Preprint manuscript No.
(will be inserted by the editor)

Fictitious Play in $2 \times n$ Games

Ulrich Berger

Vienna University of Economics, Department VW5
Augasse 2-6, A-1090 Vienna, Austria
e-mail: ulrich.berger@wu-wien.ac.at

February 2003

Abstract It is known that every continuous time fictitious play process approaches equilibrium in every nondegenerate 2×2 and 2×3 game, and it has been conjectured that convergence to equilibrium holds generally for $2 \times n$ games. We give a simple geometric proof of this. As a corollary, we obtain the same result for the discrete fictitious play process.

Journal of Economic Literature classification numbers: C72, D83.

Key words Continuous Fictitious Play, Learning Process.

1 Introduction

The idea of *fictitious play* is over half a century old. It was originally introduced by Brown (1951) as an algorithm to calculate the value of a zero-sum game. Apart from this, the fictitious play process (FP) is the prime example of a so called myopic learning process. In a fictitious play process two players are engaged in the repeated play of a finite game. Each player believes that her opponent plays a stationary mixed strategy. In each round, she estimates this strategy by the historical distribution of pure strategies and reacts with a strategy that maximizes her expected payoff in the next round, i.e., with a *myopic best response*. We say that an FP approaches equilibrium if the sequence of beliefs converges to the set of Nash equilibria of the game. A game is said to have the *fictitious play property* (FPP), if every FP approaches equilibrium in this game.

Robinson (1951) proved that every two-person zero-sum game has the FPP. Miyasawa (1961) established the same result for every 2×2 game,[1] see also Metrick and Polak (1994). Other convergence results are due to Milgrom and Roberts (1991), Monderer and Shapley (1996a, 1996b), and Sela (1999). It is well known, however, that for games with more than two strategies per player convergence need not occur. Shapley (1964) demonstrated this with a 3×3 bimatrix game for which fictitious play ends up in an asymptotically stable limit cycle. Other examples include Jordan (1993) and Foster and Young (1998). With three or more strategies per player, even chaotic behavior is possible, see Cowan (1992) or Richards (1997).

The cited results also hold for the continuous time version of the fictitious play process (CFP), or the best response dynamics (Gilboa and Matsui, 1991, Matsui, 1992), which differs from CFP only by a rescaling of time. Nonconvergence results can be found in Gaunersdorfer and Hofbauer (1995) or Krishna and Sjöström (1998), while convergence results for CFP have been established by Rosenmüller (1971), Hofbauer (1995), Harris (1998), Sela (2000), and Berger (2001, 2002).

In particular, we know that in games with more than two strategies per player CFP need not converge. On the other hand, every CFP approaches equilibrium in (nondegenerate) 2×2 games (Rosenmüller, 1971) and 2×3 games (Sela, 2000). An open question is the case of $2\times n$ games with $n \geq 4$. Monderer and Sela (1997) conjecture that convergence to equilibrium in fact extends to all nondegenerate $2\times n$ games. In the following we show that this is indeed the case.

The proof consists of two main parts. In the first part we construct a simple linear mapping from the space of mixed strategy pairs to the two-dimensional plane, which has the property that any nonconvergent CFP path is mapped to a nonconvergent path in a compact region, which does not cross itself and must therefore approach a limit cycle. The second part is to show that such a limit cycle does not exist. The proof of this latter part is straightforward and uses only the elementary geometric properties of similar triangles.

The remainder of this paper is structured as follows. Section 2 introduces the notation we use and defines FP and CFP. The main theorem is stated and proved in section 3. Section 4 extends the result to the discrete fictitious play process.

