Two Examples in a Market with Two Types of Indivisible Good

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Abstract

We consider an extension of the "permutation game" of Tijs et al (1984) in which players are endowed with and ultimately wish to consume one unit of each of *two* types of good (i.e., a house and a car). We present two examples. The first is a case where even though the "corresponding linear program" (CLP) does not solve with integers, the core of the market is not empty. The second example is a case with additively separable preferences in which there is a core vector in the market which does not correspond to *any* optimal dual solution of the CLP. Both examples demonstrate possible behavior that is impossible in many of the "standard" matching games. We show that in cases with additively separable preferences, the core-optimal dual equivalence property is recovered if each component swapping game, one with houses and the other with cars, is *convex* in the sense of Shapley (1971).

1. Introduction

One of the successes of mathematical economics has been the recent development of a detailed theory of matching games. This started with the classic models of Gale-Shapley [3], Shapley-Shubik [14], Shapley-Scarf [13], and Tijs et al [16]. These constructs are useful because they are fundamental models of markets of indivisible goods.

In this note we are primarily concerned with the relationships between the core of transferable-utility (TU) matching games[1] and linear programming. It seems as if given just about any TU matching game, it is possible to formulate a "corresponding linear program" (CLP) in which a) the variables are bounded between 0 and 1, b) if integrality is also required, the optimal objective function value is the "worth" $V(N)$ of the grand coalition N, and c) an optimal variable value of 1 (resp., 0) means that a corresponding "match" is made (resp., not made) in the obtaining of $V(N)$. Perhaps the best known example of a model and its CLP is Shapley-Shubik's assignment game, together with the "assignment linear program" (ALP) which is used to analyze it. Shapley-Shubik's fundamental result was that the core of the assignment game is precisely the set of optimal solutions to the dual of the ALP.[2]

[1] A game has transferable utility if there is a transferable resource (money), and all outcomes for all players are evaluated in terms of this transferable resource. A matching game is a game in which a major focus for players is the identity of coalition partners.

[2] For those operations researchers who are not specialists in game theory, one may think of this paper as being based on the celebrated "assignment problem" (see e.g. Winston [17], p. 372). It is well-known that this linear program always solves integrally, no matter what the coefficient matrix is. This fact is instrumental in proving that a corresponding game, called the *assignment game*, has a nonempty core. In

Other researchers have continued in a similar vein. Quint [8] considers Kaneko and Wooders' [5] TU partitioning game, sets up a "partitioning linear program" as his CLP, and proves that if the CLP solves integrally, one has a core-optimal dual solution equivalence. In another paper, the same author [10] considers "restricted houseswapping[3] games with TU", again sets up a CLP, and again proves a similar result. Finally, Curiel and Tijs [2] and Quint [9] have considered the permutation game of Tijs et al [16]. The CLP here is essentially identical to the ALP, except that the main result is that the core is equivalent to the set of vectors one can obtain by adding pairs of optimal dual variables (representing the "buyer role" and "seller role" of players). [For details, see section 2.]

From the above literature summary, we can see two types of results:

1) *The core of a TU matching game is nonempty iff its corresponding CLP solves integrally.* In the Shapley-Shubik [14], Curiel-Tijs [2], and Quint [9] papers above, the CLP always solves integrally and so these models always have nonempty cores. In the other two Quint papers [8,10] mentioned above, the CLP may solve non-integrally, and these are precisely the cases where the core is empty.

2) *If its CLP solves integrally, a TU matching game has a "core-optimal dual solution equivalence," i.e., the core of the game can be represented by the set of optimal solutions to the dual of the CLP.*[4]

In this note we are going to consider a natural extension of one of these basic models, in which 1) and 2) do not hold. In particular, we consider a TU game in which there are n players and *two* types of indivisible good (say, houses and cars). Each player is endowed with exactly one house and one car, and wishes to consume precisely one house and one car. Each player expresses his valuations of house-car bundles in terms of money, and is free to swap houses and/or cars with other players in an effort to obtain a bundle more valuable to himself. Hence, essentially what we have here is a permutation game, except with *two* types of indivisible good.

It turns out that there is an intuitive formulation of the CLP of this game. However, we present an example where the CLP does not solve integrally, but where the core is nonempty. Hence, no theorem of the type of 1) above may hold in this economy.

