

# Discretizing Evolutionary Games

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## **Abstract**

Here we examine the process of discretizing an evolutionary game. First, we change the replicator equation of game theory into a Lotka-Volterra population equation. Then we use the technique developed by Bettelheim, Agam and Shnerb [1] to turn the continuous Lotka-Volterra equation into a discrete version. This involves treating the Lotka-Volterra equation as a mean field equation and using techniques from quantum field theory and statistical mechanics.

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## 1 Game Theory Background

### 1.1 Introduction

Evolutionary game theory is concerned with how the games played between individuals in a large population affect the population itself. Since it is concerned with an average outcome of a large number of games, the concept of rational players can be dropped. That is, the individual players are no longer considered to be intelligent. This chapter is based on [5] and [7, Chapters 2 and 9]. It is perhaps best to start with an example of an evolutionary game.

**Example 1.1 (The Hawk-Dove Game)** *Assume we have a population that competes over food. Let us give a value of  $G$  to the gain an individual receives by*

eating a morsel of food. In the game that the individuals play, each player can employ one of two strategies. The Hawk strategy is to attack until one of the players is injured or retreats. We assign a cost of  $C > G$  to an injury. The Dove strategy is to bluff an attack hoping the other player retreats or, to retreat if the other player attacks.

Now, if a Hawk meets a Dove, the Hawk will attack and the Dove will retreat. The payoffs are  $G$  for the Hawk (it gets the food) and  $0$  for the Dove. If two Hawks meet we assume that each has a probability  $p = 1/2$  of injury. The average payoff to both players is then  $(G - C)/2$ . Two Doves who meet will bluff until one retreats. If each is equally likely to retreat, then the average payoff to both players is  $G/2$ . We represent this game in the following matrices:

$$M^{(1)} = \begin{bmatrix} \frac{G-C}{2} & G \\ 0 & \frac{G}{2} \end{bmatrix} \quad (1)$$

$$M^{(2)} = \begin{bmatrix} \frac{G-C}{2} & 0 \\ G & \frac{G}{2} \end{bmatrix} \quad (2)$$

The matrix  $M^{(i)}$  gives player  $i$ 's expected payoff in any game. (Note that  $M^{(2)} = (M^{(1)})^T$ , i.e.,  $M^{(2)}$  is the transpose of  $M^{(1)}$ . This is an example of a symmetric game, to be defined below.) A Hawk (Dove) strategy is given by the vector  $[1 \ 0]$  ( $[0 \ 1]$ ). Note, we will not distinguish explicitly between row and column vectors. To determine the payoffs, we represent player one's strategy as a row vector and player two's as a column vector. For example, assume that player one is a Dove and player two is a Hawk. Then the payoffs to each player are

$$\begin{aligned} u_1 &= [0 \ 1] M^{(1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 0 \\ u_2 &= [0 \ 1] M^{(2)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= G \end{aligned}$$

In general, we allow individuals in a population to use a Hawk strategy with probability  $h$  and a Dove strategy with probability  $d = (1 - h)$ . It is this situation that we will be most interested in. In particular, we want to know which  $h$ , if it exists, is "optimal" in a sense to be made precise below.

## 1.2 Game Theory

Every player in a game has a set of  $N^{(i)}$  **pure strategies**, denoted by  $\Phi^{(i)}$ . In the Hawk-Dove game, both players have the pure strategies given by Hawk and Dove, so, for player  $i$ ,  $\Phi^{(i)} = \{H^{(i)}, D^{(i)}\}$ . In addition, each player has a **payoff “matrix”**  $M^{(i)}$  where

$$M^{(i)} : \prod_{j=1}^n \Phi^{(j)} \rightarrow \mathcal{R}$$

when we have  $n$  players. The **normal form** of a game is given by

$$\Gamma \doteq \{\Phi^{(1)}, \dots, \Phi^{(n)}; M^{(1)}, \dots, M^{(n)}\} \quad (3)$$

The normal form tells us: 1) The number of players in a game; 2) Their respective strategies, and; 3) Their payoff matrices.

A **mixed strategy** is when a player assigns some probability  $p_j^{(i)}$  to each element  $\varphi_j^{(i)} \in \Phi^{(i)}$ . We let a mixed strategy be denoted by  $\mathbf{s}^{(i)} = [p_1^{(i)}, \dots, p_n^{(i)}]$ . This means that player  $i$  will use the pure strategy  $\varphi_j^{(i)}$  with probability  $p_j^{(i)}$ . The set of all possible strategies for player  $i$  is the simplex

$$S^{(i)} \doteq \left\{ \mathbf{s}^{(i)} \in \mathcal{R}^{N^{(i)}} : p_j^{(i)} \geq 0 \text{ and } \sum_{j=1}^{N^{(i)}} p_j^{(i)} = 1 \right\}$$

**Definition 1.2** The *support* of  $\mathbf{s}^{(i)} \in S^{(i)}$  is given by

$$\text{supp}(\mathbf{s}^{(i)}) \doteq \{\varphi_j^{(i)} : 1 \leq j \leq N^{(i)} \text{ and } p_j^{(i)} > 0\}$$

The *interior* of  $S^{(i)}$  is defined by

$$\text{int}(S^{(i)}) \doteq \{\mathbf{s}^{(i)} : p_j^{(i)} \neq 0 \text{ for all } 1 \leq j \leq N^{(i)}\}$$

The *boundary* of  $S^{(i)}$  is

$$\text{bd}(S^{(i)}) \doteq S^{(i)} \setminus \text{int}(S^{(i)})$$

The interior of  $S^{(i)}$  is the set of **completely mixed strategies**. The “corners” of  $S^{(i)}$  are the pure strategies of player  $i$ . The pure strategies are a subset of  $\text{bd}(S^{(i)})$ .

The set of **pure strategy combinations** of a game  $\Gamma$  is

$$\Phi \doteq \prod_{i=1}^n \Phi^{(i)}$$

Similarly, the set of all **mixed strategy combinations** is

$$S \doteq \prod_{i=1}^n S^{(i)}$$

Suppose each player uses the mixed strategy  $\mathbf{s}^{(i)}$ ,  $i = 1, \dots, n$ . We will let  $\mathbf{s} \in S$  denote the vector  $\mathbf{s} \doteq [\mathbf{s}^{(1)}, \dots, \mathbf{s}^{(n)}]$ . With these strategies, the probability that  $\varphi \doteq [\varphi_{j_1}^{(1)}, \dots, \varphi_{j_n}^{(n)}] \in \Phi$  is actually played is given by

$$s(\varphi) \doteq \prod_{i=1}^n p_{j_i}^{(i)}$$

Then the **expected payoff** for player  $i$ , given the vector  $\mathbf{s}$ , is

$$M^{(i)}(\mathbf{s}) \doteq \sum_{\varphi \in \Phi} s(\varphi) M^{(i)}(\varphi)$$

where  $M^{(i)}(\varphi)$  is the payoff to player  $i$  when  $\varphi \in \Phi$  is played.

