ALMOST ALL SOCIAL CHOICE CORRESPONDENCES ARE SUBJECT TO THE GIBBARD-SATTERTHWAITE THEOREM

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ABSTRACT

We analyse the manipulability of social choice correspondences via hyperfunctions, i.e., functions that pick a non-empty set of alternatives at each admissible preference profile over sets of alternatives. We consider a domain of lexicographic orderings of sets which allows only two orderings for every ordering over alternatives. We show that Gibbard-Satterthwaite type impossibility results prevail on this very narrow domain which is a subset of many restricted domains defined through standard axioms to extend preferences over alternatives to sets of alternatives. Hence, we are able to verify the robustness of the Gibbard-Satterthwaite theorem, showing that it holds under almost all reasonable domain restrictions. As hyperfunctions are more general objects than regular social choice correspondences, our impossibility results carry to the standard framework.

Keywords: Strategy-proofness, Manipulation, Gibbard-Satterthwaite Theorem

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1. INTRODUCTION

We know, thanks to Gibbard (1973) and Satterthwaite (1975), the non-existence of interesting non-manipulable social choice functions. This result is fairly robust. There is a literature, dating back to Pattanaik (1973), followed by Barberà (1977), Kelly (1977), Gardenfors (1978) and Feldman (1979a, 1979b), showing similar impossibility results for social choice rules, which are not necessarily singleton-valued. Their results are given in a framework where the social choice rules are modelled as social choice correspondences, which assign a set of alternatives to every preference profile over alternatives. The analysis is made under certain extension axioms which connect individual preferences over alternatives to sets of alternatives.

There is a recent trend of carrying this analysis in a framework where the manipulability question is analyzed via hyperfunctions, i.e. functions that pick a non-empty set of alternatives at each admissible preference profile over sets of alternatives. This approach has the advantage of being in a more general framework which, compared to the classical framework, allows to use a finer information about individual preferences over sets. For, standard social choice correspondences impose a strong invariance condition over social choice rules: The social outcome must be the same as far as individuals' rankings over singleton sets are the same. In other words even if individuals change their preferences over sets, the social outcome must remain unchanged as far as their ordering of singleton sets remains the same. It is clear that hyperfunctions do not have such a restriction. Hence, they are more general social choice objects. As a result, every Gibbard-Satterthwaite type of impossibility result in the world of hyperfunctions carries to the standard world of social choice correspondences.

Of course, if we consider social choice hyperfunctions defined over the full domain of preference profiles then strategy-proofness is equivalent to dictatoriality, just by the Gibbard-Satterthwaite theorem. Barberà, Dutta and Sen (2001) show that this equivalence quite robust under domain restrictions. They consider a domain of preferences orderings over sets defined through the idea of expected utility consistency. They use two versions of this concept. The former, leading to a coarser domain, ends up in a Gibbard-Satterthwaite type of result: Any unanimous social choice hyperfunction, which is non-manipulable, must be dictatorial. The latter, leading to a narrower domain allows slightly more permissive result, though still of the Gibbard-Satterthwaite spirit: Any unanimous social choice hyperfunction, which is non-manipulable, must be either dictatorial or bi-dictatorial.

We carry this result further and present Gibbard-Satterthwaite type of impossibility results for social choice hyperfunctions defined over domains restricted through extension axioms which assign to every ordering over alternatives the set of acceptable orderings over sets. We start by showing that a Gibbard-Satterthwaite type of impossibility result prevails under a lexicographic extension axiom which allows only two orderings over sets for every ordering over alternatives. This lexicographic domain is very narrow and is a subset of many domains defined through standard extension axioms. Hence, we are able to state our impossibility result for all "reasonable" restricted domains, including those obtained through extension axioms such as those used by Gardenfors (1976) and Kelly (1977).

After introducing some preliminaries in Section 2, we give the main impossibility result in Section 3 which states a Gibbard-Satterthwaite type result for our lexicographic domain. Section 4 generalizes this impossibility to the superdomains of the domain considered in the previous section. Section 5 makes some closing remarks, relating our results to the previous ones in the literature.

2. PRELIMINARIES

Taking any two integers n and m with $n \geq 2$ and $m \geq 3$, we consider a society $\mathbf{N} = \{1,\ldots, n\}$ confronting a set of alternatives $\mathbf{A} = \{a_1, a_2, \ldots, a_m\}$. We write $\underline{\mathbf{A}} = 2^{\mathbf{A}} \setminus \{\emptyset\}$ for the set of all non-empty subsets of **A**.

We let Π stand for the set of all complete, transitive and antisymmetric binary relations over **A**. Every $\rho \in \Pi$ represents an individual preference on the elements of

A in the following manner: For any a, b $\in$ **A**, a ρ b means "a is at least as good as b".[1] In case the preference ordering over **A** is specified to belong to a particular agent i $\in$ **N**, we will write it as ρ_i . A typical preference profile over **A** will be denoted by $\underline{\rho} = (\rho_1, ..., \rho_n) \in \Pi^N$.

Similarly, we let $\Re$ stand for the set of all complete and transitive orderings over $\underline{\mathbf{A}}$. Every R $\in \Re$ represents an individual preference on the elements of $\underline{\mathbf{A}}$ in the following manner: For any X, Y $\in \underline{\mathbf{A}}$, X R Y means "X is at least as good as Y". We denote P and I for the strict and indifference counterparts of R.[2] In case the preference ordering over $\underline{\mathbf{A}}$ is specified to belong to a particular agent i $\in$ **N**, we will write it as R_i, with its respective strict and indifference counterparts P_i and I_i . A typical preference profile over $\underline{\mathbf{A}}$ will be denoted by $\underline{R} = (R_1, ..., R_n) \in \Re^N$.

Given any D $\subseteq \Re$, we define a social choice hyperfunction as a mapping f: $[D]^N \to \underline{\mathbf{A}}$. Note that we only consider social choice hyperfunctions whose domains are cartesian products of some D $\subseteq \Re$, in which case we say that the social choice hyperfunction is defined over the domain D. A social choice hyperfunction f: $[D]^N \to \underline{\mathbf{A}}$ is manipulable at $\underline{R} \in D^N$ by some i $\in$ **N** if and only if there exists $\underline{R}' \in D^N$ with $R_j = R_j'$ for all j $\in N \setminus \{i\}$ such that $f(\underline{R}')\ P_i\ f(\underline{R})$. We say that f is strategy-proof if and only if there exists no $\underline{R} \in D^N$ at which f is manipulable by some i $\in$ **N**.

An immediate consequence of the Gibbard-Satterthwaite theorem is that when D = $\Re$, for social choice hyperfunctions with a range containing at least three elements, strategy-proofness is equivalent to dictatoriality.[3] We ask whether it is possible to escape this equivalence by restricting the domain D through axioms which extend preference orderings over alternatives to sets of alternatives.

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[1] As ρ is antisymmetric, we have for any distinct a, b $\in$ **A**, we have a ρ b $\Rightarrow$ not b ρ a. In other words, for distinct alternatives, a ρ b means "a is preferred to b".

[2] For any X, Y $\in \underline{\mathbf{A}}$, we write X P Y if and only if X R Y holds but Y R X does not, ie., X is preferred to Y. In case X R Y and Y R X both hold, we write X I Y, which means indifference between X and Y.

[3] A social choice hyperfunction f is *dictatorial* if and only if there exists d $\in$ **N** such that for all $\underline{R}$ within the domain of f we have $f(\underline{R}) \in \text{argmax}_{\underline{\mathbf{A}}} R_d$.

We accept that if the preference ordering over **A** is some $\rho \in \Pi$, then the preference ordering over $\underline{\mathbf{A}}$ can be some $R \in \mathfrak{R}$ which is "consistent" with ρ. Thus, we define a consistency map $\kappa: \Pi \to 2^{\mathfrak{R}} \setminus \{\emptyset\}$ which assigns to every $\rho \in \Pi$ a non-empty set $\kappa(\rho) \subseteq \mathfrak{R}$ of preference orderings on $\underline{\mathbf{A}}$ consistent with ρ. We assume that every consistency map κ satisfies a basic axiom A0, which we define as follows:

A0: Given any $\rho \in \Pi$ and any $R \in \kappa(\rho)$, we have $x \rho y \Leftrightarrow \{x\} R \{y\}$ for all $x, y \in$ **A**.

A0 requires that the ordering of individuals over singleton sets must be the same as their ordering over the basic alternatives. Remark that A0 automatically implies that $\kappa(\rho) \cap \kappa(\rho') = \emptyset$ for all distinct $\rho, \rho' \in \Pi$.

Given any consistency map κ, we write $\mathfrak{R}^{\kappa} = \cup_{\rho \in \Pi} \kappa(\rho)$ for the set of acceptable preference orderings over $\underline{\mathbf{A}}$ defined through κ. Note that $\mathfrak{R}^{\kappa}$ is always a strict subset of $\mathfrak{R}$, as every κ is assumed to satisfy our basic axiom A0.

Now take any $\rho \in \Pi$. Let $\lambda^{+}(\rho)$ be an ordering over $\underline{\mathbf{A}}$ which we call the strong lexicographic extension of ρ and define as follows: Take any two distinct $X, Y \in \underline{\mathbf{A}}$. First consider the case where $\#X = \#Y = k$ for some $k \in \{1,\ldots,m-1\}$. Let, without loss of generality, $X = \{x_1,..., x_k\}$ and $Y = \{y_1,..., y_k\}$ such that $x_j \rho x_{j+1}$ and $y_j \rho y_{j+1}$ for all $j \in \{1,\ldots,k-1\}$. We have $X \lambda^{+}(\rho) Y$ if and only if $x_h \rho y_h$ for the smallest $h \in \{1,\ldots,k\}$ such that $x_h \neq y_h$. Now consider the case where $\#X \neq \#Y$. Let, without loss of generality, $X = \{x_1,..., x_{\#X}\}$ and $Y = \{y_1,..., y_{\#Y}\}$ such that $x_j \rho x_{j+1}$ for all $j \in \{1,\ldots, \#X-1\}$and $y_j \rho y_{j+1}$ for all $j \in \{1,\ldots, \#Y-1\}$. We have either $x_h = y_h$ for all $h \in \{1,\ldots, \min\{\#X, \#Y\}\}$ or there exists some $h \in \{1,\ldots, \min\{\#X, \#Y\}\}$ for which $x_h \neq y_h$. For the first case $X \lambda^{+}(\rho) Y$ if and only if $\#X < \#Y$. For the second case, $X \lambda^{+}(\rho) Y$ if and only if $x_h \rho y_h$ for the smallest $h \in \{1,\ldots, \min\{\#X, \#Y\}\}$ such that $x_h \neq y_h$.

Note that at each $\rho \in \Pi$, this stronger version of the lexicographic extension idea, used by Kaymak and Sanver (forthcoming), determines a unique preference ordering $\lambda^{+}(\rho)$ over $\underline{\mathbf{A}}$ which is complete, transitive and antisymmetric.

We introduce the concept of a strong inverse lexicographic extension similarly: Take any $\rho \in \Pi$. The strong inverse lexicographic extension of ρ is an ordering $\lambda^{-}(\rho)$ over **A** which is defined as follows: Take any two distinct X, Y $\in$ **A**. First consider the case where #X = #Y = k for some k $\in \{1,\ldots,m-1\}$. Let, without loss of generality, X = $\{x_1,..., x_k\}$ and Y = $\{y_1,..., y_k\}$ such that $x_j \ \rho \ x_{j+1}$ and $y_j \ \rho \ y_{j+1}$ for all j $\in \{1,\ldots,k-1\}$. We have X $\lambda^{-}(\rho)$ Y if and only if $x_h \ \rho \ y_h$ for the greatest h $\in \{1,\ldots,k\}$ such that $x_h \neq y_h$. Now consider the case where #X $\neq$ #Y. Let, without loss of generality, X = $\{x_1,..., x_{\#X}\}$ and Y = $\{y_1,..., y_{\#Y}\}$ such that $x_{j+1} \ \rho \ x_j$ for all j $\in \{1,\ldots, \#X-1\}$and $y_{j+1} \ \rho \ y_j$ for all j $\in \{1,\ldots, \#Y-1\}$. We have either $x_h = y_h$ for all h $\in \{1,\ldots, \min\{\#X, \#Y\}\}$ or there exists some h $\in \{1,\ldots, \min\{\#X, \#Y\}\}$ for which $x_h \neq y_h$. For the first case X $\lambda^{-}(\rho)$ Y if and only if #X > #Y. For the second case, X $\lambda^{-}(\rho)$ Y if and only if $x_h \ \rho \ y_h$ for the smallest h $\in \{1,\ldots, \min\{\#X, \#Y\}\}$ such that $x_h \neq y_h$.

Again at each $\rho \in \Pi$, the strong inverse lexicographic extension determines a unique preference ordering $\lambda^{-}(\rho)$ over **A** which is complete, transitive and antisymmetric.

