q(i)}^i = b^i$). Therefore:

$$\text{If } L^i R^i = 1 \text{ and } \omega_{q(i)}^i \neq b^i \text{ then } P[\tilde{\omega}_{q(i)}^i = b^i | \omega] \geq \sigma. \quad (4.4)$$

$$\text{If } L^i R^i = 1 \text{ and } \omega_{q(i)}^i = b^i \text{ then } P[\tilde{\omega}_{q(i)}^i \neq b^i | \omega] \leq \alpha_2 \mu. \quad (4.5)$$

$$\text{If } L^i R^i = 1 \text{ then } P[Y^i(\tilde{\omega}) > Y^i(\omega) | \omega] \leq \alpha_2 \mu. \quad (4.6)$$

4. Using (4.3) and (4.6) we get

$$\text{If } L^i = 1 \text{ then } P[Y^i(\tilde{\omega}) > Y^i(\omega) | \omega] \leq \alpha_2 \mu. \quad (4.7)$$

5. If $R^i(\omega) = 1$, then for all the nodes $j \in N$ on the path from the root to node i , there must exist $q \in M(j)$ such that $w_q^j = a^{j,i}$ (recall that $a^{j,i}$ is

the strategy in node j which leads toward node i). To get $R^i(\tilde{\omega}) = 0$, there must be some node j on the path where no one plays $a^{j,i}$, which means that the only individual who played $a^{j,i}$ must have been chosen. Therefore (by (2.5)):

$$\text{If } R^i = 1 \text{ then } P[R^i(\tilde{\omega}) = 0 \mid \omega] \leq n\gamma_2 \frac{1}{m}. \quad (4.8)$$

6. If $\epsilon < \lambda$, then for every $\omega \in G_i(\epsilon)$, we have $L^j(\omega) = 1$ for all $j \geq i$. If $\tilde{\omega} \notin G_i(\epsilon)$ then there is some node $j \geq i$ such that $Y^j(\tilde{\omega}) \geq \epsilon > Y^j(\omega)$. Therefore (by (4.7)):

$$\text{If } \omega \in G_i(\epsilon) \text{ then } P[\tilde{\omega} \notin G_i(\epsilon) \mid \omega] \leq (n + 1 - i)\alpha_2\mu. \quad (4.9)$$

From now on, $\epsilon > 0$, $m > 2n\gamma_2$, $\mu < 1/(2\alpha_2)$ and $Q \in \Theta(m, \mu)$ are fixed.

Proposition 4.5. *If $\lambda > \epsilon$, then there is a constant $C_{1,1} = C_{1,1}(\Gamma, \epsilon)$ such that $U(\Omega_m, G_{j,i}(\epsilon)) \leq C_{1,1}(U(\Omega_m, G_j(\epsilon)) + 1/\mu)$ for all $i, j \in N$.*

Proof. Without loss of generality assume that the nodes along the path from the root to node i are $1, 2, \dots, i$.

Let $H_k = \{\omega \in G_j(\epsilon) \mid R^{k,k+1}(\omega) = 0, R^{k+1,i}(\omega) = 1\}$ for $k = 1, \dots, i - 1$; i.e., H_k is the set of all states in $G_j(\epsilon)$ such that node i is reached from node $k + 1$, but in node k no one chooses the strategy towards node $k + 1$. Let $H_0 = G_{j,i}(\epsilon)$ and let $H_i = \Omega \setminus G_j(\epsilon)$. If node i is not reached, then there is a node where no one chooses the strategy towards node i , and there is a unique maximum node like that. Therefore $\{H_k\}_{k=0}^i$ is a partition of Ω .

