Playing for Time: A Sequential Inspection Game

Rudolf Avenhaus
Fakultät für Informatik
Universität der Bundeswehr München
D-85579 Neubiberg, Germany

Morton John Canty
Forschungszentrum Jülich GmbH
D-52425 Jülich, Germany

May 6, 2003

Abstract

Inspections for timely detection of illegal activity on a finite, closed time interval and subject to first and second kind errors are modelled as a sequential, two-person game. The utilities of the players, inspector and inspectee, are assumed to be linear in the detection time with time-independent false alarm costs. Sets of Nash equilibria are obtained in which the inspectee behaves illegally or legally with probability one.

Keywords: Game theory; Modelling; Optimization; Inspections

PLAYING FOR TIME: A SEQUENTIAL INSPECTION GAME

Dr. Morton Canty
Programmgruppe Systemforschung und Technologische Entwicklung
Forschungszentrum Jülich GmbH
D-52425 Jülich
Germany

1 Introduction

There are various ways to model the detection capability of routine inspection regimes. For instance one can choose objective functions which are dependent on the time between an illegal activity and its detection, or ones which are simple dichotomies based on some imposed critical detection time goal. One can assume unobservable inspections such as might be associated with instrumental or remote surveillance or, alternatively, that the inspections are observable so that the inspectee can make his actions conditional on those of the inspector. Furthermore, statistical error probabilities may or may not be taken into account. Examples of these approaches can be found in [2, 3, 4, 6].

The inspection game that we propose and analyze here is intended to describe on-site (and thus observable) inspections, such as those carried out under treaties on arms control, disarmament or non-proliferation. Utility functions are taken to be linear in the time to detection and no *a priori* restrictions are placed on the time at which an inspection can take place. Non-detection probabilities (second kind errors) and associated false alarm probabilities (first kind errors) are included. The inspectee can react flexibly to the observed activity of the inspector, choosing illegal or legal behavior according to his own perceived self-interests. We seek the most economical behavior, both for the inspector as well as for the inspectee.

This problem has not hitherto been treated. Diamond [4] investigated a similar model for unobservable inspections, implying a non-sequential game with simultaneous choice of inspection and violation strategies. He also assumed zero-sum, thus precluding the consideration of jointly adverse costs of false accusations. Moreover the possibility of legal behavior was not taken into account. Rothenstein [6] extended Diamond's model to include errors of the first and second kind, but only for a single inspection and without associated costs of false alarms. He also considered a sequential zero-sum game of error-free inspections, which is a special case of our model, and obtained an equilibrium with randomized inspection strategies. Canty et al. [2] solved a non-sequential inspection game with first and second kind errors based on a fixed detection time goal and obtained some special solutions for the sequential case.

Specifically, we consider a single inspected object, for example a nuclear or chemical facility subject to verification in the framework of an international treaty, and a reference period of one time unit (e.g. one calendar year). In order to separate the timeliness aspect of routine inspection from the overall goal of detecting illegal activity, we assume that a thorough and unambiguous inspection takes place at the end of the reference period which will detect an illegal activity with certainty if one has occurred. In addition there are a number of less intensive and strategically placed "interim" inspections which are intended to reduce the time to detection below the length of the reference period. An interim inspection will detect a preceding or coincident illegal activity, but with some lower probability. In keeping with common notation, we call this probability $1-\beta$, where β is the probability of an error of the second kind, or *non-detection probability.* Associated with each interim inspection which is not preceded by an illegal action is a corresponding probability of an error of the first kind, the *false alarm probability* α.

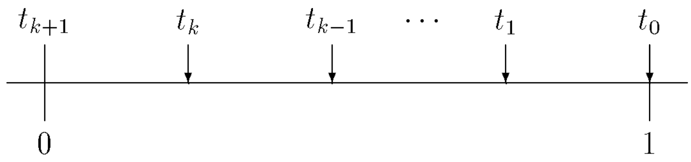

Figure 1: Sequence of inspections.

We assume that, by agreement, k interim inspections will occur within the reference period. For convenience we label the inspections backwards in time: prior to an inspection at time t_k there are k unused interim inspections available, prior to an inspection at time t_1 there is one interim inspection left and so on. It is also convenient to label the beginning of the reference period t_{k+1} and the end t_0, so we have $0 = t_{k+1} < t_k < \ldots < t_1 < t_0 = 1$ as depicted in Figure 1.

The preferences of the protagonists (inspector, inspectee) are taken to be as follows:

$(0, 0)$ for legal behavior over the reference time, and no false alarm,

$(-\ell e, -\ell f)$ legal behavior over the reference time, and ℓ false alarms, $\ell = 1, \ldots, k$,

$(-a\Delta t, d\Delta t - b)$ for detection of illegal activity after elapsed time $\Delta t \geq 0$,

where

$$0 < e < a, \quad 0 < f < b < d. \tag{1}$$

Thus the preferences are normalized to zero for legal behavior without false alarms, and the loss(profit) to the inspector(inspectee) grows proportionally with the time elapsed to detection of an illegal action. A false alarm is resolved unambiguously with time independent costs $-e$ to the inspector and $-f$ to the inspectee, whereupon the game continues. The quantity b is the cost to the inspectee of immediate detection. Note that, if $b > d$, the inspectee will behave legally even if there are no interim inspections at all. Since interim inspections introduce false alarm costs for both parties, there would be no point in performing them. Note also that the preferred outcome from the inspector's point of view is legal behavior: his primary aim is to deter the inspectee from behaving illegally.

The main result of this paper is the derivation of closed forms for the equilibrium inspection strategies and payoffs of inspector and inspectee for any number k of interim inspections and, with them, the conditions which must be met in order to induce legal behavior on the part of the inspectee. In addition we examine "saturated" equilibria which arise when false alarm costs become excessive, and recommend a procedure to avoid them.

2 One interim inspection

The game for a single interim inspection is shown in Figure 2 in extensive form. The subgames beginning at the chance nodes can be simplified easily. In particular, the situations at the inspectee's decision points labeled u_1 and u_1' are equivalent, since all payoffs following u_1' are reduced by the same amounts e resp. f relative to u_1. We obtain the reduced extensive form game shown in Figure 3.

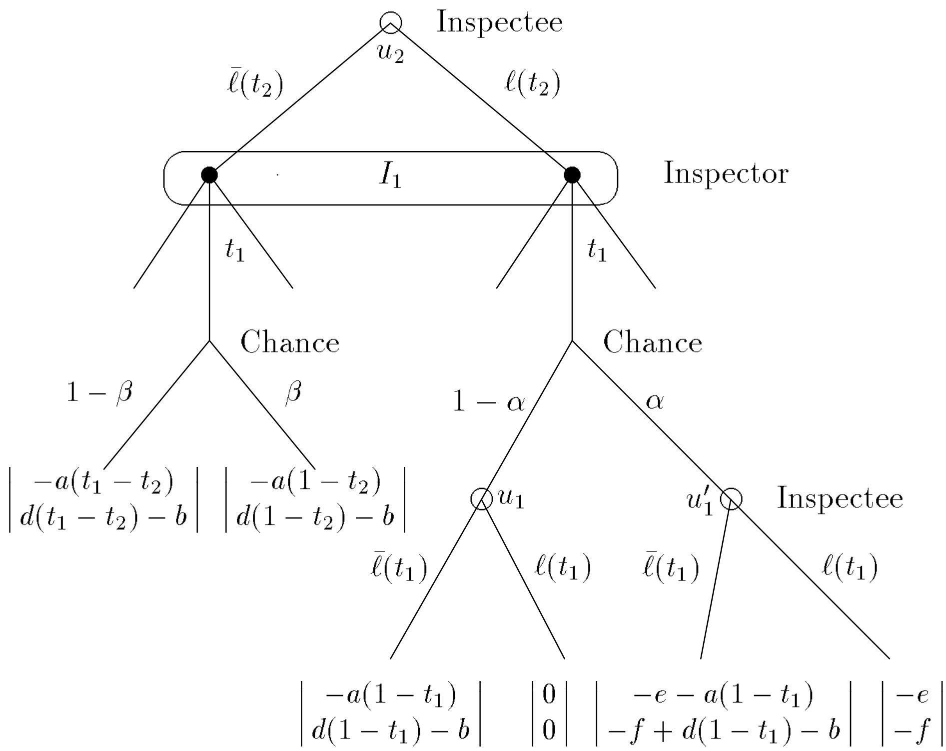

Figure 2: Extensive form of the sequential inspection game with one interim inspection. $\ell(t)$ denotes legal behavior, $\bar{\ell}(t)$ illegal behavior at time t.