2 Fictitious Play

2.1 Notation

Let (A, B) be a two-player bimatrix game where player 1 has pure strategies numbered from 1 to m, and player 2 has pure strategies $1, \ldots, n$. A is an $m\times n$

[1] He assumed a particular tie-breaking rule. Without such a rule, there are degenerate 2×2 games where FP does not converge, see Monderer and Sela (1996).

payoff matrix for player 1 and B an $n\times m$ payoff matrix for player 2. Thus, if player 1 chooses i and player 2 chooses j, the payoff to player 1 is a_{ij} and the payoff to player 2 is b_{ji}. The set of mixed strategies of player 1 is then the $m-1$ dimensional probability simplex S_m, and analogously S_n is the set of mixed strategies of player 2. The expected payoff for player 1 playing strategy i if player 2 plays the mixed strategy $\mathbf{q} = (q_1, \ldots, q_n)^t \in S_n$ (where the superscript ^{t} denotes the transpose of a vector) is $(A\mathbf{q})_i$. Analogously $(B\mathbf{p})_j$ is the expected payoff for player 2 playing strategy j against the mixed strategy $\mathbf{p} = (p_1, \ldots, p_m)^t \in S_m$. If both players use mixed strategies $\mathbf{p}$ and $\mathbf{q}$, respectively, the expected payoffs are $\mathbf{p}^t A\mathbf{q}$ to player 1 and $\mathbf{q}^t B\mathbf{p}$ to player 2. For $\mathbf{p} \in S_m$ we denote by $BR(\mathbf{p})$ the set of best responses to $\mathbf{p}$. A pair of mixed strategies $(\mathbf{p}^*, \mathbf{q}^*)$ is a Nash equilibrium if and only if $\mathbf{q}^* \in BR(\mathbf{p}^*)$ and $\mathbf{p}^* \in BR(\mathbf{q}^*)$.

2.2 Discrete Fictitious Play

For $t = 1, 2, 3, \ldots$, the sequence $(\mathbf{p}(t), \mathbf{q}(t))$ is a *discrete fictitious play process* (FP), if

$$(\mathbf{p}(1), \mathbf{q}(1)) \in S_m \times S_n \tag{1}$$

and for all t,

$$\mathbf{p}(t+1) = \frac{t\mathbf{p}(t) + \mathbf{b}^1(t)}{t+1} \text{ and } \mathbf{q}(t+1) = \frac{t\mathbf{q}(t) + \mathbf{b}^2(t)}{t+1}, \tag{2}$$

where $\mathbf{b}^1(t) \in BR(\mathbf{q}(t))$ and $\mathbf{b}^2(t) \in BR(\mathbf{p}(t))$.

2.3 Continuous Fictitious Play

If we go from discrete time steps to continuous time $t > 0$, we obtain the *continuous fictitious play process* (CFP)

$$\dot{\mathbf{p}}(t) = \frac{\mathbf{b}^1(t) - \mathbf{p}(t)}{t}, \qquad \dot{\mathbf{q}}(t) = \frac{\mathbf{b}^2(t) - \mathbf{q}(t)}{t}. \tag{3}$$

Up to a rescaling of time – which does not change the shape of the orbits – the CFP is equivalent to the *best response dynamics*

$$\dot{\mathbf{p}}(t) = \mathbf{b}^1(t) - \mathbf{p}(t), \qquad \dot{\mathbf{q}}(t) = \mathbf{b}^2(t) - \mathbf{q}(t). \tag{4}$$

Obviously, if some pure strategy i of player 1 is the *unique* best response to $\mathbf{q}(0)$, the path $\mathbf{p}(t)$ is a straight line in S_m, heading for i, as long as this strategy remains the unique best response.[2] For any pure strategy i of player 1 denote by Q_i the set of mixed strategies $\mathbf{q}$ of player 2 with

[2] More generally, Hofbauer (1995) shows that for any initial condition a piecewise linear CFP path can be constructed.

$BR(\mathbf{q}) = \{i\}$. Analogously, let P_j be the set of mixed strategies $\mathbf{p}$ of player 1 with $BR(\mathbf{p}) = \{j\}$. The sets Q_i for different pure strategies i of player 1 are (possibly empty) disjoint, open, and convex subsets of S_n (analogously for player 2).

