Nash Implementation via Hyperfunctions

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ABSTRACT

Hyperfunctions are social choice rules which assign sets of alternatives to preference profiles over sets. So, they are more general objects compared to standard (social choice) correspondences. Thus every correspondence can be expressed in terms of an equivalent hyperfunction. We postulate the equivalence between implementing a correspondence and its equivalent hyperfunction. We give a partial charaterization of Nash implementable hyperfunctions and explore the conditions under which correspondences have Nash implementable equivalent hyperfunctions. Depending on the axioms used to extend preferences over alternatives to sets, these conditions are weaker than or logically independent of Maskin monotonicity, in any case expanding the set of Nash implementable social choice rules. In fact, social choice rules such as the majority rule and the top cycle are Nash implementable through their equivalent hyperfunctions while they are not Maskin monotonic, thus not Nash implementable in the standard framework.

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1. INTRODUCTION

Maskin (1999) shows that a certain monotonicity condition, to which we refer as Maskin monotonicity, is necessary for social choice correspondences to be implemented via Nash equilibria. He also establishes Maskin monotonicty, combined with a no veto power condition suffices for Nash implementability.[1] Thus, the set of social choice correspondences which can be implemented via Nash equilibria is restricted by Maskin monotonicity.

Maskin monotonicity is a fairly demanding condition and many well-known social choice rules fail to be Nash implementable. We explore the possibility of expanding the set of Nash implementable social choice rules by carrying the implementation problem in a different setting. We consider hyperfunctions which are social choice rules with a domain consisting of orderings over non-empty sets. To every preference profile over sets, they assign a non-empty set of alternatives.[2] Note that hyperfunctions are more general objects than standard (social choice) correspondences which assign a non-empty set of alternatives to every preference profile over alternatives. In fact, correspondences are hyperfunctions satisfying the following strong invariance property: The outcome of the hyperfunction must be the same at any two preference profiles over sets where individuals' orderings of the singleton sets are the same. So every correspondence can be expressed in terms of some hyperfunction, to which we refer as the equivalent hyperfunction of the correspondence.[3]

We exploit this fact to obtain more permissive results in Nash implementation. Our point of departure is that it does not matter whether one implements a correspondence or its equivalent hyperfunction. We start by giving a partial charaterization of Nash implementable hyperfunctions. Then, using fairly strong axioms to extend preferences over alternatives to sets, we explore the conditions under which correspondences have Nash implementable hyperfunctions. Depending on the axioms used to extend preferences over alternatives to sets, these conditions are weaker than or logically independent of Maskin monotonicity, in any case expanding the set of Nash implementable social choice rules. In fact, social choice rules

[1] So this is a partial charaterization of Nash implementable social choice correspondences. It is first Moore and Repullo (1990) and then Danilov (1992) who give a full characterization. All these results are for societies with at least three people. Dutta and Sen (1991) charaterize Nash implementable social choice correspondences in two-person societies.

[2] Barberà, Dutta and Sen (2001) as well as Özyurt and Sanver (2003) use hyperfunctions to analyze dominant strategy implementable social choice correspondences.

[3] While not every hyperfunction can be expressed in terms of a correspondence.

such as the majority rule and the top cycle are Nash implementable through their equivalent hyperfunctions while they are not Maskin monotonic, thus not Nash implementable in the standard framework.

The structure of the paper is as follows: We give the preliminaries in Section 2. Section 3 gives a partial charaterization of Nash implementable hyperfunctions. Section 4 explores the conditions under which social choice correspondences have Nash implementable equivalent hyperfunctions. Section 5 gives examples of non Maskin monotonic social choice correspondences which are Nash implementable through their equivalent hyperfunctions. Concluding remarks follow in Section 6.

2. PRELIMINARIES

Taking any two integers n and m with n ≥ 2 and m ≥ 3, we consider a society **N** = {1,…, n} confronting a set of alternatives **A** = $\{a_1, a_2, \ldots, a_m\}$. We write $\underline{\mathbf{A}} = 2^{\mathbf{A}} \setminus \{\emptyset\}$ for the set of all non-empty subsets of **A**.

We let Π stand for the set of all complete, transitive and antisymmetric binary relations over **A**. Every ρ ∈ Π represents an individual preference on the elements of **A** in the following manner: For any a, b ∈ **A**, a ρ b means "a is at least as good as b".[4] In case the preference ordering over **A** is specified to belong to a particular agent i ∈ **N**, we will write it as ρ_i . A typical preference profile over **A** will be denoted by $\underline{\rho} = (\rho_1, \ldots, \rho_n) \in \Pi^N$.

Similarly, we let ℜ stand for the set of all complete and transitive orderings over $\underline{\mathbf{A}}$. Every R ∈ ℜ represents an individual preference on the elements of $\underline{\mathbf{A}}$ in the following manner: For any X, Y ∈ $\underline{\mathbf{A}}$, X R Y means "X is at least as good as Y". We denote P and I for the strict and indifference counterparts of R.[5] In case the preference ordering over $\underline{\mathbf{A}}$ is specified to belong to a particular agent i ∈ **N**, we will write it as R_i, with its respective strict and indifference

[4] As ρ is antisymmetric, we have for any distinct a, b ∈ **A**, we have a ρ b ⇒ not b ρ a. In other words, for distinct alternatives, a ρ b means "a is preferred to b".

