Bargaining, Voting and Value*

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March 27, 2003

Abstract

This paper addresses the following issue: If a set of agents bargain on a set of feasible alternatives 'under the shadow' of a voting rule, that is, any agreement can be enforced if a 'winning coalition' supports it, which general agreements are likely to arise? In other terms: Which is the influence that the voting rule used to settle agreements can project on the outcome? To give an answer we model the situation as an extension of Nash bargaining problem in which an arbitrary voting rule replaces the unanimity to settle agreements by n players. This provides a setting in which a natural extension of Nash's solution (consistent with Shapley's value) is obtained axiomatically by, basically, integrating into a single system Nash's and Shapley's characterizing systems.

Keywords: Bargaining problems, value, voting, TU and NTU games.

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*We want to thank the audience of the Sunday seminar at the Center for Rationality at the Hebrew University of Jerusalem where a preliminar version of this paper was presented. We are specially indebted to Sergiu Hart and Abraham Neyman for their insightful comments. We want also to thank David Galloway, whose inspiring interview on 23.06.02 generated some of the ideas in this work, and to the EU Commission, which under project "Searching & Teaching the EU Institutions" financed the interview. This research has been supported by the Spanish Ministerio de Ciencia y Tecnología under project BEC2000-0875, and by the Universidad del País Vasco under project UPV00031.321-H-14872/2002. The first author also acknowledges financial support from the Spanish M. C. y T. under the Ramón y Cajal Program. Part of this paper was written while the second author was visiting the Department of Economic Analysis at the University of Alicante, whose hospitality is gratefully acknowledged.

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1 Introduction

Nash (1950) proposes and characterizes axiomatically a cooperative 'solution' to the bargaining problem, in the spirit of von Neumann and Morgenstern's (1944) notion of 'value' of a zero-sum two-person game, as a rational expectation of the 'level of satisfaction' or expected utility payoff of a 'highly rational' player engaging into a bargaining situation with another rational player. In Shapley (1953) a similar notion of value for transferable utility (TU) games is also proposed and characterized. Since then, several attempts to obtain a satisfactory 'solution' or a general notion of 'value' in the more general nontransferable utility (NTU) context have been made by different authors. Harsanyi (1959, 1963), Shapley (1969), Kalai and Samet (1985), Maschler and Owen (1989, 1992) (see also Hart and Mas-Colell (1996)) have provided different proposals[1]. As bargaining problems are two-person NTU games, and TU games are a particular case of the NTU model, the way of proceeding in all cases (but in Hart and Mas-Colell (1996)) consists of looking for a notion of value for NTU games that coincides with Nash's solution and Shapley's value when restricted to bargaining problems and TU games, respectively. As is well known these extensions differ, and it is not clear on which grounds to claim superiority for any of them over the others. Perhaps there is no definite answer for this dispute. The reason for this might be an 'excess of abstraction' in the NTU model itself. The NTU model consists basically of a feasible set of utility vectors for each particular coalition. That's all. This seems to be enough in the simpler models that serve as term of reference, bargaining problems, where coalitions play no role, and TU games, where the feasible sets (or the configuration of players' preferences) are very particular. While in the general case the NTU minimalistic model seems too general to be sure ground to provide sufficient intuition.

The results presented in this paper appear to support this view. Nevertheless the original motivation of this work was somewhere else. We were interested in understanding the influence that the voting rule used to settle agreements by a set of bargainers can project on the bargaining outcome. More precisely, and this is exactly what this paper is about: If a set of agents bargain on a set of feasible alternatives 'under the shadow' of a voting rule, that is, any agreement can be enforced if a 'winning coalition' supports it, which general agreements are likely to arise? Or put into Nash's classical terms, which agreements a rational agent can expect to arise at the prospect of engaging in such a situation? The relevance of the issue is clear in many contexts. It is often the case in any kind of committee that uses a voting rule to make decisions that the final vote is

[1] And more recently, among others, Serrano (1997) and Orshan and Zarzuelo (2000). See McLean (2002) for a recent review.

but the formal settlement of a bargaining process in which the issue to vote upon has been adjusted up to gain the acceptance of all members[2]. That the voting rule by means of which agreements are settled conditions the outcome of negotiations seems intuitively obvious.

With this motivation in mind and in the spirit common to the classical results of von Neumann-Morgenstern, Nash and Shapley alluded above, in this paper we explore an extension of Nash's (1950) model of a bargaining problem. We see Nash's original model as consisting of two ingredients, a set of (two) players with von Neumann-Morgenstern (1944) preferences over a set of feasible agreements, and a voting procedure (unanimity) to settle agreements. Thus the kind of situation we are interested in can be described by a natural generalization of this model (and its traditional extension to n players), by considering arbitrary voting rules[3]. In this framework, on similar grounds as in Nash (1950) or Shapley (1953), axioms such as efficiency, anonymity, independence of irrelevant alternatives, invariance w.r.t. positive affine transformations and null-player can be adapted keeping their meaning and motivation. Then, by adding 'transfer', the lattice property used by Dubey (1975) in order to characterize the Shapley (-Shubik) value on the domain of simple games, a 'value' or a 'solution' for this class of bargaining situations is obtained axiomatically. As the model includes as particular cases the classical Nash bargaining problems (when the rule is unanimity) and the class of simple TU games (when the feasible set, or the players' preferences, meet certain conditions), it can be shown that this value is consistent with both the Nash's solution and Shapley's value. On the other hand, as this class of situations can be seen as a class of NTU games, some well known solutions can be compared with the one emerging from these axioms on this class. Finally, in order to provide arguments in support of the transfer condition, the class of problems under consideration is widened by admitting random voting rules. The result is extended to this wider domain by assuming two conditions of rationality under risk about the voting rule that entail the transfer condition.

The rest of the paper is organized as follows. In section 2 the model is precisely formulated. In section 3 the natural extension of some of Nash's and Shapley's axioms in the more general framework considered here are provided. Section 4 contains the first

[2]It was the constatation as a matter of fact by D. Galloway (in the interview mentioned in the acknowledgements) of this being the case very often in the decisions made by qualified majority by the EU Council of Ministers, along with the fact that the redistribution of weights in the Council is obviously the most problematic issue whenever new states enter the EU, which motivated this research. See Galloway (2001) for the excellent account of Nice 2000 from the point of view of a well informed and experienced insider.

[3]A setting including both a voting rule and a set of feasible agreements has been already considered by some authors (see e.g. Keiding and Peleg (2001)), but, as far as we know, with different purposes and within a completely different approach concerned with social choice issues.

characterizing result and its proof, along with the relationships with Nash's solution and the Shapley value. The case in which random voting rules are admitted is considered in section 5, where some conditions whose meaning and compellingness is argued upon justify 'transfer' and permit to extend the characterization to this wider domain. The meaning of the results is discussed in section 6 along with some lines of further research.