If player 1, say, switches from i to i', then at the switching point she is indifferent between these two pure strategies – both do equally well (and at least as good as his other pure strategies) against $\mathbf{q}$. Geometrically, the set of points $\mathbf{q} \in S_n$ where player 1 is indifferent between two given pure strategies i and i', is a (possibly empty) linear subspace of S_n. The next switch again occurs when one of the players is indifferent between two of her pure best responses. Between switches, the CFP path is a straight line heading for some pair of pure best responses.

3 CFP in $2\times n$ Games

Consider now the case $m = 2$. Actually we work with nondegenerate $2\times n$ games (A, B), i.e., games where no column of A or B contains two identical payoffs. We can now state the main result.

Theorem 1 *Every continuous fictitious play process approaches equilibrium in every nondegenerate* $2\times n$ *game.*

Proof Let us start with the contradictory assumption that there is a nondegenerate $2\times n$ game and a CFP path $(\mathbf{p}(t), \mathbf{q}(t))_{t>0}$ that does not approach equilibrium.

We can exclude the trivial cases where one of the players has a strictly dominant strategy.[3] If dominant strategies do not exist, then the set

$$H = \{(\mathbf{p}, \mathbf{q}) \in S_2 \times S_n : (A\mathbf{q})_1 = (A\mathbf{q})_2\}, \tag{5}$$

where player 1 is indifferent between both her pure strategies, is an $n-1$ dimensional linear subspace of $S_2 \times S_n$. If, on the other hand, player 2 is indifferent between the pure best replies j and j', then we have $(B\mathbf{p})_j = (B\mathbf{p})_{j'}$, which defines a particular point $\mathbf{p}^{j,j'} \in S_2$. At least one such point of indifference must exist. If at time t the CFP path $(\mathbf{p}(t), \mathbf{q}(t))$ is in $P_j \times Q_i$, then the path heads for the pure strategy pair (i, j), as long as it stays in $P_j \times Q_i$.

It is now possible to construct a linear mapping F from the state space to the two-dimensional plane, which has the important property that the images $F(P_j \times Q_i)$ of the disjoint best response regions are themselves disjoint. Consider the following linear mapping from $S_2 \times S_n$ to $\mathbb{R}^2$:

$$(x, y) = F(\mathbf{p}, \mathbf{q}) = (p_1, (A\mathbf{q})_1 - (A\mathbf{q})_2). \tag{6}$$

[3] Weakly dominant strategies are then automatically excluded by the nondegeneracy assumption.

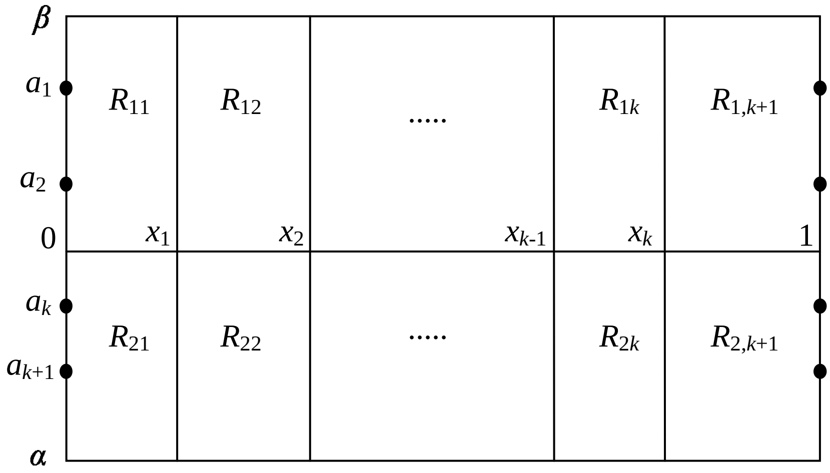

Fig. 1 The image of the state space $S_2 \times S_n$ under the linear mapping F.