**Example 1.3** *In the Hawk-Dove game we have that  $\Phi^{(1)} = \Phi^{(2)} = \{H, D\}$ . The payoff matrices were given in (1) and (2). So our normal form for this game is  $\Gamma = \{\Phi^{(1)}, \Phi^{(2)}; M^{(1)}, M^{(2)}\}$ . A typical mixed strategy for a player is  $\mathbf{s}^{(i)} = [h^{(i)}, 1 - h^{(i)}]$ , where  $0 \leq h^{(i)} \leq 1$  and we will let  $d^{(i)} = 1 - h^{(i)}$ . Then  $h^{(i)}$  ( $d^{(i)}$ ) gives the probability that player  $i$  will use the Hawk (Dove) strategy. This gives us*

$$\begin{aligned} S^{(i)} &= \left\{ \mathbf{s}^{(i)} \in \mathcal{R}^2 : h^{(i)}, d^{(i)} \geq 0 \text{ and } h^{(i)} + d^{(i)} = 1 \right\} \\ \Phi &= \left\{ H^{(1)}H^{(2)}, H^{(1)}D^{(2)}, D^{(1)}H^{(2)}, D^{(1)}D^{(2)} \right\} \\ S &= \left\{ \mathbf{s} \in \mathcal{R}^4 : s(\varphi_i) \geq 0 \text{ and } \sum_{i=1}^4 s(\varphi_i) = 1 \right\} \end{aligned}$$

If we let  $\varphi_1 \doteq H^{(1)}H^{(2)}$ ,  $\varphi_2 \doteq H^{(1)}D^{(2)}$ ,  $\varphi_3 \doteq D^{(1)}H^{(2)}$ , and  $\varphi_4 \doteq D^{(1)}D^{(2)}$ , then

$$\begin{aligned} s(\varphi_1) &= h^{(1)}h^{(2)} & s(\varphi_2) &= h^{(1)}d^{(2)} \\ s(\varphi_3) &= d^{(1)}h^{(2)} & s(\varphi_4) &= d^{(1)}d^{(2)} \end{aligned}$$

and  $\sum s(\varphi_i) = 1$ . It is easy to see that

$$M^{(i)}(\mathbf{s}) = \mathbf{s}^{(1)} \cdot M^{(i)}\mathbf{s}^{(2)}$$

where  $\mathbf{s}^{(1)} \cdot M^{(i)}\mathbf{s}^{(2)}$  is the inner product of  $\mathbf{s}^{(1)}$  with  $M^{(i)}\mathbf{s}^{(2)}$ .

Consider some  $\mathbf{s} \in S$ . Let us allow player  $i$  to freely choose any strategy  $\tilde{\mathbf{s}}^{(i)} \in S^{(i)}$  and define  $\mathbf{s} \setminus \tilde{\mathbf{s}}^{(i)} \doteq [\mathbf{s}^{(1)}, \dots, \mathbf{s}^{(i-1)}, \tilde{\mathbf{s}}^{(i)}, \mathbf{s}^{(i+1)}, \dots, \mathbf{s}^{(n)}]$ . That is,  $\mathbf{s} \setminus \tilde{\mathbf{s}}^{(i)}$  is the same as  $\mathbf{s}$  except that player  $i$ 's strategy  $\mathbf{s}^{(i)}$  is replaced by the strategy  $\tilde{\mathbf{s}}^{(i)}$ . From the strategies  $\tilde{\mathbf{s}}^{(i)} \in S^{(i)}$  there are some that are more advantageous to player  $i$ .

**Definition 1.4** We call a strategy  $\bar{\mathbf{s}}^{(i)} \in S^{(i)}$  a **best reply** of player  $i$  to  $\mathbf{s} \in S$  if

$$M^{(i)}(\mathbf{s} \setminus \bar{\mathbf{s}}^{(i)}) = \max_{\tilde{\mathbf{s}}^{(i)} \in S^{(i)}} M^{(i)}(\mathbf{s} \setminus \tilde{\mathbf{s}}^{(i)})$$

The set of all best replies for player  $i$  against  $\mathbf{s}$  is denoted by  $\beta^{(i)}(\mathbf{s})$ . The set of **pure best replies** is given by

$$B^{(i)}(\mathbf{s}) \doteq \{\varphi_j^{(i)} : \varphi_j^{(i)} \in \beta^{(i)}(\mathbf{s})\}$$

We then have that

$$\bar{\mathbf{s}}^{(i)} \in \beta^{(i)}(\mathbf{s}) \Leftrightarrow \text{supp}(\bar{\mathbf{s}}^{(i)}) \subset B^{(i)}(\mathbf{s}) \quad (4)$$

Let  $\mathbf{s}, \bar{\mathbf{s}} \in S$ . Now,  $\bar{\mathbf{s}}$  is called a **best reply** against  $\mathbf{s}$  if and only if  $\bar{\mathbf{s}}^{(i)} \in \beta^{(i)}(\mathbf{s})$  for all  $i$ . The set of **pure best replies** is given by

$$B(\mathbf{s}) \doteq \prod_{i=1}^n B^{(i)}(\mathbf{s})$$

The set of all best replies is denoted by

$$\beta(\mathbf{s}) \doteq \prod_{i=1}^n \beta^{(i)}(\mathbf{s})$$

There should be no confusion in calling  $\bar{\mathbf{s}}^{(i)}$  or  $\bar{\mathbf{s}}$  a best reply– the meaning should always be clear from the context. The same comment holds for  $\beta^{(i)}(\mathbf{s})$  and  $B^{(i)}(\mathbf{s})$  versus  $\beta(\mathbf{s})$  and  $B(\mathbf{s})$ , respectively. Note that  $B(\mathbf{s}) \in \Phi$  and  $\beta(\mathbf{s}) \in S$ .

An important case is when  $\mathbf{s} \in \beta(\mathbf{s})$ . That is, when  $\mathbf{s}$  is a best reply to itself. It is readily apparent from (4) that

$$\mathbf{s} \in \beta(\mathbf{s}) \Leftrightarrow \text{supp}(\mathbf{s}) \subset B(\mathbf{s})$$

**Definition 1.5** Given a game  $\Gamma$ , we call  $\mathbf{s} \in S$  a **Nash equilibrium (NE)** if  $\mathbf{s} \in \beta(\mathbf{s})$ . We denote the set of Nash equilibria by  $NE(\Gamma)$ , for the game  $\Gamma$ . If  $B(\mathbf{s}) = \{\mathbf{s}\}$ , we call  $\mathbf{s}$  a **strict Nash equilibrium**. If  $\text{supp}(\mathbf{s}) = B(\mathbf{s})$ , then  $\mathbf{s}$  is called a **quasi-strict Nash equilibrium**. We denote the strict and quasi-strict Nash equilibria by  $NE_s(\Gamma)$  and  $NE_q(\Gamma)$ , respectively.