Since the interim inspection is observable, the inspectee will either act illegally at the beginning of the reference period, time t_2, in which case a false alarm is excluded, or delay his decision until the interim inspection at time t_1. In the former case, the expected payoffs are

$$(-a[(t_1-t_2)(1-\beta)+(1-t_2)\beta],\ d[(t_1-t_2)(1-\beta)+(1-t_2)\beta]-b). \qquad (2)$$

If he waits for the interim inspection at t_1 and then acts illegally immediately afterward, the expected payoffs are

$$(-e\alpha+d(1-t_1),\ -f\alpha+d(1-t_1)-b). \qquad (3)$$

Finally, he can behave legally at time t_1 as well, with expected payoffs $(-e\alpha,\ -f\alpha)$.

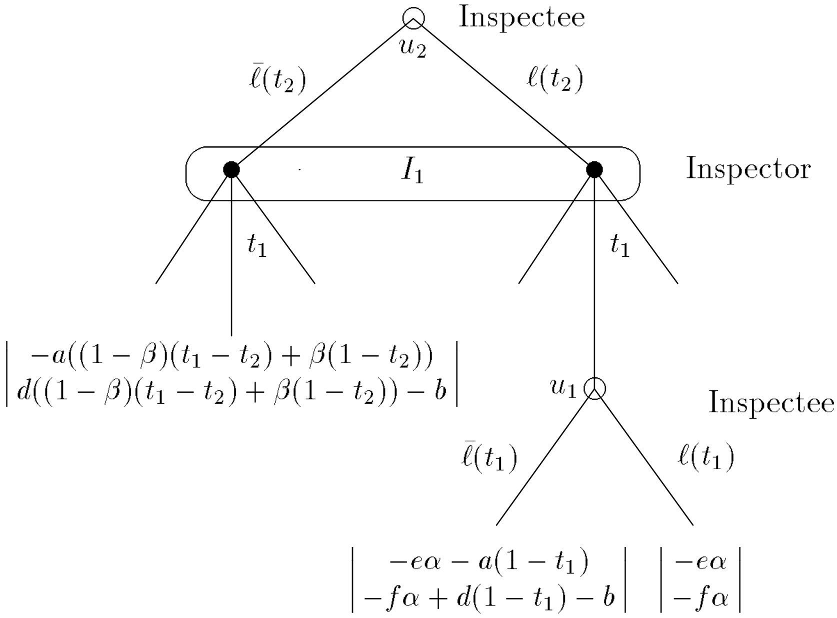

Figure 3: Reduced extensive form of the sequential inspection game with one interim inspection.

Theorem 1 *For unbiased inspection procedures, i.e. for $\alpha+\beta<1$, Nash equilibria of the inspection game represented in extensive form in Figure 3 are given as follows. Let $\{t_1 \mid t_2 \le t_1 < 1\}$ be the set of (pure) strategies of the inspector, and the probabilities g_2 and g_1 to start an illegal action at time t_2 resp. t_1 be the (mixed) behavior strategies of the inspectee at his decision points u_2 resp. u_1. Let $V_2^*(t_2)$ and $W_2^*(t_2)$ be the equilibrium payoffs to inspector and inspectee, respectively. Under the assumption*

$$\frac{b}{d} < \frac{1}{2-\beta}\left(1-t_2+\frac{f\alpha}{d}\right) =: L_2 \tag{4}$$

the inspectee acts illegally; equilibrium payoffs and strategies are

$$\begin{aligned} V_2^*(t_2) &= -aA_2(1-t_2)-e\alpha B_2 \\ W_2^*(t_2) &= dA_2(1-t_2)-f\alpha B_2 - b \\ t_1^*-t_2 &= (1-\beta)A_2(1-t_2)-\frac{f\alpha}{d}((1-\beta)B_2+\beta) \\ g_2^* &= A_2,\ g_1^*=1, \end{aligned} \tag{5}$$

where A_2 and B_2 are given by

$$A_2 = \frac{1}{2-\beta}, \quad B_2 = \frac{1-\beta}{2-\beta}. \tag{6}$$

Under the assumption

$$\frac{b}{d} \geq L_2 \tag{7}$$

the inspectee acts legally; payoffs and inspector strategy are given by

$$V_2^*(t_2) = -e\alpha, \quad W_2^*(t_2) = -f\alpha \tag{8}$$

and by the cone of deterrence [5]

$$1 - \frac{b}{d} \leq t_1^* \leq \frac{1}{1-\beta}\left(\frac{b}{d} - \frac{f\alpha}{d} - \beta + t_2\right). \tag{9}$$

Proof: Given the three pure strategies of the inspectee, namely to violate immediately at t_2, to violate at t_1, or to behave legally throughout the reference period, his Nash equilibrium conditions are, with (2) and (3),

$$\begin{aligned} W_2^*(t_2) &\geq d((1-\beta)(t_1^* - t_2) + \beta(1-t_2)) - b \\ W_2^*(t_2) &\geq -f\alpha + d(1-t_1^*) - b \\ W_2^*(t_2) &\geq -f\alpha, \end{aligned} \tag{10}$$

whereas those of the inspector are

$$\begin{aligned} \forall t_1: \; V_2^*(t_2) \geq g_2^*[&-a((1-\beta)(t_1 - t_2) + \beta(1-t_1))] \\ &+ (1-g_2^*)(-e\alpha - a(1-t_1)) \end{aligned} \tag{11}$$

in case of illegal behavior, and

$$V_2^*(t_2) \geq -e\alpha \tag{12}$$

otherwise.

Let us consider the illegal behavior equilibrium as given by (5) and (6). We see immediately that here the following indifference conditions are fulfilled for all interim inspections t_1 and for the two pure illegal strategies of the inspectee,

$$V_2^*(t_2) = g_2^*[-a((1-\beta)(t_1 - t_2) + \beta(1-t_2))] + (1-g_2^*)[-e\alpha - a(1-t_1)] \tag{13}$$

$$W_2^*(t_2) = d((1-\beta)(t_1^* - t_2) + \beta(1-t_2)) - b = -f\alpha + d(1-t_1^*) - b. \tag{14}$$

Since, furthermore,

$$W_2^*(t_2) = dA_2(1-t_2) - f\alpha B_2 - b \geq -f\alpha$$

is equivalent to (4), Nash conditions (10) and (11) are fulfilled. The unbiased test procedure condition guarantees t_1^* as given by (5) to be larger than zero.

In the case of the legal behavior equilibrium, the first two Nash conditions in (10) are just equivalent to (9), whereas the third condition in (10) as well as condition (12) are trivial. Condition (7), finally, guarantees that the cone of deterrence (9) is not empty. □

Remarks on Theorem 1:

1. For the equilibrium strategy t_1^* of the inspector given in (5), the first two inequalities in (10) are fulfilled as equalities. Therefore the strategy t_1^* in (5) also satisfies (9). This is important for practical applications.
2. The inspector's equilibrium strategy is unmixed. It could be replaced by a randomized strategy (as for example in [6]) with expected value t_1^*, but there would be no advantage in doing so.
3. The inspectee's equilibrium behavior strategy depends only on the error probabilities α and β, not on the utility parameters a and e of the inspector.