We write λ for the consistency map which at each $\rho \in \Pi$ gives the strong lexicographic and inverse lexicographic extensions of ρ, i.e., $\lambda(\rho) = \{\lambda^{+}(\rho), \lambda^{-}(\rho)\}$ for every $\rho \in \Pi$. We write $\Re^{\lambda} = \cup_{\rho \in \Pi}\lambda(\rho)$ for the set of acceptable preference orderings over **A** defined through λ.

3. THE MAIN IMPOSSIBILITY RESULT

Our main impossibility result is for the domain $\Re^{\lambda}$. We show that $\Re^{\lambda}$ is either dictatorial or bi-dictatorial, ie., unanimous and non-manipulable social choice hyperfunctions defined over $\Re^{\lambda}$ must be either dictatorial or bi-dictatorial.

Before stating our theorem, note that under the consistency map λ, the best and worst elements of every agent is a singleton set. We say that a domain D is regular if and only if D consists of orderings having singleton sets as their unique maximal and minimal elements. A regular domain D is called fully regular if every singleton set is a unique maximal and a unique minimal element for at least one ordering in D. Remark that the range of every unanimous hyperfunction defined over a fully regular domain contains all singleton sets.

Theorem 3.1: A unanimous social choice hyperfunction f: $[\Re^{\lambda}]^N \rightarrow \underline{\mathbf{A}}$ is strategy-proof if and only if f is dictatorial or bi-dictatorial.

The proof of the result benefits from the option set technique used by Barberà and Peleg (1990) as well as of the series of lemmata in Barberà, Dutta and Sen (2001). We give all these in our Appendix A.

The domain $\Re^{\lambda}$ is minimal for Theorem 3.1 to hold: The impossibility result no longer holds when the domain of the hyperfunction is further restricted to $\Re^{\lambda^+} = \cup_{\rho \in \Pi} \{\lambda^+(\rho)\}$ or to $\Re^{\lambda^-} = \cup_{\rho \in \Pi} \{\lambda^-(\rho)\}$. To see the former, consider the hyperfunction f_1: $[\Re^{\lambda^+}]^N \rightarrow \underline{\mathbf{A}}$ defined for every $\underline{R} \in [\Re^{\lambda^+}]^N$ as

$$f_1(\underline{R}) = \begin{cases} \{a\} & \text{if } \#\{i \in \mathbf{N} : \text{argmax}_{\underline{\mathbf{A}}} R_i = \{a\}\} > n/2 \text{ for some } a \in \mathbf{A} \\ \mathbf{A} & \text{otherwise} \end{cases}$$

One can check that f_1 is unanimous, non-dictatorial, neither bi-dictatorial while it is non-manipulable over $\Re^{\lambda^+}$.

To see that Theorem 3.1 fails to hold over $\Re^{\lambda^-}$, consider the hyperfunction f_2: $[\Re^{\lambda^-}]^N \rightarrow \underline{\mathbf{A}}$ defined for every $\underline{R} \in [\Re^{\lambda^-}]^N$ as $f_2(\underline{R}) = \cup_{i \in \mathbf{N}} \text{argmax}_{\underline{\mathbf{A}}} R_i$. Again one can check that f_2 is unanimous, non-dictatorial, neither bi-dictatorial while it is non-manipulable over $\Re^{\lambda^-}$.[4]

[4] Note that f_1 is manipulable over $\Re^{\lambda^-}$ and f_2 is manipulable over $\Re^{\lambda^+}$.

We also wish to remark that Theorem 3.1 fails to hold when $|\mathbf{A}| = 2$. Note that even when $|\mathbf{A}| = 2$, $|\underline{\mathbf{A}}| = 3$. So, the range condition of the Gibbard-Satterthwaite theorem holds. However, when $|\mathbf{A}| = 2$, both f_1 and f_2 defined over $\mathfrak{R}^\lambda$ are strategy-proof.

Having established the dictatoriality or bi-dictatoriality of $\mathfrak{R}^\lambda$, we ask the implication of this to the superdomains of $\mathfrak{R}^\lambda$. Although one may be tempted to think that all superdomains of $\mathfrak{R}^\lambda$ are dictatorial or bi-dictatorial, this is not the case, as the following example illustrates:

Consider a set of alternatives $\mathbf{A} = \{a, b, c\}$. We know by Theorem 3.1 that any unanimous and strategy-proof hyperfunction defined over $\mathfrak{R}^\lambda$ is dictatorial or bi-dictatorial. Now take the ordering $R \in \mathfrak{R}$ over $\underline{\mathbf{A}}$ where $\text{argmax}_{\underline{\mathbf{A}}}R = \{a, b\}$ and $\{a, b\}$ P $\{a\}$ P X for all $X \in A \setminus \{\{a, b\}, \{a\}\}$. The following hyperfunction $f : [\mathfrak{R}^\lambda \cup \{R\}]^N \rightarrow \underline{\mathbf{A}}$ defined over the domain $\mathfrak{R}^\lambda \cup \{R\}$ is unanimous, strategy-proof but neither dictatorial nor bi-dictatorial:

For all $\underline{R} \in [\mathfrak{R}^\lambda \cup \{R\}]^N$, we have

$$f(\underline{R}) = \begin{cases} \text{argmax}_{\underline{\mathbf{A}}} R_1 & \text{if } R_1 \in \mathfrak{R}^\lambda \\ \text{argmax}_{\{\{a, b\}, \{a\}\}} R_2 & \text{otherwise} \end{cases}$$

Thus, not every superdomain of $\mathfrak{R}^\lambda$ is dictatoral or bi-dictatorial. Of course, it is interesting to see which superdomains of $\mathfrak{R}^\lambda$ preserve its dictatorality or bi-dictatoriality property. We explore this in the following section.

4. MORE GENERAL IMPOSSIBILITIES

We start by a proposition which states that once we define a dictatorial function over a fully regular domain, we cannot escape Gibbard-Satterthwaite type results by extending the domain while preserving its regularity property.

Proposition 4.1: Take any D, D' $\subset \Re$ with D $\cap$ D' = $\emptyset$. Assume D is fully regular while D' is regular. Consider a hyperfunction $f : [D \cup D']^N \rightarrow \underline{\mathbf{A}}$ which is dictatorial over D, ie., for all $\underline{R} \in D^N$, $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_i$ for some i $\in$ N. If f is strategy-proof over the domain D $\cup$ D' then f exhibits i $\in$ N as the dictator over D $\cup$ D'.

The proof of Proposition 4.1 is given in Appendix B.

The next proposition gives a similar result for bi-dictatorial domains. Before stating the proposition, we need a definition that we use in the statement of the proposition. A fully regular domain D $\subseteq \Re$ is said to satisfy condition β if and only if (i) and (ii) below hold. Take any R $\in$ D. Let {a} = $\text{argmax}_{\underline{\mathbf{A}}}$ R. We have

(i) {a} P {a, b} P {b} for all b $\in$ **A** \ {a}.

(ii) {a, b} P {c, b} for all b, c $\in$ **A** \ {a}.

Proposition 4.2: Let D $\supset \Re^\lambda$ be a fully regular domain satisfying condition β. Consider a hyperfunction $f : D^N \rightarrow \underline{\mathbf{A}}$ which is bi-dictatorial over $\Re^\lambda$, ie., for all $\underline{R} \in [\Re^\lambda]^N$, $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_i \cup \text{argmax}_{\underline{\mathbf{A}}} R_j$ for some distinct i, j $\in$ N. If f is strategy-proof over the domain D then f exhibits i and j as bi-dictators over D.

The proof of Proposition 4.2 is given in Appendix C.

Propositions 4.1 and 4.2 lead to the following theorem which states that all fully regular superdomains of $\Re^\lambda$ must be dictatorial or bi-dictatorial.

Theorem 4.1: Take any fully regular domain D $\supseteq \Re^\lambda$. A unanimous social choice hyperfunction f: $[D]^N \rightarrow \underline{\mathbf{A}}$ is strategy-proof only if f is dictatorial or bi-dictatorial.

Proof: Take D and f as in the statement of the theorem. So f is strategy-proof over $\Re^\lambda$ as well. This, by Theorem 3.1, implies the dictatoriality or bi-dictatoriality of f

over $\Re^\lambda$. In the former case Proposition 4.1 implies the dictatoriality of f over D while in the latter Proposition 4.2 implies the bi-dictatoriality of f over D. ■

We will use Theorem 4.1 to obtain further impossibility results. We start by showing that domains which render bi-dictatorial hyperfunctions non-manipulable can be charaterized in terms of condition β.

Proposition 4.3: Take any fully regular domain $D \supseteq \Re^\lambda$. A bi-dictatorial hyperfunction f: $[D]^N \rightarrow \underline{\mathbf{A}}$ is strategy-proof if and only if D satisfies condition β.

Proof: We leave to check the "if" part to the reader. To show the "only if" part, consider any fully regular domain $D \supseteq \Re^\lambda$ which violates condition β and a bi-dictatorial hyperfunction f defined over D. Let, without loss of generality, individual 1 and 2 be the bi-dictators. We will show that f is manipulable. Suppose part (i) of Condition β does not hold. Consider a preference profile $\underline{R} \in D^N$ such that $\{a\}\ P_1\ \{b\}\ P_1\ \{a, b\}$ for some distinct a, b ∈ **A** where $\{a\} = \text{argmax}_{\underline{\mathbf{A}}}\ R_1$. Let also $\text{argmax}_{\underline{\mathbf{A}}}\ R_2 = \{b\}$. As f is bi-dictatorial, we have $f(\underline{R}) = \{a, b\}$. But, since $\{b\}\ P_1\ \{a, b\}$, agent 1 can manipulate f at $\underline{R}$ via some $\underline{R}$' with $\text{argmax}_{\underline{\mathbf{A}}}\ R_1' = \{b\}$ and $R_j' = R_j$ for all $j \neq 1$. Now suppose part (ii) of Condition β does not hold. Consider a preference profile $\underline{R} \in D^N$ such that $\{a\}\ P_1\ \{b\}\ P_1\ \{c\}$ while $\{b, c\}\ P_1\ \{a, c\}$ for some distinct a, b, c ∈ **A** where $\{a\} = \text{argmax}_{\underline{\mathbf{A}}}\ R_1$. Let also $\text{argmax}_{\underline{\mathbf{A}}}\ R_2 = \{c\}$. As f is bi-dictatorial, we have $f(\underline{R}) = \{a, c\}$. But, since $\{b, c\}\ P_1\ \{a, c\}$, agent 1 can manipulate f at $\underline{R}$ via some $\underline{R}$' with $\text{argmax}_{\underline{\mathbf{A}}}\ R_1' = \{b\}$ and $R_j' = R_j$ for all $j \neq 1$. ■

The following results are a direct corollary to Theorem 4.1 and Proposition 4.3.

Theorem 4.2: Take any fully regular domain $D \supseteq \Re^\lambda$ which violates condition β. A unanimous hyperfunction f: $[D]^N \rightarrow \underline{\mathbf{A}}$ is strategy-proof if and only if f is dictatorial.

Theorem 4.3: Take any fully regular domain $D \supseteq \mathfrak{R}^{\lambda}$ which satisfies condition β. A unanimous hyperfunction f: $[D]^N \to \underline{\mathbf{A}}$ is strategy-proof if and only if f is dictatorial or bi-dictatorial.

We now give our attention to various extension axioms of the literature and see whether the domain they induce leads to more permissive results. We introduce three of these right away. The first one is a dominance axiom due to Kelly (1977):

D: For any two distinct X, Y $\in \underline{\mathbf{A}}$ we have X P Y whenever

$\forall x \in X \quad \forall y \in Y \quad x \; \rho \; y$ and $\exists x \in X \; \exists y \in Y$ with $x \rho y$ but not $y \rho x$.

We write δ for the consistency map determined by D. So, for every $\rho \in \Pi$, we have $\delta(\rho) = \{R \in \mathfrak{R} : R$ satisfies D$\}$.

Our second axiom is a modified version of the monotonicity axiom of Kannai and Peleg (1984), used by Roth and Sotomayor (1990):

M: For any $X \in 2^{\mathbf{A}}$, and $x, y \in \mathbf{A} \setminus X$ we have

$$X \cup \{x\} \; R \; X \cup \{y\} \text{ if and only if } x \rho y$$

We write μ for the consistency map determined by M. So, for every $\rho \in \Pi$, we have $\mu(\rho) = \{R \in \mathfrak{R} : R$ satisfies M$\}$.

Finally, we have the Gardenfors (1976) principle defined as follows:

G: For any $X \in \underline{\mathbf{A}}$ and any $y \in \mathbf{A} \setminus X$ we have

(i) X P X ∪ {y} whenever $x^* \rho \; y$ where $x^* = \text{argmin}_X \; \rho$

(ii) X ∪ {y} P X whenever $y \rho x^*$ where $x^* = \text{argmax}_X \; \rho$

We write γ for the consistency map determined by G. So, for every $\rho \in \Pi$, we have $\gamma(\rho) = \{R \in \Re : R \text{ satisfies } G\}$.