This linear mapping has the property that it maps the state space $S_2 \times S_n$ to a rectangle $[0,1] \times [\alpha, \beta]$ with $\alpha < 0, \beta > 0$, and player 1's indifference set H to the interval $[0,1]$ on the x-axis. The indifference sets $\{\mathbf{p}^{j,j'}\} \times S_n$ of player 2 are mapped to vertical line segments $\{p_1^{j,j'}\} \times [\alpha, \beta]$, and, writing a_j short for $a_{1j} - a_{2j}$, we can also see that the pure strategy pairs $(1, j)$ and $(2, j)$ are mapped to the points $(1, a_j)$ on the right border and $(0, a_j)$ on the left border of the rectangle, respectively. The height of the rectangle is determined by $\alpha = \min\{a_j\}$ and $\beta = \max\{a_j\}$. Moreover, the piecewise linear CFP path $(\mathbf{p}(t), \mathbf{q}(t))$ is mapped to a piecewise linear path $(x(t), y(t)) = F(\mathbf{p}(t), \mathbf{q}(t))$, which we will call the *induced path.*

Renumbering the strategies of player 2 in the appropriate way, we can achieve an ordering of the indifference points $p_1^{j,j'}$ such that $0 < p_1^{1,2} < p_1^{2,3} < \cdots < p_1^{k,k+1} < 1$ for some $1 \leq k \leq n-1$, or, defining $x_i = p_1^{i,i+1}$, such that $0 < x_1 < \cdots < x_k < 1$. The image of the state space $S_2 \times S_n$ under the mapping F thus has a shape as shown in Figure 1, where the images of the sets $P_j \times Q_i$ are denoted by R_{ij}.

Note, that if the induced path $(x(t), y(t))$ is in R_{1j}, it heads for the point $(1, a_j)$, while if it is in R_{2j}, it heads for $(0, a_j)$. In short, any induced path moves to the right as long as it is above the x-axis, and to the left as long as it is below the x-axis. If the CFP path, and hence the induced path, does not converge, then the latter must move clockwise in a cyclic fashion within the rectangle. It is clear that induced paths cannot cross, and from this we can now conclude that the induced path converges to a closed limit cycle – called a *Shapley polygon* – in the rectangle. An example is depicted in Figure 2, where the 4 leftmost and the 4 rightmost edges of such a Shapley polygon are drawn.

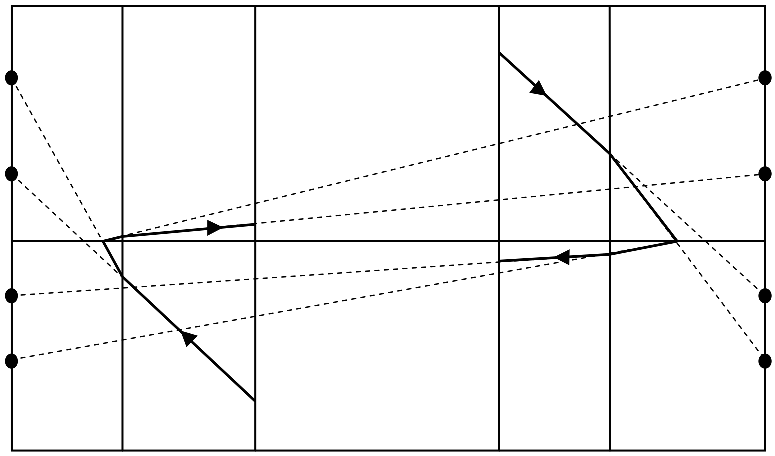

Fig. 2 The 4 leftmost and the 4 rightmost edges of a Shapley polygon in the rectangle.

We have intentionally not drawn the full polygon in Figure 2. The reason for this is that such a Shapley polygon does not exist. This is the crucial point of the analysis.