3 Two interim inspections

The game with two interim inspections beginning at time t_3 is shown in extensive form in Figure 4. Note that, in the event of a false alarm after the inspector's first inspection at t_2, the inspector knows that the inspectee behaved legally at time t_3. If there is no false alarm, he doesn't have this information. This is reflected in his information sets in the figure. The game is shown in reduced form in Figure 5. We present first of all a solution which will be generalized to arbitrarily many interim inspections in the next Section, and then consider some special solutions.

Theorem 2 *For unbiased inspection procedures, i.e. for $\alpha + \beta < 1$, Nash equilibria of the inspection game represented graphically in Figure 5 are given as follows. Let $\{t_2, t_1 \mid t_3 < t_2 < t_1 < 1\}$ be the set of (pure) strategies of the inspector, and the probabilities g_3, g_2 and g_1 be the behavior strategies of the inspectee at his decision points u_3, u_2 and u_1, respectively. Let $V_2^*(t_3)$ and $W_2^*(t_3)$ be the respective equilibrium payoffs to inspector and inspectee. Under the assumption*

$$\frac{b}{d} < \frac{1}{3-2\beta}\left(1 - t_3 + \frac{f\alpha}{d}(3-\beta)\right) =: L_3 \tag{15}$$

the inspectee behaves illegally. For

$$\frac{f\alpha}{d(1-t_3)} < \frac{1-\beta}{3-3\beta+\beta^2} \tag{16}$$

equilibrium payoffs to inspector and inspectee are, respectively,

$$\begin{aligned} V_3^*(t_3) &= -aA_3(1-t_3) - e\alpha B_3 \\ W_3^*(t_3) &= dA_3(1-t_3) - f\alpha B_3 - b. \end{aligned} \tag{17}$$

An equilibrium strategy for the inspector is

$$\begin{aligned} t_2^* - t_3 &= (1-\beta)A_3(1-t_3) - \frac{f\alpha}{d}((1-\beta)B_3 + \beta) \\ t_1^* - t_2^* &= (1-\beta)A_2(1-t_2^*) - \frac{f\alpha}{d}((1-\beta)B_2 + \beta). \end{aligned} \tag{18}$$

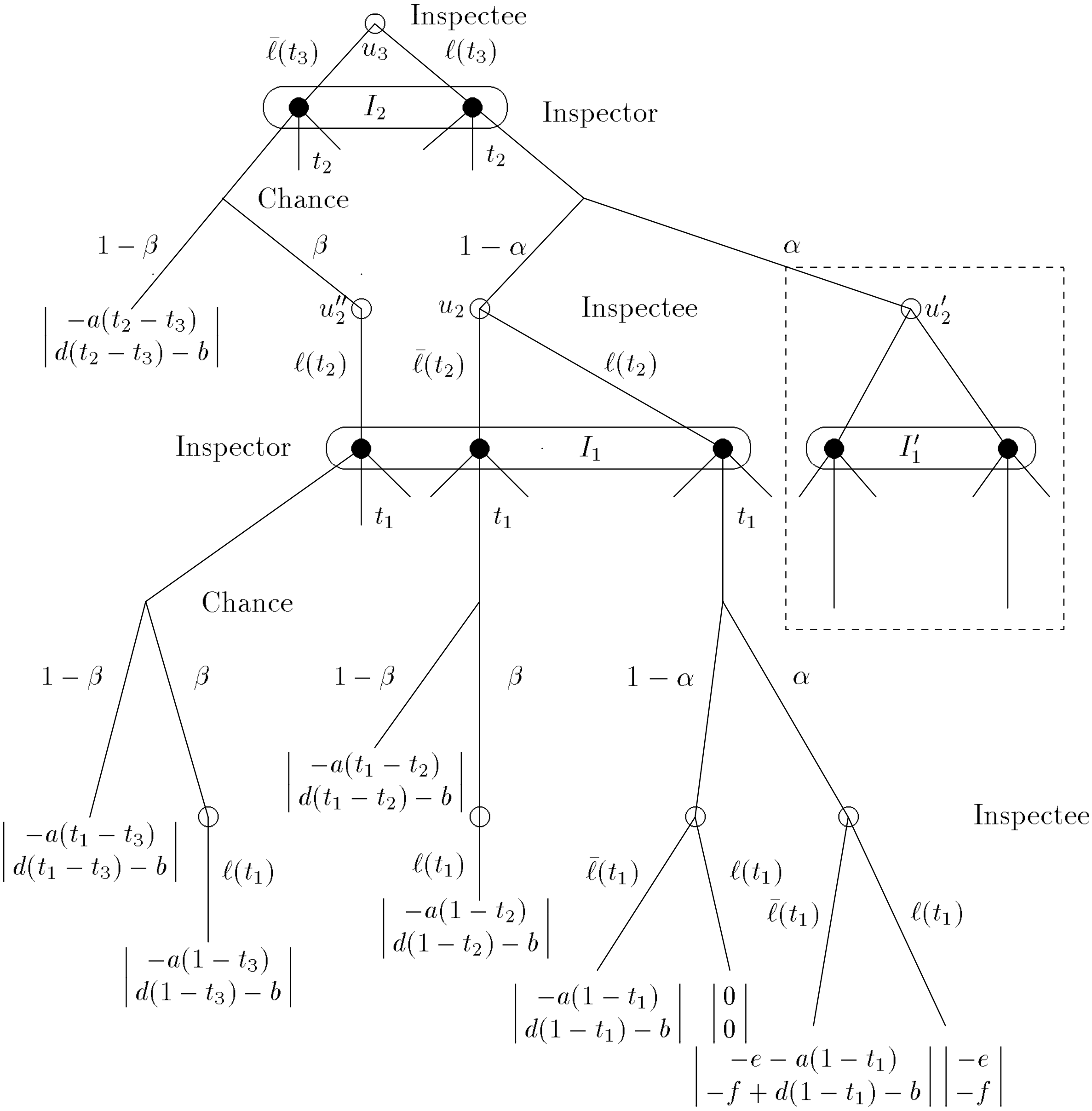

Figure 4: Extensive form of the game with two interim inspections. The proper subgame in the dashed box is identical to the game of Figure 3, except that all payoffs to the inspector and inspectee are reduced by amounts e and f, respectively.