[5] For any X, Y ∈ $\underline{\mathbf{A}}$, we write X P Y if and only if X R Y holds but Y R X does not, ie., X is preferred to Y. In case X R Y and Y R X both hold, we write X I Y, which means indifference between X and Y.

counterparts P_i and I_i . A typical preference profile over $\underline{\mathbf{A}}$ will be denoted by $\underline{R} = (R_1, ..., R_n) \in \Re^N$.

We accept that if the preference ordering over **A** is some $\rho \in \Pi$, then the preference ordering over $\underline{\mathbf{A}}$ can be some $R \in \Re$ which is "consistent" with ρ. Thus, we define a consistency map $\kappa: \Pi \rightarrow 2^{\Re} \setminus \{\emptyset\}$ which assigns to every $\rho \in \Pi$ a non-empty set $\kappa(\rho) \subseteq \Re$ of preference orderings on $\underline{\mathbf{A}}$ consistent with ρ. We assume that every consistency map κ satisfies a basic axiom A0, which we define as follows:

A0: Given any $\rho \in \Pi$ and any $R \in \kappa(\rho)$, we have $x \rho y \Leftrightarrow \{x\} R \{y\}$ for all $x, y \in \mathbf{A}$.

A0 requires that the ordering of individuals over singleton sets must be the same as their ordering over the basic alternatives. Remark that A0 automatically implies that $\kappa(\rho) \cap \kappa(\rho') = \emptyset$ for all distinct $\rho, \rho' \in \Pi$.

Given any consistency map κ, we write $\Re^\kappa = \cup_{\rho \in \Pi} \kappa(\rho)$ for the set of acceptable preference orderings over $\underline{\mathbf{A}}$ defined through κ. Note that $\Re^\kappa$ is always a strict subset of $\Re$, as every κ is assumed to satisfy our basic axiom A0.

A (social choice) correspondence is a mapping $F: \Pi^N \rightarrow \underline{\mathbf{A}}$. Let $L(a, \rho_i) = \{x \in \mathbf{A} \mid a \rho_i x\}$ be the lower contour set of an alternative $a \in \mathbf{A}$ at a given preference ρ_i of an agent i. A correspondence $F: \Pi^N \rightarrow \underline{\mathbf{A}}$ is said to be *Maskin-monotonic* if and only if given any $\underline{\rho}$, $\underline{\rho}' \in \Pi^N$ and any $a \in \mathbf{A}$, we have $a \in F(\underline{\rho}) \Rightarrow a \in F(\underline{\rho}')$ whenever $L(a; \rho_i) \subseteq L(a; \rho_i')$ for every $i \in \mathbf{N}$. A correspondence F is said to satisfy the *no veto power condition* if and only if given any $\underline{\rho} \in \Pi^N$ and any $a \in \mathbf{A}$ we have $\#\{i \in \mathbf{N} : L(a, \rho_i) = \mathbf{A}\} \geq n - 1 \Rightarrow a \in F(\underline{\rho})$.

A *mechanism for a correspondence* is an (n+1)-tuple $\mu = (\{M_i\}_{i \in \mathbf{N}}, h)$ where M_i is the non-empty message space of agent i and $h: M \rightarrow \mathbf{A}$ is the outcome function which assigns an element of **A** to each joint message $m \in M = \prod_{i=1}^{n} M_i$. At each $\underline{\rho} \in \Pi^N$, a mechanism μ induces a normal form game $\Gamma(\mu, \underline{\rho}) = \{(M_i, u_i)\}_{i \in N}$ where M_i is the strategy space of agent i and u_i is his real –valued "payoff function" defined over M such that for any $m, m' \in M$, $u_i(m) \geq$

$u_i(m')$ if and only if $h(m)\ \rho_i\ h(m')$. We write $\nu(\Gamma(\mu, \underline{\rho}))$ for the set of Nash equilibria of the game $\Gamma(\mu, \underline{\rho})$. We say that a mechanism μ *implements a social choice correspondence* $F: \Pi^N \to \underline{\mathbf{A}}$ *via Nash equilibria* if and only if given any $\underline{\rho} \in \Pi^N$ we have

$$F(\underline{\rho}) = \cup_{m \in \nu(\Gamma(\mu, \underline{\rho}))} h(m) .$$

3. IMPLEMENTABLE HYPERFUNCTIONS

Given any $D \subseteq \Re$, we define a (social choice) hyperfunction as a mapping f: $D^N \to \underline{\mathbf{A}}$. So, while we assuming that correspondences are defined over the full domain of preference orderings (over alternatives), the domains of hyperfunctions, which are preference orderings over sets, may be restricted.

A correspondence $F: \Pi^N \to \underline{\mathbf{A}}$ and a hyperfunction $f: D^N \to \underline{\mathbf{A}}$ are said to be *equivalent* if and only if for all $\underline{\rho} \in \Pi^N$ and for all $\underline{R} \in D^N$,

$$[\ x\ \rho_i\ y \Leftrightarrow \{x\}\ R_i\ \{y\} \text{ for all } i \in \mathbf{N} \text{ and for all } x, y \in \mathbf{A}] \Rightarrow F(\underline{\rho}) = f(\underline{R}).$$

So, we say that a correspondence and a hyperfunction are equivalent if at each preference profile over sets, the hyperfunction picks the set that the correspondence would pick when the alternatives are ordered in the same way as the singleton sets. Note that every correspondence has a unique equivalent hyperfunction.