2 Bargaining under the shadow of a voting rule

Thus we are concerned with a situation in which a set of agents bargain on a set of feasible alternatives 'under the shadow' of a voting procedure. That is, any agreement can be enforced if a 'winning coalition' supports it. The classical bargaining problem (Nash, 1950) corresponds to the case in which *two* agents agree by means of the *unanimity* rule. Now we list carefully the elements that configure this general two-ingredient model. The set $N = \{1, ..., n\}$ will label the *seats* of the decision procedure by means of which agreements are to be settled. A *vote configuration* is a possible result of a vote. As only yes/no voting is considered, a vote configuration can be represented by the set of 'yes'-voters. So, any $S \subseteq N$ represents the result of a vote in which players occupying seats in S voted 'yes' and those in $N \backslash S$ voted 'no'. An N-voting rule is specified by a set $W \subseteq 2^N$ of *winning* (i.e. which would lead to passing a decision) vote configurations such that (i): The unanimous 'yes' leads to the acceptance of the proposal: $N \in W$; (ii): The unanimous 'no' leads to the rejection of the proposal: $\emptyset \notin W$; (iii): If a vote configuration is winning, then any other configuration containing it is also winning: If $S \in W$, then $T \in W$ for any T containing S; (iv): If a vote configuration leads to the acceptance of a proposal, the opposite configuration will not: If $S \in W$ then $N \backslash S \notin W$. $\mathcal{W}$ will denote the set of all such N-voting rules. For voting rule W, $M(W)$ will denote the set of *minimal winning* configurations, i.e. those that do not contain any other winning configuration. For any minimal winning configuration $S \in M(W)$ $(S \neq N)$, W_S^* will denote the rule that results from W by eliminating S from the set of winning configurations, that is, $W_S^* := W \setminus \{S\}$. For any permutation $\pi : N \to N$, πW denotes the voting rule $\pi W := \{\pi(S) : S \in W\}$. A voting rule W is *symmetric* if $\pi W = W$, for any permutation π.

When a set of n voters or *players* uses an N-voting rule they will be labelled by the seats in N they occupy, and we will refer to the subset of players denoted by a subset $S \subseteq N$ as *coalition* S. Thus, depending on the context any $S \subseteq N$ will be referred to either as a vote configuration (of seats) or as a coalition (of players). We will also speak of *winning coalitions* for a given N-rule with an obvious meaning. A seat/player $i \in N$ is said to be a *null seat/player* in W, if for any coalition S, $S \in W$ if and only if $S \setminus \{i\} \in W$.

We assume also that a set of n (N-labelled) *players* makes decisions by means of rule

W in the following sense. They can reach any alternative within a set A as far as a winning coalition supports it. Among the alternatives in A a distinguished one, denoted a, represents the case of disagreement of *status quo* where players will remain in case of not reaching any agreement, that is, if no alternative obtains the support of a winning coalition. The players can also agree in the same way upon any *lottery* over this set, i.e., probability distributions over A with finite support. It is also assumed that every player has expected utility (von Neumann and Morgenstern, 1944) (vNM) preferences over this set of lotteries, denoted $\mathfrak{L}(A)$. Any lottery $l \in \mathfrak{L}(A)$ will be represented by a map $l : A \to [0,1]$ s.t. $\sum_{x \in A} l(x) = 1$, where $l(x)$ represents the probability of alternative x. The preferences of player i are denoted $\preccurlyeq_i$, and can be represented by the expected value of a utility function $u_i : A \to R$.

The situation, as specified so far, is fully described by a five-tuple

$$(N, W, A, a, \{\preccurlyeq_i\}_{i \in N}).$$

Replacing the preferences of every player $\preccurlyeq_i$ by a utility function u_i representing his/her vNM preferences, the model can be alternatively presented as

$$(N, W, A, a, \{u_i\}_{i \in N}).$$

Or further following Nash, denoting $u(l) := (u_1(l), .., u_n(l))$, where $u_i(l) := \sum_{x \in A} l(x) u_i(x)$ for any $l \in \mathfrak{L}(A)$, we can consider the set of utility vectors

$$D := u(\mathfrak{L}(A)) = \{u(l) : l \in \mathfrak{L}(A)\} \subseteq R^N,$$

together with the particular vector associated to the disagreement point $d := u(a)$, as a summary of the situation concerning the players' decision. Accepting this simplification, the situation can be summarized by a pair (B, W), where $B = (D, d)$ is a classical n-person bargaining problem, and W is the N-voting rule to enforce agreements. We assume that $D_d := \{x \in D : x \geq d\}$[4] is *compact*, and D is *convex* and *comprehensive* (i.e., $x \leq y \in D \Rightarrow x \in D$) set containing d, and such that there exists some $x \in D$ s.t. $x > d$. The first condition is quite reasonable and would be ensured, for instance, if the number of feasible deterministic agreements were finite. The second is implied by the expected utility preferences assumption. And the third can be justified by a free disposal assumption. $\mathcal{B}$ denotes the set of all such bargaining problems. For any permutation $\pi : N \to N$, $\pi B := (\pi(D), \pi(d))$ will denote the bargaining problem that results from B by π-permutation of its coordinates, so that for any $x \in R^N$, $\pi(x)$ denotes the vector in R^N s.t. $\pi(x)_{\pi(i)} = x_i$. A bargaining problem B is *symmetric* if $\pi B = B$, for any permutation π.

[4]We will write for any $x, y \in R^N$, $x \leq y$ $(x < y)$ if $x_i \leq y_i$ $(x_i < y_i)$ for all $i = 1, .., n$.

Thus, in this setting we are concerned with pairs $(B, W) \in \mathcal{B} \times \mathcal{W}$, each of which, consistently with the interpretation that accompanied its introduction, could properly be referred to as *bargaining problem B under rule W*, or for brief a *generalized bargaining problem* (B, W).

Before proceeding with the search of a 'value' or a 'solution' on this setting, let us first see how the model fits within the general NTU framework and how the classical bargaining problems and TU games fit into this model. Let $(B, W) \in \mathcal{B} \times \mathcal{W}$, we can associate with it on obvious grounds the NTU game $(N, V_{(B,W)})$, such that for any $S \subseteq N$,

$$V_{(B,W)}(S) := \begin{cases} D^S & \text{if } S \in W, \\ ch(d)^S & \text{if } S \notin W, \end{cases}$$

where $x^S \in R^S$ denotes the S-projection of any $x \in R^N$, $D^S := \{x^S : x \in D\} \subseteq R^S$, and $ch(d)$ denotes the comprehensive hull of $\{d\}$. The n-person classical bargaining problem corresponds to the case in which W is the unanimity rule $W = \{N\}$, with N as the only winning coalition. While when the bargaining ingredient in the generalized model is the bargaining problem $\Lambda := (\Delta, 0)$, where $\Delta := \left\{x \in R^{\dot{N}} : \sum_{i \in N} x_i \leq 1\right\}$, the associated NTU game $V_{(\Lambda,W)}$ is equivalent to a TU game. Note that in preferences terms the situation behind Λ is the following. There is no feasible agreement which Pareto dominates any lottery over agreements b^i $(i = 1, 2, .., n)$, where b^i denotes an agreement which is optimal for player i and indifferent to the status quo for the other players (i.e., $u_i(b^i) = 1$, and $u_j(b^i) = 0$ for $j \neq i$, for suitably chosen utilities). In this case $V_{(\Lambda,W)}$ is equivalent to the *simple* TU game representing the rule v_W, given by

$$v_W(S) := \begin{cases} 1 & \text{if } S \in W, \\ 0 & \text{if } S \notin W. \end{cases}$$

Thus the family of associated NTU games $\{(N, V_{(B,W)}) : (B, W) \in \mathcal{B} \times \mathcal{W}\}$ includes all classical bargaining problems and all simple monotonic proper games. We will refer to $\Lambda = (\Delta, 0)$ as the (normalized) TU-bargaining problem.