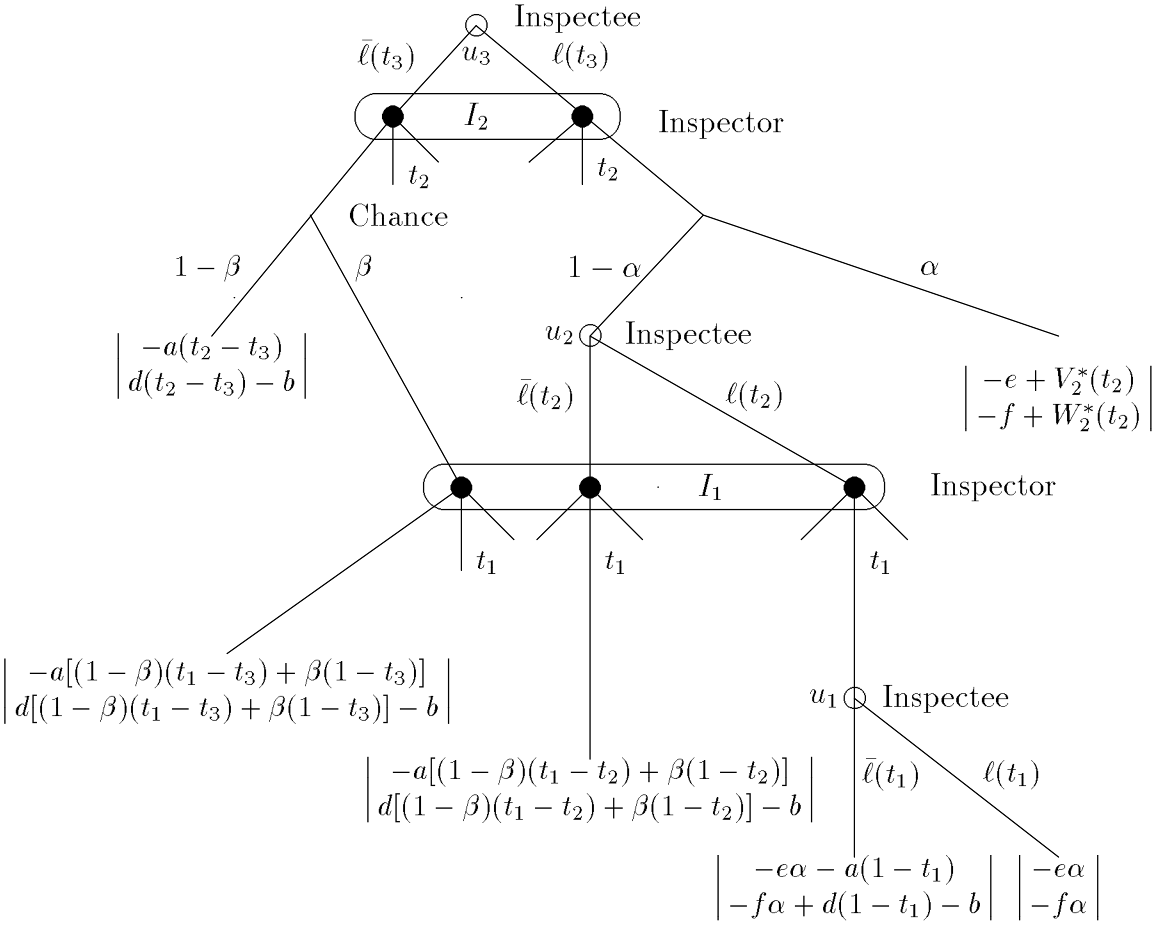

Figure 5: Reduced extensive form of the game of Figure 4.

The inspectee's equilibrium behavior strategies at his decision nodes u_3, u_2 and u_1 are, respectively,

$$g_3^* = A_3, \quad g_2^* = \frac{2(1-\alpha)-\beta}{2(1-\alpha)} A_2, \quad g_1^* = 1. \tag{19}$$

The quantities A_3 and B_3 are given by

$$\begin{aligned} A_3 &= \frac{A_2}{1+(1-\beta)A_2} = \frac{1}{3-2\beta} \\ B_3 &= \frac{B_2+(1-\beta)A_2}{1+(1-\beta)A_2} = \frac{3(1-\beta)}{3-2\beta}. \end{aligned} \tag{20}$$

Under the assumption

$$\frac{b}{d} \geq L_3 \tag{21}$$

the inspectee behaves legally; the equilibrium payoffs and the equilibrium strategies of the inspector are given by

$$V_3^*(t_3) = -2e\alpha, \quad W_3^*(t_3) = -2f\alpha, \tag{22}$$

$$\begin{aligned} \frac{b}{d} - \frac{2f\alpha}{d} &\geq \beta((1-\beta)(t_1^* - t_2^*) + \beta(1-t_2^*)) + t_2^* - t_3 \\ \frac{b}{d} - \frac{f\alpha}{d} &\geq (1-\beta)(t_1^* - t_2^*) + \beta(1-t_2^*) \\ \frac{b}{d} &\geq 1 - t_1^*. \end{aligned} \tag{23}$$

The complement of (16) *is treated in Theorem 3 below.*

Proof: The pure behavior strategies of the inspectee are $\{g_3 = 1, g_2 = g_1 = 0\}$, $\{g_3 = 0, g_2 = 1, g_1 = 0\}$, $\{g_3 = g_2 = 0, g_1 = 1\}$ and $\{g_3 = g_2 = g_1 = 0\}$. His corresponding Nash equilibrium conditions are

$$\begin{aligned} W_3^*(t_3) &\geq d(1-\beta)(t_2^* - t_3) + d\beta((1-\beta)(t_1^* - t_3) + \beta(1-t_3)) - b \\ W_3^*(t_3) &\geq (1-\alpha)(d(1-\beta)(t_1^* - t_2^*) + d\beta(1-t_2^*) - b) + \alpha(-f + W_2^*(t_2^*)) \\ W_3^*(t_3) &\geq (1-\alpha)(-f\alpha + d(1-t_1^*) - b) + \alpha(-f + W_2^*(t_2^*)) \\ W_3^*(t_3) &\geq (1-\alpha)(-f\alpha) + \alpha(-f + W_2^*(t_2^*)). \end{aligned} \tag{24}$$

First of all we note that, in the event of legal behavior at t_3 and false alarm, the inspectee will behave illegally in the subsequent proper subgame, since we have with (15)

$$L_2 = \frac{1}{2-\beta}\left(1 - t_2^* + \frac{f\alpha}{d}\right) = L_3 > \frac{b}{d},$$

so that illegal behavior condition (4) in Theorem 1 is satisfied. With (14), (17) and (18) the first 3 conditions in (24) are easily seen to be satisfied as equality and the last to be equivalent to (15). Condition (16) guarantees that $t_2^* - t_3 > 0$.

The Nash equilibrium condition for the inspector in case of illegal behavior is

$$\begin{aligned} \forall t_1, t_2 : \; V_3^*(t_3) \geq g_3^*(-a)\Big[(1-\beta)(t_2 - t_3) + \beta((1-\beta)(t_1 - t_3) + \beta(1-t_3))\Big] \\ + (1-g_3^*)\Big[(1-\alpha)[g_2^*(-a)((1-\beta)(t_1 - t_2) + \beta(1-t_2)) \\ + (1-g_2^*)(-e\alpha - a(1-t_1))] + \alpha[-e + V_2^*(t_2)]\Big] \end{aligned} \tag{25}$$

Substituting for $V_2^*(t_2)$ as given by (5), the coefficient of t_2 in this expression is

$$a[-g_3^*(1-\beta) + (1-g_3^*)(1-\alpha)g_2^* + (1-g_3^*)\alpha A_2],$$

which vanishes by virtue of (19). Similarly, the coefficient of t_1 is

$$a[-g_3^*\beta(1-\beta) - (1-g_3^*)((1-\alpha)(1-\beta)g_2^* + (1-\alpha)(1-g_2^*))]$$

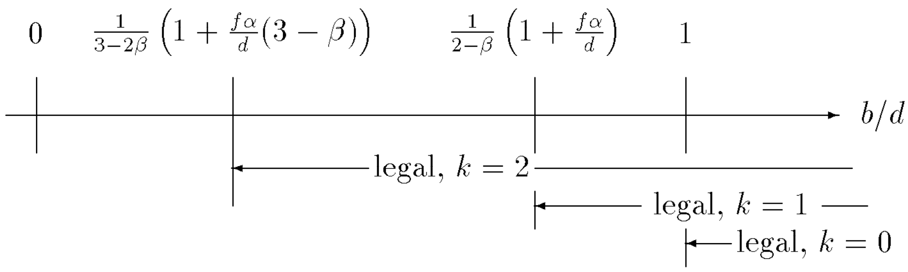

Figure 6: Graphical representation of (6) and (15).

which again vanishes. Therefore the right hand side of (25) is, with (5),

$$g_3^*\beta(-aA_2 - e\alpha B_2 + at_3) + (1 - g_3^*)(-aA_2 - e\alpha - e\alpha B_2) = V_3^*(t_3)$$

as given by (17). Therefore the condition (25) is fulfilled as equality.

Finally, let us consider the solution (21)-(23). The Nash equilibrium condition for the inspector is fulfilled as equality. The first three conditions in (24) for the inspectee are just equivalent to the "cone of deterrence", i.e. inequalities (23). The last condition in (24) is fulfilled as equality. □

Remarks on Theorem 2:

1. Given condition (16), condition (15) with $t_3 = 0$ is a smaller bound for b/d than (4) with $t_2 = 0$, so that we have the situation shown in Figure 6 for the $k = 1$ and $k = 2$ games each played over the full reference period.
2. According to (18) the inspector's behavior strategy at his information set I_1 is the same as his equilibrium strategy in the game with one interim inspection, Eq. (5). On the other hand we see from (19) that this is not the case for the inspectee at his decision point u_2. This phenomenon is a consequence of the game's information structure, as has been observed in another context by von Stengel [7].