A *mechanism for a hyperfunction* is an (n+1)-tuple $\mu = (\{M_i\}_{i \in \mathbf{N}}, H)$ where M_i is the non-empty message space of agent i and $H: M \to \underline{\mathbf{A}}$ is the outcome correspondence which assigns a nonempty-subset of $\mathbf{A}$ to each joint message $m \in M$. At each $\underline{R} \in \Re^N$, a mechanism μ induces a normal form game $\Gamma(\mu, \underline{R}) = \{(M_i, u_i)\}_{i \in N}$ where M_i is the strategy space of agent i and u_i is his real –valued "payoff function" defined over M such that for any m, m' $\in$ M, $u_i(m) \geq u_i(m')$ if and only if $H(m)\ R_i\ H(m')$. We write $\nu(\Gamma(\mu, \underline{R}))$ for the set of Nash equilibria of the game $\Gamma(\mu, \underline{R})$. We say that a mechanism μ *implements a hyperfunction* f: $D^N \rightarrow \underline{\mathbf{A}}$ *via Nash equilibria* if and only if given any $\underline{R} \in D^N$ we have

$$f(\underline{R}) = \cup_{m \in \nu(\Gamma(\mu, \underline{R}))} H(m).$$

Remark that, the above definition of implementation is a rather weak one, as the standart concept would require $f(\underline{R}) = H(m)$ for every $m \in \nu(\Gamma(\mu, \underline{R}))$. On the other hand our aim is not to implement hyperfunctions but to implement correspondences through hyperfunctions and the definition we use suffices for this. For it does not matter whether one implements a correspondence or its equivalent hyperfunction according to the definition of implementation we will be using.

We now introduce a weak monotonicity condition for hyperfunctions which is necessary for their implementability.

Write $L(X, R_i) = \{X \in \underline{\mathbf{A}} \mid X\ R_i\ Y\}$ for the lower contour set of a set $X \in \underline{\mathbf{A}}$ at a given preference R_i of an agent i. A hyperfunction f: $D^N \rightarrow \underline{\mathbf{A}}$ is said to be *weakly monotonic* if and only if given any $\underline{R} \in D^N$, there exists a cover C of $f(\underline{R})$ such that for all $X \in C$ and all $\underline{R}' \in D^N$ with $L(X; R_i) \subseteq L(X; R_i')$ for all $i \in \mathbf{N}$, we have $X \subseteq f(\underline{R}')$.

Proposition 3.1. Take any $D \subseteq \Re$. A hyperfunction f: $D^N \rightarrow \underline{\mathbf{A}}$ is Nash implementable only if f is weakly monotonic.

Proof: Take a hyperfunction f: $D^N \rightarrow \underline{\mathbf{A}}$ which is Nash implementable. Let $\mu = (\{M_i\}_{i \in \mathbf{N}}, H)$ be the mechanism which Nash implements f. Take any $\underline{R} \in D^N$. As μ Nash implements f, we have $f(\underline{R}) = \cup_{m \in \nu(\Gamma(\mu, \underline{R}))} H(m)$. Clearly, $\{H(m)\}_{m \in \nu(\Gamma(\mu, \underline{R}))}$ is a cover C of $f(\underline{R})$. Now take any $H(m) \in C$ and any $\underline{R}' \in D^N$ with $L(H(m); R_i) \subseteq L(H(m); R_i')$ for all $i \in \mathbf{N}$. Thus $m \in \nu(\Gamma(\mu, \underline{R}'))$ which implies that $H(m) \subseteq f(\underline{R}')$ as μ Nash implements f, showing that $\{H(m)\}_{m \in \nu(\Gamma(\mu, \underline{R}))}$ is the cover mentioned by the proposition. Q.E.D.

A hyperfunction f: $D^N \rightarrow \underline{\mathbf{A}}$ is said to satisfy the *set no veto power condition* if and only if given any $\underline{R} \in D^N$ and any $X \in \underline{\mathbf{A}}$ we have $\#\{i \in \mathbf{N} : L(X, R_i) = \mathbf{A}\} \geq n - 1 \Rightarrow X \subseteq f(\underline{R})$.

Proposition 3.2. Let $\#N \geq 3$. Take any $D \subseteq \Re$. A weakly monotonic hyperfunction $f: D^N \rightarrow \underline{\mathbf{A}}$ satisfying the set no veto power condition is Nash implementable.

Proof: Our proof will be constructive. Let $\#N \geq 3$. Take any $D \subseteq \Re$. Consider any weakly monotonic hyperfunction $f: D^N \rightarrow \underline{\mathbf{A}}$ satisfying the set no veto power condition. For any $\underline{R} \in D^N$, write $C(\underline{R})$ for the cover of $f(\underline{R})$ which renders f weakly monotonic. We will construct a Maskin (1999) type mechanism $\mu=(\{M_i\}_{i\in N}, H)$ where the message space of an agent i will be $M_i = D^N \times \underline{\mathbf{A}} \times \mathbb{N}$. Thus a message $m_i = (\underline{R}^i, X^i, n^i)$ of an agent i will consist of some preference profile $\underline{R}^i$ in D, some non-empty set of alternatives X^i and some integer $n^i \in \mathbb{N}$. Now, for any $j \in \mathbf{N}$, we define two subsets of the joint message space M as follows:

$M_{2j} = \{m \in M : \forall\, i \in \mathbf{N} \setminus \{j\}, m_i = (\underline{R}^*, X^*, n^i)$ for some $(\underline{R}^*, X^*) \in D^N \times \underline{\mathbf{A}}$ with $X^* \in C(\underline{R}^*)$ and $m_j = (\underline{R}^{**}, X^{**}, n^j)$ such that $X^* \not\subset f(\underline{R}^{**})$ and $X^{**} \in L(X^*; R_j^*)$ but $X^{**} \notin L(X^*; R_j^{**})\}$.