We then consider another example in which each player has *additively separable preferences*. This means that he/she has a monetary evaluation of each house and each car, and the cardinal utility derived from any house-car bundle is just his/her monetary evaluation of the house plus his/her monetary evaluation of the car. It is known that this condition causes the CLP to solve integrally, and so the core is nonempty. However, we show that there is a core payoff which cannot be formed as a sum of optimal dual variables in the

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fact, the core coincides with the set of optimal solutions to the dual of the assignment problem.

We can then try to duplicate this analysis for generalizations of the assignment problem/game. If we start with a transportation problem (with integer "right-hand-sides"), again the linear program always solves with integers. The corresponding "transportation game" therefore again has a nonempty core, except this time the core does *not* coincide with the set of dual solutions to the transportation problem.

In this paper we consider a *different* generalization of the assignment problem. There are now *two* types of sink, and each source is assigned to one sink of each type. This time the "corresponding linear program" (CLP) does not necessarily solve with integers – however, we present an example to show that even in cases where the CLP does not solve with integers, the core can still be nonempty. And we present another example which shows that we again do not necessarily have a core-optimal dual solution coincidence in this model.

[3] Regarding the word "houseswapping": in some of Shapley's unpublished undergraduate lecture notes, the well-known Shapley-Scarf [13] model is referred to as "the houseswapping game".

[4] Of course, we qualify this statement depending on cases. For the permutation game, for example, the equivalence is between core vectors and sums of optimal dual variables (see Section 2).

natural way. Hence, no theorem of type 2) above may hold in this economy either.

At this point, let us remark on two types of model for which similar counterexamples (to "type 2 theorems") *have* been found. One is the linear production game of Owen [7]. Linear production games are not true "matching games" in the sense that there are no indivisible goods in the model; however there is a "CLP" for which a) the optimal objective function value is $V(N)$, and b) optimal dual variables can be added together in a natural way to get core vectors. However, not all core vectors can be obtained this way. The other type is certain *two*-sided TU market games, featuring buyers and sellers of multiple units of indivisible good. Examples here include the generalized assignment game of Kaneko [4], the exchange market game of Ma [6][5], and the "transportation game" of Sánchez-Soriano, López, and García-Jurado [11]. Again one may find a core vector by setting up an appropriate CLP, but core-optimal dual solution equivalence does not necessarily hold.[6] Hence our game, which is a *one*-sided market with two types of indivisible good (and additively separable preferences), represents a third such model type.[7]

It would be interesting if one could identify some common characteristic of linear production games, transportation games and our games, which causes this non-equivalence of core and optimal dual solution set. For our house and car swapping games with additive separability, we will see (in Section 4) that the non-equivalence occurs only if the core of the house swapping game "added to" the core of the car swapping game does not coincide with the core of the whole game of house and car swapping.

2. Background: The Permutation Game

We begin with a review of Tijs et al's permutation game [16]. Let $N = \{1, ..., n\}$ be the player set. Each player is originally endowed with a house, with "house i " meaning the house originally owned by player i. The players also place nonnegative monetary valuations upon each of the houses – so we define the $n \times n$ matrix A in which a_{ij} represents the monetary valuation that player i has for house j. We also assume that players desire to consume exactly one house, i.e., the utility to player i for a bundle of two or more houses is just the maximum of the a_{ij}'s over the houses j in the bundle.

The allowable moves for the players are to swap houses and transfer monetary utility amongst themselves. A redistribution of the houses is thus described by a **permutation** of N, i.e., a bijection $\pi : N \to N$, with $\pi(i) = j$ meaning that player i receives house j. Let Π be the set of permutations of N. If $\pi \in \Pi$ and $S \subseteq N$, the notation $\pi(S)$ means $\cup_{i \in S} \pi(i)$.

The above gives rise to a TU game $G^P = (N, V)$, called a **permutation game,** in which the characteristic function $V : 2^N \to \mathbf{R}$ is defined by $V(\emptyset) = 0$ and

$$V(S) = \max_{\pi \in \Pi \ s.t. \ \pi(S) = S} \sum_{i \in S} a_{i\pi(i)} \quad \text{for all } S \in 2^N \setminus \{\emptyset\}.$$

In words, the worth $V(S)$ of coalition S is just the maximum of the surpluses generated over all ways in which the players in S swap their own houses amongst themselves.

The **core** of G^P is the set of payoff vectors $x \in \mathbf{R}^n$ such that (a) $x(N) \equiv \Sigma_{i \in N} x_i = V(N)$

[5] Ma's main result is that in the model of Bikhchandani and Mamer [1], *competitive equilibrium* solutions always correspond to dual solutions of a CLP. He also constructs an example with *core vectors* which *don't* correspond to any such dual solution. Now Bikhchandani and Mamer's model certainly contains much more than just "two-sided" games; however, Ma's example of core/dual solution nonequivalence is essentially an instance of a two-sided game.