Lemma 1 *There exists no Shapley polygon in the rectangle.*

Proof We prove this by contradiction. Starting with the assumption that there exists a Shapley polygon with $2(k+1)$ edges (a Shapley $2(k+1)$-gon), where $k \geq 2$, we show that there also exists a Shapley $2k$-gon. This implies that there exists a Shapley 4-gon. It remains to show that a Shapley 4-gon does not exist. For the first part let us assume that there is a Shapley $2(k+1)$-gon, as indicated in Figure 3. Note that we have $0 < x < x_1 < x_{k-1} < x_k < x' < 1$ and $y_1, y_{k-1}, y_k > 0$, as well as $z_1, z_{k-1}, z_k < 0$. Also, necessarily, $a_1 > 0$ and $a_{k+1} < 0$, but a_k may have any sign.

With the notation in this figure, by comparing the similar triangles in the left half, we can derive the following:

$$\frac{y_1}{x_1 - x} = \frac{a_1}{1 - x} \quad \text{and} \quad \frac{-z_1}{x_1 - x} = \frac{a_1 - z_1}{x_1}. \tag{7}$$

Noting that $a_1 > 0, z_1 < 0$, and $x < x_1$, these equations yield

$$-\frac{z_1}{y_1} = \frac{a_1 - z_1}{a_1} \frac{1 - x}{x_1} > \frac{1 - x}{x_1} > \frac{1 - x_1}{x_1}. \tag{8}$$

By comparing the similar triangles on the right side of the polygon in Figure 3 we can also find the identities

$$\frac{y_k}{x' - x_k} = \frac{y_k - a_{k+1}}{1 - x_k} \quad \text{and} \quad \frac{-z_k}{x' - x_k} = -\frac{a_{k+1}}{x'}. \tag{9}$$

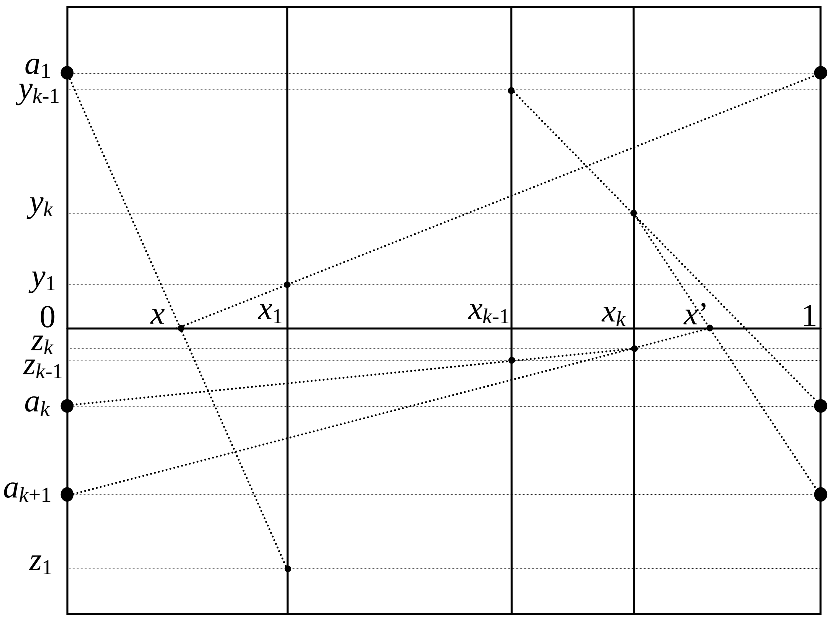

Fig. 3 Notation for the proof of Lemma 1.

This yields

$$-\frac{z_k}{y_k} = \frac{1-x_k}{x_k - y_k/a_{k+1}} < \frac{1-x_k}{x_k}, \tag{10}$$

since y_k and $-a_{k+1}$ are positive. On the other hand, for arbitrary $y_k > 0, z_k < 0$ the inequality $-z_k/y_k < (1-x_k)/x_k$ is sufficient for the existence of a number $a_{k+1} < 0$, for which the equality in (10) holds. This can be seen, if we let a_{k+1} go from $-\infty$ to 0. The middle term in (10) then falls from $(1-x_k)/x_k$ to 0, and by the intermediate value theorem, for some number $a_{k+1} < 0$ it is equal to $-z_k/y_k$.