Next we consider some special solutions which arise when condition (16) in Theorem 2 is not fulfilled. Since these solutions will not be required for the generalization to follow, we will set $t_3 = 0$ for convenience.

Theorem 3 *For the game of Figure 5 with $t_3 = 0$, if*

$$\frac{b}{d} < \frac{1}{2-\beta}\left(1 + \frac{f\alpha}{d}\right) =: L_2', \tag{26}$$

the inspectee behaves illegally. For

$$\frac{1-\beta}{3-3\beta+\beta^2} \le \frac{f\alpha}{d} < \frac{(1-\beta)(2-\alpha+\beta-\beta^2)}{4-\alpha-2\beta}, \tag{27}$$

equilibrium payoffs are given by

$$\begin{aligned} V_3^*(0) &= -\beta A_2 \left(\frac{a(2-\alpha-\beta) + e\alpha(1-\beta)(4-\alpha-2\beta)}{1-\alpha+\beta-\beta^2} \right) \\ W_3^*(0) &= \beta A_2 \left(\frac{d(2-\alpha-\beta) - f\alpha(1-\beta)(4-\alpha-2\beta)}{1-\alpha+\beta-\beta^2} \right) - b. \end{aligned} \tag{28}$$

Equilibrium strategies are

$$\begin{aligned} t_2^* &= 0 \\ t_1^* &= \frac{W_3^*(0) + b - d\beta^2}{d\beta(1-\beta)} \end{aligned} \tag{29}$$

$$g_3^* = \frac{1-\alpha}{1-\alpha+\beta-\beta^2}, \quad g_2^* = 0, \quad g_1^* = 1. \tag{30}$$

For

$$\frac{f\alpha}{d} \geq \frac{(1-\beta)(2-\alpha+\beta-\beta^2)}{4-\alpha-2\beta}, \tag{31}$$

equilibrium payoffs and strategies are given by

$$\begin{aligned} V_3^*(0) &= -\beta^2 a \\ W_3^*(0) &= \beta^2 d - b \\ t_2^* &= t_1^* = 0 \\ g_3^* &= 1, \quad g_2^* = 0, \quad g_1^* = 0. \end{aligned} \tag{32}$$

Proof: The pure behavior strategies of the inspectee are, as before, $\{g_3 = 1, g_2 = g_1 = 0\}$, $\{g_3 = 0, g_2 = 1, g_1 = 0\}$, $\{g_3 = g_2 = 0, g_1 = 1\}$ and $\{g_3 = g_2 = g_1 = 0\}$. His corresponding Nash equilibrium conditions are

$$\begin{aligned} W_3^*(0) &\geq d\beta((1-\beta)t_1^* + \beta(1-t_3)) - b \\ W_3^*(0) &\geq (1-\alpha)(d(1-\beta)t_1^* + d\beta - b) + \alpha(-f + W_2^*(0)) \\ W_3^*(0) &\geq (1-\alpha)(-f\alpha + d(1-t_1^*) - b) + \alpha(-f + W_2^*(0)) \\ W_3^*(0) &\geq (1-\alpha)(-f\alpha) + \alpha(-f + W_2^*(0)), \end{aligned} \tag{33}$$

where $W_2^*(0)$ is given by (5) with $t_2 = 0$.

Consider first the case in which (27) is satisfied. With (28) and substituting for t_1^* from (29), we see that the first and third conditions are fulfilled as equality, whereas the second condition is equivalent to the left-hand inequality in (27). The fourth condition, after some algebra, is equivalent to

$$\frac{f\alpha}{d}(4-\alpha-2\beta) \geq (1-\alpha+\beta-\beta^2)(2-\beta)\frac{b}{d} + \alpha - 2\beta - \beta^2.$$

Replacing b/d on the right hand side of the above inequality with its maximum value L_2' as given in (26) we obtain the condition

$$\frac{f\alpha}{d} \geq \frac{1-\beta}{3-3\beta+\beta^2},$$

which is again the left-hand inequality in (27). The Nash equilibrium condition for the inspector in case of illegal behavior is

$$\begin{aligned} V_3^*(0) \geq g_3^*(-a)\Big[&(1-\beta)t_2 + \beta((1-\beta)t_1 + \beta)\Big] \\ &+ (1-g_3^*)\Big[(1-\alpha)[g_2^*(-a)((1-\beta)(t_1-t_2) + \beta(1-t_2)) \\ &\qquad + (1-g_2^*)(-e\alpha - a(1-t_1))] + \alpha[-e + V_2^*(t_2)]\Big] \end{aligned} \tag{34}$$

Substituting from (30), the coefficient of t_1 in the above expression vanishes. For any t_1, the inspector's payoff is maximized for $t_2 = 0$, since the inspectee does not violate at t_2. Setting $t_2 = 0$ in (34) we see that it is satisfied as equality. Finally, the right-hand inequality in (27) guarantees that t_1^* is positive.

Next consider the case in which (31) is satisfied. With (32) the first condition in (33) is satisfied as equality. The third condition is exactly equivalent to (31) while the second condition is equivalent to

$$\frac{f\alpha}{d} \geq \frac{(1-\beta)(\alpha(1-\beta) + \beta(2-\beta))}{2-\beta+\alpha(1-\beta)}, \tag{35}$$

which is also fulfilled by (31) since

$$\begin{aligned} &\frac{(1-\beta)(2-\alpha+\beta-\beta^2)}{4-\alpha-2\beta} - \frac{(1-\beta)(\alpha(1-\beta)+\beta(2-\beta))}{2-\beta+\alpha(1-\beta)} \\ &= \frac{(1-\alpha)(2-3\beta+\beta^2)^2}{(2+\alpha(1-\beta)-\beta)(4-\alpha-2\beta)} > 0. \end{aligned}$$

The fourth condition, after some algebra, is equivalent to

$$\frac{f\alpha}{d}(4-\alpha-2\beta) \geq (2-\beta)\left((1-\alpha)\frac{b}{d} - \beta^2\right) + \alpha.$$

Replacing b/d on the right hand side of the above inequality with its maximum value L_2' as given in (26) we obtain the condition

$$\frac{f\alpha}{d} \geq \frac{1-\beta^2(2-\beta)}{3-2\beta},$$

which is also fulfilled by virtue of (31) since

$$\frac{(1-\beta)(2-\alpha+\beta-\beta^2)}{4-\alpha-2\beta} - \frac{1-\beta^2(2-\beta)}{3-2\beta} = \frac{(1-\alpha)(2-\beta)(1-\beta)^2}{(3-2\beta)(4-\alpha-2\beta)} > 0.$$

Finally, $t_2^* = t_2^* = 0$ are obviously best replies to the inspectee's equilibrium strategy $g_3^* = 1$. □

Remarks on Theorem 3:

1. The equilibrium payoffs (28) are quite complicated. The analytical solutions were in fact obtained by brute force using vertex enumeration of the convex polyhedra associated with an equivalent bimatrix game, programmed on a computer-algebra system [1].
2. When condition (26) is not fulfilled the inspectee behaves legally in the proper subgame which arises after a false alarm, but not necessarily in the rest of the game tree. The solutions can also be computed, but as we argue at the beginning of the next section, these situations are of less interest and will therefore not be pursued further.

4 Any number of interim inspections

The special equilibria of Theorem 3 are of questionable practical value, since placing *interim* inspections at the beginning of the reference period is a contradiction in terms. In solution (32) for example, the comparatively large false alarm costs to the inspectee (condition (31)) compel him to violate immediately in order to avoid false alarms altogether, and the inspector must react by also inspecting immediately. Although justifiable from the theoretical point of view, this is not likely to be an acceptable inspection strategy in real situations. By reducing the number of inspections by one the chance of a false alarm is reduced, leading to solution (5) with a "genuine" interim inspection. In the sequel, we take the point of view that the number of interim inspections should always be chosen such that, given the inspectee's utilities d, f and error probabilities α, β, the equilibrium inspection times are positive. For unbiased inspection procedures the number of interim inspections satisfying this requirement will never be less than one, see Theorem 1. We shall therefore generalize only the "unsaturated" equilibria (Theorems 1 and 2) to an arbitrary number of interim inspections. Condition (36) in the following theorem guarantees that $t_k > 0$.