Let $1 \leq k < i$ and $\omega \in H_k$. Then $\omega \in G_j(\epsilon)$ and $R^l(\omega) = 0$, $R^{l,l+1}(\omega) = 1$ for all $k \leq l < i$. To get $\tilde{\omega} \in H_l$ for some $k < l < i$ there must be $R^{l,l+1}(\tilde{\omega}) = 0$, which can happen only by mutation (by (4.3)). By (4.9) we get $P[\tilde{\omega} \in H_l \mid \omega] = P[\tilde{\omega} \notin G_j(\epsilon) \mid \omega] \leq O(\mu)$. Therefore there is a constant g_k (which depends

on the game and not on m or μ) such that

$$Q(\omega, \cup_{l=k+1}^i H_l) = P[\tilde{\omega} \in \cup_{l=k+1}^i H_l \mid \omega] \leq g_k \mu. \quad (4.10)$$

To get $\tilde{\omega} \in H_l$ for some $0 \leq l < k$, it is enough that all the following will happen:

- $\tilde{\omega} \in G_j(\epsilon)$, and by (4.9) we get $P[\tilde{\omega} \in G_j(\epsilon) \mid \omega] \geq (1 - \alpha_2 \mu)$.
- $R^{l', l'+1}(\tilde{\omega}) = 1$ for all $k < l' < i$, and by (4.3) we get $P[R^{l', l'+1}(\tilde{\omega}) = 1 \mid \omega] \geq (1 - \alpha_2 \mu)$.
- $R^{k, k+1}(\tilde{\omega}) = 1$, which happens if $q(k)$ choose by mutation the strategy towards node i , and by (4.2) we get $P[R^{k, k+1}(\tilde{\omega}) = 1 \mid \omega] \geq \alpha_1 \mu$.

As each of those things depends on different nodes, we get (using (2.4)) that there is a constant $f_k > 0$ such that

$$Q(\omega, \cup_{l=0}^{k-1} H_l) = P[\tilde{\omega} \in \cup_{l=0}^{k-1} H_l \mid \omega] \geq (1 - \alpha_2 \mu)^{i-k} \alpha_1 \mu \geq f_k \mu. \quad (4.11)$$

Using Proposition 4.3 on $\{H_k\}_{k=0}^i$ with (4.10) and (4.11) we get

$$\begin{aligned} U(\Omega_m, G_{j,i}(\epsilon)) &= U(\Omega_m, H_0) \leq \left(U(\Omega_m, \Omega_m \setminus H_i) + \sum_{k=1}^{i-1} \frac{1}{f_k \mu} \right) \prod_{k=1}^{i-1} \left(1 + \frac{g_k \mu}{f_k \mu} \right) \\ &\leq C_{1,1} \left(U(\Omega_m, G_j(\epsilon)) + \frac{1}{\mu} \right). \quad \square \end{aligned}$$

Proposition 4.6. *If $\lambda > \epsilon$, then there is a constant $C_{1,2} = C_{1,2}(\Gamma, \epsilon)$ such that $U(\Omega_m, G_i(\epsilon)) \leq C_{1,2} \cdot (U(\Omega_m, G_{i+1,i}(\epsilon)) + m) e^{C_{1,2} m \mu}$ for all $i \in N$.*

Proof. Without loss of generality assume that ϵm is an integer.

Let $i \in N$. Note that by the assumption on the numbering of the nodes, for every node $j \in N(i)$ we have $j > i$. Therefore $N(i) \subseteq \{i+1, \dots, n\}$.

Let $K = m - \epsilon m + 1$. Let $H_K = \Omega \setminus G_{i+1,i}(\epsilon)$, let $H_k = \{\omega \in G_{i+1,i}(\epsilon) \mid Y^i(\omega) = \frac{k + \epsilon m}{m}\}$ for $k = 1, \dots, K-1$, and let $H_0 = \{\omega \in G_{i+1,i}(\epsilon) \mid Y^i(\omega) \leq \epsilon\}$.

Then $\{H_k\}_{k=0}^{K-1}$ is a partition of $G_{i+1,i}(\epsilon)$ according to the value of Y^i , and $\{H_k\}_{k=0}^K$ is a partition of Ω .