From Figure 3 we can derive a third pair of identities. These are

$$\frac{y_{k-1}-a_k}{1-x_{k-1}} = \frac{y_k - a_k}{1-x_k} \quad \text{and} \quad \frac{-z_{k-1}+a_k}{x_{k-1}} = \frac{-z_k + a_k}{x_k}, \tag{11}$$

which yield

$$y_{k-1} = \frac{y_k(1-x_{k-1}) - a_k(x_k - x_{k-1})}{1-x_k} \tag{12}$$

and

$$-z_{k-1} = \frac{-z_k x_{k-1} - a_k(x_k - x_{k-1})}{x_k}. \tag{13}$$

Note that the nominator $-z_k x_{k-1} - a_k(x_k - x_{k-1}) > 0$ since $z_{k-1} < 0$. Taken together we have

$$-\frac{z_{k-1}}{y_{k-1}} = \frac{-z_k x_{k-1} - a_k(x_k - x_{k-1})}{y_k(1 - x_{k-1}) - a_k(x_k - x_{k-1})} \frac{1 - x_k}{x_k}. \tag{14}$$

From (10) it follows that $-z_k x_k < y_k(1 - x_k)$, and $x_{k-1} < x_k$ implies $1 - x_k < 1 - x_{k-1}$, so we get the inequality

$$-z_k x_{k-1} < y_k(1 - x_{k-1}). \tag{15}$$

Adding $-a_k(x_k - x_{k-1})$, we have

$$0 < -z_k x_{k-1} - a_k(x_k - x_{k-1}) < y_k(1 - x_{k-1}) - a_k(x_k - x_{k-1}), \tag{16}$$

or

$$0 < \frac{-z_k x_{k-1} - a_k(x_k - x_{k-1})}{y_k(1 - x_{k-1}) - a_k(x_k - x_{k-1})} < 1. \tag{17}$$

Furthermore, $x_k > x_{k-1}$ implies $(1-x_k)/x_k < (1-x_{k-1})/x_{k-1}$, and together with (14) and (17) this yields

$$-\frac{z_{k-1}}{y_{k-1}} < \frac{1 - x_{k-1}}{x_{k-1}}. \tag{18}$$

Note that from the consideration below (10) we can now conclude that there exists a number $a < 0$ such that the line connecting the points (x_{k-1}, y_{k-1}) and $(1, a)$ and the line connecting (x_{k-1}, z_{k-1}) and $(0, a)$ intersect on the x-axis at some point x' between x_{k-1} and 1. In short, this means that there exists a Shapley $2k$-gon. Repeating the same argument, we get

$$-\frac{z_{k-2}}{y_{k-2}} < \frac{1 - x_{k-2}}{x_{k-2}}, \tag{19}$$

and so on, until we reach

$$-\frac{z_1}{y_1} < \frac{1 - x_1}{x_1}, \tag{20}$$

implying the existence of a Shapley 4-gon. However, this contradicts inequality (8). Hence the Shapley 4-gon and all of the “larger” Shapley polygons do not exist. □

This concludes the proof of the main theorem. □

4 FP in $2\times n$ Games

There are no general results relating the behavior of discrete and continuous time ficititious play. However, by invoking a result from Hofbauer and Sorin (2002, Proposition 8) – see also Hofbauer (1995) – we can easily derive convergence to equilibrium in nondegenerate $2\times n$ games also for FP.