A quasi-strict NE requires that every player assigns some positive probability to all of their best replies. A strategy  $\mathbf{s}$  is a strict NE if and only if it is the unique best reply to itself. Then  $\mathbf{s} \in \Phi$ , which means that each player has one, and only one, pure strategy best reply to  $\mathbf{s}$ . That is,  $\beta^{(i)}(\mathbf{s}) = \{\varphi_{j_i}^{(i)}\} = B^{(i)}(\mathbf{s})$ , for all  $i$ . Hence,  $\beta(\mathbf{s}) = \{\varphi_j\} = B(\mathbf{s})$  where  $\varphi_j = [\varphi_{j_1}^{(1)}, \dots, \varphi_{j_n}^{(n)}]$ .

**Example 1.6** For the Hawk-Dove game, every  $\mathbf{s} \in S$  is given by  $\mathbf{s} = [\mathbf{s}^{(1)}, \mathbf{s}^{(2)}]$  where  $\mathbf{s}^{(i)} = [h^{(i)}, 1 - h^{(i)}]$ . Then  $\mathbf{s} \setminus \tilde{\mathbf{s}}^{(i)}$  represents the situation where we allow player  $i$  to vary  $h^{(i)}$  while the other player's  $h^{(j)}$  is held constant. To find player one's best replies, we need to find the  $\bar{h}^{(1)}$ 's such that

$$\begin{bmatrix} \bar{h}^{(1)} & 1 - \bar{h}^{(1)} \end{bmatrix} \begin{bmatrix} \frac{G-C}{2} & G \\ 0 & \frac{G}{2} \end{bmatrix} \begin{bmatrix} h^{(2)} \\ 1 - h^{(2)} \end{bmatrix}$$

is maximized for a fixed  $h^{(2)}$ . Now

$$\begin{aligned} \begin{bmatrix} \frac{G-C}{2} & G \\ 0 & \frac{G}{2} \end{bmatrix} \begin{bmatrix} h^{(2)} \\ 1 - h^{(2)} \end{bmatrix} &= \begin{bmatrix} \frac{G}{2} (1 - h^{(2)}) + \frac{G}{2} (1 - h^{(2)}) \frac{C}{G} \\ \frac{G}{2} (1 - h^{(2)}) \end{bmatrix} \\ &:= \begin{bmatrix} A^{(1)} \\ B^{(1)} \end{bmatrix} \end{aligned}$$

If  $A^{(1)} > B^{(1)}$ , then player one should use the Hawk strategy. So  $A^{(1)} > B^{(1)} \Rightarrow \bar{h}^{(1)} = 1$ . Similarly,  $A^{(1)} < B^{(1)} \Rightarrow \bar{h}^{(1)} = 0$  and  $A^{(1)} = B^{(1)} \Rightarrow \bar{h}^{(1)} \in [0, 1]$ .

Now,  $A^{(1)} \geq B^{(1)} \Leftrightarrow G/C \geq h^{(2)}$ . Then, since  $C > G$ ,

$$\begin{aligned} \beta^{(1)}(\mathbf{s}) = B^{(1)} = \{H\} & \quad \text{if } G/C > h^{(2)} \\ \beta^{(1)}(\mathbf{s}) = B^{(1)} = \{D\} & \quad \text{if } G/C < h^{(2)} \\ \beta^{(1)}(\mathbf{s}) = S^{(1)}, B^{(1)} = \{H, D\} & \quad \text{if } G/C = h^{(2)} \end{aligned}$$

By the symmetry of the game, we can let  $\bar{h}^{(1)} \rightarrow \bar{h}^{(2)}$  and  $h^{(2)} \rightarrow h^{(1)}$  in the above analysis to find player two's best replies

$$\beta^{(2)}(\mathbf{s}) = B^{(2)} = \{H\} \quad \text{if } G/C > h^{(1)}$$



$$\begin{aligned} \beta^{(2)}(\mathbf{s}) &= B^{(2)} = \{D\} && \text{if } G/C < h^{(1)} \\ \beta^{(2)}(\mathbf{s}) &= S^{(2)}, B^{(2)} = \{H, D\} && \text{if } G/C = h^{(1)} \end{aligned}$$

Now we will find the NE's of the Hawk-Dove game. If  $h^{(2)} < G/C$  then  $\beta^{(1)}(\mathbf{s}) = \{H\}$ . This implies that  $h^{(1)} = 1 > G/C$  so that  $\beta^{(2)}(\mathbf{s}) = \{D\}$ . Hence,  $h^{(2)} = 0 < G/C$ . So,  $(H, D)$  is the unique best reply to itself. Also, it is easy to show that  $(D, H)$  is a second NE. Note that  $(H, D)$  and  $(D, H)$  are strict NE's. These equilibria tell us that it is always best to use the opposite pure strategy of your opponent's pure strategy. (In the context of evolutionary game theory, these equilibria are irrelevant since we will not allow a player to condition their strategy on the other player's strategy.)

A third NE is when  $h^{(1)} = h^{(2)} = G/C$ . This is a quasi-strict, but not strict, NE. Since this equilibrium is a **symmetric NE** (i.e., it does not depend on which player is called player one, for example), we may expect a population to adopt this strategy. It is this idea that we now examine.

### 1.3 Evolutionarily Stable Strategies

Evolutionary game theory largely examines games between two players. These games, called **bimatrix games**, are given by  $\Gamma = \{\Phi^{(1)}, \Phi^{(2)}; M^{(1)}, M^{(2)}\}$ . We say that  $\Gamma$  is a **symmetric game** if the conditions  $\Phi^{(1)} = \Phi^{(2)}$  and  $M^{(1)} = (M^{(2)})^T$  hold. This is the case we will be most interested in. For symmetric games,  $\Phi^{(2)}$  and  $M^{(2)}$  are redundant. For example, if player one uses strategy  $\mathbf{q}$  and player two uses  $\mathbf{p}$ , we have for player two's payoff

$$\begin{aligned} \mathbf{q} \cdot M^{(2)} \mathbf{p} &= \mathbf{p} \cdot (M^{(2)})^T \mathbf{q} \\ &= \mathbf{p} \cdot M^{(1)} \mathbf{q} \end{aligned}$$

Each player then uses the same pure strategy space  $\phi = \Phi^{(1)} = \{\phi_1, \dots, \phi_m\}$ . The payoff of one pure strategy versus another is given by the corresponding element of the **fitness matrix**  $M = M^{(1)}$ . So strategy  $\phi_i$  versus  $\phi_j$  has payoff  $\phi_i \cdot M \phi_j = M_{ij}$  and strategy  $\phi_j$  versus  $\phi_i$  has payoff  $M_{ji}$ . All of this allows us to identify the symmetric game  $\Gamma$  in (3) with the fitness matrix  $M$ . Hence, instead of referring to the game  $\Gamma = \{\Phi^{(1)}, \Phi^{(2)}; M^{(1)}, M^{(2)}\}$ , we will, in this case, refer to the game  $M$  instead.

