From Set -Valued Solutions to Single -Valued Solutions: the Centroid *

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Abstract

In the framework of cooperative game theory there are several solution concepts that give rise to different ways of dividing the worth of the grand coalition $v(N)$ among the players. We work on the set of transferable utility games, shortly, TU games. Although solution concepts admit different classifications, to our purpose, we divide them into two groups: set-valued solutions and single-valued solutions. Roughly speaking, set-valued solutions provide a subset of the set of all assignments, this subset is chosen according to some specific criteria, which depend on the kind of solution we are aiming for. Examples of this type are the stable sets (von Neumann and Morgenstern, 1944), the core (Gillies, 1953), the kernel (Davis and Maschler, 1965), the bargaining sets (Aumann and Maschler, 1964; among others), etc. On the other hand, one can establish some desirable properties or axioms that determine a unique solution for each game, and that is known as a single-valued solution. The Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969) are solutions of this type.

Each solution concept has its interpretation and attends to specific principles (fairness, stability...) and all of them enrich the setting of cooperative game theory. Besides, there are many papers that research on relations between single-valued solutions and set-valued solutions. To mention some of the results, when the core is non-empty, the nucleolus selects an imputation in it, and for the class of convex games, the Shapley value is in the core.

Our work establishes a direct connection between a set-valued solution and a particular singled-valued solution obtained from it.

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In this paper, given a non-empty set-valued solution and a probability distribution over it, we provide a single-valued solution that inherits a large number of properties of the set. So, we give a procedure that selects a unique outcome from a set-valued solution having, at least, two good properties: it is in the convex hull of the set and it responds to some principle of fairness if players agree on the probability measure. A particular case of our solution is the Shapley value if we consider the set-valued solution given by the set of all marginal contributions vectors with their corresponding weights. Besides, recently González-Díaz et al. (2003) proved that the $\tau - value$ (Tijs, 1981) corresponds to the mean of the uniform distribution defined over the core cover boundary.

Our first result provides a characterization of this single-valued solution in a general framework, i.e. given a set S, endowed with a probability measure we select a specific point in the convex hull of S. Next, we pay special attention when the set-valued solution is the core of a TU game, and so we analyze in detail the class of balanced games. A new solution concept that selects a unique outcome in the core, called the centroid, is obtained. Easily we observe that it does not coincide with the classical singled-valued solutions. The centroid has a nice interpretation and we obtain an axiomatic characterization using well known properties in game theory. More precisely, we obtain a characterization that allows to compare it with the Shapley value. Our next step is to study special subclasses of games, such as convex games, and we get direct relations with the Shapley value in some specific cases. To finish our paper, we provide some tools for the computation of the solution in most of the situations handled along the paper.

Keywords: cooperative games, TU games, set-valued solutions, single-valued solutions, Shapley value.

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