Corollary 1 *Every nondegenerate* $2\times n$ *game has the fictitious play property.*

Proof By Theorem 1, every CFP path approaches equilibrium in a nondegenerate $2\times n$ game. This means that the maximal invariant set of the CFP dynamics (3) is the set of Nash equilibria of the game. By Proposition 8 of Hofbauer and Sorin (2002), the set of limit points of a discrete fictitious play process is contained in this Nash equilibrium set. In other words, every limit point of an FP is a Nash equilibrium. It follows that every FP approaches equilibrium. □

References

1. Berger, U. (2001). Best response dynamics for role games. *International Journal of Game Theory* **30**, 527-538.
2. Berger, U. (2002). Continuous fictitious play via projective geometry. Mimeo, Vienna University of Economics.
3. Brown, G. W. (1951). Iterative solution of games by fictitious play. *In* "Activity Analysis of Production and Allocation" (T.C. Koopmans, Ed.), pp. 347-376, New York, Wiley.
4. Cowan, S. (1992). *Dynamical systems arising from game theory.* Dissertation, UC Berkeley.
5. Foster, D. P. and Young, H. P. (1998). On the nonconvergence of fictitious play in coordination games. *Games and Economic Behavior* **25**, 79-96.
6. Gaunersdorfer, A. and Hofbauer, J. (1995). Fictitious play, Shapley polygons, and the replicator equation. *Games and Economic Behavior* **11**, 279-303.
7. Gilboa, I. and Matsui, A. (1991). Social stability and equilibrium. *Econometrica* **59**, 859-867.
8. Harris, C. (1998). On the rate of convergence of continuous-time fictitious play. *Games and Economic Behavior* **22**, 238-259.
9. Hofbauer, J. (1995). Stability for the best response dynamics. Mimeo, University of Vienna.
10. Hofbauer, J. and Sorin, S. (2002). Best response dynamics for continuous zero-sum games. Mimeo, Ecole Polytechnique.
11. Jordan, J. S. (1993). Three problems in learning mixed-strategy Nash equilibria. *Games and Economic Behavior* **5**, 368-386.
12. Krishna, V. and Sjöström, T. (1998). On the convergence of fictitious play. *Mathematics of Operations Research* **23**, 479-511.
13. Matsui, A. (1992). Best response dynamics and socially stable strategies. *Journal of Economic Theory* **57**, 343-362.
14. Metrick, A. and Polak, B. (1994). Fictitious play in 2×2 games: a geometric proof of convergence. *Economic Theory* **4**, 923-933.

15. Milgrom, P. and Roberts, J. (1991). Adaptive and sophisticated learning in normal form games. *Games and Economic Behavior* **3**, 82-100.
16. Miyasawa, K. (1961). On the convergence of the learning process in a 2×2 non-zero-sum two person game. Economic Research Program, Princeton University, Research Memorandum No. 33.
17. Monderer, D. and Sela, A. (1996). A 2×2 game without the fictitious play property. *Games and Economic Behavior* **14**, 144-148.
18. Monderer, D. and Sela, A. (1997). Fictitious play and no-cycling conditions. Mimeo, The Technion.
19. Monderer, D. and Shapley, L. S. (1996a). Fictitious play property for games with identical interests. *Journal of Economic Theory* **68**, 258-265.
20. Monderer, D. and Shapley, L. S. (1996b). Potential games. *Games and Economic Behavior* **14**, 124-143.
21. Richards, D. The geometry of inductive reasoning in games. *Economic Theory* **10**, 185-193.
22. Robinson, J. (1951). An iterative method of solving a game. *Annals of Mathematics* **54**, 296-301.
23. Rosenmüller, J. (1971). Über Periodizitätseigenschaften spieltheoretischer Lernprozesse. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **17**, 259-308.
24. Sela, A. (1999). Fictitious play in 'one-against-all' multi-player games. *Economic Theory* **14**, 635-651.
25. Sela, A. (2000). Fictitious play in 2×3 games. *Games and Economic Behavior* **31**, 152-162.
26. Shapley, L. S. (1964). Some topics in two-person games. *Annals of Mathematical Studies* **5**, 1-28.