When we examined the Hawk-Dove game above, we found a symmetric NE. We would like to think that the population would settle to this strategy *and* would

be stable. By stable we mean that a mutant strategy can not invade a population using the NE. But, since the NE was not strict, it is not obvious that it is stable against invasion. The invaders could use a strategy that does as well as the NE strategy against the NE strategy. It is apparent that we need to refine our concept of an equilibrium so that it reflects our intuition about stability. We do this presently.

Suppose a population  $Q$  consists of two types of individuals,  $I$  and  $J$ . Then  $Q = xJ + (1 - x)I$  where  $x$  and  $(1 - x)$  are the **frequencies** of  $J$  and  $I$  individuals, respectively. Now, let  $I$  use the strategy  $\hat{\mathbf{p}}$  and  $J$  the strategy  $\mathbf{p}$ . Then  $Q$  corresponds to the strategy mixture  $x\mathbf{p} + (1 - x)\hat{\mathbf{p}}$ . Letting the simplex

$$S^m \doteq \left\{ \mathbf{p} \in \mathcal{R}^m : p_j \geq 0 \text{ and } \sum_{j=1}^m p_j = 1 \right\}$$

be the set of all possible strategies in our game  $M$ , we make the following definition.

**Definition 1.7**  $\hat{\mathbf{p}} \in S^m$  is *evolutionarily stable* if, for all  $\mathbf{p} \in S^m$ ,  $\mathbf{p} \neq \hat{\mathbf{p}}$ , and all  $\epsilon(\hat{\mathbf{p}}) > 0$  sufficiently small, we have that

$$\mathbf{p} \cdot M(\epsilon\mathbf{p} + (1 - \epsilon)\hat{\mathbf{p}}) < \hat{\mathbf{p}} \cdot M(\epsilon\mathbf{p} + (1 - \epsilon)\hat{\mathbf{p}}) \quad (5)$$

In (5) we have, after letting  $\epsilon \rightarrow 0$ ,

$$\mathbf{p} \cdot M\hat{\mathbf{p}} \leq \hat{\mathbf{p}} \cdot M\hat{\mathbf{p}}$$

That is,  $\hat{\mathbf{p}}$  is a best reply to itself— so  $\hat{\mathbf{p}}$  is a symmetric NE. We can also rewrite (5) as

$$(1 - \epsilon)(\hat{\mathbf{p}} \cdot M\hat{\mathbf{p}} - \mathbf{p} \cdot M\hat{\mathbf{p}}) + \epsilon(\hat{\mathbf{p}} \cdot M\mathbf{p} - \mathbf{p} \cdot M\mathbf{p}) > 0 \quad (6)$$

So the fact that  $\hat{\mathbf{p}}$  is a NE is not enough to guarantee that it is evolutionarily stable (i.e., that it can not be invaded). We need to add a second condition that  $\hat{\mathbf{p}}$  must satisfy. Equation (6) motivates the following definition.

**Definition 1.8** A strategy  $\hat{\mathbf{p}} \in S^m$  is an *evolutionarily stable strategy (ESS)* if it satisfied the following two conditions:

- *equilibrium condition*

$$\mathbf{p} \cdot M\hat{\mathbf{p}} \leq \hat{\mathbf{p}} \cdot M\hat{\mathbf{p}}$$

for all  $\mathbf{p} \in S^m$ . That is,  $\hat{\mathbf{p}}$  is a symmetric NE.

- *stability condition*

$$\mathbf{p} \cdot M\mathbf{p} < \hat{\mathbf{p}} \cdot M\mathbf{p}$$

when  $\mathbf{p} \neq \hat{\mathbf{p}}$  and  $\mathbf{p} \cdot M\hat{\mathbf{p}} = \hat{\mathbf{p}} \cdot M\hat{\mathbf{p}}$ . That is,  $\hat{\mathbf{p}}$  is stable against invasion.

The stability condition states that when  $\mathbf{p}$  and  $\hat{\mathbf{p}}$  fare equally well against  $\hat{\mathbf{p}}$ ,  $\hat{\mathbf{p}}$  will fare better against  $\mathbf{p}$  than  $\mathbf{p}$  will. This seems reasonable. If  $\mathbf{p}$  tries to invade,  $\hat{\mathbf{p}}$  will *on average* fare better in the perturbed population  $Q = \epsilon J + (1 - \epsilon)I$ . Thus, the  $I$  individuals' fitness will be higher than the  $J$  individuals' fitness, and they will be able to resist the invasion by  $J$ .

We may ask when an ESS exists? (Note, if  $\hat{\mathbf{p}}$  is a strict NE then it is an ESS since the stability condition is already met.) Let us first state a theorem proved by Nash [6].

**Theorem 1.9 (Nash, 1951)** *Every symmetric bimatrix game has a symmetric NE given by  $(\hat{\mathbf{p}}, \hat{\mathbf{p}})$ ,  $\hat{\mathbf{p}} \in S^m$ .*

So the equilibrium condition is not much of a restriction. It seems intuitive that we should be able to characterize the ESS by some form of stability condition. This is in fact possible.

**Theorem 1.10** *The strategy  $\hat{\mathbf{p}} \in S^m$  is an ESS of the game  $M$  if and only if there exists a neighborhood  $U$  of  $\hat{\mathbf{p}}$ ,  $U \subset S^m$ , such that*

$$\mathbf{p} \cdot M\mathbf{p} < \hat{\mathbf{p}} \cdot M\mathbf{p}$$

for all  $\mathbf{p} \in U$  with  $\mathbf{p} \neq \hat{\mathbf{p}}$ .

The final question we will ask is when is an ESS unique? The following theorem gives us a partial answer to this question.

**Theorem 1.11** *Let  $\hat{\mathbf{p}}$  be an ESS of the game  $M$  and  $(\mathbf{p}, \mathbf{p})$  be a symmetric NE. If  $\text{supp}(\mathbf{p}) \subset B(\hat{\mathbf{p}})$  then  $\mathbf{p} = \hat{\mathbf{p}}$ .*

We can see this by assuming that  $(\mathbf{p}, \mathbf{p})$  and  $(\hat{\mathbf{p}}, \hat{\mathbf{p}})$ ,  $\mathbf{p} \neq \hat{\mathbf{p}}$ , are both symmetric NE of the game  $M$ . Now let  $\text{supp}(\mathbf{p}) \subset B(\hat{\mathbf{p}})$  which means that  $\mathbf{p}$  is a best

reply against itself and  $\hat{\mathbf{p}}$ . So  $\mathbf{p}$  and  $\hat{\mathbf{p}}$  fare equally well against  $\hat{\mathbf{p}}$ , and  $\mathbf{p}$  fares at least as well against  $\mathbf{p}$  as  $\hat{\mathbf{p}}$  does. Hence, the stability condition is not met. So  $\hat{\mathbf{p}}$  is not an ESS of the game  $M$ . Now take the contrapositive of this to prove the theorem. This gives us a partial uniqueness corollary. Assume  $\hat{\mathbf{p}} \in \text{int}(S^m)$ , so that  $\text{supp}(\hat{\mathbf{p}}) = \phi$ . This implies that  $B(\hat{\mathbf{p}}) = \phi$ . So  $\text{supp}(\mathbf{p})$  is always a subset of  $B(\hat{\mathbf{p}})$ . Then theorem 1.11 holds, and we have