We write $\Re^{\delta} = \cup_{\rho \in \Pi} \delta(\rho)$, $\Re^{\mu} = \cup_{\rho \in \Pi} \mu(\rho)$, $\Re^{\gamma} = \cup_{\rho \in \Pi} \gamma(\rho)$ for the domains determined by the respective consistency maps δ, μ, and γ. Note that we have $\Re^{\gamma} \subset \Re^{\delta}$, as G is stronger than D. Note also that M and G as well as M and D are logically independent.

Theorem 4.4: Consider any unanimous hyperfunction f: $[D]^N \rightarrow \underline{\mathbf{A}}$.

(i) Let $D = \Re^{\delta}$. The hyperfunction f is strategy-proof if and only if f is dictatorial.

(ii) Let $D = \Re^{\gamma}$. The hyperfunction f is strategy-proof if and only if f is dictatorial.

(iii) Let $D = \Re^{\delta} \cap \Re^{\mu}$. The hyperfunction f is strategy-proof if and only if f is dictatorial or bi-dictatorial.

(iv) Let $D = \Re^{\gamma} \cap \Re^{\mu}$. The hyperfunction f is strategy-proof if and only if f is dictatorial or bi-dictatorial.

Proof: We first show (i) and (ii). Check that $\Re^{\lambda}$ is a subset of $\Re^{\delta}$ and $\Re^{\gamma}$. Moreover both $\Re^{\delta}$ and $\Re^{\gamma}$ are fully regular and violate Condition β. Hence by Theorem 4.2, any unanimous and strategy-proof hyperfunction defined over those domains must be dictatorial. Needless to say dictatorial hyperfunctions are always strategy-proof. We now show (iii) and (iv). Again check that both $\Re^{\delta} \cap \Re^{\mu}$ and $\Re^{\gamma} \cap \Re^{\mu}$ are supersets of $\Re^{\lambda}$. Moreover they are fully regular and satisfy Condition β. Hence by Theorem 4.3, any unanimous and strategy-proof hyperfunction defined over those domains must be dictatorial or bi-dictatorial. The strategy-proofness of dictatorial functions is obvious and we know by Proposition 4.3 that bi-dictatorial functions defined over domains satisfying Condition β are strategy-proof. ■

We close the section by stating an impossibility result on obtaining single-peaked domains via reasonable extension axioms.[5]

Theorem 4.5: Take any consistency map κ (satisfying A0) and let $\mathfrak{R}^{\kappa} = \cup_{\rho \in \Pi} \kappa(\rho)$ be the set of acceptable preference orderings over $\underline{\mathbf{A}}$ defined through κ. If $\mathfrak{R}^{\kappa}$ is fully regular, then the Cartesian product domain $[\mathfrak{R}^{\kappa}]^N$ is not single-peaked.

Proof: Assume $\mathfrak{R}^{\kappa}$ is fully regular and suppose for a contradiction that $[\mathfrak{R}^{\kappa}]^N$ is single peaked. Hence, there exists a total order T over $\underline{\mathbf{A}}$ according to which every $\underline{R} \in [\mathfrak{R}^{\kappa}]^N$ is single peaked. Take any such $\underline{R} \in [\mathfrak{R}^{\kappa}]^N$. As $\mathfrak{R}^{\kappa}$ is regular, the peak of every individual is a singleton set. Now consider the preference profile $\underline{\rho} \in \Pi^N$ where individuals order alternatives same as their ordering of singleton sets in $\underline{R}$, ie., $R_i \in \kappa(\rho_i)$ for every $i \in \mathbf{N}$. It is easy to check that $\underline{\rho}$ is single-peaked with respect to the total order θ over **A** which is defined through T as follows: For all x, y ∈ **A**, x θ y if and only if {x} T {y}. So the domain $\{\underline{\rho} \in \Pi^N : \exists\, \underline{R} \in [\mathfrak{R}^{\kappa}]^N \text{ with } R_i \in \kappa(\rho_i) \text{ for all } i \in \mathbf{N}\}$ is single-peaked with respect to θ. However, as $\mathfrak{R}^{\kappa}$ is fully regular, this domain is the full domain of preference profiles over **A** which is well-known of not being single-peaked, hence giving the desired contradiction. ■

5. CONCLUDING REMARKS

Our paper is inspired by Barberà, Dutta and Sen (2001) and it is thanks to their lemmata that we are able to prove our main Theorem 3.1. However our results are more general. Theorem 3.1 shows the dictatorality or bi-dictatoriality of a particular domain $\mathfrak{R}^{\lambda}$ based on some lexicographic extension idea, which is consistent with many standard axioms of the literature. So, by exploiting this, we show the dictatorality or bi-dictatorality of many superdomains of $\mathfrak{R}^{\lambda}$.

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[5] There is an extensive literature about strategy-proof social choice rules defined over single-peaked domains, among which we can non-exhaustively list Barberà, Sonnenschein and Zhou (1991), Serizawa (1995), Barberà, Massò and Neme (1999), Berga (2002).

In particular, we know by Theorems 4.2 and 4.3 that all superdomains of $\Re^{\lambda}$ where the best and worst elements are singleton sets are dictatorial or bi-dictatorial.[6] Note that $\Re^{\lambda}$ is very restricted and the orderings it contains are very mild in the sense of satisfying many standard extension axioms. Moreover the requirement of having singleton sets as best and worst elements is very natural when we conceive sets as first refinements from which a unique alternative will be finally chosen. That is why, we interpret our results as the "almost" impossibility of escaping the Gibbard-Satterthwaite theorem for social choice hyperfunctions by domain restrictions.[7]

In fact, many previous impossibility results of the literature where sets are interpreted as first refinements, can be obtained through our theorems. Going back to Gärdenfors (1976), Barberà (1977) and Kelly (1977), we know that our Theorem 4.4 covers the environment they define. Most recently, Barberà, Dutta and Sen (2001) consider domains restricted through the idea of conditional expected utility consistency (CEUC) and conditional expected utility consistency with equal probabilities (CEUCEP). The domain obtained through CEUC is dictatorial while the narrower one obtained through CEUCEP is dictatorial or bi-dictatorial. It is clear that both domains are fully regular and our lexicographic domain is their strict subset. Thus, our Theorems 4.2 and 4.3 imply the impossibility results of Barberà, Dutta and Sen (2001).[8]

We also wish to note that Feldman (1980), Duggan and Schwartz (2000), Ching and Zhou (2002), Benoit (forthcoming) are other papers to which our work is related.

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6 Specifically, they are dictatorial when they violate condition β.

7 At this point, let us recall that this impossibility translates into the standard world of social choice correspondences.

8 We also wish to note that some of our results in Theorem 4.4 can be obtained through Barberà, Dutta and Sen (2001). For, their domains obtained through CEUC and CEUCEP, which we denote respectively $\Re^{CEUC}$ and $\Re^{CEUCEP}$, are related to the ones we use in this theorem. First of all, $\Re^{\delta}$ is a superset of $\Re^{CEUC}$, hence $\Re^{CEUCEP}$. So part (i) of Theorem 4.4 is somewhat expected, given the Barberà, Dutta and Sen (2001) results. Although $\Re^{\gamma}$ is a superset of $\Re^{CEUCEP}$, it is unrelated to $\Re^{CEUC}$. Thus part (i) of Theorem 4.4 is not a direct corollary to the impossibility results of Barberà, Dutta and Sen (2001). Similarly, $\Re^{\mu}$ is a superset of $\Re^{CEUCEP}$, but unrelated to $\Re^{CEUC}$. So parts (iii) and (iv) of Theorem 4.4 can again be seen as expected under the current literature.

Barberà, Dutta and Sen (2001) give an extensive discussion about the relationship of their results and the ones in these cited papers, which also clarifies the place of what we are doing here among this plethora of contributions.

APPENDIX A

Theorem 3.1: A unanimous social choice hyperfunction f: $[\Re^{\lambda}]^{N} \rightarrow \underline{\mathbf{A}}$ is strategy-proof if and only if f is dictatorial or bi-dictatorial.

Proof: We omit the proof of the "if" part which is obvious and only show the "only if" part. The proof comes out as a result of a series of lemmata.

We start by remarking that every $P \in \Re^{\lambda}$ is complete, transitive and antisymmetric. Our first lemma is about the properties of the orderings in $\Re^{\lambda}$.

Lemma A1: The domain $\Re^{\lambda}$ satisfies the following four properties:

(i) For all $P \in \Re^{\lambda}$ and for all x, y $\in$ **A**, we have $\{x\}\ P\ \{y\} \Rightarrow \{x\}\ P\ \{x, y\}\ P\ \{y\}$.

(ii) For all $P \in \Re^{\lambda}$, for all $X \in 2^{\mathbf{A}}$ and for all x, y $\in \mathbf{A} \setminus X$ we have and $X \cup \{x\}\ P\ X \cup \{y\} \Leftrightarrow \{x\}\ P\ \{y\}$.

(i) For all x, y $\in$ **A**, there exists $P \in \Re^{\lambda^-}$ such that $\text{argmax}_{\underline{\mathbf{A}}}\ P = \{x\}$ and $\{x\}\ P\ \{x, y\}\ P\ \{y\}\ P\ X$ for all $X \in \underline{\mathbf{A}} \setminus \{\{x\}, \{x, y\}, \{y\}\}$.

(ii) For all x, y $\in$ **A**, there exists $P \in \Re^{\lambda^+}$ such that $\text{argmin}_{\underline{\mathbf{A}}}\ P = \{x\}$ and $X\ P\ \{y\}\ P\ \{x, y\}\ P\ \{x\}$ for all $X \in \underline{\mathbf{A}} \setminus \{\{x\}, \{x, y\}, \{y\}\}$.

Proof of Lemma A1: (i), (ii), (iii) and (iv) are all immediate consequences of the definitions of strong lexicographic and strong inverse lexicographic extensions. ■

We first prove Theorem 3.1 for the case of two agents and then generalize the result by induction.

Let f: $[\Re^{\lambda}]^2 \rightarrow \underline{\mathbf{A}}$ be a two-person social choice hyperfunction. We write $O_2(P_1) = \{f(P_1, P_2) \mid P_2 \in \Re^{\lambda}\}$ for the option set of individual 2 given $P_1 \in \Re^{\lambda}$, which is the set of alternatives that individual 2 may enforce by pretending some of his admissible orderings while agent 1 is pretending P_1. Similarly, for all $P_2 \in \Re^{\lambda}$, we write $O_1(P_2) = \{f(P_1, P_2) \mid P_1 \in \Re^{\lambda}\}$ for the option set of individual 1. The following result, to which we will refer as "Fact", is an immediate consequence of the definitions of option sets and of strategy-proofness.

Fact: If f: $[\Re^{\lambda}]^2 \rightarrow \underline{\mathbf{A}}$ is strategy-proof, then we have $f(P_1, P_2) \in \text{argmax}_{O_2(P_1)} P_2 = \text{argmax}_{O_1(P_2)} P_1$ for all $P_1, P_2 \in \Re^{\lambda}$.

We continue our proof by stating results about the structure of option sets. By a slight abuse of notation, given any $X \subset \underline{\mathbf{A}}$, we denote $O_2(P_1, X) = O_2(P_1) \cap X$ and write $O_2(P_1) = O_2(P_1, \underline{\mathbf{A}})$ for sake of consistency.

The following five lemmata is for two-person social choice hyperfunctions f: $[\Re^{\lambda}]^2 \rightarrow \underline{\mathbf{A}}$ which are unanimous and strategy-proof. For any integer k with $1 \leq k < m$, $\mathbf{A_k} = \{X \subseteq \mathbf{A} : \#X = k\}$ stands for the set of all k element subsets of **A**.

Lemma A2: For any $P_1, P_1' \in \Re^{\lambda}$ with $\text{argmax}_{\underline{\mathbf{A}}} P_1 = \text{argmax}_{\underline{\mathbf{A}}} P_1' = \{a\}$ for some $a \in \mathbf{A}$ we have $O_2(P_1, \underline{\mathbf{A}}_1) = O_2(P_1', \underline{\mathbf{A}}_1)$.