$M_{1j} = \{m \in M \setminus M_{2j} : \forall\, i \in \mathbf{N} \setminus \{j\}, m_i = (\underline{R}^*, X^*, n^i)$ for some $(\underline{R}^*, X^*) \in D^N \times \underline{\mathbf{A}}$ with $X^* \in C(\underline{R}^*)\}$

Define $M_1 = \cup_{j\in \mathbf{N}} M_{1j}$, $M_2 = \cup_{j\in \mathbf{N}} M_{2j}$ and $M_3 = M \setminus (M_1 \cup M_2)$.

For any $m \in M_3$, we denote $j^{***} = \operatorname{argmax}_{j \in \mathbf{N}} n^j$ (ties can be broken arbitrarily) and let $X^{***} \in \underline{\mathbf{A}}$ be the outcome announced by agent j^{***}.

The outcome function $H: M \rightarrow \underline{\mathbf{A}}$ is defined, according to the above notation, as follows:

$$H(m) = \begin{cases} X^* & \text{whenever } m \in M_1 \\ X^{**} & \text{whenever } m \in M_2 \\ X^{***} & \text{whenever } m \in M_3 \end{cases}$$

So, if all agents announce the same preference profile $\underline{R}^*$ and the same outcome $X^* \in C(\underline{R}^*)$, then the outcome X^* is implemented. If all but one of the agents agree on announcing $\underline{R}^*$ and

X^* and the remaining agent j announces a different preference profile $\underline{R}^{**}$ then X^* is still implemented, unless $X^* \not\subset f(\underline{R}^{**})$ and j happens to be the agent for whom there exists some $X^{**} \in \underline{\mathbf{A}}$ with $X^{**} \in L(X^*; R_j^*)$ but $X^{**} \notin L(X^*; R_j^{**})$, and announces X^{**} in his message, in which case X^{**} is implemented.[6] Otherwise, implement the outcome announced by the agent who announces the highest integer (ties can be broken arbitrarily).

Now take any $\underline{R} \in D^N$. Writing $\Gamma(\mu, \underline{R})$ for the game that the mechanism μ induces at $\underline{R}$, we claim that $\bigcup_{m \in \nu(\Gamma(\mu, \underline{R}))} H(m) = f(\underline{R})$.

We will first show that $f(\underline{R}) \subseteq \bigcup_{m \in \nu(\Gamma(\mu, \underline{R}))} H(m)$. Take any $X^* \in C(\underline{R})$ and consider the joint message $m \in M_1$ where $m_i = (\underline{R}, X^*, n^i)$ for every $i \in \mathbf{N}$. It is clear that no agent i, by unilaterally changing his message, can obtain a message in M_3. In fact the only thing he can do is to stay in M_1 without being able to change the final outcome or to fall in M_2 where the new outcome he can enforce will be in $L(X^*; R_j)$. So $m \in \nu(\Gamma(\mu, \underline{R}))$. Thus, for any $X^* \in C(\underline{R})$, there exists some $m \in \nu(\Gamma(\mu, \underline{R}))$ with $H(m) = X^*$, showing that $f(\underline{R}) \subseteq \bigcup_{m \in \nu(\Gamma(\mu, \underline{R}))} H(m)$.

We will now show that $\bigcup_{m \in \nu(\Gamma(\mu, \underline{R}))} H(m) \subseteq f(\underline{R})$. Take any $m \in \nu(\Gamma(\mu, \underline{R}))$. We claim that $H(m) \subseteq f(\underline{R})$.

Consider first the case where $m \in M_1$ and thus $H(m) = X^*$. Suppose that $X^* \not\subset f(\underline{R})$. By weak monotonicity of f, $\exists\, j \in N$ and $\exists\, X^{**} \in \underline{\mathbf{A}}$ such that $X^{**} \in L(X^*; R_j^*)$ but $X^{**} \notin L(X^*; R_j)$, which contradicts the fact that $m \in \nu(\Gamma(\mu, \underline{R}))$, as this agent j, by altering his message m_j, could switch the outcome from X^* to X^{**} and be better off, as $X^{**} \notin L(X^*; R_j)$.

Now consider the case where $m \in M_2$ and thus $H(m) = X^{**}$. We want to show that $X^{**} \subseteq f(\underline{R})$. We know in this case that any agent $i \in \mathbf{N} \setminus \{j\}$ by altering his message m_i, can obtain his most preferred outcome. But as $m \in \nu(\Gamma(\mu, \underline{R}))$, it must be the case that $X^{**} \in \operatorname{argmax}_{X \in \underline{\mathbf{A}}} R_i$ for every $i \in \mathbf{N} \setminus \{j\}$. As f satisfies the set no veto power condition, we have $X^{**} \subseteq f(\underline{R})$.

[6] As f is weakly monotonic, $X^* \subseteq f(\underline{R}^*)$ and $X^* \not\subset f(\underline{R}^{**})$ for some $X^* \in C(\underline{R}^*) \Rightarrow \exists\, i \in N$ and $\exists\, X^{**} \in \underline{\mathbf{A}}$ such that $X^{**} \in L(X^*; R_j^*)$ but $X^{**} \notin L(X^*; R_j^{**})$.