**Corollary 1.12** *If  $\hat{\mathbf{p}} \in \text{int}(S^m)$  is an ESS of the game  $M$ , then it is the unique ESS.*

**Example 1.13** *Now we examine the symmetric NE of the Hawk-Dove game. The equilibrium condition is already met. With  $h^{(2)} = G/C$ , we have  $\hat{\mathbf{p}} = [G/C (1 - G/C)]$ , so that*

$$\begin{bmatrix} \frac{G-C}{2} & G \\ 0 & \frac{G}{2} \end{bmatrix} \begin{bmatrix} \frac{G}{C} \\ 1 - \frac{G}{C} \end{bmatrix} = \begin{bmatrix} \frac{G}{2}(1 - \frac{G}{C}) \\ \frac{G}{2}(1 - \frac{G}{C}) \end{bmatrix}$$

*Then, for every  $\mathbf{p} \in S^m$ ,  $\mathbf{p} \neq \hat{\mathbf{p}}$ , we have that  $\mathbf{p} \cdot M\hat{\mathbf{p}} = \hat{\mathbf{p}} \cdot M\hat{\mathbf{p}}$ . Let  $\mathbf{p} = [p_1 (1 - p_1)]$ . It follows that*

$$\begin{aligned} \mathbf{p} \cdot M\mathbf{p} &= -\frac{C}{2}p_1^2 + \frac{G}{2} \\ \hat{\mathbf{p}} \cdot M\mathbf{p} &= -Gp_1 + \frac{G}{2} + \frac{G^2}{2C} \end{aligned}$$

so

$$\mathbf{p} \cdot M\mathbf{p} - \hat{\mathbf{p}} \cdot M\mathbf{p} = -\frac{C}{2}p_1^2 + Gp_1 - \frac{G^2}{2C}$$

*We need to check that*

$$-\frac{C}{2}p_1^2 + Gp_1 - \frac{G^2}{2C} < 0$$

*or, equivalently, if*

$$p_1^2 - 2\frac{G}{C}p_1 + \left(\frac{G}{C}\right)^2 > 0$$

*But,*

$$\begin{aligned} p_1^2 - 2\frac{G}{C}p_1 + \left(\frac{G}{C}\right)^2 &= \left(p_1 - \frac{G}{C}\right)^2 \\ &> 0 \end{aligned}$$

for all  $\mathbf{p} \neq \hat{\mathbf{p}}$ . Hence,  $\hat{\mathbf{p}}$  is an ESS of the game  $M$ . Since  $\hat{\mathbf{p}} \in \text{int}(S^m)$ , it is the unique ESS.

As we stated earlier, we would expect the population playing the Hawk-Dove game to utilize the strategy  $\hat{\mathbf{p}}$  in example 1.13. Not only will it receive an optimal payoff from the game, but it will be stable against invasion also. What if the population is not utilizing  $\hat{\mathbf{p}}$ ? How can it evolve to the strategy  $\hat{\mathbf{p}}$ ? We turn our attention to these questions in Section 2.

## 2 Replicator Dynamics

### 2.1 Introduction

So far we have only examined static games between two players. The concept of an ESS implies that some sort of dynamics is occurring in the population though. Here we will add in temporal dynamics in order to achieve a more realistic model of games that are repeatedly played. We then examine the connections between the game theoretic (frequency) dynamics and the dynamics arising from a density model.

### 2.2 The Replicator (Frequency) Equation

In a game  $M$ , we can imagine having strategy types  $\mathbf{p}_i \in S^m$ . We let  $x_i$  denote the **frequency** of the  $i$ -type individuals in a population, where  $x_1 + \dots + x_n = 1$ . Letting  $\mathbf{x} = [x_1, \dots, x_n]$ , the **state** of a population is given by  $\mathbf{x} \in S^n$ . If an  $i$ -type individual plays against a  $j$ -type individual, the payoff to the  $i$ -type player is given by  $A_{ij} = \mathbf{p}_i \cdot M\mathbf{p}_j$ . We can then construct a new fitness matrix  $A$  from the underlying game  $M$ .

Given a population state  $\mathbf{x}$  and a fitness matrix  $A$ , the fitness (i.e., expected payoff) of an  $i$ -type individual is  $(A\mathbf{x})_i$ , the  $i$ th row of  $A\mathbf{x}$ . The average fitness is  $\mathbf{x} \cdot A\mathbf{x}$ . Assuming  $i$ -types only produce  $i$ -type offsprings, the frequency of the  $i$ -types in the next generation will be (by basic Darwinian theory)

$$x_i' = x_i \frac{(A\mathbf{x})_i}{\mathbf{x} \cdot A\mathbf{x}}$$

or

$$x'_i - x_i = x_i \frac{(Ax)_i - \mathbf{x} \cdot A\mathbf{x}}{\mathbf{x} \cdot A\mathbf{x}} \quad (7)$$

If we assume the changes in  $x_i$  are small between generations, we can replace (7) with

$$\dot{x}_i = x_i \frac{(Ax)_i - \mathbf{x} \cdot A\mathbf{x}}{\mathbf{x} \cdot A\mathbf{x}} \quad (8)$$

Equation (8) can be further simplified by using the following theorem.

**Theorem 2.1** *Let  $B(\mathbf{x}, t) > 0$ . Then the solutions  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$  and  $\dot{\mathbf{x}}' = B(\mathbf{x}', t)\mathbf{f}(\mathbf{x}', t)$  can be mapped into each other through a change in the time variable,  $\tau = \theta(t)$ , where  $\theta$  is a strictly monotonic function. If  $B$  and  $\mathbf{f}$  are time-independent, then the two systems have the same orbits.*

So we finally have that our **replicator (frequency) equation**

$$\dot{x}_i = x_i [(Ax)_i - \mathbf{x} \cdot A\mathbf{x}] \quad (9)$$

has the same trajectories as (8). From (9), note that  $x_i = 0 \Rightarrow \dot{x}_i = 0$ . So a strategy type that is extinct remains so. Also, it is easy to show that  $\dot{x}_1 + \dots + \dot{x}_n = 0$ . Then  $\mathbf{x} \in S^n$  for all time.

We can define the symmetric Nash equilibria and evolutionarily stable *states* as before.