Theorem 4 *For unbiased inspection procedures, Nash equilibria of the inspection game with k interim inspections, the extensive form of which is a straightforward generalization of Figure 5, are given as follows: Let*

$$\{t_k, \ldots, t_1 \mid 0 = t_{k+1} < t_k < \ldots < t_1 < 1\}$$

be the set of pure strategies for the inspector and $g_{k+1}, \ldots, g_1$ the inspectee's corresponding behavior strategies.

Under the assumption

$$\frac{f\alpha}{d} < \frac{A_{k+1}}{B_{k+1} + \frac{\beta}{1-\beta}}, \tag{36}$$

provided

$$\frac{b}{d} < A_{k+1} - \frac{f\alpha}{d}(B_{k+1} - k) =: L_{k+1} \tag{37}$$

the inspectee behaves illegally. Equilibrium payoffs to inspector and inspectee are, respectively,

$$\begin{aligned} V^*_{k+1}(t_{k+1}) &= -aA_{k+1}(1 - t_{k+1}) - e\alpha B_{k+1} \\ W^*_{k+1}(t_{k+1}) &= dA_{k+1}(1 - t_{k+1}) - f\alpha B_{k+1} - b. \end{aligned} \tag{38}$$

A (positive) equilibrium strategy of the inspector is given by

$$t^*_j - t^*_{j+1} = (1-\beta)A_{j+1}(1 - t^*_{j+1}) - \frac{f\alpha}{d}\Big((1-\beta)B_{j+1} + \beta\Big) \tag{39}$$

for $j = 1, \ldots, k$ and $t^*_{k+1} = 0$. The inspectee's equilibrium strategy is given by

$$g^*_{k+1} = A_{k+1}, \quad g^*_j = \frac{j(1-\alpha) - (j-1)\beta}{j(1-\alpha)} A_j, \quad j = 1, \ldots, k, \tag{40}$$

where A_j and B_j are given by

$$A_j = \frac{1}{1 + (j-1)(1-\beta)}, \quad B_j = \frac{j}{2}(1 - A_j),\ j = 1, 2, \ldots\ . \tag{41}$$

Provided

$$\frac{b}{d} \geq L_{k+1} \tag{42}$$

the inspectee behaves legally with payoff

$$W^*_{k+1}(t_{k+1}) = -kf\alpha. \tag{43}$$

The equilibrium strategy of the inspector is determined by the cone of deterrence, as given by (with the convention $\sum_{i=0}^{-1} = 0$)

$$\frac{b}{d} \geq (k-1)\frac{f\alpha}{d} + (1-\beta)\sum_{i=0}^{k-j-1}(t_{k-i-j} - t_{k-j+1})\beta^i + (1 - t_{k-j-1})\beta^{k-j}, \tag{44}$$

for $j = 0, \ldots, k$, with (39) being an element of this cone. His equilibrium payoff is

$$V^*_{k+1}(t_{k+1}) = -ke\alpha. \tag{45}$$

Proof: In terms of his undominated pure strategies, the inspectee's Nash equilibrium conditions are determined by the following set of inequalities:

$$
\begin{aligned}
W^*_{k+1}(t_{k+1}) &\geq d\Big[(1-\beta)(t^*_k - t_{k+1}) + \beta\Big[(1-\beta)(t^*_{k-1} - t_{k+1}) + \\
&\qquad \beta[(1-\beta)(t^*_{k-2} - t_{k+1}) + \ldots + \beta(1-t_{k+1})]\ldots\Big]\Big] - b \\
W^*_{k+1}(t_{k+1}) &\geq -f\alpha + d\Big[(1-\beta)(t^*_{k-1} - t^*_k) + \beta\Big[(1-\beta)(t^*_{k-2} - t^*_k) + \\
&\qquad \beta[(1-\beta)(t^*_{k-3} - t^*_k) + \ldots + \beta(1-t^*_k)]\ldots\Big]\Big] - b \\
&\ \vdots \\
W^*_{k+1}(t_{k+1}) &\geq -kf\alpha + d(1-t^*_1) - b \\
W^*_{k+1}(t_{k+1}) &\geq -kf\alpha.
\end{aligned}
\tag{46}
$$

Let us consider first the illegal equilibrium. There the last inequality of (46) is satisfied by virtue of (37). We show by induction that all other inequalities are satisfied as equalities. For $k = 2$ we have proved this in Theorem 2. Now assume that for $k-1$ interim inspections the set of inequalities corresponding to (46) holds as a set of equalities. Then we can write (46) as follows:

$$
\begin{aligned}
W^*_{k+1}(t_{k+1}) &\geq \beta W_k(t^*_k) + d(t^*_k - t_{k+1}) - (1-\beta)b \\
W^*_{k+1}(t_{k+1}) &\geq -f\alpha + W^*_k(t^*_k) \\
W^*_{k+1}(t_{k+1}) &\geq -2f\alpha + W^*_{k-1}(t^*_{k-1}) \\
&\ \vdots \\
W^*_{k+1}(t_{k+1}) &\geq -(k-1)f\alpha + d(1-t^*_1) - b,
\end{aligned}
\tag{47}
$$

with $W^*_j(t^*_j)$ given by

$$
W^*_j(t^*_j) = dA_j(1-t^*_j) - f\alpha B_j - b \tag{48}
$$

and $t^*_j - t^*_{j+1}$ given by (39) for $j = 1,\ldots,k+1$. Now we have with (39), for $j = 2,\ldots,k+1$,

$$
\begin{aligned}
-f\alpha + W_{j-1}(t^*_{j-1}) &= -f\alpha + dA_{j-1}(1-t^*_{j-1}) - f\alpha B_{j-1} - b \\
&= dA_{j-1}(1-t^*_j) - dA_{j-1}(t^*_{j-1} - t^*_j) - f\alpha(B_{j-1}+1) - b \\
&= dA_{j-1}(1-(1-\beta)A_j)(1-t^*_j) \\
&\quad - f\alpha(-A_{j-1}(1-\beta)B_j + \beta) + B_{j-1} + 1) - b \\
&= dA_j(1-t^*_j) - f\alpha B_j - b \\
&= W^*_j(t^*_j),
\end{aligned}
\tag{49}
$$

since the A_j and B_j as given by (41) satisfy the recursive relations

$$A_j = \frac{A_{j-1}}{1+(1-\beta)A_{j-1}}, \quad B_j = \frac{B_{j-1}+1-\beta A_{j-1}}{1+(1-\beta)A_{j-1}}.$$

Therefore we see that the second inequality in (47) is fulfilled as equality. Furthermore using (49) for $j = k$ we see that the third inequality in (47) in identical to the one preceding it and therefore again fulfilled as equality. Similarly the remaining inequalities in (47) are shown to hold as equalities. It remains to demonstrate that the first inequality in (47) also holds as equality. This claim is, together with the second equality, equivalent to

$$W_k^*(t_k^*) = \frac{1}{1-\beta}(d(t_k^* - t_{k+1}) + f\alpha) - b.$$

With (48), however, this is equivalent to $t_k^* - t_{k+1}$ as given by (39) for $j = k$ and $t_{k+1}^* = t_{k+1}$, and with $W_{k+1}^*(t_{k+1})$ given by (38). Finally, condition (36) guarantees that $t_k^* > 0$.

Turning to the legal equilibrium, we see immediately that (43) and (44) satisfy the Nash equilibrium conditions (46).