Finally consider the case where $m \in M_3$ and thus $H(m) = X^{***}$. We want to show that $X^{***} \subseteq f(\underline{R})$. We know in this case that any agent $i \in \mathbf{N}$, by altering his message m_i, can obtain his most preferred outcome. But as $m \in \nu(\Gamma(\mu, \underline{R}))$, it must be the case that $X^{***} \in \text{argmax}_{X \in \underline{\mathbf{A}}} R_i$ for every $i \in \mathbf{N}$. Again as f satisfies the set no veto power condition, we have $X^{***} \subseteq f(\underline{R})$. Q.E.D.

In the next section, we explore the conditions which ensure correspondences to have Nash implementable equivalent hyperfunctions.

4. IMPLEMENTING CORRESPONDENCES THROUGH HYPERFUNCTIONS

We start by introducing two new definitions.

We say that a consistency map κ satisfies the *strong solidarity condition* if and only if given any $\rho, \rho' \in \Pi$, any $R \in \kappa(\rho)$, any $R' \in \kappa(\rho')$ and any $X \in \underline{\mathbf{A}}$, we have $L(X; R) \subseteq L(X; R') \Rightarrow L(x; \rho) \subseteq L(x; \rho')$ for all $x \in X$.

Strong solidarity imposes that the improvement of a set in an ordering must require the improvement of all the elements of this set in the corresponding orderings over alternatives. This is a fairly demanding condition while there are consistency maps which satisfy it.[7] We will turn to this issue later in Section 5, where we give examples of correspondences which are Nash implementable through hyperfunctions while they are not Maskin-monotonic, thus not Nash implementable in the standard framework.

A correspondence $F: \Pi^N \rightarrow \underline{\mathbf{A}}$ is said to be *group monotonic* if and only if given any $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $L(x; \rho_i) \subseteq L(x; \rho_i')$ for all $i \in \mathbf{N}$ and for all $x \in F(\underline{\rho})$, we have $F(\underline{\rho}) \subseteq F(\underline{\rho}')$.

Group monotonicity imposes the following restriction over correspondences: Pick a preference profile $\underline{\rho}$ where some set X is the social choice. Let $\underline{\rho}'$ be some other preference profile where for every agent the lower contour set of every element of X does not shrink. Then the social choice at $\underline{\rho}$' must include all the elements of X. Note that group monotonicity is weaker than Maskin monotonicity.

Theorem 4.1: Let κ be a consistency map satisfying the strong solidarity condition. Take any correspondence F: $\Pi^N \to \underline{\mathbf{A}}$ and its equivalent hyperfunction f : $[\Re^\kappa]^\mathbf{N} \to \underline{\mathbf{A}}$. If F is group monotonic then f is weakly monotonic.

Proof: Take κ, F and f as in the statement of the theorem. Assume F is group monotonic. To show that f is weakly monotonic, take any $\underline{R} \in [\Re^\kappa]^\mathbf{N}$. For every i ∈ **N**, let $R_i \in \kappa(\rho_i)$. As f is the equivalent hyperfunction of F, we have $f(\underline{R}) = F(\underline{\rho})$. Consider the cover C = {f($\underline{R}$)} of f($\underline{R}$). Now take some $\underline{R}' \in [\Re^\kappa]^\mathbf{N}$ with $L(f(\underline{R}); R_i) \subseteq L(f(\underline{R}); R_i')$ for every i ∈ **N**. Let $R_i' \in \kappa(\rho_i')$ for all i ∈ **N**. As κ satisfies the strong solidarity condition, for every i ∈ **N**, we have $L(x; \rho_i) \subseteq L(x; \rho_i')$ for all $x \in f(\underline{R}) = F(\underline{\rho})$. This implies $f(\underline{R}) \subseteq F(\underline{\rho}')$ by the group monotonicity of F. Again, as f is the equivalent hyperfunction of F, we have $F(\underline{\rho}') = f(\underline{R}')$. Hence, given any $\underline{R} \in [\Re^\kappa]^\mathbf{N}$, the weak monotonicity of f can be established through the cover C = {f($\underline{R}$)}. Q.E.D.

Call a domain D ⊆ ℜ regular if given any R ∈ D we have argmax R = {x} for some x ∈ **A**, i.e., every ordering in D has a unique best element which is a singleton set. Regularity is a fairly weak conditions. In fact, almost all consistency maps which conceive sets as first refinements of the original sets of alternatives lead to regular domains.[8]

Theorem 4.2: Take any correspondence F: $\Pi^N \to \underline{\mathbf{A}}$ and its equivalent hyperfunction f: $[\Re^\kappa]^\mathbf{N} \to \underline{\mathbf{A}}$ where κ is any consistency map leading to a domain $\Re^\kappa$ which is regular. If F satisfies the no veto power condition, then f satisfies the set no veto power condition.

Proof: Take F, f and κ as in the statement of the theorem. Assume F satisfies the no veto power condition. We will show that f satisfies the set no veto power condition as well. Now

[7] For example, the strong lexicographic and strong inverse lexicographic extensions used by Kaymak and Sanver (forthcoming) and Özyurt and Sanver (2003) are consistency maps satisfying the strong solidarity condition.
[8] To give a few examples, the consistency maps based on the extension axioms of Gärdenfors (1976), Barberà (1977) and Kelly (1977) will all lead to regular domains.

take any $\underline{R} \in [\Re^\kappa]^N$ and any $X \in \underline{\mathbf{A}}$ such that $\#\{i \in \mathbf{N} : L(X, R_i) = \mathbf{A}\} \geq n - 1$. As $\Re^\kappa$ is regular, $X = \{x\}$ for some $x \in \mathbf{A}$. Now let $R_i \in \kappa(\rho_i)$ for all $i \in \mathbf{N}$. So given any $i \in \mathbf{N}$, $L(\{x\}, R_i) = \mathbf{A} \Rightarrow L(x, \rho_i) = \mathbf{A}$. Thus, $\#\{i \in \mathbf{N} : L(x, \rho_i) = \mathbf{A}\} \geq n - 1$. As F satisfies the no veto power condition, $x \in F(\underline{\rho})$, showing that $\{x\} \subseteq f(\underline{R})$ as f and F are equivalent, completing the proof. Q.E.D.