[6] Kaneko [4] gave a sufficient condition for core-optimal dual solution equivalence in his model, which requires that each active seller has at least one competitor.

[7] A *two*-sided matching game is a game with *two* types of players (usually buyers and sellers), whereas a *one*-sided game has only one type of player (who usually plays the role of both buyer and seller).

and (b) $x(S) \equiv \Sigma_{i \in S} x_i \geq V(S)$ for all $S \subseteq N$. Hence the core is the set of feasible payoff vectors which exhibit a certain kind of stability. It can be characterized by considering the "corresponding linear program" (CLP):

$$\overline{m} = \max_{p} \sum_{i \in N} \sum_{j \in N} a_{ij} p_{ij} \tag{P}$$
$$\begin{aligned} s.t. \sum_{i \in N} p_{ij} &= 1 \qquad \text{for all } j \in N, \\ \sum_{j \in N} p_{ij} &= 1 \qquad \text{for all } i \in N, \\ p_{ij} &\geq 0 \qquad \text{for all } (i,j) \in N^2, \end{aligned}$$

and its dual:

$$\underline{m} = \min_{u,v} \sum_{i \in N} (u_i + v_i) \tag{D}$$
$$s.t.\ u_i + v_j \geq a_{ij} \quad \text{for all } (i,j) \in N^2.$$

Theorem 2.1 (Curiel and Tijs [2] and Quint [9]).[8] *Let G^P be any permutation game. Then x is a core vector of $G^P \iff x = u+v$, where (u,v) is an optimal solution to linear program (D).*

Curiel and Tijs interpreted this result as follows: We think of each player as being composed of a "buyer part" and a "seller part". Then, for player i, a core payoff x_i consists of the payoff that he/she obtains as a buyer (u_i) plus that obtained as a seller (v_i). Under this interpretation, it is natural that the vector v is also an equilibrium price vector (cf. Quint [9]).

It should be noted that a proof of the theorem above depends heavily upon the fact that linear program (P) must solve integrally, no matter what the matrix A is. This also implies that both optimal objective function values $\overline{m}$ and $\underline{m}$ of (P) and (D) are equal to $V(N)$, the worth of the grand coalition.

3. A TU House and Car Swapping Game

We now present an extension of the permutation game, which is a market game with *two* types of good. Let us call these types houses and cars. The player set is again $N = \{1, ..., n\}$. Each player i is endowed with one house and one car, denoted by "house i" and "car i", respectively.

The allowable moves for the players are to exchange houses and cars, and to transfer monetary utility amongst themselves. Similarly to the permutation game, we assume that each player's utility functions are such that he/she would always wish to consume precisely one house and one car. Hence, the important redistributions of the indivisible goods, called **allocations**, are those in which each player receives one house and one car. Formally, an allocation is a function $\pi : N \to N \times N$ with the property that there exist two permutations π_1 and π_2 of the set N with $\pi(i) = (\pi_1(i), \pi_2(i))$ for all $i \in N$. So $\pi(i) = (j,k)$ means that player i ends up with house j and car k. Let $\mathbf{\Pi}$ be the set of all allocations.

The players evaluate all possible house-car bundles in terms of money. Specifically, d_{ijk} represents player i's monetary valuation of the bundle of house j and car k. These valuations are denoted by the three-dimensional $n \times n \times n$ matrix $D = (d_{ijk})$.

[8] Curiel and Tijs proved this theorem in the $\Longleftarrow$ direction, while Quint proved the converse.

The description above gives rise to a TU game $G^{HC} = (N, V)$, whose characteristic function V is defined by $V(\varnothing) = 0$ and

$$V(S) = \max_{\pi\in\mathbf{\Pi}:\pi_1(S)=\pi_2(S)=S} \sum_{i\in S} d_{i\pi_1(i)\pi_2(i)} \quad \text{for all } S \in 2^N \setminus \{\varnothing\}.$$

So now $V(S)$ is just the maximum of the surpluses generated over all ways in which the players in S swap their own houses *and cars* amongst themselves. The **core** of game G^{HC} is again the set of vectors $x \in \mathbf{R}^n$ with $x(N) = V(N)$ and $x(S) \geq V(S)$ for all $S \subseteq N$.