Proof of Lemma A2: Take any $P_1, P_1' \in \Re^{\lambda}$ as in the statement of the lemma and suppose $O_2(P_1, \underline{\mathbf{A}}_1) \neq O_2(P_1', \underline{\mathbf{A}}_1)$. Assume, without loss of generality, that $\{x\} \in O_2(P_1, \underline{\mathbf{A}}_1)$ but $\{x\} \notin O_2(P_1', \underline{\mathbf{A}}_1)$. Note that $\{x\} \neq \{a\}$ since f is unanimous. Now pick $P_2 \in \Re^{\lambda}$ such that $\text{argmax}_{\underline{\mathbf{A}}} P_2 = \{x\}$ and $\{x\}\ P_2\ \{a, x\}\ P_2\ \{a\}\ P_2\ X$ for all $X \in \underline{\mathbf{A}} \setminus \{\{a\}, \{a, x\}, \{x\}\}$. We know by part (iii) of Lemma A1 that such a P_2 exists. However, as $\{x\} \in O_2(P_1, \underline{\mathbf{A}}_1)$, in order not to contradict our Fact, we must have $f(P_1, P_2) = \{x\}$. Moreover, since $\{a\} \in O_2(P_1', \underline{\mathbf{A}}_1)$ and $\{x\} \notin O_2(P_1', \underline{\mathbf{A}}_1)$, the strategy-proofness of f implies that $f(P_1', P_2)$ is either $\{a, x\}$ or $\{a\}$. But, as part (i) of Lemma

A1 says, we have {a} P_1 {a, x} P_1 {x}. Thus, in either case, agent 1 will manipulate *f* at (P_1, P_2) by pretending P_1', contradicting its strategy-proofness. ■

Lemma A2 says that the singleton sets in the option set of an agent -say agent 2- must be the same for all preferences of the other agent -say agent 1- where the top element does not change. The next lemma shows that the option set of an agent, given the preference of the other, must either contain one singleton or all of them.

Lemma A3: For all $P_1 \in \Re^{\lambda}$, either $O_2(P_1, \underline{\mathbf{A}}_1) = \text{argmax}_{\underline{\mathbf{A}}} P_1$ or $O_2(P_1, \underline{\mathbf{A}}_1) = \underline{\mathbf{A}}_1$.

Proof of Lemma A3: Take any $P_1 \in \Re^{\lambda}$ and suppose for a contradiction that the statement of the lemma does not hold. Let $\text{argmax}_{\underline{\mathbf{A}}} P_1$ = {a} for some a ∈ **A**. Let also {b}, {c} ∈ $\underline{\mathbf{A}}$ \ {a} be such that {b} ∈ $O_2(P_1, \underline{\mathbf{A}}_1)$ and {c} ∉ $O_2(P_1, \underline{\mathbf{A}}_1)$. Since Lemma A2 implies that $O_2(P_1, \underline{\mathbf{A}}_1)$ depends only on the maximal element of P_1, we may assume without loss of generality that {c} P_1 {b}. Using Lemma A1 (iii), take $R_2 \in \Re^{\lambda^-}$ such that $\text{argmax}_{\underline{\mathbf{A}}} P_2$ = {c} and {c} P_2 {b, c} P_2 {b} P_2 X for all X ∈ $\underline{\mathbf{A}}$ \ {{b}, {b, c}, {c}}. Thus, *f*(R_1, R_2) = {b, c} or {b}, as otherwise agent 2 can manipulate at (P_1, P_2). But we know by Lemma A1 (i) that {c} P_1 {b, c} P_1 {b}. Since $\text{argmax}_{\underline{\mathbf{A}}} P_2$ = {c}, by unanimity of *f*, agent 1 can enforce {c} by pretending P_1' where $\text{argmax}_{\underline{\mathbf{A}}} P_1$' = {c}. Therefore, agent 1 can manipulate at (P_1, P_2) by pretending P_1'. ■

Now, we show that if the option set of an agent –say agent 2- contains all singletons for some preference ordering of agent 1, then agent 2 can enforce all singletons for every preference ordering of agent 1.

Lemma A4: If $O_2(P_1, \underline{\mathbf{A}}_1) = \underline{\mathbf{A}}_1$ for some $P_1 \in \Re^{\lambda}$, then $O_2(P_1', \underline{\mathbf{A}}_1) = \underline{\mathbf{A}}_1$ for all $P_1' \in \Re^{\lambda}$.

Proof of Lemma A4: Suppose, for a contradiction that $O_2(P_1, \underline{\mathbf{A}}_1) = \underline{\mathbf{A}}_1$ for some $P_1 \in \Re^{\lambda}$ while $O_2(P_1', \underline{\mathbf{A}}_1) \neq \underline{\mathbf{A}}_1$ for some $P_1' \in \Re^{\lambda}$. Let $\text{argmax}_{\underline{\mathbf{A}}} P_1$ = {a} for some a ∈ **A**. We know, by Lemma A3, that $O_2(P_1', \underline{\mathbf{A}}_1)$ = {b} where {b} = $\text{argmax}_{\underline{\mathbf{A}}} P_1'$. Of course,

$\{b\} \neq \{a\}$, as otherwise it would contradict Lemma A2. Now take some $c \in \mathbf{A} \setminus \{a, b\}$ and assume, thanks to Lemma 3.2, that $\{a\}\ P_1\ \{b\}\ P_1\ \{c\}$. Pick $P_2 \in \Re^{\lambda^-}$ with $\text{argmax}_{\underline{\mathbf{A}}}\ P_2 = \{c\}$ and $\{c\}\ P_2\ \{b, c\}\ P_2\ \{b\}\ P_2\ X$ for all $X \in \underline{\mathbf{A}} \setminus \{\{b\}, \{b, c\}, \{c\}\}$. Observe that $f(P_1, P_2) = \{c\}$ while $f(P_1', P_2)$ is either $\{b,c\}$ or $\{b\}$. But since we have $\{b\}\ P_1\ \{c, b\}\ P_1\ \{c\}$ from Lemma A1 (i), agent 1 can manipulates at (P_1, P_2) by pretending P_1'. ■

Remark that $O_2(P_1, \underline{\mathbf{A}}_1) = \underline{\mathbf{A}}_1$ for all $P_1 \in \Re^{\lambda}$ implies the dictatorality of agent 2 for regular domains. Of course $O_1(P_2, \underline{\mathbf{A}}_1) = \underline{\mathbf{A}}_1$ for all $P_2 \in \Re^{\lambda}$ implies the dictatorality of agent 1. So given Lemma A3 and Lemma A4, either one of the agents is the dictator, or the only singleton in the option set of one agent is the maximal element of the other agent. That is, $O_2(P_1, \underline{\mathbf{A}}_1) = \text{argmax}_{\underline{\mathbf{A}}}\ P_1$ and $O_1(P_2, \underline{\mathbf{A}}_1) = \text{argmax}_{\underline{\mathbf{A}}}\ P_2$ for every $P_1, P_2 \in \Re^{\lambda}$. We will refer to this latter case as (*).

If $O_2(P_1, \underline{\mathbf{A}}_1) = \text{argmax}_{\underline{\mathbf{A}}}\ P_1$ and $O_1(P_2, \underline{\mathbf{A}}_1) = \text{argmax}_{\underline{\mathbf{A}}}\ P_2$ for every $P_1, P_2 \in \Re^{\lambda}$, ie. (*) holds, then the two-element sets in the option set of an agent –say agent 2- must consist of those which contain the best element of agent 1. We state and show this in the following lemma.

Lemma A5: Assume (*) holds. For any $P_1 \in \Re^{\lambda}$ with $\text{argmax}_{\underline{\mathbf{A}}}\ P_1 = \{a\}$ for some $a \in \mathbf{A}$, we must have $O_2(P_1, \underline{\mathbf{A}}_2) = \{\{a, b\} \mid b \in A\}$.

Proof of Lemma A5: Assume (*) holds. Take any $P_1 \in \Re^{\lambda}$ with $\text{argmax}_{\underline{\mathbf{A}}}\ P_1 = \{a\}$ for some $a \in \mathbf{A}$. We first show that $\{\{a, b\} \mid b \in A\} \subseteq O_2(P_1, \underline{\mathbf{A}}_2)$. Suppose that there exists $b \in \mathbf{A}$ such that $\{a, b\} \notin O_2(P_1, \underline{\mathbf{A}}_2)$. Pick $P_2 \in \Re^{\lambda^-}$ with $\text{argmax}_{\underline{\mathbf{A}}}\ P_2 = \{b\}$ and $\{b\}\ P_2\ \{a, b\}\ P_2\ \{a\}\ P_2\ X$ for all $X \in \underline{\mathbf{A}} \setminus \{\{a\}, \{a, b\}, \{b\}\}$. Since (*) holds, $f(P_1,P_2)$ can be neither $\{a\}$ nor $\{b\}$. Moreover, $f(P_1,P_2) \neq \{a, b\}$ also, as by assumption $\{a, b\} \notin O_2(P_1, \underline{\mathbf{A}}_2)$. Thus, $f(P_1,P_2) = X$ for some X with $\{a\}\ P_2\ X$. But agent 2 can then manipulate at (P_1, P_2) by pretending some P_2' with $\text{argmax}_{\underline{\mathbf{A}}}\ P'_2 = \{a\}$ and obtain $\{a\}$ by the unanimity of f.

We now show that $O_2(P_1, \underline{\mathbf{A}}_2) \subseteq \{\{a, b\} | b \in \mathbf{A}\}$. Suppose not, ie., there exists $\{b, c\} \in O_2(P_1, \underline{\mathbf{A}}_2)$ where a, b, c are all distinct. Pick some $P_2 \in \Re^{\lambda^-}$ with $\text{argmax}_{\underline{\mathbf{A}}} P_2 = \{b\}$ and $\{b\}\ P_2\ \{b, c\}\ P_2\ \{c\}\ P_2\ X$ for all $X \in \underline{\mathbf{A}} \setminus \{\{b\}, \{b, c\}, \{c\}\}$. As (*) holds, we have $\{b\} \notin O_2(P_1, \underline{\mathbf{A}}_1)$. So, it must be the case that $f(P_1, P_2) = \{b, c\}$. But we also have $\{a, b\} \in O_1(P_2, \underline{\mathbf{A}}_2)$ from the result of the first part of this lemma and $\{a, b\}\ P_1\ \{b, c\}$ from Lemma A1 (ii). Therefore, agent 1 can manipulate at (P_1, P_2). ■

The following lemma shows that when (*) holds, option sets of the agents does not contain sets with cardinality greater than 2.

Lemma A6: Assume (*) holds. For all $P_1 \in \Re^{\lambda}$, we have $O_2(P_1, \underline{\mathbf{A}}) = O_2(P_1, \underline{\mathbf{A}}_1) \cup O_2(P_1, \underline{\mathbf{A}}_2)$.

Proof of Lemma A6: Assume (*) holds. Take any $P_1 \in \Re^{\lambda}$ and assume for a contradiction that there exists some $X \in O_2(P_1, \underline{\mathbf{A}})$ with $\#X > 2$. Assume, without loss of generality, that $X = \{b_1, b_2, \ldots, b_L\}$ is the set of smallest cardinality among those in $O_2(P_1, \underline{\mathbf{A}})$ with cardinality exceeding 2. In case such a set is not unique, choose one arbitrarily.

Let $\text{argmax}_{\underline{\mathbf{A}}} P_1 = \{a\}$ for some $a \in \mathbf{A}$. We first show that $a \in X$. Suppose not. Let $\underline{X}$ denote the set of all singleton subsets of X and let $\{b_1\} = \text{argmax}_{\underline{X}} P_1$. We can pick $P_2 \in \Re^{\lambda^-}$ such that $\text{argmax}_{\underline{\mathbf{A}}} P_2 = \{b_1\}$, $\text{argmin}_{\underline{\mathbf{A}}} P_2 = \{a\}$ and $\{b_1\}\ P_2\ \{b_2\}, \ldots, P_2\ \{b_L\}\ P_2\ \{x\}$ for all $x \notin X$. Moreover, for all $Y, Z \in \underline{\mathbf{A}}$, if $a \notin Y$ and $a \in Z$, then $Y\ P_2\ Z$. From Lemma A5, $O_2(P_1, \underline{\mathbf{A}}_2) = \{\{a, x\} | x \in \mathbf{A}\}$. Since $X\ P_2\ \{a, b_1\}$, we must have $X\ P_2\ Y$ for all $Y \in O_2(P_1, \underline{\mathbf{A}}_2)$. In addition, $X\ P_2\ Y$ for all Y such that $|Y| \geq L$. Since X is the set of smallest cardinality greater than 2 in $O_2(P_1, \underline{\mathbf{A}})$, this ensures that $f(P_1, P_2) = X$. But, since $\{b_1\}\ P_1\ X$ and $f(P_1', P_2) = \{b_1\}$ when $\text{argmax}_{\underline{\mathbf{A}}} P_1' = \{b_1\}$, agent 1 can manipulate (P_1, P_2) by pretending P_1'. Therefore, $\text{argmax}_{\underline{\mathbf{A}}} P_1 \in X$.

From now on, we assume, without loss of generality, that $\text{argmax}_{\underline{X}} P_1 = \{b_1\}$ and $\text{argmax}_{\underline{X} \setminus \{\{b_1\}\}} P_1 = \{b_2\}$. Therefore, $\{b_1, b_2\}\ P_1\ X$. Now, pick $P_2 \in \Re^{\lambda^+}$ with

(i) $\text{argmax}_{\underline{\mathbf{A}}}$ $P_2 = \{b_2\}$ and $\{b_k\}$ P_2 $\{b_{k+1}\}$ for all $k \in \{2, 3,\ldots, L-1\}$

(ii) $\text{argmin}_{\underline{X}}$ $P_2 = \{b_1\}$ and $\{b_1\}$ P_2 $\{x\}$ for all $x \notin X$.