We know by Theorem 4.1 that a correspondence which is group monotonic (but possibly not Maskin monotonic) has an equivalent hyperfunction which satisfies the necessary condition for Nash implementability. Moreover, Theorem 4.2 tells that the no veto power condition carries in a much easier way, as regularity of domains is a fairly mild condition. So we interpret Theorem 4.1 and 4.2 as a positive result for Nash implementation as they herald the weakening of the necessary (and almost sufficient) conditions for the implementation of correspondences. In fact, we now know that every weakly monotonic correspondence satisfying the no veto power condition can be Nash implemented through hyperfunctions under an appropriately chosen consistency map. We state this as a corollary.

Corollary 4.1. Assume $\#N \geq 3$. Let κ be a consistency map satisfying the strong solidarity condition and leading to a domain $\Re^\kappa$ which is regular. Every group monotonic correspondence $F: \Pi^N \rightarrow \underline{\mathbf{A}}$ satisfying the no veto power condition has a Nash implementable equivalent hyperfunction $f : [\Re^\kappa]^N \rightarrow \underline{\mathbf{A}}$.

Remark that our results are based on a balance between the strenghts of the solidarity condition imposed on consistency maps and the monotonicity condition imposed on correspondences. To preserve this balance, weakening the strong solidarity condition of consistency maps requires strenghtening the group monotonicity condition. Hence, one can obtain results similar to those of Theorem 4.1 and 4.2 under more general consistency maps at the expense of narrowing the set of Nash implementable hyperfunctions.

Consider for example the following solidarity condition for consistency maps:

A consistency map κ is said to satisfy the *solidarity condition* if and only if given any $\rho, \rho' \in \Pi$, any $R \in \kappa(\rho)$, any $R' \in \kappa(\rho')$ and any $X \in \underline{\mathbf{A}}$, we have $L(X; R) \subseteq L(X;R') \Rightarrow L(x^*; \rho) \subseteq$

$L(x^*; \rho')$ where $x^* = \text{argmax}_X \rho$.

This clearly weakens the strong solidarity condition. The issue of finding consistency maps satisfying this condition will be handled in the next section.[9]

A correspondence $F: \Pi^N \rightarrow \underline{\mathbf{A}}$ is said to be *strongly group monotonic* if and only if given any $\underline{\rho}$, $\underline{\rho}' \in \Pi^N$ with $L(x^*; \rho_i) \subseteq L(x^*; \rho_i')$ for all $i \in \mathbf{N}$ where $x^* = \text{argmax}_{F(\underline{\rho})} \rho_i$, we have $F(\underline{\rho}) \subseteq F(\underline{\rho}')$.

This clearly strenghtens the the group monotonicity condition. Note also that strong group monotonicty neither implies nor is implied by Maskin monotonicty. We can easily check that Theorems 4.1, 4.2 and Corollary 4.1 can easily be adopted to these new definitions. We state the final result without proof.

Corollary 4.2. Assume $\#N \geq 3$. Let κ be a consistency map satisfying the solidarity condition and leading to a domain $\Re^\kappa$ which is regular. Every strong group monotonic correspondence $F: \Pi^N \rightarrow \underline{\mathbf{A}}$ satisfying the NVP condition has a Nash implementable equivalent hyperfunction $f : [\Re^\kappa]^{\mathbf{N}} \rightarrow \underline{\mathbf{A}}$.

Which social choice rules are covered by Corollaries 4.1 and 4.2? We devote the next section to examples of social choice correspondences which are not Maskin monotonic (hence not Nash implementable in the classical sense) while they can be Nash implemented via hyperfunctions.

5. EXAMPLES OF CORRESPONDENCES IMPLEMENTABLE THROUGH HYPERFUNCTIONS

We start by the top-cycle, a social choice rule introduced by Schwartz (1972). At any $\underline{\rho} \in \Pi^N$, define a binary relation $\tau(\underline{\rho})$ over **A** through the majority relation as follows: For all $x, y \in \mathbf{A}$, $x \, \tau(\underline{\rho}) \, y$ if and only if $\# \{i \in \mathbf{N} : x \, \rho_i \, y\} \geq n/2$. Assuming an odd number of agents, $\tau(\underline{\rho})$ is a

[9] See also footnote 6.

tournament, i.e., a complete and antisymmetric binary relation. Given any $\underline{\rho} \in \Pi^N$, the top cycle TC($\underline{\rho}$) is the smallest subset X of **A** with respect to set inclusion satisfying x $\tau(\underline{\rho})$ y for all x ∈ X and for all y ∈ **A** \ X. Note that TC($\underline{\rho}$) is non-empty at each $\underline{\rho} \in \Pi^N$.[10]

We first note that the top cycle, as a social choice correspondence, is not Maskin monotonic. To see this, take a society **N** = {1, 2, 3} confronting a set of alternatives **A** = {x, y, z} and let $\underline{\rho}, \underline{\rho}' \in \Pi^N$ be as follows:

ρ_1	ρ_2	ρ_3	ρ_1'	ρ_2'	ρ_3'
x	y	z	x	y	x
y	z	x	y	z	z
z	x	y	z	x	y

Example 5.1.