Let $1 \leq k < K$ and $\omega \in H_k$. Then $\omega \in G_{i+1}(\epsilon)$, $R^i(\omega) = 1$ and $L^i(\omega) = 1$. To get $\tilde{\omega} \in H_l$ for some $k < l < K$, there must be $Y^i(\tilde{\omega}) > Y^i(\omega)$. To get $\tilde{\omega} \in H_K$ there must be $\tilde{\omega} \notin G_{i+1,i}(\epsilon)$, which means that either $R^i(\tilde{\omega}) = 0$ or $\tilde{\omega} \notin G_{i+1}(\epsilon)$. Therefore (using (4.6), (4.8) and (4.9)) there is a constant g (which does not depend on k) such that

$$\begin{aligned} Q(\omega, \cup_{l=k+1}^K H_l) &= P[\tilde{\omega} \in \cup_{l=k+1}^K W_l \mid \omega] \\ &\leq P[Y^i(\tilde{\omega}) > Y^i(\omega) \mid \omega] + P[R^i(\tilde{\omega}) = 0 \mid \omega] + P[\tilde{\omega} \notin G_{i+1}(\epsilon) \mid \omega] \\ &\leq g(\mu + \frac{1}{m}). \end{aligned} \quad (4.12)$$

To get $\tilde{\omega} \in H_{k-1}$, it is enough that all the following will happen:

- $\tilde{\omega} \in G_{i+1}(\epsilon)$, and by (4.9) we get $P[\tilde{\omega} \in G_{i+1}(\epsilon) \mid \omega] \geq (1 - \alpha_2\mu)$.
- $R^i(\tilde{\omega}) = 1$, and by (4.8) we get $P[R^i(\tilde{\omega}) = 1 \mid \omega] \geq (1 - n\gamma_2\frac{1}{m})$.
- $Y^i(\tilde{\omega}) = Y^i(\omega) - \frac{1}{m}$, and that happens if $q(i)$ is chosen from $\{q \in M(i) \mid w_q^i \neq b^i\}$, and changes his strategy to b^i . By (2.5) and (4.4) we get $P[Y^i(\tilde{\omega}) = Y^i(\omega) - \frac{1}{m} \mid \omega] \geq \sigma\gamma_1 \frac{|\{q \in M(i) \mid w_q^i \neq b^i\}|}{m} = \sigma\gamma_1 Y^i(\omega) \geq \sigma\gamma_1\epsilon$.

As each of those things depends on different nodes, we get (using (2.4)) that there is a constant $f > 0$ such that

$$Q(\omega, \cup_{l=0}^{k-1} H_l) = P[\tilde{\omega} \in \cup_{l=0}^{k-1} H_l \mid \omega] \geq (1 - \alpha_2\mu)(1 - n\gamma_2\frac{1}{m})\sigma\gamma_1\epsilon \geq f\epsilon. \quad (4.13)$$

Using Proposition 4.3 on $\{H_k\}_{k=0}^K$ with (4.12) and (4.13) we get

$$\begin{aligned}
U(\Omega_m, G_i(\epsilon)) &= U(\Omega_m, H_0) \leq \left(U(\Omega_m, \Omega_m \setminus H_K) + \sum_{k=1}^{K-1} \frac{1}{f\epsilon} \right) \prod_{k=1}^{K-1} \left(1 + \frac{g(\mu+1/m)}{f\epsilon} \right) \\
&\leq \left(U(\Omega_m, G_{i+1,i}(\epsilon)) + \frac{m}{f\epsilon} \right) \left(1 + \frac{g}{f\epsilon} \frac{m\mu+1}{m} \right)^m \\
&\leq \frac{1}{f\epsilon} \cdot \left(U(\Omega_m, G_{i+1,i}(\epsilon)) + m \right) \cdot e^{\frac{g}{f\epsilon}(m\mu+1)}. \quad \square
\end{aligned}$$

Corollary 4.7. $U(\Omega_m, G_1(\epsilon)) \leq C_1(m + 1/\mu)e^{C_1m\mu}$.