3 Conditions for a 'value' or a 'solution'

In the class of situations described by this model the question addressed by Nash (1950) and Shapley (1953) in their respective cases can also be addressed: Which is the 'level of satisfaction' or utility that a rational player can expect from such a situation, or, in the classical terms, which is the *value* for any player of the prospect of engaging into a situation like this? We will proceed as in the two seminal papers, by stating reasonable conditions that will narrow the possible options up to a unique solution or value is characterized.

But previously a precise formulation of what we are looking for is needed. In a bargaining situation described by a pair (B, W), where $B = (D, d)$, unless no agreement is reached, the natural outcome will be a payoff vector, i.e. a point x in D (associated with the -random or not- alternative agreed upon) and a winning coalition $S \subseteq N$ supporting it. Such pairs would in fact be the outcomes to be considered in a detailed noncooperative analysis. But here, as already stated, we adopt the 'short-cut' of the cooperative approach in the classical sense. That is, we look for a vector of utilities that can be arguably supported as representing the worth for every player of the prospect of engaging in a bargaining situation under a giving voting rule specified by a pair (B, W). Or, in other terms, a vector of utilities that can be seen as a reasonable agreement by rational players in such a situation[5].

We will impose some conditions on a map $\Phi : \mathcal{B} \times \mathcal{W} \to R^N$, for vector $\Phi(B, W) \in R^N$ to be considerable as a rational agreement, or better as a reasonable expectation of utility levels of an agreement in a bargaining situation B under voting rule W. To begin with we build up into the very notion of value or solution the requirements of being feasible and no worse than the *status quo* for any player, that is, we impose as prerequisites: $\Phi(B, W) \in D$, and $\Phi(B, W) \geq d$, if $B = (D, d)$. In addition to this we require the following conditions, all but one direct adaptations of Nash's and Shapley's characterizing properties:

1. (*Eff*) *Efficiency:* For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, $\Phi(B, W)$ is not strictly dominated by any point in D. That is, there is no $x \in D$, such that $x > \Phi(B, W)$.

2. (*An*) *Anonymity:* For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, and any permutation $\pi{:}N \to N$, and any $i \in N$,

$$\Phi_{\pi(i)}(\pi(B, W)) = \Phi_i(B, W),$$

where $\pi(B, W) := (\pi B, \pi W)$. That is, $\pi(B, W)$ is the generalized bargaining problem that results from (B, W) by a consistent relabelling of seats and players.

3. (*IIA*) *Independence of irrelevant alternatives:* Let $B, B' \in \mathcal{B}$, with $B = (D, d)$ and $B' = (D', d')$, such that $d' = d$, $D' \subseteq D$ and $\Phi(B, W) \in D'$, then $\Phi(B', W) = \Phi(B, W)$, for any $W \in \mathcal{W}$.

4. (*IAT*) *Invariance w.r.t. positive affine transformations:* For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, and all $\alpha \in R^N_{++}$ and $\beta \in R^N$,

$$\Phi(\alpha * B + \beta, W) = \alpha * \Phi(B, W) + \beta,$$

[5] In the formerly alluded interview with D. Galloway (see footnote 1), in reference to the way in which negotiations in the EU's Council use to proceed, he also pointed out at the capacity of experienced negotiators to 'guess', at a certain stage of the bargaining process after some negotiating rounds, 'where more or less the final agreement will lie'.

where $\alpha * B + \beta = (\alpha * D + \beta, \alpha * d + \beta)$, denoting $\alpha * x := (\alpha_1 x_1, .., \alpha_n x_n)$, and $\alpha * D + \beta := \{\alpha * x + \beta : x \in D\}$.

5. (*NP*) *Null player*: For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, if $i \in N$ is a null player in W, then $\Phi_i(B, W) = d_i$.

6. (*T*) *Transfer:* For any two rules $W, W' \in \mathcal{W}$, and all $S \in M(W) \cap M(W')$ $(S \neq N)$:

$$\Phi(\Lambda, W) - \Phi(\Lambda, W_S^*) = \Phi(\Lambda, W') - \Phi(\Lambda, W_S'^*). \qquad (1)$$

The reader can see by herself the precise correspondence of axioms 1 to 5 with some of Nash's and Shapley's axioms. But it is worth noting some subtle differences arising in this setup. Observe that *Eff*, *IIA* and *IAT* (adaptations of Nash's axioms) concern the feasible set, while *NP* (from Shapley's system) concerns the voting rule to settle agreements, and *An* (integrating Nash's symmetry and Shapley's anonymity*)* concerns both. Note also that Nash's symmetry is not required, for a pairwise formulation of this condition in this setup, concerning at once the feasible set and the voting rule, whatever it be, is implied by *An*. In particular, fixing the rule to be the unanimity, Nash's symmetry is implied. We omit the arguments in support of every one of these conditions, which can be found in Nash's and Shapley's papers[6]. It may be worth though remarking that it seems natural in our setting to set null players's expectations to zero, or more precisely to the *status quo* level, given their null capacity to influence the outcome given the voting rule according to which final agreements are settled. Thus only 'additivity' from Shapley's system is missing. But this condition, the most criticized in Shapley's system, does not even make sense in this setting. In order to complete the axiomatic system we adapt 'transfer', a weak form of additivity or lattice property, used by Dubey (1975) in order to characterize the Shapley (-Shubik) value on the domain of simple games. In fact (1) is the adaptation of the more transparent formulation of this condition in terms of voting rules provided in Laruelle and Valenciano (2001). In words, this condition postulates that the effect of eliminating a minimal winning coalition from the set of winning ones is the same whatever the voting rule (one of whose minimal winning coalition is that coalition) as far as the bargaining component is the normalized TU-bargaining problem[7]. In comparison with the previous

[6] For a careful discussion of Nash's axioms see, e.g., chapter 1 in Binmore (1998).