Check that TC($\underline{\rho}$) = **A** and TC($\underline{\rho}$') = {x}. So, y ∈ TC($\underline{\rho}$) but y ∉ TC($\underline{\rho}$'), while the lower contour set of y is the same for all agents at both preference profiles, hence showing the non-monotonicity of the top cycle.

We now show that the equivalent hyperfunction of the top cycle is Nash implementable over the domain obtained through the following lexicographic extension axiom: Take any ρ ∈ Π and any two distinct X, Y ∈ $\underline{\mathbf{A}}$. First consider the case where #X = #Y = k for some k ∈{1,…,m-1}. Let, without loss of generality, X = {x_1,..., x_k} and Y = {y_1,..., y_k} such that x_j ρ x_{j+1} and y_j ρ y_{j+1} for all j ∈ {1,…,k-1}. We have X R Y if and only if x_h ρ y_h for the smallest h ∈ {1,…,k} such that $x_h \neq y_h$. Now consider the case where #X ≠ #Y. Let, without loss of generality, X = {x_1,..., $x_{\#X}$} and Y = {y_1,..., $y_{\#Y}$} such that x_j ρ x_{j+1} for all j ∈ {1,…, #X-1}and y_j ρ y_{j+1} for all j ∈ {1,…, #Y-1}. We have either $x_h = y_h$ for all h ∈ {1,…, min{#X, #Y}} or there exists some h ∈ {1,…, min{#X, #Y}} for which $x_h \neq y_h$. For the first case, we have X R Y if and only if #X < #Y. For the second case, X R Y if and only if x_h ρ y_h for the smallest h ∈ {1,…, min{#X, #Y}} such that $x_h \neq y_h$.

[10] Moreover, Miller (1977) shows that when $\tau(\underline{\rho})$ is antisymmetric, TC($\underline{\rho}$) is a singleton consisting of the unique Condorcet winner, when it exists; otherwise it is a set with at least three elements, over which there is a majority cycle.

Kaymak and Sanver (forthcoming) show that to each $\rho \in \Pi$, the lexicographic extension axiom determines a unique complete, transitive and antisymmetric ordering $\lambda(\rho)$ over **A**. We hence have $\Re^{\lambda} = \cup_{\rho \in \Pi} \{\lambda(\rho)\}$ as the domain determined through λ.

Proposition 5.1: Assume an odd number of agents with $\#N \geq 3$. The top cycle is Nash implementable through its equivalent hyperfunction defined over $\Re^{\lambda}$.

Proof: We will prove the proposition through Corollary 4.1. First check that λ satisfies the strong solidarity condition of Corollary 4.1 and leads to a domain $\Re^{\lambda}$ which is regular. It is clear that the top cycle satisfies the no veto power condition. To complete the proof, we must show that it is also group monotonic.

Take any $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $L(x; \rho_i) \subseteq L(x; \rho_i')$ for all $i \in \mathbf{N}$ and all $x \in TC(\underline{\rho})$. First note that for all $x \in TC(\underline{\rho})$ and for all $y \in \mathbf{A} \setminus TC(\underline{\rho})$, we have $x\ \tau(\underline{\rho})\ y \Rightarrow x\ \tau(\underline{\rho}')\ y$. Hence, $TC(\underline{\rho}') \subseteq TC(\underline{\rho})$. Note also that for all $x, y \in TC(\underline{\rho})$, we have $x\ \tau(\underline{\rho})\ y \Leftrightarrow x\ \tau(\underline{\rho}')\ y$, implying $TC(\underline{\rho}) \subseteq TC(\underline{\rho}')$, which in turn implies $TC(\underline{\rho}) = TC(\underline{\rho}')$, establishing the group monotonicity of the top cycle. Q.E.D.

We also wish to note that the top cycle fails to satisfy the strong group monotonicity condition of Corollary 4.2, disabling us to Nash implement it under milder extension maps, satisfying the (weaker) solidarity condition.[11]

[11] One can check this through the following three-by-three social choice problem with $\underline{\rho}, \underline{\rho}' \in \Pi^N$ as follows:

ρ_1	ρ_2	ρ_3	ρ_1'	ρ_2'	ρ_3'
x	y	z	x	y	z
y	z	x	y	z	y
z	x	y	z	x	x

We now consider the *majority rule* which picks the alternative considered as the best by a strict majority[12] when such alternative exists and picks all alternatives otherwise. For each $\underline{\rho} \in \Pi^N$, we define the majority rule $M : \Pi^N \rightarrow \underline{\mathbf{A}}$ as

$$M(\underline{\rho}) = \begin{cases} \{x\} & \text{if } \#\{i \in \mathbf{N} : L(x; \rho_i\} = \mathbf{A}\} > n/2 \\ \mathbf{A} & \text{otherwise} \end{cases}$$

One can check through Example 5.1 above that M is not Maskin monotonic. We now show that the equivalent hyperfunction of M is Nash implementable over the domain obtained through the following two extension axioms:

Our first axiom that we call A1 requires that the best element of a set must be preferred to the set itself. In other words, for all $\rho \in \Pi$, for all $R \in \kappa(\rho)$, and for all $X \in \underline{\mathbf{A}}$ with $\#X > 1$ we have $\{x\}$ P X where $x = \text{argmax}_X\, \rho$.