The CLP for a game G^{HC} is

$$\begin{aligned} \overline{m}_{HC} = \max_p \sum_{i\in N} \sum_{j\in N} \sum_{k\in N} d_{ijk}p_{ijk}, & \qquad (P2)\\ s.\ t. \sum_{i\in N} \sum_{j\in N} p_{ijk} = 1 \quad & \text{for all } k \in N,\\ \sum_{i\in N} \sum_{k\in N} p_{ijk} = 1 \quad & \text{for all } j \in N,\\ \sum_{j\in N} \sum_{k\in N} p_{ijk} = 1 \quad & \text{for all } i \in N,\\ p_{ijk} \geq 0 \quad & \text{for all } (i, j, k) \in N^3. \end{aligned}$$

We note two things. First, unlike in the permutation game, it is possible for CLP $(P2)$ of a game G^{HC} to solve non-integrally (see Example 3.1 below). Second, by the result of Bikhchandani and Mamer [1], if $(P2)$ *does* solve integrally, then core of game G^{HC} must be nonempty.[9] However, as the following example shows, it is possible for $(P2)$ to solve non-integrally but still have the core be nonempty:

Example 3.1. $N = \{1, 2\}$, $d_{111} = d_{122} = d_{212} = d_{221} = 1$, $d_{112} = d_{121} = d_{211} = d_{222} = 0$.

It is relatively straightforward to see that with this particular $D = (d_{ijk})$, CLP $(P2)$ has a unique non-integral solution, namely $p_{111} = p_{122} = p_{212} = p_{221} = \frac{1}{2}$ and $p_{112} = p_{121} = p_{211} = p_{222} = 0$. Yet the characteristic function of the resulting TU game is $V(\{1\}) = 1$, $V(\{2\}) = 0$, and $V(\{1, 2\}) = 1$, which clearly has the core point $(1, 0)$.[10]

One condition for guaranteeing the integrality of CLP $(P2)$ is **additive separability** of matrix D. Additive separability means that there exist two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ik})$ such that $d_{ijk} = a_{ij} + b_{ik}$ for all $(i, j, k) \in N^3$. The interpretation here is that a_{ij}

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[9] Bikhchandani and Mamer actually proved that the assumption of integral solution is equivalent to the set of competitive equilibrium allocations being nonempty, but this in turn implies that the core is nonempty, since the core contains the set of competitive equilibrium allocations.

Alternatively, we can directly prove core nonemptiness as follows. Let $p^* = (p^*_{ijk})$ be an optimal integral solution to $(P2)$. Then there exist two permutations π_1 and π_2 of N such that $\pi_1(i) = j$ and $\pi_2(i) = k$ iff $p_{ijk} = 1$. Let (u^*, v^*, w^*) be an optimal solution to the dual LP of $(P2)$ (program $(D2)$ in the text), and define vector x^* by $x^*_i = u^*_i + v^*_i + w^*_i$ for $i = 1, ..., n$. Since $V(N) = \overline{m}_{HC} = \Sigma_{i\in N} d_{i\pi_1(i)\pi_2(i)} = \underline{m}_{HC}$, we have $x^*(N) = V(N)$. We also have $x^*(S) \geq V(S)$ for each $S \subseteq N$. This is because, letting μ_1 and μ_2 be permutations of S with $V(S) = \Sigma_{i\in S} d_{i\mu_1(i)\mu_2(i)}$, we have $x^*(S) = \Sigma_{i\in S}(u^*_i + v^*_{\mu_1(i)} + w^*_{\mu_2(i)}) \geq \Sigma_{i\in S} d_{i\mu_1(i)\mu_2(i)} = V(S)$. Thus x^* is a core vector, and so the core is nonempty.

In some TU matching models (e.g. Shapley-Shubik [14] or Tijs et al [16]), the Birkhoff-Von Neumann theorem (BVNT) can be used to show that the CLP must solve integrally, and this in turn is used to show core nonemptiness. In our model, Curiel-Tijs [2] have already pointed out that we can never hope to use this technique to show core nonemptiness, because the BVNT theorem doesn't generalize in the right way. But that doesn't seem to be as relevant here, because Example 3.1 shows that the CLP solving integrally is not equivalent to core nonemptiness.

[10] Alternatively (following the comment of Curiel-Tijs [2] p. 330) we could note that we have a superadditive two-person game, and therefore the core is nonempty.

and b_{ik} represent player i's monetary evaluations for house j and car k, respectively. Thus $d_{ijk} = a_{ij} + b_{ik}$ means that i's evaluation of the house-car bundle (j, k) is just his evaluation a_{ij} of house j added to his evaluation b_{ik} of car k. It is easily seen that Example 3.1 does not satisfy additive separability.