[7] As shown in Laruelle and Valenciano (2001) transfer can be replaced by the weaker (in the presence of anonymity) condition of 'symmetric gain-loss', which in the present setting can be stated like this: For any rule $W \in \mathcal{W}$, and all $S \in M(W)$ $(S \neq N)$:

$$\Phi_i(\Lambda, W) - \Phi_i(\Lambda, W_S^*) = \Phi_j(\Lambda, W) - \Phi_j(\Lambda, W_S^*)$$

for all $i, j \in S$ and for all $i, j \in N \setminus S$. Consequently this condition can replace T in our characterizing system.

conditions some doubts may arise concerning the compellingness of this condition for a value. In section 5 we come back to this point and provide some arguments in support of it.

4 Characterization of a solution or a value

As we will presently see, conditions 1-6 characterize a 'value' or a 'solution' $\Phi : \mathcal{B} \times \mathcal{W} \to R^N$. But previously let us precisely establish the relationships of such a solution/value with Nash's bargaining solution and the Shapley value, showing in anticipation the consistency of this solution/value with both classical notions. We do so *on axiomatic grounds*, showing that conditions 1-6 yield back Nash's and Shapley-Dubey's characterizing systems when restricted to classical bargaining problems and TU-problems, respectively.

Denote by $Nash(B)$ the Nash (1950) bargaining solution of an n-person bargaining problem $B = (D, d)$, that is,

$$Nash(B) = \arg \max_{x \in D_d} \prod_{i \in N} (x_i - d_i),$$

and by $Nash^w(B)$ the w-weighted asymmetric Nash bargaining solution (Kalai, 1977) of the same problem for a vector of nonnegative weights $w = (w_i)_{i \in N}$, such that $\sum_{i \in N} w_i = 1$,

$$Nash^w(B) = \arg \max_{x \in D_d} \prod_{i \in N} (x_i - d_i)^{w_i}.$$

In fact, if any of the weights is zero, the w-weighted Nash solution may not be unique under the conditions assumed on the bargaining problems. This difficulty can be overcome in two ways. Either assuming a condition of 'non-levelness' on the feasible set, or imposing the disagreement 'payoff' for players whose weight is zero. That is, redefining

$$Nash_i^w(B) := \begin{cases} \arg_i \max_{x \in D_d^J} \prod_{i \in J} (x_i - d_i)^{w_i} & \text{if } i \in J, \\ d_i & \text{if } i \in N \setminus J. \end{cases} \tag{2}$$

where $J = \{i \in N : w_i > 0\}$. In what follows we will always refer to definition (2).

Proposition 1 *Let $\Phi : \mathcal{B} \times \mathcal{W} \to R^N$ be a solution/value that satisfies Eff, An, IIA and IAT, then if W is a symmetric voting rule (and for the unanimity rule in particular), for any bargaining problem $B \in \mathcal{B}$, it holds*

$$\Phi(B, W) = Nash(B).$$

Proof. Let W be a symmetric rule. Then *Eff, IIA* and *IAT* become for $\Phi(-, W) : \mathcal{B} \to R^N$ exactly Nash's (1950) corresponding characterizing conditions of his solution in the domain $\mathcal{B}$ of classical bargaining problems. As to the symmetry condition, note that for any permutation π, as W is symmetric, it holds $W = \pi W$, and also for any symmetric bargaining problem $B = \pi B$. Therefore for any symmetric problem B, $\pi(B, W) = (B, W)$, and consequently, by *An,*

$$\Phi_i(B, W) = \Phi_{\pi(i)}(\pi(B, W)) = \Phi_{\pi(i)}(B, W).$$

Hence $\Phi(-, W)$'s symmetry follows immediately. Thus, from Nash (1950) it must hold $\Phi(B, W) = Nash(B)$ for any bargaining problem $B \in \mathcal{B}$. ■

Note that in view of its proof and the fact that *NP* and *T* become empty requirements when W is any fixed symmetric rule, Proposition 1 can be rephrased like this: the characterizing axioms 1-6 when restricted to $\Phi(-, W) : \mathcal{B} \to R^N$ for any fixed symmetric rule, *become Nash's axiomatic system.*

Now we turn our attention to the relationship with the Shapley value. Denote by $Sh(v)$ the Shapley (1953) value of a TU game v, given by

$$Sh_i(v) = \sum_{S:S \subseteq N} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S \backslash i)),$$

and by $Sh(W)$ the Shapley-Shubik (1954) index of a voting rule W, i.e., the Shapley value of the associated simple game v_W. We have the following result.

Proposition 2 *Let $\Phi : \mathcal{B} \times \mathcal{W} \to R^N$ be a solution/value that satisfies Eff, An, NP and T, then for any voting rule $W \in \mathcal{W}$, $\Phi(\Lambda, W) = Sh(W)$.*

Proof. Let $W \in \mathcal{W}$. As stated in section 2, in the case of the TU-bargaining problem Λ the associated NTU game $V_{(\Lambda,W)}$ is equivalent to the simple TU game v_W s.t. $v_W(S) = 1$ iff $S \in W$. And, as is well known, efficiency, anonymity, null player and transfer characterize the Shapley (-Shubik) value in the domain of simple (superadditive or not) games (Dubey (1975), see also Laruelle and Valenciano (2001)). It is then easily checked that in our setting conditions *Eff, An* (mind Λ is symmetric)*, NP* and *T* become their homonymous for $\Phi(\Lambda, -) : \mathcal{W} \to R^N$. Thus $\Phi(\Lambda, W) = Sh(v_W) = Sh(W)$ for any voting rule. ■

Notice that in the context of Proposition 2, i.e. when $B = \Lambda$, conditions *IIA* and *IAT* become empty requirements. Thus also Proposition 2 can be rephrased like this: the characterizing axioms 1-6 when restricted to $\Phi(\Lambda, -) : \mathcal{W} \to R^N$ become Dubey's characterizing system of the Shapley value (or Shapley-Shubik index) in $\mathcal{W}$. Thus Propositions 1 and 2 show the consistency of a solution or value satisfying characterizing conditions 1-6 with both the Nash bargaining solution and the Shapley value.

Before proceeding with the main result we state the following lemma, whose simple proof is omitted, which will be of use.

Lemma 1 *For any vector of nonnegative weights* $w = (w_i)_{i \in N}$*, s.t.,* $\sum_{i \in N} w_i = 1$,

$$Nash^w(\Lambda) = w.$$

We have now the main result of this section, whose proof follows similar steps to that of Nash of his classical result.