**Definition 2.2** *A state  $\hat{\mathbf{x}} \in S^n$  is a **Nash Equilibrium (NE)** of the game  $A$  if, for all  $\mathbf{x} \in S^n$*

$$\mathbf{x} \cdot A\hat{\mathbf{x}} \leq \hat{\mathbf{x}} \cdot A\hat{\mathbf{x}}$$

*We call  $\hat{\mathbf{x}}$  an **evolutionarily stable state (ESS)** if, for all  $\mathbf{x}$  in some neighborhood  $U \subset S^n$  of  $\hat{\mathbf{x}}$ ,  $\mathbf{x} \neq \hat{\mathbf{x}}$ , we have*

$$\mathbf{x} \cdot A\mathbf{x} < \hat{\mathbf{x}} \cdot A\mathbf{x}$$

We will use ESS to denote both evolutionarily stable *strategies* and *states*. This should not cause confusion as the meaning will be clear from the context.

Before stating some results about the NE's and ESS's of a game  $A$ , we need to define some stability concepts.

**Definition 2.3** A solution  $\mathbf{u}(t)$  of a differential equation is **Lyapunov stable** if, for a given  $\epsilon > 0$ , there exists a  $\delta(\epsilon) > 0$  such that, if the solution  $\mathbf{v}(t)$  satisfies  $\|\mathbf{u} - \mathbf{v}\| < \delta$  at  $t = t_0$ , then  $\|\mathbf{u} - \mathbf{v}\| < \epsilon$  for every  $t > t_0$ . If, in addition, we have

$$\lim_{t \rightarrow \infty} \|\mathbf{u} - \mathbf{v}\| \rightarrow 0$$

then  $\mathbf{u}(t)$  is called **asymptotically stable**.

The following theorem collects some basic results concerning ESS's.

**Theorem 2.4** The following results hold for  $\hat{\mathbf{x}}$ :

1. If  $\hat{\mathbf{x}} \in S^n$  is a NE of the game  $A$ , then  $\hat{\mathbf{x}}$  is a fixed point of (9).
2. If  $\hat{\mathbf{x}}$  is a Lyapunov stable solution of (9), then  $\hat{\mathbf{x}}$  is a NE of the game  $A$ .
3. If  $\hat{\mathbf{x}}$  is an ESS of the game  $A$ , then it is an asymptotically stable fixed point of (9).
4. If  $\hat{\mathbf{x}} \in \text{int}(S^n)$  is an ESS of the game  $A$ , then it is a globally stable fixed point of (9).

The **mean population strategy** is given by  $\tilde{\mathbf{p}} \doteq \sum x_i \mathbf{p}_i$ . Now consider some other average population strategy  $\hat{\mathbf{p}} = \sum y_i \mathbf{p}_i$ . We call  $\hat{\mathbf{p}}$  **strongly stable** if, given an  $\epsilon > 0$ ,  $\tilde{\mathbf{p}} \rightarrow \hat{\mathbf{p}}$  under the evolution given in (9) whenever  $\|\hat{\mathbf{p}} - \tilde{\mathbf{p}}\| < \epsilon$  at  $t = t_0$ . That is, as long as the mean population strategy starts sufficiently close to  $\hat{\mathbf{p}}$ , it will converge to  $\hat{\mathbf{p}}$  provided  $\hat{\mathbf{p}}$  is a possible mean population strategy. The concept of strong stability formalizes the behavior that we intuitively would want an ESS to have. We end this section with the following important theorem.

**Theorem 2.5**  $\hat{\mathbf{p}} \in S^n$  is an ESS of the game  $A$  if and only if it is strongly stable under the evolution given in (9).

## 2.3 The Lotka-Volterra (Density) Equation

The last section dealt with the evolution of the frequencies in a population. Here we show that this is closely connected to the evolution of the population densities

(to be defined). It is this connection that we will be most concerned about. The replicator equation (9) can be mapped onto the Lotka-Volterra equation (to be defined below). First note that (9) is invariant under the addition of a constant to any column of  $A$ . So we can always assume that  $A$  has all zeros in the last row. We define new variables

$$y_i \doteq \frac{x_i}{x_n} \quad (10)$$

for  $i = 1, \dots, n$ . Since (10) gives us

$$x_i = \frac{y_i}{\sum_{j=1}^n y_j}$$

we refer to the  $y_i$  as the **densities** of the population. Equation (10) leads to

$$\begin{aligned} \dot{y}_i &= \frac{d}{dt} \left( \frac{x_i}{x_n} \right) \\ &= \left( \frac{x_i}{x_n} \right) [(A\mathbf{x})_i - (A\mathbf{x})_n] \\ &= y_i (A\mathbf{x})_i \\ &= y_i (A\mathbf{y})_i x_n \end{aligned} \quad (11)$$

Employing Theorem 2.1, we have that

$$\begin{aligned} \dot{y}_i &= y_i (A\mathbf{y})_i \\ &= y_i \left( A_{in} + \sum_{j=1}^{n-1} A_{ij} y_j \right) \end{aligned} \quad (12)$$

where, in the second equality, we used the fact that  $y_n = 1$  and  $A_{ni} = 0$ . Equation (12) is the **Lotka-Volterra (density) equation**. While we will not explicitly represent  $y_n$  in (12), it is very important that we remember that *there is always an "extra variable"*,  $y_n = 1$ , associated with the density equation. Since  $y_n = 1$ , we can define a new matrix  $A_I := A_{ij}$ ,  $i, j = 1, \dots, n-1$ . We call  $A_I$  the **interaction matrix**. For simplicity, rewrite (12) as

$$\dot{y}_i = y_i [r_i + (A_I \mathbf{y})_i] \quad (13)$$

where  $i = 1, \dots, n-1$  and  $r_i \doteq A_{in}$ .

Let us review what we have done so far. We started with a static game that individuals in a population play. This game naturally leads to a dynamical system



for the evolution of the strategies in a population (i.e., the replicator equation). By defining new variables  $y_i$  that give the relative frequencies of the  $\mathbf{p}_i$  strategies versus the  $\mathbf{p}_n$  strategy, we mapped the replicator equation onto the Lotka-Volterra equation. As we will now show, the Lotka-Volterra equation naturally arises from a completely different way of analyzing populations.

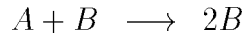
Consider two competing populations: a predator population  $y_1$ , and a prey population  $y_2$ . In the absence of prey, the predator's population will decline by a rate given by  $r_1 + A_{11}y_1$ , where  $r_1, A_{11} \leq 0$ . When prey are present, the predator's population increases by a rate  $A_{12}y_2$ ,  $A_{12} \geq 0$ . When predators are not present, the prey's population increases (or decreases) by a factor of  $r_2 + A_{22}y_2$ ,  $r_2 \geq 0$  and  $A_{22} \leq 0$ . The  $A_{22}$  represents the prey's intraspecific competition for food. Predators will reduce the prey population with rate  $A_{21}y_1$ ,  $A_{21} \leq 0$ . This gives us the equations

$$\dot{y}_1 = y_1 [r_1 + A_{11}y_1 + A_{12}y_2] \quad (14)$$

$$\dot{y}_2 = y_2 [r_2 + A_{21}y_1 + A_{22}y_2] \quad (15)$$

which is just (13) for  $n = 2$ .