The next extension axiom, A2, assumes optimistic individuals who prefers a set X to a set Y if they prefers the best outcome of X to the best outcome of Y. In other words, for all $\rho \in \Pi$, for all $R \in \kappa(\rho)$, and for all $X, Y \in \underline{\mathbf{A}}$ we have :

X P Y whenever x p y where $X \subseteq L(x; \rho\}$ and $Y \subseteq L(y; \rho)$.

We denote κ for the consistency map which assigns to each ordering over alternatives, all orderings over sets which satisfy axioms A1 and A2, hence defining $\mathfrak{R}^\kappa = \cup_{\rho \in \Pi} \kappa(\rho)$.

Proposition 5.2: Let $\# N \geq 3$. Given the consistency map κ determined by axioms A1 and A2, the majority rule is Nash implementable through its equivalent hyperfunction defined over $\mathfrak{R}^\kappa$.

Proof: We will prove the proposition through Corollary 4.2. First check that κ satisfies the solidarity condition of Corollary 4.2 and leads to a domain $\mathfrak{R}^\kappa$ which is regular. It is clear that

[12] A strict majority is a coalition of agents whose cardinality exceeds its complement coalition.

the majority rule satisfies the no veto power condition. To complete the proof, we must show that it is also strongly group monotonic. To see this, take any $\underline{\rho}$, $\underline{\rho}' \in \Pi^N$ such that for all $i \in \mathbf{N}$, we have $L(x^*; \rho_i) \subseteq L(x^*; \rho_i')$ where $x^* = \text{argmax}_{M(\underline{\rho})} \rho_i$. We have to show that $M(\underline{\rho}) \subseteq M(\underline{\rho}')$. By definition of M, $M(\underline{\rho})$ is either a singleton or equals $\mathbf{A}$.

First consider the case where $M(\underline{\rho}) = \{x^*\}$ for some $x^* \in \mathbf{A}$. By definition of M, we have $\#\{i \in \mathbf{N} : L(x^*; \rho_i) = \mathbf{A}\} > n/2$. As $L(x^*; \rho_i) \subseteq L(x^*; \rho_i')$ for all $i \in \mathbf{N}$, we have $\#\{i \in \mathbf{N} : L(x^*; \rho'_i) = \mathbf{A}\} > n/2$ as well which, again by definition of M, implies $M(\underline{\rho}') = \{x^*\}$.

Now consider the case where $M(\underline{\rho}) = \mathbf{A}$. By definition of M, for any $x \in \mathbf{A}$, we have $\#\{i \in \mathbf{N} : L(x; \rho_i) = \mathbf{A}\} \leq n/2$. Recall that for all $i \in \mathbf{N}$, $L(x^*; \rho_i) \subseteq L(x^*; \rho_i')$ where $x^* = \text{argmax}_{\mathbf{A}} \rho_i$. Thus, for each $i \in \mathbf{N}$, $\text{argmax}_{\mathbf{A}} \rho_i = \text{argmax}_{\mathbf{A}} \rho_i'$, impliying $\#\{i \in \mathbf{N} : L(x; \rho'_i) = \mathbf{A}\} \leq n/2$ for all $x \in \mathbf{A}$, which, again by definition of M, implies $M(\underline{\rho}') = \mathbf{A}$. Hence, M is strongly group monotonic, completing the proof. Q.E.D.

As a further example, consider the *plurality rule* when we have only three voters. The plurality rule picks the alternatives considered as best by the highest number of voters. For each $\underline{\rho} \in \Pi^N$, we define the plurality rule $PL : \Pi^N \to \underline{\mathbf{A}}$ as the subset X of $\mathbf{A}$ such that for all $x \in X$ and for all $y \in \mathbf{A}$ we have $\#\{i \in N : L(x; \rho_i\} = \mathbf{A}\} \geq \#\{i \in N : L(y; \rho_i\} = \mathbf{A}\}$.

The plurality rule is a refinement of the majority rule. One can easily check that it is not Maskin monotonic, but strongly group monotonic when we have only three voters. Thus, we state the following result without proof.

Proposition 5.3: Let #N = 3. Given the consistency map κ determined by axioms A1 and A2, the plurality rule is Nash implementable through its equivalent hyperfunction defined over $\Re^\kappa$.

6. CONCLUDING REMARKS

Our results on Nash implementability are positive. We show that group monotonic social choice correspondences (satisfying the no veto power condition) can be Nash implemented via their equivalent hyperfunctions. The strenght of the group monotonicity condition depends on the strenght of the axioms used to extend preferences over alternatives to sets. In any case, it is not stronger than Maskin monotonicity. Thus, there are social choice rules which are not Maskin monotonic but group monotonic, which expands the set of Nash implementable social choice rules. In particular, the top cycle and qualified majority rules are Nash implementable via their equivalent hyperfunctions, while they fail to be Maskin monotonic.

This positive result is based on our postulating the equivalence between implementing a correspondence and its equivalent hyperfunction. This is an acceptable claim when outcomes of correspondences are conceived as a first refinement of the set of alternatives from which a final unique choice will be made.

We wish to close by mentioning the possibility of defining the concept of a minimal group monotonic extension, similar to the minimal monotonic extension of Sen (1995). Clearly, the minimal group monotonic extension of a social choice correspondence will be a subset of its minimal monotonic extension. Thus, non Maskin monotonic social choice rules can be Nash implemented at a "lower cost". So it would be interesting to look for the minimal group monotonic extensions of various social choice correspondences, in the same manner as Thomson (1999) does for minimal monotonic extensions.

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