Theorem 1 *There exists a unique solution/value* $\Phi : \mathcal{B} \times \mathcal{W} \to R^N$ *that satisfies efficiency (Eff), anonymity (An), independence of irrelevant alternatives (IIA), invariance w.r.t. affine transformations (IAT), null player (NP) and transfer (T), and it is given by*

$$\Phi(B, W) = Nash^{Sh(W)}(B).$$

Proof. *Existence:* For any $(B, W) \in \mathcal{B} \times \mathcal{W}$, $Nash^{Sh(W)}(B)$ exists from the compactness of D_d, whose convexity makes it unique under definition (2). It is easy to see then that the solution $\Phi(B, W) := Nash^{Sh(W)}(B)$ satisfies *Eff, An, IIA, IAT and NP*. As to *T*, let $W, W' \in \mathcal{W}$, and $S \in M(W) \cap M(W')$ $(S \neq N)$. In view of Lemma 1 and the Shapley value satisfying transfer, we have

$$\Phi(\Lambda, W) - \Phi(\Lambda, W_S^*) = Nash^{Sh(W)}(\Lambda) - Nash^{Sh(W_S^*)}(\Lambda) = Sh(W) - Sh(W_S^*)$$

$$= Sh(W') - Sh(W_S'^*) = Nash^{Sh(W')}(\Lambda) - Nash^{Sh(W_S'^*)}(\Lambda) = \Phi(\Lambda, W') - \Phi(\Lambda, W_S'^*).$$

Uniqueness: Let $\Phi : \mathcal{B} \times \mathcal{W} \to R^N$ be a value or solution that satisfies *Eff, An, IIA, IAT, NP* and *T*. Let any problem (B, W), with $B = (D, d)$. Without loss of generality in view of *IAT* we can assume $d = 0$. Denote $x^* := Nash^{Sh(W)}(B)$, which exists and is unique. Let us now choose the utility functions so that the above-mentioned point is transformed by a linear map $T : R^N \to R^N$ into the point $Sh(W)$. That is,

$$T_i(x) := \begin{cases} \frac{Sh_i(W)}{x_i^*}\, x_i & \text{if} \quad Sh_i(W) \neq 0, \\ x_i & \text{if} \quad Sh_i(W) = 0. \end{cases}$$

Thus, $T(x^*) = Sh(W)$. Since this involves the multiplication of the utilities by constants, $Sh(W)$ will now be the point that maximizes $\prod_{i \in N} {x_i}^{Sh_i(W)}$ in the transformed problem $T(B)$, that is $Sh(W) = Nash^{Sh(W)}(T(B))$. For no points of the set $T(D)$ will $\sum_{i \in N} x_i > 1$ now, since if there were a point of the set with $\sum_{i \in N} x_i > 1$ at some point of the line segment between $Sh(W)$ and that point, there would be a value of $\prod_{i \in N} x_i^{Sh_i(W)}$ greater than its value at $Sh(W)$, which by Proposition 2 and Lemma 1 is

$$Sh(W) = Nash^{Sh(W)}(\Lambda) = \Phi(\Lambda, W).$$

Now, as by *IIA* it must be $\Phi(\Lambda, W) = \Phi(T(B), W)$, and by *IAT*, $\Phi(T(B), W) = T(\Phi(B, W))$, we have

$$T(\Phi(B, W)) = Sh(W),$$

which yields

$$\Phi(B, W) = T^{-1}(Sh(W)) = x^* = Nash^{Sh(W)}(B),$$

which establishes the assertion. ■

5 Rationality under risk on the voting rule

The axiomatic system of Theorem 1 is basically the result of amalgamating Nash's and Shapley-Dubey's systems. In this setting efficiency (*Eff*), anonymity (*An*), independence of irrelevant alternatives (*IIA*), invariance w.r.t. affine transformations (*IAT*) and null player (*NP*) can be motivated from the 'value' point of view as compelling conditions for the expectations of rational players, but the transfer (*T*) assumption may give rise to some doubts. To begin with the transfer condition is assumed only on the subdomain of problems of the type (Λ, W), in which the bargaining problem is TU-like. In fact, as it is easy to check, the solution characterized in Theorem 1 does not satisfy (1) for arbitrary bargaining situations (B, W) [8]. In this section some arguments in favor of the transfer condition will be given in the form of stronger conditions in a wider setting, but again restricted to situations in which the bargaining problem is TU-like. As we will see, there seems to exist a deep basic limitation concerning this point.

In order to provide additional support to the transfer condition (and at the same time extending the characterizing result) we will enrich the model admitting as well *random* voting rules. There are two points of view to motivate this extension of the domain. On the one hand, it is a fact that in many cases a committee uses different voting rules to decide upon different classes of issues, and sometimes there may exist uncertainty about which rule should be used to decide upon an issue, so that uncertainty about the rule may arise. On the other hand, at the very foundations of Nash's model, which is our basic term of reference, is the consideration of risk on the feasible agreements, inherent to the description of vNM-rational players (though in practice negotiators tend to avoid random agreements) and which constrains the admissible utility functions. The situation under consideration here is an extension of the one modeled by Nash in which the new element is the voting rule. Introducing risk on the voting rule will allow us similarly to further

[8] The same can be said about the condition of 'symmetric gain-loss', alluded in footnote 7, which can replace transfer in Theorem 1.

constraining the degrees of freedom in the choice of the 'value' function, which can be interpreted as an extension of the players' vNM utility functions.

Thus, the second ingredient in the model will now be in general some lottery over the set of all N-voting rules, denoted $\mathfrak{L}(\mathcal{W})$. In this wider setting (we identify at all effects every deterministic voting rule with the corresponding degenerated lottery that assigns probability 1 to that rule) a generalized bargaining problem will consist of a pair $(B, \lambda) \in \mathcal{B} \times \mathfrak{L}(\mathcal{W})$. Any lottery $\lambda \in \mathfrak{L}(\mathcal{W})$ will be represented by a map $\lambda : \mathcal{W} \to [0, 1]$ s.t. $\sum_{W \in \mathcal{W}} \lambda(W) = 1$, where $\lambda(W)$ represents the probability of voting rule W. Note that any given $\lambda \in \mathfrak{L}(\mathcal{W})$ induces a probability of each coalition being winning. We will also use the following notation: For any $S \subseteq N$,

$$\lambda_S := \sum_{W : S \in W \in \mathcal{W}} \lambda(W),$$

that is, λ_S is the probability of S being winning for the random procedure λ. The distinction between a voting rule W and its associated simple game v_W, will allow us to avoid ambiguity: $\mu W + (1 - \mu)W'$ represents a random voting rule, while $\mu v_W + (1 - \mu)v_{W'}$ is a TU game.

As in the case of deterministic voting rules, for any $(B, \lambda) \in \mathcal{B} \times \mathfrak{L}(\mathcal{W})$, we can associate with it the NTU game $(N, V_{(B,\lambda)})$, such that for any $S \subseteq N$,

$$V_{(B,\lambda)}(S) := \begin{cases} \lambda_S D^S + (1 - \lambda_S)d^S & \text{if } \lambda_S \neq 0, \\ ch(d)^S & \text{if } \lambda_S = 0. \end{cases}$$

Observe this is consistent with the meaning of points in D as vectors of vNM utilities: if λ_S is the probability of S being winning, this coalition can guarantee expected utilities for its members within $\lambda_S D^S + (1 - \lambda_S)d^S$. While for the TU-bargaining case (Λ, λ), the associated NTU game $V_{(\Lambda,\lambda)}$ is equivalent to the TU game

$$v_\lambda(S) := \lambda_S, \text{ for all } S \subseteq N.$$

Note also that

$$v_\lambda = \sum_{W \in \mathcal{W}} \lambda(W)\ v_W. \tag{3}$$

In fact, properly speaking, any generalized bargaining problem in which the bargaining component is equivalent to Λ *up to a positive affine transformation,* can be considered as a TU game (up to a change of zero and unit of scale of every player's utility). But mind that in this way *not* all n-person TU games are generated. In view of (3), only those in the cone generated by the convex-hull (in the space R^{2^n-1}) of the set of simple games are generated as particular cases of our model[9].