Hence, we have $\{b_2\} \notin O_2(R_1, \underline{\mathbf{A}})$ from (*) and $\{b_1, b_2\} \in O_2(R_1, \underline{\mathbf{A}})$ from Lemma A5.

Therefore, by referring to our Fact, we claim that $f(P_1, P_2) = X$, as otherwise individual 2 can manipulate. To see that this claim holds, suppose $f(P_1, P_2) = Y$ for some $Y \neq X$. In case where $\#Y \geq \#X$, there exists $y \in Y$ such that $y \notin X$. But since $P_2 \in \Re^{\lambda^+}$ and $\{b_1\}$ P_2 $\{x\}$ for all $x \notin X$, we can conclude that X P_2 Y. The second and last case is that either $\#Y = 1$ or $\#Y = 2$, since X is set of smallest cardinality among those in $O_2(P_1, \underline{\mathbf{A}})$ with cardinality exceeding 2. If $\#Y = 1$ then $Y = \{b_1\}$ in order not to contradict (*) and X P_2 $\{b_1\}$. If $\#Y = 2$, we know from Lemma A5 that $b_1 \in Y$, ie., $Y = \{x, b_1\}$ for some $x \in \mathbf{A}$ and X P_2 Y, independent of whatever x is.

Now consider $P_1' \in \Re^{\lambda^-}$ with $\text{argmax}_{\underline{\mathbf{A}}}$ $P_1' = \{b_1\}$ and $\{b_1\}$ P_1' $\{b_1, b_2\}$ P_1' $\{b_2\}$ P_1' X for all $X \in \underline{\mathbf{A}} \setminus \{\{b_1\}, \{b_1,b_2\}, \{b_2\}\}$. Observe that $\{b_1\} \notin O_1(P_2,\underline{\mathbf{A}})$ by (*) and $\{b_1, b_2\} \in O_1(P_2,\underline{\mathbf{A}})$ by Lemma A5. Therefore, by using our Fact, we must have $f(P_1', P_2) = \{b_1, b_2\}$, as otherwise agent 1 can manipulate. But since $\{b_1,b_2\}$ P_1 X, agent 1 can manipulate at (P_1, P_2) by pretending P_1'. ■

Lemma A5 and Lemma A6 ensure the bi-dictatorality of f when (*) holds. We have already seen, by Lemma A3 and Lemma A4, that either one of the agents is the dictator, or (*) holds. Thus, with two agents, any unanimous and strategy-proof f defined over $\Re^{\lambda}$ is either dictatorial or bi-dictatorial.

We will complete the proof by induction, showing that if the impossibility expressed by Theorem 3.1 holds over the domain $[\Re^{\lambda}]^k$ for some $k \in \{2,..., n-1\}$, then it also holds over the domain $[\Re^{\lambda}]^{k+1}$. Now take any $k \in \{2,..., n-1\}$ and assume unanimous and strategy-proof hyperfunctions defined over $[\Re^{\lambda}]^k$ have to be dictatorial or bi-dictatorial. Consider any unanimous and strategy-proof hyperfunction f: $[\Re^{\lambda}]^{k+1} \rightarrow \underline{\mathbf{A}}$. We will show that f is dictatorial or bi-dictatorial.

To see this, fix agents k, k+1 and define a hyperfunction g : $[\Re^{\lambda}]^{k} \rightarrow \underline{\mathbf{A}}$ through f as follows: For all $(P_1,..., P_k) \in [\Re^{\lambda}]^{k}$, $g(P_1,..., P_k) = f(P_1,..., P_k, P_k)$. In other words, at every k-person preference profile $(P_1,..., P_k)$, g picks what f would have picked at the k+1 person preference profile $(P_1,..., P_k, P_k)$, which differs from $(P_1,..., P_k)$ by an additional agent k+1 who has the same preference as agent k. It is clear that g is unanimous. Moreover, no agent $i \in \{1, 2, \ldots, k\text{-}1\}$ can manipulate g as this would contradict the strategy-proofness of f. Also observe that for all $(P_1,..., P_k) \in [\Re^{\lambda}]^{k}$ and for all $P_k' \in \Re^{\lambda}$, we have $g(P_1,..., P_k) = f(P_1,..., P_k, P_k)\ P_k\ f(P_1,..., P_k', P_k)\ P_k\ f(P_1,..., P_k', P_k') = g(P_1,..., P_{k-1}, P_k')$. Thus, agent k cannot manipulate g either. So g is unanimous and strategy-proof and hence dictatorial or bi-dictatorial as we have assumed that unanimous and strategy-proof hyperfunctions defined over $[\Re^{\lambda}]^{k}$ have to be dictatorial or bi-dictatorial.

We will show that f inherits the dictatoriality or bi-dictatoriality of g, in each of the following four exhaustive cases:

CASE 1: g is dictatorial and some agent i who differs from k is the dictator.

CASE 2: g is bi-dictatorial with agent k not being one of the bi-dictators.

CASE 3: g is dictatorial with agent k being the dictator.

CASE 4: g is bi-dictatorial with agent k being one of the bi-dictators.

CASE 1: We claim that if g is dictatorial and some agent i who differs from k and k+1 is the dictator, then the same agent i is also the dictator of f. Assume, without loss of generality, that agent 1 is the dictator of g. Take any preference profile $\underline{P} = (P_1, P_2,\ldots, P_{k+1}) \in [\Re^{\lambda}]^{k+1}$ with $\text{argmax}_{\underline{\mathbf{A}}}\ P_1 = \{a\}$ for some $a \in \mathbf{A}$. We will show that $f(\underline{P}) = \{a\}$. Pick $P_k' \in \Re^{\lambda}$ such that $\text{argmin}_{\underline{\mathbf{A}}}\ P_k' = \{a\}$. Check that $f(P_1, P_2,\ldots, P_{k-1},$

P_k', P_k') = g (P_1, P_2,…, P_{k-1}, P_k') = {a}, because agent 1 is the dictator of g. Since f is strategy proof, we have f(P_1, P_2,…, P_{k-1}, P_k', P_k') P_k' f(P_1, P_2,…, P_{k-1}, P_k', P_{k+1}) P_k' f(P_1, P_2,…, P_{k-1}, P_k, P_{k+1}). As f(P_1, P_2,…, P_{k-1}, P_k', P_k') = $\text{argmin}_{\underline{\mathbf{A}}}$ P_k'= {a}, this implies f(P̲) = {a}.

CASE 2: We claim that if g is bi-dictatorial and agents i and j who differ from k are the bi-dictators, then the same agents i and j are also the bi-dictators of f. Assume, without loss of generality, that agents 1 and 2 are the bi-dictators of g. Take any preference profile P̲ = (P_1, P_2,…, P_{k+1}) ∈ $[\Re^{\lambda}]^{k+1}$ with $\text{argmax}_{\underline{\mathbf{A}}}$ P_1 = {a} and $\text{argmax}_{\underline{\mathbf{A}}}$ P_2 = {b} for some a, b ∈ **A**. We will show that f(P̲) = {a, b}. Pick, thanks to Lemma A1 (iv), P_k' ∈ $\Re^{\lambda^+}$ with X P_k' {a} P_k' {a, b} P_k' {b} for all X ∈ **A** \ {{a}, {a, b}, {b}}. Let P_k' = P_{k+1}'. We have f(P_1, P_2,…, P_k', P_{k+1}') = g (P_1, P_2,…, P_{k-1}, P_k') = {a, b} as g is bi-dictatorial. Observe that f(P_1, P_2,…, P_k, P_{k+1}') is either {a, b} or {b}, as otherwise agent k can manipulate at (P_1, P_2,…, P_k', P_{k+1}'). Similarly, f(P_1, P_2,…, P_k, P_{k+1}) is either {a, b} or {b}, as otherwise agent k can manipulate at (P_1, P_2,…, P_k', P_{k+1}). Now, pick P_k" ∈ $\Re^{\lambda^+}$ such that X P_k" {b} P_k" {a, b} P_k" {a} for all X ∈ **A** \ {{a}, {a, b}, {b}}. Let P_k" = P_{k+1}" and observe that f(P_1, P_2,…, P_k'', P_{k+1}'') = {a, b}. By applying the previous arguments, it follows that f(P_1, P_2,…, P_k, P_{k+1}) is either {a, b} or {a}. Thus, we have f(P̲) is either {a, b} or {b}, as well as f(P̲) is either {a, b} or {a}, thus implying f(P̲) = {a, b}.

CASE 3: We claim that if g is dictatorial and agent k is the dictator, then either agent k or agent k+1 is the dictator of f or agents k and k+1 are the bi-dictators of f.

Now, fix some (Q_1, Q_2, …,Q_{k-1}) ∈ $[\Re^{\lambda}]^{k-1}$ and define a two-person social choice hyperfunction h: $[\Re^{\lambda}]^2 \rightarrow \underline{\mathbf{A}}$ for the two person society {k, k+1} as follows: For all (P_k, P_{k+1}) ∈ $[\Re^{\lambda}]^2$, h (P_k, P_{k+1}) = f(Q_1,…,Q_{k-1}, P_k, P_{k+1}). The strategy-proofness of f implies the strategy proofness of h, while the fact that agent k is a dictator of g implies that h is unanimous. As we have already established Theorem 3.1 for the case of two agents, h is either dictatorial or bi-dictatorial.

We now show that

(i) if $i \in \{k, k+1\}$ is the dictator of h, then i is the dictator of f.

(ii) if k and k+1 are bi-dictators of h, then they are bi-dictators of f.

To show (i), assume without loss of generality, that k is the dictator of h. Now, fix some $(Q_1', Q_2', \ldots, Q_{k-1}') \in [\Re^\lambda]^{k-1}$ with $Q_j' = Q_j$ for all $j \in \{1,\ldots,k-2\}$ and $Q_{k-1}' \neq Q_{k-1}$ where $\{b\}\ Q_{k-1}'\ \{a\}$ for some $a, b \in \mathbf{A}$. Define $h': [\Re^\lambda]^2 \rightarrow \underline{\mathbf{A}}$ in an identical manner as h, i.e., for all $(P_k, P_{k+1}) \in [\Re^\lambda]^2$, $h'(P_k, P_{k+1}) = f(Q_1', Q_2', \ldots, Q_{k-1}', P_k, P_{k+1})$. Note again that, h' is dictatorial or bi-dictatorial. In particular, we claim that agent k, who is the dictator of h, is the dictator of h' as well. Suppose not, i.e., either agent k+1 is the dictator of h' or agents k and k+1 are bi-dictators of h'. To see that this leads to a contradiction, take $P_k, P_{k+1} \in \Re^\lambda$ such that $\text{argmax}_{\underline{\mathbf{A}}} P_k = \{b\}$ and $\text{argmax}_{\underline{\mathbf{A}}} P_{k+1} = \{a\}$. Therefore, $h'(P_k, P_{k+1})$ is either $\{a\}$ or $\{a, b\}$. Thus, $f(Q_1,\ldots,Q_{k-2}, Q_{k-1}', P_k, P_{k+1}) = h'(P_k, P_{k+1})$ which is either $\{a\}$ or $\{a, b\}$, while $f(Q_1,\ldots,Q_{k-1}, P_k, P_{k+1}) = h(P_k, P_{k+1}) = \{b\}$. But, since $\{b\}\ Q_{k-1}'\ \{a, b\}\ Q_{k-1}'\ \{a\}$, agent k-1 can manipulate $(Q_1,\ldots,Q_{k-2}, Q_{k-1}', P_k, P_{k+1})$ by pretending Q_{k-1}. Hence, agent k is dictator of h' as well. Repeating the argument above shows that agent k is the dictator of the function h, independent of the choice of $(Q_1, Q_2, \ldots, Q_{k-1}) \in [\Re^\lambda]^{k-1}$, which implies that agent k is the dictator of f.