It is difficult to give a game theoretic interpretation to this derivation of the Lotka-Volterra equation. The two populations are not really competing over anything like food. Rather, the prey are trying to avoid the predators and the predators are trying to capture the prey. Also, underlying (14) and (15) is a certain discretized situation. If a predator captures the prey, the prey's population is reduced by a definite amount, namely one. We can also imagine that this consumption of prey leads to an increase in the reproductive success of the predator. For example, maybe the consumption of one prey leads to an additional predator in the next generation. This is perhaps clearer if we think not about populations but, instead, think of a simple chemical reaction [4, page 7]. Let two chemicals  $A$  and  $B$  react as



If  $a$  and  $b$  are the concentrations of  $A$  and  $B$ , respectively, we have, as an approximation, that

$$\dot{a} = a[-b] \quad (16)$$

$$\dot{b} = b[a] \quad (17)$$

Equations (16) and (17) are a special case of (14) and (15). The discrete nature of the underlying reaction should now be clear.

Let us return to the Hawk-Dove game. Using (1), our matrix will be given by

$$M' = \begin{bmatrix} \frac{G-C}{2} & \frac{G}{2} \\ 0 & 0 \end{bmatrix} \quad (18)$$

Then our Lotka-Volterra equation is

$$\dot{y}_1 = y_1 \left[ \frac{G}{2} + \frac{G-C}{2} y_1 \right] \quad (19)$$

$$y_2 \equiv 1 \quad (20)$$

where  $y_1$  is the relative frequency of the Hawk strategy in a population. Equation (19) is the smooth approximation of the discrete process

$$y_1 \xrightarrow{\frac{G}{2}} 2y_1 \quad (21)$$

$$y_1 + y_1 \xrightarrow{\frac{C-G}{2}} \emptyset \quad (22)$$

What happens to  $x_1$  and  $x_2$  if the process in (21) occurs? We have

$$y_1 \longrightarrow y_1' = y_1 + 1 \quad (23)$$

Then,

$$x_1 \longrightarrow x_1' = \frac{y_1'}{1 + y_1'} \quad (24)$$

$$= \frac{y_1 + 1}{y_1 + 2}$$

$$x_2 \longrightarrow x_2' = \frac{1}{1 + y_1'} \quad (25)$$

$$= \frac{1}{y_1 + 2}$$

So, if  $y_1$  is initially small, an increase in  $y_1$  leads to a relatively large increase (decrease) in  $x_1$  ( $x_2$ ). For  $y_1$  large, any change in  $y_1$  results in a small change in  $x_1$  and  $x_2$ .

### 3 Discretizing Evolutionary Games

As we have previously stated, the density equations are a smooth approximation to the true dynamics. The same can be said about the frequency equations because we assumed that the changes in  $x_i$  were small when deriving (8). In this section we examine the effects of this discretization.

The discretization effects on the predator-prey model

$$\dot{y}_1 = [-\mu + \lambda y_2]y_1 \quad (26)$$

$$\dot{y}_2 = [\sigma - \lambda y_1]y_2 \quad (27)$$

$\mu, \sigma, \lambda > 0$ , have already been explored in [1] (see also [2]). Here we summarize some of the results obtained in this procedure. The first step is to derive the Master equations for the predator-prey model

$$y_1 \xrightarrow{\mu} \emptyset \quad (28)$$

$$y_2 \xrightarrow{\sigma} 2y_2 \quad (29)$$

$$y_1 + y_2 \xrightarrow{\lambda} 2y_1 \quad (30)$$

Let  $P_{m,n}$  be the probability that there are  $m$  predators and  $n$  prey. Then the rate of change in  $P_{m,n}$  is the rate at which probability “flows into” this state less the rate at which probability “flows out” of this state. The Master equations give the equations for the changes in the  $P_{m,n}$ ’s. For (28) to (30) we have

$$\begin{aligned} \dot{P}_{m,n} = & -(\mu m + \sigma n + \lambda mn)P_{m,n} + \mu(m+1)P_{m+1,n} \\ & + \sigma(n-1)P_{m,n-1} + \lambda(m-1)(n+1)P_{m-1,n+1} \end{aligned} \quad (31)$$

The first term in (31) gives the probability of leaving state  $P_{m,n}$ . This can happen through annihilation (i.e., death) of a predator, creation (i.e., birth) of a prey, or consumption of a prey by a predator resulting in the birth of a new predator (equations (28), (29) and (30), respectively). The other three terms in (31) give the probabilities of entering state  $P_{m,n}$  through the reactions in (28) through (30).

Now let  $|m, n\rangle$  be the state where there are  $m$  predators and  $n$  prey. We define

$$|\psi\rangle \doteq \sum_{m,n} P_{m,n} |m, n\rangle \quad (32)$$

as the state of our system. Let the creation of a predator be denoted by the operator  $\hat{a}^\dagger$ , so that

$$\hat{a}^\dagger |m, n\rangle \doteq |m+1, n\rangle \quad (33)$$

The annihilation of a predator will be given by the operator  $\hat{a}$  defined by

$$\hat{a}|m, n\rangle \doteq m|m-1, n\rangle \quad (34)$$

We similarly define the creation and annihilation of the prey with the operators  $\hat{b}^\dagger$  and  $\hat{b}$  :

$$\hat{b}^\dagger|m, n\rangle \doteq |m, n+1\rangle \quad (35)$$

$$\hat{b}|m, n\rangle \doteq n|m, n-1\rangle \quad (36)$$

The operators satisfy the commutation relations

$$[\hat{a}, \hat{a}^\dagger] \doteq \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \quad (37)$$

$$= 1$$

$$[\hat{b}, \hat{b}^\dagger] \doteq \hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b} \quad (38)$$

$$= 1$$

with all other commutators being zero. We can see this from, for example,

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] |m, n\rangle &= (\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})|m, n\rangle \\ &= \hat{a}|m+1, n\rangle - m\hat{a}^\dagger|m-1, n\rangle \\ &= [(m+1) - m]|m, n\rangle \\ &= 1|m, n\rangle \end{aligned}$$

and

$$\begin{aligned} [\hat{a}, \hat{b}^\dagger] |m, n\rangle &= (\hat{a}\hat{b}^\dagger - \hat{b}^\dagger\hat{a})|m, n\rangle \\ &= \hat{a}|m, n+1\rangle - m\hat{b}^\dagger|m-1, n\rangle \\ &= (m - m)|m-1, n+1\rangle \\ &= 0|m-1, n+1\rangle \end{aligned}$$