Now the issue is how to extend in this wider setting the notion of a solution/value $\Phi : \mathcal{B} \times \mathfrak{L}(\mathcal{W}) \to R^N$. Note that the conditions introduced in the previous section do not involve random voting procedures. Nevertheless, although from a technical point of view it is not necessary, the reader can easily check that there is no difficulty in extending *Eff, An, IIA, IAT* and *NP* to the wider domain $\mathcal{B} \times \mathfrak{L}(\mathcal{W})$, where they keep their meaning and motivation. In order to extend the characterization to this wider domain we will replace transfer by two *rationality assumptions concerning the behavior under risk on the decision rule.*

First, as it is well-known, different lotteries over voting rules may assign exactly the same probability of being winning to every coalition. It seems reasonable to assume that only these probabilities should influence the expectations of vNM-rational players. Thus we impose a first rationality condition concerning random voting rules:

7. (*CED*) *Coalitional expectations dependence*: For all $\lambda, \lambda' \in \mathfrak{L}(\mathcal{W})$, such that for any $S \subseteq N$, $\lambda_S = \lambda'_S$ (i.e. the probability of any coalition being winning is the same in both random procedures), $\Phi(B, \lambda) = \Phi(B, \lambda')$ for all B.

Now consider the special case in which B is the TU-bargaining problem Λ. As commented in section 2, this means that *any* agreement above the status quo is indifferent to a lottery on the best agreements for each player, b_i, (each of them indifferent to the status quo for the other players) and the status quo. Thus, for every voting rule W, the prospect of engaging into (Λ, W) must be indifferent to a lottery of this kind, with associated level of utility $\Phi(\Lambda, W) \in \Delta_0$. It seems then consistently rational, in case of bargaining under a random voting rule λ, to assess the prospect of engaging into (Λ, λ) as indifferent to the lottery over the preferred agreements b_i and the status quo that results by composing lottery λ and the corresponding equivalent to each (Λ, W). In other words, the following condition *amounts as to assuming* Φ *to define vNM preferences over the subdomain* $\{\Lambda\} \times \mathfrak{L}(\mathcal{W})$.

8. (*RR*) *Rationality under random voting rules:* For all $\lambda \in \mathfrak{L}(\mathcal{W})$,

$$\Phi(\Lambda, \lambda) = \sum_{W \in \mathcal{W}} \lambda(W) \Phi(\Lambda, W).$$

[9]Nevertheless, the model can be generalized meaningfully so as to yield all TU games, or at least significant classes of them, as monotonic or superadditive games. This is possible by means of generalizing the notion of random voting procedure beyond lotteries on them. Namely, admitting any TU game v such that $v(N) = 1$ and satisfying certain minimal conditions for a consistent interpretation in this sense, as describing (in the sense that simple games and lotteries over them do) the probability of every coalition being winning. But this extension, inverse to the usual interpretation of simple games as a particular case of TU game, will not be discussed here for it falls outside the central goal of this paper.

A slightly weaker version of this condition (equivalent to it under *Eff*), which results from replacing 'indifferent to' by 'no worse than', is the following:

(*WRR*) *Weak rationality under random voting rules:* For all $\lambda \in \mathfrak{L}(\mathcal{W})$,

$$\Phi(\Lambda, \lambda) \geq \sum_{W \in \mathcal{W}} \lambda(W)\Phi(\Lambda, W). \tag{4}$$

We will use the weaker *WRR* for extending Theorem 1, yielding as an obvious corollary the same result for *RR*. Note that, as it was the case with transfer in the context of deterministic voting rules, both conditions concern only situations in which the bargaining element is TU-like. We will discuss later the possibility of extending these conditions to arbitrary bargaining situations. The following lemma shows the connection of this condition (and indirectly of *RR*) with transfer.

Lemma 2 *Let $\Phi : \mathcal{B} \times \mathfrak{L}(\mathcal{W}) \to R^N$ be a solution/value that satisfies Eff, CED and WRR, then $\Phi(\Lambda, -) : \mathcal{W} \to R^N$ satisfies T.*

Proof. Let $W, W' \in \mathcal{W}$, and $S \in M(W) \cap M(W')$ $(S \neq N)$. It is immediate to check that the lotteries $\frac{1}{2}W + \frac{1}{2}W_S'^*$ and $\frac{1}{2}W_S^* + \frac{1}{2}W'$ assign the same probability of being winning to every coalition. On the other hand, the Pareto boundary of Δ is a hyperplane. This along with Φ's efficiency yields that inequality (4), implied by *WRR,* becomes an *equality*, so that, combining *CED* and *WRR* we have

$$\frac{1}{2}\Phi(\Lambda, W) + \frac{1}{2}\Phi(\Lambda, W_S'^*) = \Phi(\Lambda, \frac{1}{2}W + \frac{1}{2}W_S'^*)$$

$$= \Phi(\Lambda, \frac{1}{2}W_S^* + \frac{1}{2}W') = \frac{1}{2}\Phi(\Lambda, W_S^*) + \frac{1}{2}\Phi(\Lambda, W').$$

Which yields transfer (1) for $\Phi(\Lambda, -)$. ■

Thus, under efficiency, the transfer condition used in Theorem 1 is implied by the rationality conditions *CED* and *WRR* (or *RR*) on the behavior facing risk on the voting rule.

Now let $Sh(v_\lambda)$ be for any random voting rule $\lambda \in \mathfrak{L}(\mathcal{W})$ the Shapley value of the associated TU game v_λ. We have the following extension of Proposition 2.

Proposition 3 *Let $\Phi : \mathcal{B} \times \mathfrak{L}(\mathcal{W}) \to R^N$ be a solution/value that satisfies Eff, An, NP, CED and WRR, then for any random voting rule $\lambda \in \mathfrak{L}(\mathcal{W})$, $\Phi(\Lambda, \lambda) = Sh(v_\lambda)$.*

Proof. Let $\Phi : \mathcal{B} \times \mathfrak{L}(\mathcal{W}) \to R^N$ satisfy *Eff, An, NP, CED* and *WRR*. By Lemma 2, $\Phi(\Lambda, -) : \mathcal{W} \to R^N$ satisfies transfer. Thus for the case of a deterministic voting rule

$W \in \mathcal{W}$, Proposition 2 implies that $\Phi(\Lambda, W) = Sh(W)$. Now let $\lambda \in \mathfrak{L}(\mathcal{W})$ be any random voting rule. By *Eff*, as seen in Lemma 2, inequality (4), implied by *WRR,* becomes an equality. Then, by the relation established by Proposition 2 for deterministic voting rules, together with (3) and the well known linearity of the Shapley value, the result extends to any random voting rule:

$$\Phi(\Lambda, \lambda) = \sum_{W \in \mathcal{W}} \lambda(W)\ \Phi(\Lambda, W) = \sum_{W \in \mathcal{W}} \lambda(W)\ Sh(v_W)$$

$$= Sh(\sum_{W \in \mathcal{W}} \lambda(W)\ v_W) = Sh(v_\lambda). \blacksquare$$

Then we have the following extension of Theorem 1 for the case of random voting rules, whose proof, entirely similar to that of Theorem 1, is omitted.