Proof. If $\epsilon_1 \geq \epsilon_2 > 0$ then $G_1(\epsilon_2) \subseteq G_1(\epsilon_1)$ and therefore $U(\Omega_m, G_1(\epsilon_2)) \geq U(\Omega_m, G_1(\epsilon_1))$. Thus, assume that $\epsilon < \lambda$. Using Proposition 4.5 with Proposition 4.6 we get, for all $i \in N$,

$$\begin{aligned}
U(\Omega_m, G_i(\epsilon)) &\leq C_{1,2} \cdot \left(U(\Omega_m, G_{i+1,i}(\epsilon)) + m \right) e^{C_{1,2}m\mu} \\
&\leq C_{1,2} \cdot \left(C_{1,1} \cdot \left(U(\Omega_m, G_{i+1}(\epsilon)) + 1/\mu \right) + m \right) e^{C_{1,2}m\mu}.
\end{aligned}$$

As $U(\Omega_m, G_{n+1}(\epsilon)) = 0$, we get $U(\Omega_m, G_1(\epsilon)) \leq C_1 \cdot (m + 1/\mu)e^{C_1m\mu}$. \square

Proposition 4.8. *Let $\lambda > \epsilon_1 > \epsilon_2 > 0$. Then there is a constant $C_2 = C_2(\Gamma, \epsilon_1 - \epsilon_2) > 0$ such that $U(\omega, \Omega_m \setminus G_1(\epsilon_1)) > C_2(m/\mu)$ for every $\omega \in G_1(\epsilon_2)$.*

Proof. Let $\omega \in G_1(\epsilon_1)$ such that $Y^i(\omega) \leq k/m$ for some $i \in N$ and $k \leq \epsilon_1 m$. By (4.7) and (2.6) $P[Y^i(\tilde{\omega}) = \frac{k+1}{m} \mid \omega] = P[Y^i(\tilde{\omega}) > Y^i(\omega) \mid \omega] \leq \alpha_2 \mu$. Therefore, for $\omega \in G_1(\epsilon_1)$ such that $\max_{i \in N} Y^i(\omega) \leq k/m$, we get $P[\max_{i \in N} Y^i(\tilde{\omega}) = \frac{k+1}{m} \mid \omega] \leq n\alpha_2 \mu$ and $U(\{\max_{i \in N} Y^i \leq \frac{k}{m}\}, \{\max_{i \in N} Y^i = \frac{k+1}{m}\}) \geq \frac{1}{n\alpha_2 \mu}$ for all $k \leq \epsilon_1 m$. Let $X_k = \{\omega \mid \max_{i \in N} Y^i(\omega) = \frac{k}{m}\}$ for all $k \leq \epsilon_1 m$ and $X_K = \{\omega \mid \max_{i \in N} Y^i(\omega) > \epsilon_1\} = \Omega_m \setminus G_1(\epsilon_1)$, where $K = \lfloor \epsilon_1 m \rfloor + 1$. Thus $U(X_k, X_{k+1}) \geq \frac{1}{n\alpha_2 \mu}$ and $U(X_k, X_K) = \sum_{l=k}^{K-1} U(X_l, X_{l+1})$ for all $k \leq \epsilon_1 m$. Therefore, for all $\omega \in G_1(\epsilon_2)$,

$$U(\omega, X_K) \geq \sum_{l=\lfloor \epsilon_2 m \rfloor}^{\lfloor \epsilon_1 m \rfloor} \frac{1}{n\alpha_2 \mu} \geq \frac{(\epsilon_1 - \epsilon_2)m - 2}{n\alpha_2 \mu}. \quad \square$$

Now we can prove our main result:

Theorem 4.9. *For every $\epsilon > 0$ and $\infty > D > 0$,*

$$\lim_{\substack{\mu \rightarrow 0 \\ m \rightarrow \infty \\ \mu m \leq D}} \hat{\pi}_{\mu, m}[BI_\epsilon] = 1.$$

Proof. Without loss of generality assume that $\epsilon < \lambda$. For¹⁷ $m\mu \leq D$ and $Q \in \Theta(m, \mu)$, there is, by Corollary 4.7, a constant C_1 such that $U(\Omega_m, G_1(\epsilon/2)) \leq C_1(m + 1/\mu)$, and by Proposition 4.8, there is a constant $C_2 > 0$ such that for every $\omega \in G_1(\epsilon/2)$, we have $U(\omega, \Omega_m \setminus G_1(\epsilon)) > C_2(m/\mu)$. Let π_Q be the invariant distribution of Q ; then, by Proposition 4.4,

$$\hat{\pi}_Q\left[\frac{X \setminus BI_\epsilon}{BI_\epsilon}\right] = \pi_Q\left[\frac{\Omega_m \setminus G_1(\epsilon)}{G_1(\epsilon)}\right] \leq \frac{C_1(m + \frac{1}{\mu})}{C_2(m/\mu)} = \frac{C_1}{C_2}\left(\mu + \frac{1}{m}\right).$$

Therefore,

$$\hat{\pi}_{m, \mu}[BI_\epsilon] \geq \frac{1}{1 + \frac{C_1}{C_2}\left(\mu + \frac{1}{m}\right)} \xrightarrow[\mu \rightarrow 0, m \rightarrow \infty]{} 1. \quad \square$$

Proof of Theorem 3.4. Follows immediately from Theorems 4.9 and 3.2 of Hart [2002]. □

5 Extensions

We now present some related questions.

1. As seen in Remark 2 in Subsection 3.1, we must take a neighborhood of b in order to assure that the limit (3.1) exists and is positive, and therefore, as seen in Corollary 3.5, for every $\epsilon > 0$, the iterative limit $\liminf_m \liminf_\mu$ is positive. However, we do not know whether we need to take a neighborhood of b in order to assure that the iterative limit is positive.

¹⁷We may assume, of course, that $m > 2n\gamma_2$ and $\mu < 1/(2\alpha_2)$.

Thus, it is possible that the backward induction equilibrium is uniformly evolutionarily stable; i.e., the probability, as $\mu \rightarrow 0$, of reaching BI is bounded away from zero for all m .

2. In order to prove the Main Theorem, we assume that the probability of switching by selection to a better strategy is positive (see Remark 3 in Subsection 2.3). It seems that without this assumption, the limit (3.1) may well be strictly less than 1, but this does not necessarily imply that the backward induction equilibrium is not ESLP.
3. We assume in our proof that $m\mu \leq D$ for some $0 < D < \infty$. This assumption is needed in order to show, using Proposition 4.6, that

$$U(\Omega_m, G_i(\epsilon)) \leq O(U(\Omega_m, G_{i,i+1}(\epsilon)) + m);$$

i.e., if node i is a reached node, and in all nodes $j > i$ most of the individuals play their backward induction strategy, then after m periods, most of the individuals in node i will play their backward induction strategy.

It seems possible that the inequality

$$U(\Omega_m, G_i(\epsilon)) \leq O(U(\Omega_m, G_{i,i+1}(\epsilon/2)) + m + 1/\mu)$$

can be proven without any assumption on $m\mu$; i.e., if we start from a state in which at all the nodes $j > i$ the proportion of individuals who do not play the backward induction strategy is small enough (no more than $\epsilon/2$), then the expected time to reach $G_i(\epsilon)$ is bounded. In this case, one obtains a unified proof of the Main Theorem and the two cases ($m\mu \leq D$ here and $m\mu \geq \delta$ in Hart [2002]) need not be considered separately.

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