If matrix D satisfies additive separability, then CLP $(P2)$ decomposes into two linear programs of type (P), one with matrix A and the other with matrix B. Both of these will solve integrally, and so $(P2)$ will also solve integrally. Thus the optimal value $\overline{m}_{HC}$ of CLP $(P2)$ becomes $V(N)$ of game G^{HC}. This in turn implies that the core is nonempty.[11] But we then have a question:

> *Is there a coincidence between core vectors and optimal solutions to the dual of $(P2)$, as in Theorem 2.1 for the permutation game?*

The answer to this question is *negative,* even under the assumption of additive separability. We show this by the example below, but first let us write down the dual of $(P2)$:

$$\underline{m}_{HC} = \min_{u,v,w} \sum_{i \in N} (u_i + v_i + w_i) \qquad (D2)$$
$$s.\ t.\ u_i + v_j + w_k \geq d_{ijk} \quad \text{for all } (i, j, k) \in N^3.$$

Readers will note the similarity between this linear program and program (D) in the previous section.

Example 3.2. *$N = \{1, 2, 3\}$ and*

$A = (a_{ij})$	$j=1$	$j=2$	$j=3$
$i=1$	0	20	15
$i=2$	15	0	20
$i=3$	20	15	0

$B = (b_{ik})$	$k=1$	$k=2$	$k=3$
$i=1$	5	20	0
$i=2$	0	5	20
$i=3$	20	0	5

Hence $D = (d_{ijk})$ is as follows:

d_{1jk}	$k=1$	$k=2$	$k=3$
$j=1$	5	20	0
$j=2$	25	40	20
$j=3$	20	35	15

d_{2jk}	$k=1$	$k=2$	$k=3$
$j=1$	15	20	35
$j=2$	0	5	20
$j=3$	20	25	40

d_{3jk}	$k=1$	$k=2$	$k=3$
$j=1$	40	20	25
$j=2$	35	15	20
$j=3$	20	0	5

This example gives a TU game G_0^{HC} with the following characteristic function:

$$V(\varnothing) = 0,\ V(\{1\}) = V(\{2\}) = V(\{3\}) = 5,$$
$$V(\{12\}) = d_{122} + d_{211} = 55,\ V\ (\{23\}) = d_{233} + d_{322} = 55,$$
$$V(\{13\}) = d_{133} + d_{311} = 55,$$
$$V(N) = d_{122} + d_{233} + d_{311} = 120.$$

Since D is additively separable, CLP $(P2)$ for this example must solve integrally, and so the optimal objective function value must be equal to $V(N)$. Indeed, it is easy to see that

[11] We remark that Curiel and Tijs [2] were the first to prove nonemptiness of the core of our game, in the case of additive separability.

an optimal solution to $(P2)$ is $p^*_{122} = p^*_{233} = p^*_{311} = 1$ and all other p^*_{ijk}'s equal to zero, and $\overline{m}_{HC} = \underline{m}_{HC} = V(N) = 120$.

For any given game G^{HC} associated with an additively separable D, it is easy to prove that for any optimal solution (u^*, v^*, w^*) to $(D2)$, the vector x^* given by $x^* = u^* + v^* + w^*$ is in the core.[12] So the analogue of Theorem 2.1 in the "backwards direction" holds. However, the "forward direction" is not necessarily true. Indeed,

Claim: *Consider game G_0^{HC}, which is associated with an additively separable matrix. Then a) $x = (10, 55, 55)$ is in the core of G_0^{HC}, but b) x cannot be represented as a sum $u + v + w$ for any optimal solution (u, v, w) to $(D2)$.*

Proof. First, it is easy to verify that x is in the core of G_0^{HC}. Now suppose that there exists an optimal solution (u, v, w) to $(D2)$ with

$$u_1 + v_1 + w_1 = 10, \tag{3.1}$$
$$u_2 + v_2 + w_2 = 55, \tag{3.2}$$
$$u_3 + v_3 + w_3 = 55. \tag{3.3}$$

Then this implies that dual program $(D2)$ solves with $\underline{m}_{HC} = \Sigma_{i\in N}(u_i + v_i + w_i) = 120$, and thus we must have

$$u_1 + v_2 + w_2 = d_{122} = 40, \tag{3.4}$$
$$u_2 + v_3 + w_3 = d_{233} = 40, \tag{3.5}$$
$$u_3 + v_1 + w_1 = d_{311} = 40. \tag{3.6}$$