From (32) we see that

$$\begin{aligned} |\dot{\psi}\rangle &= \sum_{m,n} \dot{P}_{m,n} |m, n\rangle \\ &= \sum_{m,n} [-(\mu m + \sigma n + \lambda mn)P_{m,n} + \mu(m+1)P_{m+1,n} \\ &\quad + \sigma(n-1)P_{m,n-1} + \lambda(m-1)(n+1)P_{m-1,n+1}] |m, n\rangle \end{aligned} \quad (39)$$

Using the creation and annihilation operators, we can rewrite (39) as

$$|\dot{\psi}\rangle = -H|\psi\rangle \quad (40)$$

where

$$H \doteq \mu(\hat{\mathbf{a}}^\dagger - 1)\hat{\mathbf{a}} + \sigma(1 - \hat{\mathbf{b}}^\dagger)\hat{\mathbf{b}}^\dagger\hat{\mathbf{b}} + \lambda\hat{\mathbf{a}}^\dagger(\hat{\mathbf{b}}^\dagger - \hat{\mathbf{a}}^\dagger)\hat{\mathbf{a}}\hat{\mathbf{b}} \quad (41)$$

This is easily checked since, for example,

$$\begin{aligned} \mu(\hat{\mathbf{a}}^\dagger - 1)\hat{\mathbf{a}}|\psi\rangle &= \sum_{m,n} P_{m,n} \mu(\hat{\mathbf{a}}^\dagger - 1)\hat{\mathbf{a}}|m, n\rangle \\ &= \sum_{m,n} P_{m,n} \mu(\hat{\mathbf{a}}^\dagger - 1)m|m-1, n\rangle \\ &= \sum_{m,n} P_{m,n} (\mu m|m, n\rangle - \mu m|m-1, n\rangle) \\ &= \sum_{m,n} (\mu m P_{m,n} - \mu(m+1)P_{m+1,n}) |m, n\rangle \end{aligned}$$

We can do a similar analysis of the other terms in (41) to recover (39).

Now let us apply this formalism to the relatively simpler equation (19). The discrete process here is given by (21) and (22)

$$y_1 \xrightarrow{\frac{G}{2}} 2y_1 \quad (42)$$

$$y_1 + y_1 \xrightarrow{\frac{C-G}{2}} \emptyset \quad (43)$$

Our Master equations are given by

$$\begin{aligned} \dot{P}_m &= - \left[ \frac{G}{2}m + \frac{C-G}{4}m(m-1) \right] P_m + \frac{G}{2}(m-1)P_{m-1} \\ &\quad + \frac{C-G}{4}(m+2)(m+1)P_{m+2} \end{aligned} \quad (44)$$

With the creation and annihilation operators defined as above, we have

$$\begin{aligned} |\dot{\psi}\rangle &= \sum_m \dot{P}_m |m\rangle \\ &= -H_{HD}|\psi\rangle \end{aligned} \quad (45)$$

where

$$H_{HD} \doteq \frac{G}{2}(1 - \hat{\mathbf{a}}^\dagger)\hat{\mathbf{a}}^\dagger\hat{\mathbf{a}} + \frac{C-G}{4}(\hat{\mathbf{a}}^\dagger\hat{\mathbf{a}}^\dagger - 1)\hat{\mathbf{a}}\hat{\mathbf{a}} \quad (46)$$

So what does this discretization of the density equation say about the frequency equation? We will look at the process in (28) to (30). First note that  $y_1 = m$ ,  $y_2 = n$  and  $y_3 = 1$  corresponds to one, and only one,  $x_1 = q$ ,  $x_2 = r$  and  $x_3 = s$ , where  $q, r, s \in \mathcal{Q}$  (that is,  $q, r$ , and  $s$  are members of the rational numbers  $\mathcal{Q}$ ). Also, for any allowed  $x_1, x_2$  and  $x_3$ , we have a unique  $m$  and  $n$  that they correspond to. Not all  $q, r$  and  $s$  are possible. Since  $x_1 + x_2 + x_3 = 1$  and  $y_i = x_i/x_3$ , we must have

$$q + r + s = 1 \quad (47)$$

$$q = ms \quad (48)$$

$$r = ns \quad (49)$$

By (47),  $s$  is redundant as a label. So we have the unique correspondence

$$P_{m,n} \longleftrightarrow P_{q,r} \quad (50)$$

where

$$q = \frac{m}{m+n+1} \quad (51)$$

$$r = \frac{n}{m+n+1} \quad (52)$$

$\dot{P}_{m,n}$  are the Master equations in the density formalism and  $\dot{P}_{q,r}$  are the Master equations in the frequency formalism. We have not derived an explicit expression for  $\dot{P}_{q,r}$ , but let us see what (39) can tell us. Let  $P_{i,j}(0) = \delta_{im}\delta_{jn}$ , where  $i, j, m, n \in \mathcal{N}_0$  and  $\delta_{lk}$  is the Kronecker delta defined by

$$\delta_{lk} \doteq \begin{cases} 1 & \text{if } l = k \\ 0 & \text{if } l \neq k \end{cases}$$

Then (39) says that, for  $t > 0$ ,  $P_{m,n}(t) < 1$ . That is, the probability that our individual is in state  $P_{m,n}$  begins to diffuse away. So, for  $t > 0$ , we need to use (39) to give us the probability that our individual is in state  $|m', n'\rangle$ . But this means, in the frequency formalism, that an individual that starts at  $|q, r\rangle$ , with  $|q, r\rangle$  having the obvious definition, may not remain in  $|q, r\rangle$ . That is, *the individual may mutate* (see [5, Chapter 20]).

Now we need to ask: “How does this discretized evolution relate to the frequency and density equations?” We are really asking if the concept of an ESS

has any significance when we look at the “microstructure” of the evolution of a population through mutation. As shown in [1], it does from an average viewpoint. They find that the evolution of the population mean follows the average evolution of the mutating individuals. It is this population mean that gives meaning to the ESS concept. Specifically, we define the average, for the Hawk-Dove game,

$$\bar{m} \doteq \sum_m m P_m \quad (53)$$

where  $\bar{m}$  is the average relative frequency of the hawk strategy in a population. Then (53) evolves according to the equation in (19)

$$\dot{\bar{m}} = \bar{m} \left[ \frac{G}{2} + \frac{G-C}{2} \bar{m} \right] \quad (54)$$

## 4 Discussion

We have briefly reviewed the process of discretizing an evolutionary game. This process allows us to consider the replicator equation as the evolution of the average strategy of a large population. The Master equations that arise in the discrete process allow us to examine the evolution of the probability density of a population’s strategies. Thus, we can examine a game on two levels: 1) at the level of the population average (a “global” approach), and; 2) at the level of the individual (a “local” approach).

The article by Bettelheim, Agam and Shnerb [1] goes into a more detailed and extensive analysis of this process. In particular, they also examine the case where there is a spatial dimension. This may be useful in applying the discretization process to the games considered in [3].

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