To show (ii), assume k and k+1 are bi-dictators of h. We claim that this again implies the bi-dictatoriality of k and k+1 independent of the choice of $(Q_1, Q_2, \ldots, Q_{k-1}) \in [\Re^\lambda]^{k-1}$. To see this, take $(Q_1', Q_2', \ldots, Q_{k-1}') \in [\Re^\lambda]^{k-1}$ with $Q_j' = Q_j$ for all $j \in \{1,\ldots,k-2\}$ and $Q_{k-1}' \neq Q_{k-1}$ where $\{a\}\ Q_{k-1}'\ \{b\}$ as well as $\{a\}\ Q_{k-1}\ \{b\}$ for some $a, b \in \mathbf{A}$. Define $h': [\Re^\lambda]^2 \rightarrow \underline{\mathbf{A}}$ as above. Note that, h' is dictatorial or bi-dictatorial. In particular, we claim that h' is bi-dictatorial. Suppose not, i.e., either agent k or agent k+1 is the dictator of h'. To see that this leads to a contradiction, take $P_k, P_{k+1} \in \Re^\lambda$ such that $\text{argmax}_{\underline{\mathbf{A}}} P_k = \{a\}$ and $\text{argmax}_{\underline{\mathbf{A}}} P_{k+1} = \{b\}$. Thus, $f(Q_1,\ldots,Q_{k-2}, Q_{k-1}', P_k, P_{k+1}) = h'(P_k, P_{k+1})$ is either $\{a\}$ or $\{b\}$, while $f(Q_1,\ldots,Q_{k-1}, P_k, P_{k+1}) = h(P_k, P_{k+1}) = \{a, b\}$. But, since $\{a\}\ Q_{k-1}\ \{b\}$ and $\{a\}\ Q_{k-1}'\ \{b\}$, agent k-1 can either manipulate $(Q_1,\ldots,Q_{k-2}, Q_{k-1}', P_k, P_{k+1})$ by pretending Q_{k-1} or manipulate $(Q_1,\ldots,Q_{k-2}, Q_{k-1}, P_k, P_{k+1})$ by pretending Q_{k-1}'. Hence, agents k and k+1 are bi-dictators of h' as well. Repeating the argument above shows that agents k and k+1 are bi-dictators of the function h independent of the choice of $(Q_1, Q_2, \ldots, Q_{k-1}) \in [\Re^{\lambda}]^{k-1}$, which implies that they are bi-dictators of *f* as well.

CASE 4: We claim that if g is bi-dictatorial with agent k being one of the bi-dictators and say, without loss of generality, some $i \in \{1, \ldots, k-1\}$ the other bi-dictator, then either {i, k} or {i, k+1} are bi-dictators of *f*.

Now consider the k-person social choice hyperfunction g': $[\Re^{\lambda}]^{k} \rightarrow \underline{\mathbf{A}}$ defined through *f* as follows: For all $\underline{P}_{-ik} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{k-1}, P_{k+1}) \in [\Re^{\lambda}]^{k-1}$, $g'(\underline{P}_{-ik}, P_i) = f(\underline{P}_{-i}, P_i, P_k)$ with $P_i = P_k$. Therefore, we know that one of the cases 1 through 4 must hold for g', given that i and k are bi-dictators in g.

We first show that neither CASE 1, nor CASE 2 can hold for g'.

Suppose for a contradiction that CASE 1 holds for g', i.e., there exists an agent j different from i and k who is the dictator in g'. First consider the case where j = k+1. Take $\underline{P} \in [\Re^{\lambda}]^{k+1}$ such that $P_i = P_k$ with $\text{argmax}_{\underline{\mathbf{A}}} P_k = \{a\}$ and $\text{argmax}_{\underline{\mathbf{A}}} P_{k+1} = \{b\}$ for some $a, b \in \mathbf{A}$. We have $f(\underline{P}_{-ki}, P_i, P_k) = g'(\underline{P}_{-ki}, P_i) = \{b\}$, since agent k+1 is the dictator in g'. Now take $P_k' = P_{k+1}$ and observe that $f(\underline{P}_{-ki}, P_i, P_k') = g(\underline{P}_{-kk+1}, P_k') = \{a, b\}$, because agents i and k are bi-dictators in g. Since, $\{a\}\ P_k\ \{a, b\}\ P_k\ \{b\}$, agent k can manipulate $(\underline{P}_{-ki}, P_i, P_k)$ by pretending P_k'. Now, consider the case where $j \neq k+1$. Again take $\underline{P} \in [\Re^{\lambda}]^{k+1}$ such that $P_i = P_k$ with $\text{argmax}_{\underline{\mathbf{A}}} P_k = \{a\}$, $\text{argmax}_{\underline{\mathbf{A}}} P_j = \{b\}$ and $\text{argmax}_{\underline{\mathbf{A}}} P_{k+1} = \{a\}$ for some $a, b \in \mathbf{A}$. We have $f(\underline{P}_{-ki}, P_i, P_k) = g'(\underline{P}_{-ki}, P_i) = \{b\}$, since agent j is the dictator in g'. Now let $P_k' = P_{k+1}$. We have $f(\underline{P}_{-ki}, P_i, P_k') = g(\underline{P}_{-k\,k+1}, P_k') = \{a\}$, because agent i and k are bi-dictators in g. Since $\{a\}\ P_k\ \{b\}$, agent k can manipulate $(\underline{P}_{-ki}, P_i, P_k)$ by pretending P_k', showing that CASE 1 cannot hold for g'.

Now suppose for a contradiction that CASE 2 holds for g' given that agent i and k are bi-dictators in g, i.e., there exists agents j and j' both different from i and k such that j and j' are bi-dictators of g'. Take $\underline{P} \in [\Re^{\lambda}]^{k+1}$ such that $P_i = P_k$ with $\mathrm{argmax}_{\underline{\mathbf{A}}}P_k = \{a\}$, $\mathrm{argmax}_{\underline{\mathbf{A}}}P_j = \{b\}$, $\mathrm{argmax}_{\underline{\mathbf{A}}}P_{j'} = \{a\}$ and $\mathrm{argmax}_{\underline{\mathbf{A}}}P_{k+1} = \{a\}$ for some a, b $\in$ **A**. We have $f(\underline{P}_{-ik}, P_i, P_k) = g'(\underline{P}_{-ik}, P_i) = \{a, b\}$, since agents j and j' are bi-dictators of g'. Letting $P_k' = P_{k+1}$, we have $f(\underline{P}_{-ik}, P_i, P_k') = g(\underline{P}_{-k\ k+1}, P_k') = \{a\}$, because agents i and k are bi-dictators in g. Since $\{a\}\ P_k\ \{a, b\}$, agent k can manipulate $(\underline{P}_{-ik}, P_i, P_k)$ by pretending P_k', showing that CASE 2 cannot hold for g' either.

Therefore, only one of the following two cases holds for g':

(i) Agent i is the dictator of g'.

(ii) Agents i and j are the bi-dictators of g', where j differs from k .

Suppose case (i) holds. We use for g' the arguments that we have used about g in CASE 3, which shows that either i or k dictates in f or i and k are bi-dictators of f. But, we have assumed that i and k are bi-dictators of g. Therefore, i and k are bi-dictators in f,.as otherwise f will not be strategy-proof.

Suppose case (ii) holds. Since i and k are bi-dictators of g, we must have j = k+1, as otherwise we will contradict the strategy-proofness of f. Therefore, i and k+1 are bi-dictators of f.

Now, consider the k-person social choice hyperfunction g'': $[\Re^{\lambda}]^k \rightarrow \underline{\mathbf{A}}$ defined through f as follows: For all $\underline{P}_{-i\ k+1} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{k-1}, P_k) \in [\Re^{\lambda}]^{k-1}$, $g''(\underline{P}_{-i\ k+1}, P_{k+1}) = f(\underline{P}_{-i\ k+1}, P_{k+1}, P_i)$ with $P_{k+1} = P_i$. By arguments identical to those about g', it follows that CASE 1 and CASE 2 will not apply to g''. So, again there are two cases that may hold for g'': The first one is that agent k+1 is the dictator of g''. But then, k+1 and i would be bi-dictators of f, as otherwise we would have contradicted the bi-dictatoriality of i and k for g or the bi-dictatoriality of i and k+1 to g'. The second case is that agent k+1 and some agent j who differs from agent i are bi-dictators of g''. Hence we are only left with the case below to which we refer as (**):

(**) {i, k} are bi-dictators of g, {i, k+1} are bi-dictators of g’, and {k, k+1} are bi-dictators of g”.

We complete the proof of Theorem 3.1 by showing that (**) cannot hold.

Suppose (**) holds. Pick a, b, c $\in \underline{\mathbf{A}}$ and $P_{k+1}, P_{k+1}'' \in \mathfrak{R}^{\lambda^-}$ satisfying the following for all X $\in$ A \ {{a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}

(i) {a} P_{k+1} {a, b} P_{k+1} {b} P_{k+1} {a, c} P_{k+1} {a, b, c} P_{k+1} {b, c} P_{k+1} {c} P_{k+1} X

(ii) {b} P_{k+1}'' {a, b} P_{k+1}'' {a} P_{k+1}'' {b, c} P_{k+1}'' {a, b, c} P_{k+1}'' {a, c} P_{k+1}'' {c} P_{k+1}'' X

let $(P_i, P_k) \in [\mathfrak{R}^{\lambda}]^2$ be such that $\text{argmax}_{\underline{\mathbf{A}}} P_i$ = {c} and $\text{argmax}_{\underline{\mathbf{A}}} P_k$ = {a}. Let $\underline{P}_{-i\,k\,k+1} \in [\mathfrak{R}^{\lambda}]^{k-2}$ be an arbitrary profile for k-2 person society. Since i and k are bi-dictators in g, we have f ($\underline{P}_{-i\,k\,k+1}$, P_i, P_{k+1}, P_k) = {a, c}. We claim that f ($\underline{P}_{-i\,k\,k+1}$, P_i, P_k, P_{k+1}'') must either be {a, c}, {a, b, c} or {b, c}. To see this, observe that if this outcome is in the set {{a}, {a, b}, {b}}, then agent k+1 will manipulate at ($\underline{P}_{-i\,k\,k+1}$, P_i, P_k, P_{k+1}) via P_{k+1}''. If it is not in this set nor in {{a, c}, {a, b, c}, {b, c}}, then agent k+1 will manipulate at ($\underline{P}_{-i\,k\,k+1}$, P_i, P_k P_{k+1}'') via P_{k+1} .

CASE A : f ($\underline{P}_{-i\,k\,k+1}$, P_i, P_k, P_{k+1}'') = {b, c}.

Let $P_i'' = P_{k+1}''$ and $P_i' \in \mathfrak{R}^{\lambda^-}$ be such that {b} P_i' {b, c} P_i' {c} P_i' X for all X $\in \underline{\mathbf{A}}$ \ {{b}, {b, c}, {c}}. Since k and k+1 are bi-dictators in g”, we must have f ($\underline{P}_{-i\,k\,k+1}$, P_i'', P_k, P_{k+1}'') = {a, b}. We claim that f ($\underline{P}_{-i\,k\,k+1}$, P_i', P_k, P_{k+1}'') = {a, b}. Suppose not . Since k+1 can force the outcome to be {a, b} by announcing $P_{k+1}' = P_k$, and since {b} P_{k+1}' {a, b} we must have f ($\underline{P}_{-i\,k\,k+1}$, P_i', P_k, P_{k+1}'') = {b} otherwise agent k+1 will manipulate. But, then i will manipulate at ($\underline{P}_{-i\,k\,k+1}$, P_i'', P_k, P_{k+1}'') via P_i'. Therefore, f ($\underline{P}_{-i\,k\,k+1}$, P_i', P_k, P_{k+1}'') = {a, b}. Since {b, c} P_i' {a, b}, i will manipulate at ($\underline{P}_{-i\,k\,k+1}$, P_i', P_k, P_{k+1}'') via P_i. Thus, f will not be strategy-proof if case A were to hold.

CASE B : f ($\underline{P}_{-i\,k\,k+1}$, P_i , P_k, P_{k+1}'') = {a, c}.

Let P_i" = P_k and let $P_i^* \in \mathfrak{R}^{\lambda^-}$ be such that {a} P_i^* {a, c} P_i^* {c} P_i^* X for all X ∈ $\underline{\mathbf{A}}$ \ {{a}, {a, c}, {c}}. Since i and k+1 are bi-dictators in g', we have *f* ($\underline{P}_{-i\ k\ k+1}$, P_i", P_k, P_{k+1}") = {a, b}. By replicating the appropriate arguments in case A, it follows that *f* ($\underline{P}_{-i\ k\ k+1}$, P_i^*, P_k, P_{k+1}") = {a, b}. But {a, c} P_i^* {a, b}. Therefore, i will manipulate at ($\underline{P}_{-i\ k\ k+1}$, P_i^*, P_k, P_{k+1}") via P_i. So, *f* would not be strategy-proof if case B were to hold.

CASE C: *f* ($\underline{P}_{-i\ k\ k+1}$, P_i , P_k, P_{k+1}") = {a, b, c}.

Let P_i" = P_{k+1}" and P_i' ∈ $\mathfrak{R}^{\lambda^+}$ be such that {b} P_i' {c} P_i' {a} P_i' {x} for all {x} ∈ $\underline{\mathbf{A}}_1$ \ {{a}, {b}, {c}}. Since P_i' ∈ $\mathfrak{R}^{\lambda^+}$ then {a, b, c} P_i' {a, b}. Since k and k+1 are bi-dictators in g", we must have *f* ($\underline{P}_{-i\ k\ k+1}$, P_i", P_k, P_{k+1}") = {a, b}. We claim that *f* ($\underline{P}_{-i\ k\ k+1}$, P_i', P_k, P_{k+1}") = {a, b}. Suppose not. Since k+1 can force the outcome to be {a, b} by announcing P_{k+1}' = P_i' we must have *f* ($\underline{P}_{-i\ k\ k+1}$, P_i', P_k, P_{k+1}") = {b}. But, then i will manipulate at ($\underline{P}_{-i\ k\ k+1}$, P_i", P_k, P_{k+1}") via P_i' since {b} P_i' {a, b}. Therefore, *f* ($\underline{P}_{-i\ k\ k+1}$, P_i', P_k, P_{k+1}") = {a, b}. Hence, since {a, b, c} P_i' {a, b}, i will manipulate at ($\underline{P}_{-i\ k\ k+1}$, P_i', P_k, P_{k+1}") via P_i. Thus, *f* would not be strategy-proof if case C were to hold.