Adding equations (3.4) and (3.6), and then subtracting (3.1) gives

$$u_3 + v_2 + w_2 = 70. \tag{3.7}$$

On the other hand, since (u, v, w) is a solution to $(D2)$, it must be that $u_1 + v_3 + w_1 \geq d_{131} = 20$ and $u_2 + v_1 + w_3 \geq d_{213} = 35$. These inequalities in conjunction with $\Sigma_{i\in N}(u_i + v_i + w_i) = 120$ imply that $u_3 + v_2 + w_2 \leq 65$. This contradicts (3.7), and so the proof of the claim is completed. ■

4. Discussion

In this section we will examine a condition which guarantees the "core-optimal dual solution" equivalence discussed in the last section. Needless to say, this condition does not hold in Example 3.2.

Let (N, V^{HC}) be a TU house-and-car swapping game (or "HCS game") with valuation matrix $D = (d_{ijk})$. Throughout this section we assume that matrix D is additively separable, and let $A = (a_{ij})$ and $B = (b_{ik})$ be valuation matrices for houses and cars with $d_{ijk} = a_{ij} + b_{ik}$ for all $i, j, k \in N$. We define (N, V^H) and (N, V^C) to be the *component* permutation games associated with matrices A and B, respectively. Let $C(V)$ denote the core of a given game (N, V).

[12] In our example one optimal solution to $(D2)$ is $(u^*_1, u^*_2, u^*_3,\ v^*_1, v^*_2, v^*_3,\ w^*_1, w^*_2, w^*_3) = (25, 0,\ 25,\ 5, 0, 10, 10, 15, 30)$. Forming the vector $x^* = u^* + v^* + w^*$, we have $x^* = (40, 15, 65)$. Indeed, x^* is in the core of G_0^{HC}.

In general, one may interpret the optimal dual variables u^*, v^*, and w^* as the shadow prices of the players' "matching abilities". In this sense the result that "$x^* = u^* + v^* + w^*$ is in the core" means that if one gives the players "what they deserve in terms of matching ability", the result is a core vector. However, Example 3.2 shows that not all core vectors can necessarily be formed this way.

From the additive separability of matrix D, we have $V^{HC}(S) = V^H(S) + V^C(S)$ for each $S \subseteq N$. However, this additivity of characteristic functions does not imply the **additivity-of-cores property**, i.e., it is not necessarily true that

$$C(V^H) + C(V^C) = C(V^H + V^C).$$

Here, $C(V^H + V^C) = \{x \in \mathbf{R}^N | x(N) = V^H(N) + V^C(N) = V^{HC}(N)$, and $x(S) \geq V^H(S) + V^C(S) = V^{HC}(S)$ for all $S \subseteq N\}$. In other words, $C(V^H + V^C) = C(V^{HC})$. On the other hand, the sum $C(V^H) + C(V^C)$ means $\{x + y | x \in C(V^H)$ and $y \in C(V^C)\}$. It is clear that $C(V^H) + C(V^C) \subseteq C(V^H + V^C)$. However, it is possible for there to be vectors in $C(V^H + V^C)$ which are not in $C(V^H) + C(V^C)$.

Finally, we say that the HCS game has the **core-optimal dual solution equivalence (CODSE) property** if every element of $C(V^{HC})$ can be written as $u+v+w$, where (u, v, w) is an optimal solution to program $(D2)$ of Section 3.

Proposition 4.1. *Suppose (N, V^{HC}) is an HCS game with (additively separable) valuation matrix $D = (d_{ijk})$. Then if the game has the additivity-of-cores property, it has the CODSE property.*

Proof. As defined by $(D2)$ in Section 3, the dual of the CLP of game (N, V^{HC}) is

$$\begin{aligned} \underline{m}_{HC} = \min_{u,v,w} \sum_{i \in N} (u_i + v_i + w_i) \\ s.t.\ u_i + v_j + w_k \geq d_{ijk} \quad \text{for all } (i, j, k) \in N^3. \end{aligned} \tag{4.1}$$