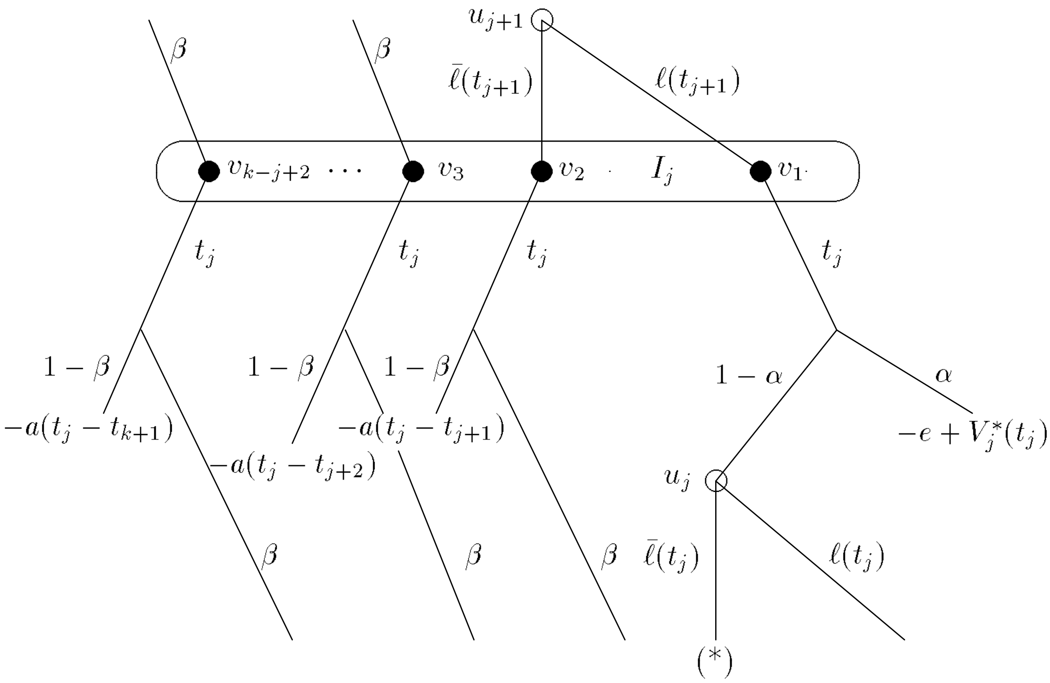

Figure 7: The inspector's information set I_j and his payoffs involving t_j.

To show the Nash condition is satisfied for the inspector, consider the inspector's information set I_j and the edges leading to and from it as shown in Figure 7. The terminal nodes are labeled with the inspector's payoff. The symbol (*) denotes the continuation in which the inspector's payoff is a function of t_j. It corresponds to payoff

$$-a[\ (1-\beta)(t_{j-1}-t_j)+\beta[(1-\beta)(t_{j-2}-t_j)+\beta[\ \ldots+\beta(1-t_j)]\ldots\ ]\ ]$$

where the number of square bracket pairs is equal to j and where the combined coefficient of t_j is simply $+a$.

For $k=2$ interim inspections it was proved in Theorem 2 that the equilibrium behavior strategy of the inspectee makes the inspector indifferent to his choice of t_2 and t_1 and that his equilibrium payoff is

$$V_2^*(t_2)=-aA_2(1-t_2)-e\alpha B_2.$$

We shall therefore assume inductively that for $k-1$ interim inspections the inspector is indifferent with respect to his choice of t_j, $j=k-1,\ldots,1$, and that his equilibrium payoff in any proper subgame beginning at time t_j is

$$V_j^*(t_j)=-aA_j(1-t_j)-e\alpha B_j,\quad j=k-1,\ldots,1. \tag{50}$$

At equilibrium, the realization probabilities $\rho_{k+1}(v_i)$ of the decision points v_i in I_j for $i=1,\ldots,k+2-j$ are given by

$$\begin{aligned}
\rho_{k+1}(v_{k-j+2}) &= A_{k+1}\beta^{k-j}\\
\rho_{k+1}(v_i) &= (1-A_{k+1})(1-g_k^*)\cdots(1-g_{i+j}^*)g_{i+j-1}^*(1-\alpha)^{k-i-j+2}\beta^{i-2},\\
&\qquad\qquad i=k-j+1,\ldots,2\\
\rho_{k+1}(v_1) &= (1-A_{k+1})(1-g_k^*)\cdots(1-g_{j+1}^*)(1-\alpha)^{k-j}.
\end{aligned} \tag{51}$$

From Figure 7 and making use of (50), the coefficient of t_j in the inspector's expected payoff is

$$-a(1-\beta)\sum_{i=2}^{k-j+2}\rho_{k+1}(v_i)+a[(1-\alpha)g_j^*+\alpha A_j]\rho_{k+1}(v_1).$$

If this coefficient vanishes, then the inspector is indifferent as to his choice of t_j. We therefore wish to demonstrate that

$$[(1-\alpha)g_j^*+\alpha A_j]\rho_{k+1}(v_1)=(1-\beta)\sum_{i=2}^{k-j+2}\rho_{k+1}(v_i). \tag{52}$$

According to the induction assumption, we have

$$[(1-\alpha)g_j^*+\alpha A_j]\rho_k(v_1)=(1-\beta)\sum_{i=2}^{k-j+1}\rho_k(v_i). \tag{53}$$

We can write (51) in the form

$$\begin{aligned}\rho_{k+1}(v_{k-j+2}) &= A_{k+1}\beta^{k-j}\\ \rho_{k+1}(v_{k-j+1}) &= (1-A_{k+1})g_k^*(1-\alpha)\beta^{k-j-1}\\ \rho_{k+1}(v_i) &= \frac{1-A_{k+1}}{1-A_k}(1-g_k^*)(1-\alpha)\rho_k(v_i),\ \ i=k-j,\ldots,1.\end{aligned} \tag{54}$$

Thus (52) is equivalent to

$$[(1-\alpha)g_j^*+\alpha A_j]\frac{1-A_{k+1}}{1-A_k}(1-g_k^*)(1-\alpha)\rho_k(v_1)=(1-\beta)\sum_{i=2}^{k-j+2}\rho_{k+1}(v_i),$$

or to

$$\begin{aligned}&[(1-\alpha)g_j^*+\alpha A_j]\frac{1-A_{k+1}}{1-A_k}(1-g_k^*)(1-\alpha)\rho_k(v_1)=\\ &\qquad (1-\beta)\Bigg[A_{k+1}\beta^{k-j}+(1-A_{k+1})g_k^*(1-\alpha)\beta^{k-j-1}\\ &\qquad +\frac{1-A_{k+1}}{1-A_k}(1-g_k^*)(1-\alpha)\left(\sum_{i=2}^{k-j+1}\rho_k(v_i)-\rho_k(v_{k-j+1})\right)\Bigg].\end{aligned}$$

With the induction assumption (53) this becomes

$$0=A_{k+1}\beta+(1-A_{k+1})g_k^*(1-\alpha)-\frac{1-A_{k+1}}{1-A_k}(1-g_k^*)(1-\alpha)A_k,$$

which is fulfilled by (40) and (41). Thus the inspector is indifferent to his choice of t_j, $j=k,\ldots,1$.