Hence (**) cannot hold, which completes the proof of Theorem 3.1. ■

APPENDIX B

Proposition 4.1: Take any D, D' $\subset \Re$ with D $\cap$ D' = $\varnothing$. Assume D is fully regular while D' is regular. Consider a hyperfunction $f : [D \cup D']^N \rightarrow \underline{\mathbf{A}}$ which is dictatorial over D, ie., for all $\underline{R} \in D^N$, $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_i$ for some i $\in$ N. If f is strategy-proof over the domain D $\cup$ D' then f exhibits i $\in$ N as the dictator over D $\cup$ D'.

Proof: Take D, D' and f as in the statement of the proposition. Assume, without loss of generality that the dictator over D is agent 1. We will prove the proposition by a series of lemmata.

Lemma B1: For all $\underline{R} \in [D \cup D']^N$ with $R_1 \in D$ and $\#\{i \in N \setminus \{1\} : R_i \in D'\} = 1$ we have $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_1$.

Proof of Lemma B1: Take any $\underline{R} \in [D \cup D']^N$ as in the statement of the lemma. Assume, without loss of generality that $R_2 \in D'$. We want to show that $f(\underline{R}) = \{a\}$ where $\{a\} = \text{argmax}_{\underline{\mathbf{A}}} R_1$. Suppose not, ie., $f(\underline{R}) = X$ for some $X \in \underline{\mathbf{A}} \setminus \{\{a\}\}$. Now, take some $R_2' \in D$ with $\text{argmin}_{\underline{\mathbf{A}}} R_2 = \{a\}$. Such an R_2' exists as D is fully regular. Consider the profile $\underline{R}' \in D^N$ where $R_j' = R_j'$ for all j $\in$ N $\setminus$ {2}while the preference of agent 2 is R_2'. As $\underline{R}' \in D^N$, we have $f(\underline{R}') = \{a\}$, because agent 1 is the dictator over D. But, since X P_2' {a}, agent 2 will manipulate $\underline{R}'$ by pretending $R_2 \in D'$, contradicting that f is strategy-proof. ■

Lemma B2: For all $\underline{R} \in [D \cup D']^N$ with $R_1 \in D$ and $\#\{i \in N \setminus \{1\} : R_i \in D'\} = k$ for some k $\in$ {1,..., n-1} we have $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_1$.

Proof: We will prove the lemma by induction. In fact, Lemma B1 shows that Lemma B2 holds for k = 1. Now, suppose that we have $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_1$ whenever $\#\{i \in N \setminus \{1\} : R_i \in D'\} = k-1$. We claim that $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_1$ whenever $\#\{i \in N \setminus \{1\} : R_i \in D'\} = k$ as well. To see this, let $\#\{i \in N \setminus \{1\} : R_i \in D'\} = k$ and assume without loss of generality that $\{j \in N : 2 \leq j \leq k + 1\}$ is the set of agents whose preferences belong to D'. We wish to show that $f(\underline{R}) = \{a\} = \text{argmax}_{\underline{\mathbf{A}}} R_1$. Suppose not, ie., $f(\underline{R})$

= X for some X ∈ $\underline{\mathbf{A}}$ \ {{a}}. Now, take some R_{k+1}' ∈ D with $\text{argmin}_{\underline{\mathbf{A}}}$ R_{k+1}' = {a}. Such an R_{k+1}' exists as D is fully regular. Consider the profile $\underline{R}$' ∈ $[D \cup D']^N$ where R_j' = R_j for all j ∈ N \ {k+1}while the preference of agent k+1 is R_{k+1}'. Note that at the profile $\underline{R}$', the number of agents whose preferences belong to the set D' is precisely k-1, which implies that f($\underline{R}$') = {a}, as agent 1 is the dictator in this case. But, since X P_{k+1}' {a}, agent 2 will manipulate $\underline{R}$' by pretending R_2 ∈ D', contradicting that f is strategy-proof. ■

Lemma B3: For all $\underline{R}$ ∈ $[D \cup D']^N$ with R_1 ∈ D' we have f($\underline{R}$) = $\text{argmax}_{\underline{\mathbf{A}}}$ R_1.

Proof of Lemma B3: Take any $\underline{R}$ ∈ $[D \cup D']^N$ with R_1 ∈ D' and suppose for a contradiction that f($\underline{R}$) = X for some X ∈ $\underline{\mathbf{A}}$ \ {{a}} while $\text{argmax}_{\underline{\mathbf{A}}}$ R_1 = {a}. Take any R_1' ∈ D with $\text{argmax}_{\underline{\mathbf{A}}}$ R_1' = {a} and consider $\underline{R}$' ∈ $[D \cup D']^N$ with R_j' = R_j for all j ∈ N \ {1} while the preference of agent 1 is R_1'. We know by Lemma B2 that f($\underline{R}$') = {a}, hence agent 1 can manipulate $\underline{R}$ by pretending R_1' ∈ D, contradicting that f is strategy-proof. ■

The fact that f is dictatorial over D ∪ D' is a direct consequence of Lemmata B1, B2, and B3. ■

APPENDIX C

Proposition 4.2: Let $D \supset \mathfrak{R}^{\lambda}$ be a fully regular domain satisfying condition β. Consider a hyperfunction $f : D^N \to \underline{\mathbf{A}}$ which is bi-dictatorial over $\mathfrak{R}^{\lambda}$, ie., for all $\underline{R} \in [\mathfrak{R}^{\lambda}]^N$, $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_i \cup \text{argmax}_{\underline{\mathbf{A}}} R_j$ for some distinct $i, j \in N$. If f is strategy-proof over the domain D then f exhibits i and j as bi-dictators over D.

Proof: Take D and f as in the statement of the proposition. Assume, without loss of generality that the bi-dictators over $\mathfrak{R}^{\lambda}$ are agents 1 and 2. We will prove the proposition by a series of lemmata.

Lemma C1: For all $\underline{R} \in D^N$ with $R_1, R_2 \in \mathfrak{R}^{\lambda}$ and $\#\{i \in N \setminus \{1, 2\} : R_i \in D \setminus \mathfrak{R}^{\lambda}\} = 1$ we have $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_1 \cup \text{argmax}_{\underline{\mathbf{A}}} R_2$.

Proof: Take any $\underline{R} \in D^N$ as in the statement of the lemma. Assume, without loss of generality that $R_3 \in D \setminus \mathfrak{R}^{\lambda}$. We want to show that $f(\underline{R}) = \{a_1, a_2\}$ where $\{a_1\} = \text{argmax}_{\underline{\mathbf{A}}} R_1$ and $\{a_2\} = \text{argmax}_{\underline{\mathbf{A}}} R_2$. Suppose for a contradiction that $f(\underline{R}) = X$ for some $X \in \underline{\mathbf{A}} \setminus \{\{a_1, a_2\}\}$.

Consider first the case where $X \neq \{a_2\}$. Take $\underline{R}' \in [\mathfrak{R}^{\lambda}]^N$ with $R_j' = R_j$ for all $j \in N \setminus \{3\}$ while $R_3' \in \mathfrak{R}^{\lambda}$ is as follows: $Y\ P_3'\ \{a_1\}\ P_3'\ \{a_1, a_2\}\ P_3'\ \{a_2\}$ for all $Y \in \underline{\mathbf{A}} \setminus \{\{a_1\}, \{a_1, a_2\}, \{a_2\}\}$. As $\underline{R}' \in [\mathfrak{R}^{\lambda}]^N$ and f is bi-dictatorial over $\mathfrak{R}^{\lambda}$, we have $f(\underline{R}') = \{a_1, a_2\}$. Since $X\ P_3'\ \{a_1,a_2\}$, agent 3 will manipulate $\underline{R}'$ by pretending $R_3 \in D \setminus \mathfrak{R}^{\lambda}$, contradicting that f is strategy-proof.

Now consider the case where $X \neq \{a_1\}$. Take $\underline{R}' \in [\mathfrak{R}^{\lambda}]^N$ with $R_j' = R_j$ for all $j \in N \setminus \{3\}$ while $R_3' \in \mathfrak{R}^{\lambda}$ is as follows: $Y\ P_3'\ \{a_2\}\ P_3'\ \{a_1, a_2\}\ P_3'\ \{a_1\}$ for all $Y \in \underline{\mathbf{A}} \setminus \{\{a_1\}, \{a_1, a_2\}, \{a_2\}\}$. As $\underline{R}' \in [\mathfrak{R}^{\lambda}]^N$ and f is bi-dictatorial over $\mathfrak{R}^{\lambda}$, we have $f(\underline{R}') = \{a_1, a_2\}$. Since $X\ P_3'\ \{a_1,a_2\}$, agent 3 will manipulate $\underline{R}'$ by pretending $R_3 \in D \setminus \mathfrak{R}^{\lambda}$, contradicting that f is strategy-proof.

$X \neq \{a_2\}$ and $X \neq \{a_1\}$ exhaust all possible cases, hence completing the proof of the lemma. ■

Lemma C2: For all $\underline{R} \in D^N$ with $R_1, R_2 \in \Re^\lambda$ and $\#\{i \in N \setminus \{1, 2\} : R_i \in D \setminus \Re^\lambda\} = k$ for some $k \in \{1,..., n\text{-}2\}$, we have $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}}\, R_1 \cup \text{argmax}_{\underline{\mathbf{A}}}\, R_2$.

Proof: We will prove the lemma by induction. In fact, Lemma C1 shows that Lemma C2 holds for $k = 1$. Now, suppose that we have $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}}\, R_1 \cup \text{argmax}_{\underline{\mathbf{A}}}\, R_2$ whenever $\#\{i \in N \setminus \{1, 2\} : R_i \in D \setminus \Re^\lambda\} = k\text{-}1$. We claim that $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}}\, R_1 \cup \text{argmax}_{\underline{\mathbf{A}}}\, R_2$ whenever $\#\{i \in N \setminus \{1, 2\} : R_i \in D \setminus \Re^\lambda\} = k$ as well. To see this, let $\#\{i \in N \setminus \{1, 2\} : R_i \in D \setminus \Re^\lambda\} = k$ and assume without loss of generality that $\{j \in N : 3 \leq j \leq k + 2\}$ is the set of agents whose preferences belong to $D \setminus \Re^\lambda$. We wish to show that $f(\underline{R}) = \{a_1, a_2\}$ where $\{a_1\} = \text{argmax}_{\underline{\mathbf{A}}}\, R_1$ and $\{a_2\} = \text{argmax}_{\underline{\mathbf{A}}}\, R_2$. Suppose not, ie., $f(\underline{R}) = X$ for some $X \in \underline{\mathbf{A}} \setminus \{\{a_1, a_2\}\}$.

Consider first the case where $X \neq \{a_2\}$. Take $\underline{R}' \in D^N$ with $R_j' = R_j$ for all $j \in N \setminus \{k+2\}$ while $R_{k+2}' \in \Re^\lambda$ is as follows: $Y\ P_{k+2}'\ \{a_1\}\ P_{k+2}'\ \{a_1, a_2\}\ P_{k+2}'\ \{a_2\}$ for all $Y \in \underline{\mathbf{A}} \setminus \{\{a_1\}, \{a_1, a_2\}, \{a_2\}\}$. Note that in $\underline{R}'$ the number of agents whose preferences are in $D \setminus \Re^\lambda$ is precisely k-1 which, by the induction hypothesis, implies $f(\underline{R}') = \{a_1, a_2\}$. Since $X\ P_{k+2}'\ \{a_1, a_2\}$, agent k+2 can manipulate $\underline{R}'$ by pretending R_{k+2}, contradicting that f is strategy-proof.

Now consider the case where $X \neq \{a_1\}$. Take $\underline{R}' \in D^N$ with $R_j' = R_j$ for all $j \in N \setminus \{k+2\}$ while $R_{k+2}' \in \Re^\lambda$ is as follows: $Y\ P_{k+2}'\ \{a_2\}\ P_{k+2}'\ \{a_1, a_2\}\ P_{k+2}'\ \{a_1\}$ for all $Y \in \underline{\mathbf{A}} \setminus \{\{a_1\}, \{a_1, a_2\}, \{a_2\}\}$. Note that in $\underline{R}'$ the number of agents whose preferences are in $D \setminus \Re^\lambda$ is precisely k-1 which, by the induction hypothesis, implies $f(\underline{R}') = \{a_1, a_2\}$. Since $X\ P_{k+2}'\ \{a_1, a_2\}$, agent k+2 can manipulate $\underline{R}'$ by pretending R_{k+2}, contradicting that f is strategy-proof.