Theorem 2 *There exists a unique solution/value* $\Phi : \mathcal{B} \times \mathfrak{L}(\mathcal{W}) \rightarrow R^N$ *that satisfies efficiency (Eff), anonymity (An), independence of irrelevant alternatives (IIA), invariance w.r.t. affine transformations (IAT), null player (NP), coalitional expectations dependence (CED) and weak rationality under random voting rules (WRR), and it is given by*

$$\Phi(B, \lambda) = Nash^{Sh(v_\lambda)}(B).$$

Even if *RR* (as *WRR*) seems compellingly motivated, it appears as desirable to have instead a general condition applicable to all situations (B, λ) whatever B. We examine now some possibilities. First observe that *RR* cannot be asked for arbitrary problems (B, λ). The reason is that in NTU bargaining problems this condition would possibly clash with efficiency, as convex combinations of Pareto efficient points will not in general be efficient. As to the weaker *WRR*, it may seem appealing at first sight claiming that rational players, at the prospect of using a random voting rule, should reach a better agreement if it were feasible *before* applying the random rule. Namely:

(S*RR*) *Strong rationality under random voting rules:* For all $B \in \mathcal{B}$, and all $\lambda \in \mathfrak{L}(\mathcal{W})$,

$$\Phi(B, \lambda) \geq \sum_{W \in \mathcal{W}} \lambda(W) \Phi(B, W).$$

Obviously this condition implies *WRR* (and, along with *Eff*, *RR*).

Another stronger (under *Eff*) condition than *RR* or *WRR* is the following adaptation to this setting of the conditions in the NTU context of 'conditional additivity' used by Aumann (1985) and by Hart (1985) to characterize the NTU Shapley value and Hasanyi's solution respectively. In the present context[10] a similar condition can be stated like this (denoting by ∂D the Pareto boundary of D):

(CA) *Conditional additivity:* For all $(B,\lambda)\in\mathcal{B}\times\mathfrak{L}(\mathcal{W})$ s.t. $\sum_{W\in\mathcal{W}}\lambda(W)\Phi(B,W)\in\partial D$,

$$\Phi(B,\lambda)=\sum_{W\in\mathcal{W}}\lambda(W)\Phi(B,W).$$

It is easy to check that the solution characterized in Theorem 2 satisfies also RR, and consequently this condition can replace WRR in Theorem 2. But this is not so for the also stronger and at first sight appealing SRR, nor the above adaptation of conditional additivity CA. Any of these conditions along with *Eff, An, IIA, IAT, NP* and *CED*, yield an empty value: no solution satisfies them all. We have the following somewhat frustrating double impossibility result:

Theorem 3 *There does not exist a solution/value $\Phi:\mathcal{B}\times\mathfrak{L}(\mathcal{W})\rightarrow R^N$ that satisfies efficiency (Eff), anonymity (An), independence of irrelevant alternatives (IIA), invariance w.r.t. affine transformations (IAT), null player (NP) and coalitional expectations dependence (CED), and also either strong rationality under random voting rules (SRR) or conditional additivity (CA).*

As SRR (as CA under *Eff*) implies WRR, in view of the uniqueness result of Theorem 2 it suffices to provide an example (B,λ) for which

$$Nash^{Sh(v_\lambda)}(B)\not\geq\sum_{W\in\mathcal{W}}\lambda(W)Nash^{Sh(W)}(B), \tag{5}$$

and one example in which $\sum_{W\in\mathcal{W}}\lambda(W)Nash^{Sh(W)}(B)\in\partial D$, but

$$Nash^{Sh(v_\lambda)}(B)\neq\sum_{W\in\mathcal{W}}\lambda(W)Nash^{Sh(W)}(B). \tag{6}$$

The following example serves to both purposes[11] exemplifying (5) and (6).

Counterexample: Let $N=\{1,2,3\}$, and $B=(D,0)$ the 3-person bargaining problem in which D is the comprehensive hull of the convex hull of the set:

$$\left\{(1,0,0),(0,1,0),(\frac{2}{3},0,\frac{1}{3}),(0,\frac{2}{3},\frac{1}{3})\right\}.$$

[10]The direct translation of this condition consisting of its requirement for the associated NTU problems $V_{(B,\lambda)}$ does not make sense in this smaller domain, in which the sum $V_{(B,\lambda)}+V_{(B',\lambda')}$ would not be in general associated to any problem in our domain.

[11]We want to thank to Abraham Neyman, who provided a first example for (5).

And let λ be the random voting rule that assigns probability 1/3 to each of the three unanimity rules W^{12}, W^{13}, and W^{23}, where $W^{i,j}$ denotes the rule whose only minimal winning coalition is $\{i,j\}$. Then, as $Sh(v_\lambda) = (1/3, 1/3, 1/3)$, we have

$$Nash^{Sh(v_\lambda)}(B) = Nash(B) = (1/3, 1/3, 1/3).$$

While

$$\sum_{W \in \mathcal{W}} \lambda(W) Nash^{Sh(W)}(B) = \frac{1}{3}(\frac{2}{3}, 0, \frac{1}{3}) + \frac{1}{3}(0, \frac{2}{3}, \frac{1}{3}) + \frac{1}{3}(\frac{1}{2}, \frac{1}{2}, 0) = (\frac{3.5}{9}, \frac{3.5}{9}, \frac{2}{9}).$$

Thus we have (5), and also (6) in spite of $(\frac{3.5}{9}, \frac{3.5}{9}, \frac{2}{9}) \in \partial D$.

In sum, conditions *SRR* and *CA* are incompatible with the remaining conditions. Thus, including any of them would force to eliminate some of the others. On the other hand, they imply the weaker *WRR* and *RR* respectively, which on the one hand along with *CED* imply *T*, and on the other, as Theorem 2 shows, along with the other conditions have characterizing power.