Let $z \in C(V^{HC})$. The proposition is proved if we show that there exists an optimal solution (u, v, w) to dual CLP (4.1) such that

$$z_i = u_i + v_i + w_i \quad \text{for all } i \in N.$$

To show this, we first note that since matrix D is additively separable, the primal CLP of (N, V^{HC}) solves integrally; its optimal objective function value is $V^{HC}(N)$; and $V^{HC}(N) = V^H(N) + V^C(N)$. Thus, by the duality theorem of linear programming

$$V^H(N) + V^C(N) = V^{HC}(N) = \underline{m}_{HC}. \tag{4.2}$$

Next, since $C(V^H) + C(V^C) = C(V^{HC})$, there exist core vectors $x \in C(V^H)$ and $y \in C(V^C)$ with $z = x + y$. Let $A = (a_{ij})$ and $B = (b_{ik})$ be valuation matrices that define component permutaion games (N, V^H) and (N, V^C), respectively. As defined by (D) in section 2, the dual CLP's of (N, V^H) and (N, V^C) are as follows:

$$\begin{aligned} \underline{m}_H = \min_{u,v} \sum_{i \in N} (u_i + v_i) \\ s.t.\ u_i + v_j \geq a_{ij} \quad \text{for all } (i, j) \in N^2, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \underline{m}_C = \min_{u,w} \sum_{i \in N} (u_i + w_i) \\ s.t.\ u_i + w_k \geq b_{ik} \quad \text{for all } (i, k) \in N^2. \end{aligned} \tag{4.4}$$

It follows from Theorem 2.1 that there exist optimal solutions (u', v') to dual CLP (4.3) and (u'', w'') to (4.4) such that $x = u' + v'$ and $y = u'' + w''$.

Define (u^*, v^*, w^*) to be a $3n$-dimensional vector with $u_i^* = u_i' + u_i''$, $v_i^* = v_i'$, and $w_i^* = w_i''$ for all $i \in N$. Since the optimal objective function value of the CLP of a permutation game (N, V) is $V(N)$, we have $\underline{m}_H = V^H(N)$ and $\underline{m}_C = V^C(N)$. This, together with (4.2), implies that $\Sigma_{i\in N}(u_i^* + v_i^* + w_i^*) = \Sigma_{i\in N}(u_i' + v_i') + \Sigma_{i\in N}(u_i'' + w_i'') = \underline{m}_H + \underline{m}_C = \underline{m}_{HC}$.

Finally, since $d_{ijk} = a_{ij} + b_{ik}$ for all $i, j, k \in N$, and since (u', v') and (u'', w'') are feasible solutions to (4.3) and (4.4), respectively, it holds that $u_i^* + v_j^* + w_k^* = (u_i' + v_j') + (u_i'' + w_k'') \geq a_{ij} + b_{ik} = d_{ijk}$ for all $(i, j, k) \in N^3$. Hence (u^*, v^*, w^*) is an optimal solution to dual CLP (4.1) with the property that $z_i = u_i^* + v_i^* + w_i^*$ for all $i \in N$. The proof is complete.■

Proposition 4.1 shows that core-optimal dual solution equivalence holds for a HCS game if the game has the additivity-of-cores property. However, we feel it is rather rare for HCS games to have such additivity. A typical case is Example 3.2, where we demonstrated there was no core-optimal dual solution equivalence, and so there is no additivity-of-cores property.[13]

It should be noted, however, that there *is* a well-known class of TU games whose cores have the additivity. These are the convex games, first studied by Shapley [12]. A TU game (N, V) is **convex** if $V(\varnothing) = 0$ and $V(S) + V(T) \leq V(S \cup T) + V(S \cap T)$ for any $S,\ T \in 2^N \backslash \{\varnothing\}$. For each $\pi \in \Pi$, define $S_{\pi,k} = \{i \in N | \pi(i) \leq k\}$ for all $k \in N$, and let $a^\pi = (a_i^\pi)_{i\in N}$ be the payoff vector with $a_i^\pi = V(S_{\pi,\pi(i)}) - V(S_{\pi,\pi(i)-1})$ for all $i \in N$. We imagine the players "entering a room" according to ordering π, with a_i^π being player i's contribution when he/she joins coalition $S_{\pi,\pi(i)-1}$. Shapley [12] proved that the cores of convex games have a nice geometric property.

Theorem 4.2 (Shapley [12]).[14] *For any convex game (N, V), the core is nonempty, and it is the convex hull of $\{a^\pi | \pi \in \Pi\}$.*

Given two convex games (N, V^1) and (N, V^2), the *sum* game (N, V) with $V(S) = V^1(S) + V^2(S)$ for all $S \subseteq N$ is also convex. This, together with Theorem 4.2, implies the following proposition.

Proposition 4.2. *For any two convex games (N, V^1) and (N, V^2),*

$$C(V^1) + C(V^2) = C(V^1 + V^2).$$

Proof. It is clear from the definition of cores that $C(V^1) + C(V^2) \subseteq C(V^1 + V^2)$. We show that $C(V^1 + V^2) \subseteq C(V^1) + C(V^2)$.