$X \neq \{a_2\}$ and $X \neq \{a_1\}$ exhaust all possible cases, hence completing the proof of the lemma. ■

Lemma C3: For all $\underline{R} \in D^N$ with $R_1 \in \Re^\lambda$ if and only if $R_2 \in D \setminus \Re^\lambda$ we have $f(\underline{R})$ = $\text{argmax}_{\underline{\mathbf{A}}} R_1 \cup \text{argmax}_{\underline{\mathbf{A}}} R_2$.

Proof: Take any $\underline{R} \in D^N$ as in the statement of the lemma. Assume without loss of generality that $R_1 \in D \setminus \Re^\lambda$ while $R_2 \in \Re^\lambda$. We want to show that $f(\underline{R}) = \{a_1, a_2\}$ where $\{a_1\} = \text{argmax}_{\underline{\mathbf{A}}} R_1$ and $\{a_2\} = \text{argmax}_{\underline{\mathbf{A}}} R_2$. Suppose not, ie., $f(\underline{R}) = X$ for some $X \in \underline{\mathbf{A}} \setminus \{\{a_1, a_2\}\}$. We now consider ten cases about the value that X can take:

CASE 1: $X = \{a_1\}$. Consider $\underline{R}' \in D^N$ with $R_1' \in \Re^\lambda$ with $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{a_1\}$ and $R_i' = R_i$ for all $i \in N \setminus \{1\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}'$ i.e., $f(\underline{R}') = \{a_1, a_2\}$. As $R_1' \in \Re^\lambda$, we have $\{a_1\}\ P_1'\ \{a_1, a_2\}$, hence agent 1 can manipulate $\underline{R}'$ by pretending R_1, contradicting that f is strategy-proof.

CASE 2: $X = \{a_2\}$. Since D' satisfies condition β, we have $\{a_1\}\ P_1\ \{a_1,a_2\}\ P_1\ \{a_2\}$. Consider $\underline{R}' \in D^N$ with $R_1' \in \Re^\lambda$ with $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{a_1\}$ and $R_i' = R_i$ for all $i \in N \setminus \{1\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}'$ i.e., $f(\underline{R}') = \{a_1, a_2\}$. Since $\{a_1,a_2\}\ P_1\ \{a_2\}$, agent 1 can manipulate $\underline{R}$ by pretending R_1', contradicting that f is strategy-proof.

CASE 3: $X = \{x\}$ for some $x \in \mathbf{A} \setminus \{a_1, a_2\}$. Consider $\underline{R}' \in D^N$ with $R_1' \in \Re^\lambda$ with $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{x\}$ and $R_i' = R_i$ for all $i \in N \setminus \{1\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}'$ i.e., $f(\underline{R}') = \{x, a_2\}$. Since $\{x\}\ P_1'\ \{x, a_2\}\ P_1'\ \{a_2\}$, agent 1 can manipulate $\underline{R}'$ by pretending R_1, contradicting that f is strategy-proof.

CASE 4: $X = \{a_1, x\}$ for some $x \in \mathbf{A} \setminus \{a_1, a_2\}$. Consider $\underline{R}' \in D^N$ with $R_1' \in \Re^\lambda$ with $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{x\}$ and $\{x\}\ P_1'\{a_1\}\ P_1'\ \{a_2\}$, while $R_i' = R_i$ for all $i \in N \setminus \{1\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}'$ i.e., $f(\underline{R}') = \{x, a_2\}$. Since $\{x, a_1\}\ P_1'\ \{x, a_2\}$, agent 1 can manipulate $\underline{R}'$ by pretending R_1, contradicting that f is strategy-proof.

CASE 5: $X = \{a_2, x\}$ for some $x \in \mathbf{A} \setminus \{a_1, a_2\}$. As D satisfies condition β and $\text{argmax}_{\underline{\mathbf{A}}} R_1 = \{a_1\}$, we have $\{a_1, a_2\} P_1 \{a_2, x\}$. Consider $\underline{R}' \in D^N$ with $R_1' \in \mathfrak{R}^{\lambda}$ with $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{a_1\}$ and $R_i' = R_i$ for all $i \in N \setminus \{1\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}'$ i.e., $f(\underline{R}') = \{a_1, a_2\}$. Hence, agent 1 can manipulate $\underline{R}$ by pretending R_1', contradicting that f is strategy-proof.

CASE 6: $X = \{x, y\}$ for some $x, y \in \mathbf{A} \setminus \{a_1, a_2\}$. Consider $\underline{R}' \in D^N$ with $R_1' \in \mathfrak{R}^{\lambda^-}$ where $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{a_1\}$ and $\{a_1\} P_1' \{x\} P_1' \{y\} P_1' \{a_2\}$, while $\{x, y\} P_1' \{a_1, a_2\}$. Let $R_i' = R_i$ for all $i \in N \setminus \{1\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}'$ i.e., $f(\underline{R}') = \{a_1, a_2\}$. Thus agent 1 can manipulate $\underline{R}'$ by pretending R_1, contradicting that f is strategy-proof.

CASE 7: $\# X > 2$ with $a_1 \in X$ but $a_2 \notin X$. Consider $\underline{R}' \in D^N$ with $R_1' \in \mathfrak{R}^{\lambda}$ where $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{a_1\}$, $\text{argmin}_{\underline{\mathbf{A}}} R_1' = \{a_2\}$ while $X P_1' \{a_1, a_2\}$. Let $R_i' = R_i$ for all $i \in N \setminus \{1\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}'$ i.e., $f(\underline{R}') = \{a_1, a_2\}$. Since $X P_1' \{a_1, a_2\}$, agent 1 can manipulate $\underline{R}'$ by pretending R_1, contradicting that f is strategy-proof.

CASE 8: $\# X > 2$ with $a_2 \in X$ but $a_1 \notin X$. Consider $\underline{R}' \in D^N$ with $R_1' \in \mathfrak{R}^{\lambda^+}$ where $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{x\}$ for some $x \in X \setminus \{a_2\}$, while $\text{argmin}_{\underline{\mathbf{A}}} R_1' = \{a_2\}$. Let $R_i' = R_i$ for all $i \in N \setminus \{1\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}'$ i.e., $f(\underline{R}') = \{x, a_2\}$. Since $R_1' \in \mathfrak{R}^{\lambda^+}$ we have $X P_1' \{x, a_2\}$, thus agent 1 can manipulate $\underline{R}'$ by pretending R_1, contradicting that f is strategy-proof.

CASE 9: $\# X > 2$ with $a_1, a_2 \notin X$. Consider $\underline{R}' \in D^N$ with $R_1' \in \mathfrak{R}^{\lambda^-}$ where $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{a_1\}$ and $\text{argmin}_{\underline{\mathbf{A}}} R_1' = \{a_2\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}'$ i.e., $f(\underline{R}') = \{a_1, a_2\}$. Since $R_1' \in \mathfrak{R}^{\lambda^-}$ we have $X P_1' \{a_1, a_2\}$, thus agent 1 can manipulate $\underline{R}'$ by pretending R_1, contradicting that f is strategy-proof.

CASE 10: $\# X > 2$ with $a_1, a_2 \in X$. Consider $\underline{R}' \in D^N$ with $R_1' \in \mathfrak{R}^{\lambda^+}$ where $\text{argmax}_{\underline{\mathbf{A}}} R_1' = \{a_1\}$ and $\text{argmin}_{\underline{\mathbf{A}}} R_1' = \{a_2\}$. We know by Lemma C2 that agents 1 and 2 are bi-dictators at $\underline{R}$' i.e., $f(\underline{R}') = \{a_1, a_2\}$. Since $R_1' \in \Re^{\lambda^+}$, we have X P_1' $\{a_1, a_2\}$, thus agent 1 can manipulate $\underline{R}$' by pretending R_1, contradicting that f is strategy-proof.

The ten cases above about the value of X are exhaustive, thus completing the proof. ■

Lemma C4: For all $\underline{R} \in D^N$ with $R_1, R_2 \in D \setminus \Re^{\lambda}$ we have $f(\underline{R}) = \text{argmax}_{\underline{\mathbf{A}}} R_1 \cup \text{argmax}_{\underline{\mathbf{A}}} R_2$.

Proof: Take any $\underline{R} \in D^N$ as in the statement of the lemma. We want to show that $f(\underline{R}) = \{a_1, a_2\}$ where $\{a_1\} = \text{argmax}_{\underline{\mathbf{A}}} R_1$ and $\{a_2\} = \text{argmax}_{\underline{\mathbf{A}}} R_2$. Suppose not, ie., $f(\underline{R})$ = X for some X $\in \underline{\mathbf{A}} \setminus \{\{a_1, a_2\}\}$. The value that X can belongs to one of the ten cases described in the proof of Lemma C3. In fact, Lemma C4 can be shown exactly in the same way as Lemma C3, with the sole difference that whenever Lemma C3 benefits from Lemma C2, Lemma C4 benefits from Lemma C3. ■

REFERENCES

Aswal, N., S. Chatterji, and A. Sen (forthcoming), "Dictatorial Domains", *Economic Theory*

Barberà, S. (1977), "The Manipulability of Social Choice Mechanisms that do not Leave Too Much to Chance", *Econometrica*, 45: 1572-1588

Barberà, S., B. Dutta, and A. Sen (2001), "Strategy-Proof Social Choice Correspondences", *Journal of Economic Theory*, 101: 374-394.

Barberà, S., J. Massò, and A. Neme (1999), "Maximal Domains of Preferences Preserving Strategy-Proofness for Generalized Median Voter Schemes", *Social Choice and Welfare*, 16: 321-336.

Barberà, S. and P. K. Pattanaik (1984), "Extending an Order on a Set to the Power Set: Some Remarks on Kannai-Peleg's Approach", *Journal of Economic Theory*, 32: 185-191.

Barberà, S. and B. Peleg (1990), "Strategy-Proof Voting Schemes with Continuous Preferences", *Social Choice and Welfare*, 7: 31-38.

Barberà, S., H. Sonnenschein and L. Zhou (1991), "Voting by Committees", *Econometrica*, 59: 595-609.

Benoit, J. P. (2000), "The Gibbard-Sattertwaite Theorem: A Simple Proof", *Economic Letters*, 69: 319-322.

Benoit, J. P. (forthcoming), Strategic Manipulation in Voting Games When Lotteries and Ties are Permitted, *Journal of Economic Theory*

Berga, D. (2002), "Single-Peakedness and Strategy-Proofness of Generalized Median Voter Schemes", *Social Choice and Welfare*, 19, 175-192.

Ching, S. and L. Zhou (2002), “Multi-Valued Strategy-Proof Social Choice Rules”, *Social Choice and Welfare*, 19: 569-580.

Duggan, J. and T. Schwartz (2000), Strategic Manipulability without Resoluteness or Shared Beliefs: Gibbard-satterthwaite Generalized, *Social Choice and Welfare*, 17: 85-93.

Feldman, A. (1979a): “Nonmanipulable Multi-Valued Social Decision Functions”, *Public Choice*, 34: 177-188

Feldman, A. (1979b): “Manipulation and the Pareto Rule”, *Journal of Economic Theory*, 21: 473-482.

Feldman, A. M. (1980) Strongly Nonmanipulable Multi-Valued Collective Choice Rules, *International Review for Social Sciences*, 33

Gärdenfors, P. (1976): “Manipulation of Social Choice Functions”, *Journal of Economic Theory*, 13: 217-228.

Gärdenfors, P. (1978), “On Definitions of Manipulation of Social Choice Functions”, in J.J. Laffont (ed.), Aggregation and Revelation of preferences,. Amsterdam: North Holland.

Gibbard, A. (1973), “Manipulation of Voting Schemes: A General Result”, *Econometrica*, 41, 587-601.

Kannai Y. and B. Peleg (1984), “A Note on the Extension of an Order on a Set to the Power Set”, *Journal of Economic Theory*, 32: 172-175

Kaymak, B. and M. R. Sanver (forthcoming), “Sets of Alternatives as Condorcet Winners”, *Social Choice and Welfare*

Kelly, J. (1977), “Strategy-Proofness and Social Choice Functions without Single-Valuedness”, *Econometrica*, 45: 439-4

Pattanaik, P.K. (1973), “On the Stability of Sincere Voting Situations”, *Journal of Economic Theory*, 6: 558-574.

Roth, A. and M. A. O. Sotomayor (1990), “Two-Sided Matching: A Study in Game Theoretic Modelling and Analysis”, Cambridge University Press.

Satterhwaite, M. (1975), “Strategy-proofness and Arrow’s Conditions : Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions”, *Journal of Economic Theory*. 10, 187-217.

Serizawa, S. (1995), “Power of Voters and Domain of Preferences Where Voting by Committees Is Strategy-Proof”, Journal of Economic Theory, 67, 599-608.