6 Interpretation and concluding remarks

We have provided a model-synthesis that extends in a natural and consistent way Nash's classical bargaining model and solution, as well as his axiomatic characterization. The characterization has been achieved by integrating two axiomatic systems into a single and consistent one, which when restricted to classical bargaining problems becomes Nash's system, and when restricted to simple games becomes a characterizing system of the Shapley value. As a result the solution or value emerging turns out to be consistent with both Nash's solution and Shapley's value. It is worth remarking the meaningfulness of the axioms we have used in the specific context that underlies the model: bargaining under the shadow of a voting rule. *Eff, IIA* and *IAT* are natural adaptations of Nash's axioms, *An* integrates both Nash's symmetry and Shapley's anonymity, while *NP* comes from Shapley's system. Finally, in our system two rationality assumptions (*CED* and *RR* or *WRR*) concerning behavior under *risk on the decision rule*, replace the often criticized additivity in Shapley's system. These latter conditions imply the transfer condition, which completes a characterizing system if random voting rules are no admitted.

A specific point worth remarking concerns the symmetry assumption in the previous related literature. Our model provides a new point of view to appreciate Kalai's (1977) nonsymmetric bargaining solutions. These solutions emerge when symmetry is dropped in Nash's system, or still assuming symmetry in an adequately 'replicated' problem (Kalai, 1977), as if each player negotiated on behalf of a number of players, as is usually the underlying situation when a nonsymmetric rule is used to make decisions. As to their justification it is still nowadays often pointed out to the 'unequal bargaining skill' of the bargainers as a reason to drop symmetry. Curiously enough this same reason but in positive terms (i.e., in terms of the 'equal bargaining skill') was first invoked by Nash (1950) in support of symmetry, and later rejected by Nash himself in (1953) as inconsistent with his model. Nash then argues in terms of isomorphism in support of symmetry. While Binmore (1998, p. 78) justifies the asymmetric Nash solutions as reflecting the different 'bargaining power' of the players 'determined by the strategic advantages conferred on players by the circumstances under which they bargain', and uses the term 'bargaining power' to refer to the players' weights[12]. Our more general model conciliates what is consistent in these interpretations, symmetry and nonsymmetry in particular. In our model asymmetry: (i) is the result of a more general requirement of symmetry (anonymity) involving the feasible set *and the voting rule*, based on an isomorphism motivation (that yields symmetry only if the voting rule is symmetric); *and* (ii) reflects the actually different bargaining capacity due to the possibly nonsymmetric voting rule under whose shadow bargaining takes place. It is also worth remarking that in our model the 'weights' are endogenously generated from the axioms, and have a clear meaning.

A very interesting outcome of this model is that it yields two different meanings to the Shapley-Shubik index of a voting rule (random or not). First, it is the generalized Nash solution of a very particular (TU) bargaining problem in which such rule is used (Propositions 2 and 3), and, second and more interesting, it determines the weights of different bargainers when that voting rule is used to settle agreements in arbitrary bargaining problems (Theorems 1 and 2). The first meaning is well known, but the second is new and much more interesting. This result has important consequences for the voting power theory, where in fact, as it has been mentioned, the issue that motivated this research was raised. Since Shapley-Shubik (1954) and Banzhaf (1965) there has been a still alive debate about the merits of either 'power index'. The lack, still today, of a unanimous result of this never ending debate is based in the lack of conceptual rigor and clarity concerning which notion of power one is talking about. If one refers to power as the capacity (or probability) to be influential or crucial in a formal sense when decisions are to be made according to a voting rule, then Banzhaf's criticism (see also Coleman (1971)) of the Shapley-Shubik index (widely ignored apparently due to the 'nice' -however uncompelling in that context- mathematical properties of the Shapley value) has not yet been satisfactorily answered, neither presumably will ever be as far as power is understood in these terms. More recently,

[12]This interpretation is consistent with the outcome of Rubinstein (1982) alternating offers model within the noncooperative approach. It is also consistent with the interpretation proposed by Valenciano and Zarzuelo (1994) within Rubinstein, Safra and Thomson (1992) preference-based model.

Felsenthal and Machover (1998) and ourselves (Laruelle and Valenciano, 2000, 2002) have supported this claim. But the result of this paper provides a model in which the Shapley-Shubik index acquires a new, unexpected and relevant meaning: it yields the (bargaining) weights of the bargainers when they use a voting rule to settle or enforce agreements, and this *with independence of the feasible set* (under minimal regularity conditions). Thus, in a completely different sense from usual interpretations, the Shapley-Shubik index makes sense as a measure of power after all, or better as a measure of *the bargaining power under a voting rule* in a very precise sense. Or, in other terms, Theorems 1 and 2 provide of a rigorous meaning and an evaluation axiomatically founded for the obscure notion of 'P-power', introduced by Felsenthal and Machover (1998) (in opposition of 'I-power' or power to influence), but devoid so far of a precise formulation[13].

It has been shown that a significant class of NTU games with a clear meaning is associated with the class of two-ingredient models considered by us, and that the solution axiomatically characterized is consistent with both the Nash's solution, and the Shapley value within the class of TU games included in our model. But in general no solution in the NTU literature coincides with the one obtained here on axiomatic grounds[14]. This seems to corroborate the impression of the excessive abstraction and consequent lack of intuitive basis in the nude NTU model: unless you put something else in it it does not provide sufficient sure ground for a solution. Is is also worth remarking the uniqueness of the solution characterized here and its proof, unlike is usually the case with NTU solutions, not requiring any fix point argument, but being entirely similar to Nash's proof of his classical result.

As matter of interesting further research we can mention several lines. First, and possibly the most important, the 'Nash program' challenge: the noncooperative foundation of the cooperative solution characterized. Rubinstein (1982) seems the natural term of reference as a starting point, but the achievement of this goal does not appear to be obvious at all. Possibly a noncooperative approach could also account for something that the cooperative does not: it has been shown that for any symmetric rule the axioms yield

[13] Felsenthal and Machover suggest the Shapley-Shubik index as a measure of 'P-power', though not very enthusiastically in view of the lack of a clear basis for that choice, in what looks more like a concession to the weight of tradition in the game theory literature.

[14] The solution obtained depends on the bargaining problem and the players' weights (given by the Shapley-Shubik index of the voting rule). Consequently, the bargaining problem can be modified in many ways so that the feasible set for some coalition(s) changes without the solution changing, unlike other solutions of the associated NTU game. Nevertheless, as Sergiu Hart pointed out, the solution characterized here can be seen as the Shapley NTU value-like of an NTU game by associating to every winning coalition the *whole* feasible set, but this means a double ad hoc extension of the Shapley NTU value notion and of the very notion of NTU game.

the Nash solution, insensitive to which symmetric rule is used, while intuitively speaking it seems easier to make agreements by means of a simple majority rule than by unanimity. Still within the cooperative approach, it would be interesting to extend Rubinstein, Safra and Thomson (1992) preference-based model so as to cover and reinterpret the results presented in this paper. Finally, an obvious line of work is examining the effects of replacing independence of irrelevant alternatives by some of the alternative conditions proposed in the literature. Nevertheless, in this respect it is worth stressing that in spite of the *IIA* condition being the most criticized in Nash's original system, we find it preferable to any of the suggested alternatives, as better founded in the strict rationality terms used by Nash.

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