Finally we determine inspector's equilibrium payoff. Since he is indifferent as to choice of t_j, $j=k,\ldots,1$, let $1-\epsilon<t_k<t_{k-1}<\ldots t_1<1$ for ϵ arbitrarily small. Then, apart from terms involving ϵ, the inspector's payoff V_{k+1} is given by

$$\begin{aligned}V_{k+1}=-aA_{k+1}+(1-A_{k+1})\Big[&\alpha(-e+V_k^*(t_k))+\\ &+(1-\alpha)(1-g_k^*)[\alpha(-e+V_{k-1}^*(t_{k-1}))+\\ &+\ \ldots\\ &+(1-\alpha)(1-g_2^*)(-e\alpha)]\ \cdots\ \Big]\end{aligned} \tag{55}$$

It follows from Theorem 2 that $V_3=V_3^*(0)$. Therefore we assume inductively that

$$V_k=V_k^*(0), \tag{56}$$

or, equivalently, that

$$\begin{aligned}\frac{V_k^*(0)+aA_k}{1-A_k}=\ &\alpha(-e+V_{k-1}^*(t_{k-1}))+\\ &+(1-\alpha)(1-g_{k-2}^*)\Big[\alpha(-e+V_{k-1}^*(t_{k-1}))+\\ &+\ \ldots\\ &+(1-\alpha)(1-g_2^*)(-e\alpha)]\ \cdots\ \Big]\end{aligned} \tag{57}$$

Substituting this into (55), we have

$$V_{k+1} = -aA_{k+1} + (1 - A_{k+1})\left[\alpha(-e + V_k^*(t_k)) + \frac{(1-\alpha)(1-g_k^*)}{1-A_k}(V_k^*(0) + aA_k)\right]$$

which, with (34) and $t_k \to 1$, is equivalent to

$$V_{k+1} = -aA_{k+1} - e\alpha(1 - A_{k+1})\left[(1 - \alpha B_k) + \frac{(1-\alpha)(1-g_k^*)}{1-A_k}B_k\right] = V_{k+1}^*(0).$$

where the last equality follows from (40) and (41). □

5 Discussion

For (39) to be an equilibrium we require $t_k^* - t_{k+1} > 0$, or equivalently,

$$R(k+1) := A_{k+1} - \frac{f\alpha}{d}\left(B_{k+1} + \frac{\beta}{1-\beta}\right) > 0. \tag{58}$$

We have $R(k+1) < R(k)$, so that an upper limit k_0 on the number of interim inspections for the solution of Theorem 4 to be valid is given by $R(k_0) = 0$. On the other hand, the inspectee will behave legally for $k < k_0$ interim inspections given by the condition

$$W_k^*(t_{k+1}) < -kf\alpha. \tag{59}$$

The situation is illustrated in Figure 8 for appropriate values of α, β, b, d and f. Numerical calculations indicate that $k_1 > k_0$ for reasonable values of these parameters.

It is interesting to compare the equilibria we have obtained with the situation in which the inspections are unobservable. The game is then simultaneous rather than sequential. Diamond [4] gives the solution for this case, but treated as a zero-sum game in which the payoff to player 2 is the time to detection and $\alpha = \beta = 0$. However it is straightforward to generalize his solution to match the assumptions of our nonzero sum model for the simplest case $k = 1$. The generalization is as follows [9]:

An equilibrium strategy for the inspector is to choose his single interim inspection time t_1 on an interval $0 \le t_1 \le \kappa < 1$ according to the distribution function

$$F^*(t_1) = -\frac{1}{1-\beta} \cdot \log\left(\frac{(1-t_1)(1-\beta) - \frac{f\alpha}{d}}{1 - \beta - \frac{f\alpha}{d}}\right), \tag{60}$$

where κ is given by

$$\kappa = (1 - e^{\beta - 1})\left(1 - \frac{f\alpha}{d} \cdot \frac{1}{1-\beta}\right). \tag{61}$$

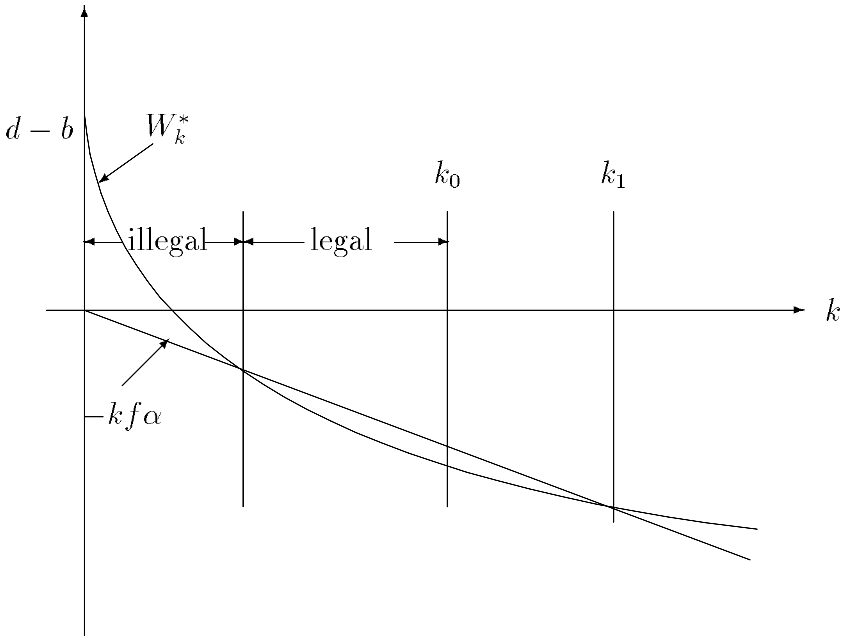

Figure 8: Legal and illegal behavior as determined by k.

The inspectee randomizes similarly, whereby his distribution function has an atom at $t = 0$. His equilibrium payoff is

$$\tilde{W}_2^* = d(1 - \kappa) - f\alpha - b. \tag{62}$$

For $\tilde{W}_2^* < -f\alpha$, or equivalently for

$$\frac{b}{d} > 1 - \kappa \tag{63}$$

the inspectee will behave legally. Thus, unlike the sequential illegal game, *both* players play mixed (randomized) strategies. Comparing (62) with (5), we see that, not surprisingly, the unobservability places the inspectee at a disadvantage. That is, for $\alpha + \beta < 1$ and $f < d$, we have

$$\tilde{W}_2^* - W_2^*(0) = d\left(\frac{1}{e^{1-\beta}} - \frac{1}{2-\beta}\right)\left(1 - \frac{f}{d} \cdot \frac{\alpha}{1-\beta}\right) < 0. \tag{64}$$

Consistent with this, the limit for b/d to induce the inspectee to legal behavior is lower in the non-sequential model.

An often-discussed proposal to reduce routine inspection effort while maintaining the timeliness of an inspection regime is to replace scheduled inspections with a smaller number of randomly chosen, unannounced inspections. The unpredictability aspect of such measures is appealing, as they would seem to place the potential violator in a permanent state of uncertainty and thus serve to deter illegal activity. In the context of routine verification under the Nuclear Weapons Non-Proliferation Treaty, Sanborn [8] contrasts the intuitive attractiveness of unannounced, random inspections with the substantial practical difficulties of implementing them and with the burden to the inspected party in trying to accommodate them. Significantly, in the model presented here, the inspector's equilibrium strategy is not mixed. This is a consequence of the modelling assumption that the inspections are observable and that they can occur at any time on the reference interval. Thus the inspection schedule can be common knowledge.

Acknowledgement

The authors would like to thank Bernhard von Stengel for his very helpful suggestions on the material presented here.

References

[1] M. J. Canty, *Resolving Conflicts with Mathematica: An computer-oriented introduction to the theory of two-person games*, Academic Press, London, 2003.

[2] M. J. Canty, D. Rothenstein, R. Avenhaus, Timely inspection and deterrence, *European Journal of Operational Research* 131 (2001) 108–223.

[3] M. J. Canty, D. Rothenstein, R. Avenhaus, A sequential attribute sampling inspection game for item facilities, *Naval Research Logistics* 48 (2001) 496–505.

[4] H. Diamond, Minimax policies for unobservable inspections, *Mathematics of Operations Research* 7(1) (1982) 139–153.

[5] D. M. Kilgour, Site selection for on-site inspection in arms control, *Arms Control* 13(3) (1992) 439.

[6] D. Rothenstein, Imperfect inspection games over time, Discussion Paper 151, Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, 1997.

[7] B. von Stengel, Recursive inspection games, IASFOR Report No. S-9106, Institut für Angewandte Systemforschung und Operations Research, Universität der Bundeswehr München, August, 1991.

[8] J. Sanborn, Considerations regarding the scheduling of unannounced random inspections, Proceedings of the IAEA Symposium on International Safeguards, Vienna, 2001, IAEA-SM-367/12/07, CD-ROM.

[9] K. Sohrweide, Diamond's Inspektionsspiel mit Fehlern 1. und 2. Art, Diploma Thesis, Universität der Bundeswehr München, Germany, 2002.