Let $z \in C(V^1 + V^2)$. Let (N, V) be the game with $V(S) = V^1(S) + V^2(S)$ for all $S \subseteq N$. For each permutation $\pi \in \Pi$, define payoff vectors $d^\pi = (d_i^\pi)_{i\in N}$, $a^\pi = (a_i^\pi)_{i\in N}$ and $b^\pi = (b_i^\pi)_{i\in N}$ by $d_i^\pi = V(S_{i,\pi(i)}) - V(S_{i,\pi(i)-1})$, $a_i^\pi = V^1(S_{i,\pi(i)}) - V^1(S_{i,\pi(i)-1})$, and $b_i^\pi = V^2(S_{i,\pi(i)}) - V^2(S_{i,\pi(i)-1})$ for each $i \in N$. We note that $d^\pi = a^\pi + b^\pi$ for all $\pi \in \Pi$. Since (N, V) is convex, it follows from Theorem 4.2 that there exists a vector $(p_\pi)_{\pi\in\Pi}$ with $\Sigma_{\pi\in\Pi}\, p_\pi = 1$ and $p_\pi \geq 0$ for all $\pi \in \Pi$ such that $z = \Sigma_{\pi\in\Pi}\, p_\pi d^\pi = \Sigma_{\pi\in\Pi}\, p_\pi a^\pi + \Sigma_{\pi\in\Pi}\, p_\pi b^\pi$. Let $x = \Sigma_{\pi\in\Pi}\, p_\pi a^\pi$ and $y = \Sigma_{\pi\in\Pi}\, p_\pi b^\pi$. Since games (N, V^1) and (N, V^2) are also convex, we have that $x \in C(V^1)$ and $y \in C(V^2)$. This means that $C(V^1 + V^2) \subseteq C(V^1) + C(V^2)$. Hence $C(V^1) + C(V^2) = C(V^1 + V^2)$. ■

[13]For Example 3.2, characteristic function V^H is defined as $V^H(\{1\}) = V^H(\{2\}) = V^H(\{3\}) = 0$, $V^H(\{1,2\}) = V^H(\{1,3\}) = V^H(\{2,3\}) = 35$, $V^H(N) = 60$. Characteristic function V^C is defined as $V^C(\{1\}) = V^C(\{2\}) = V^C(\{3\}) = 5$, $V^C(\{1,2\}) = V^C(\{1,3\}) = V^C(\{2,3\}) = 20$, $V^C(N) = 60$. We saw the vector $(10, 55, 55)$ is in the core of G_0^{HC}, i.e., $C(V^H + V^C)$, but it cannot be written as $y + z$ with $y \in C(V^H)$ and $z \in C(V^C)$.

[14]This theorem follows from Shapley's original theorems 3, 4, and 5 in his paper [12].

Propositions 4.1 and 4.2 imply that if the component permutation games of a given HCS game are convex, then the HCS game has the CODSE property. However, not much is known about sufficient conditions for convexity in permutation games.

Proposition 4.3. *Let (N, V) be a permutation game with valuation matrix $A = (a_{ij})$. If there exists a permutation $\pi \in \Pi$ such that*

$$\begin{aligned} a_{i\pi(i)} &\geq 0 \text{ for all } i \in N, \\ a_{ij} &= 0 \text{ for all } i, j \in N \text{ with } j \neq \pi(i), \end{aligned} \tag{4.5}$$

then permutation game (N, V) is convex.

Proof. Let (N, V) be a permutation game with valuation matrix $A = (a_{ij})$ and permutation π satisfying condition (4.5). Define $M_S = \{i \in S | \pi(i) \in S\}$ for $S \in 2^N \backslash \{\varnothing\}$.

It is clear that the worth of any coalition is attained by matching as many of its members as possible according to π. Hence,

$$V(S) = \sum_{i \in M_S} a_{i\pi(i)} \quad \text{for each } S \in 2^N \backslash \{\varnothing\}. \tag{4.6}$$

We now show that game (N, V) is convex. Let S and T be any nonempty coalitions. If $i \in M_S \cup M_T$, then both i and $\pi(i)$ belong to $S \cup T$. If $i \in M_S \cap M_T$, then both i and $\pi(i)$ belong to $S \cap T$. Thus $M_S \cup M_T \subseteq M_{S \cup T}$ and $M_S \cap M_T \subseteq M_{S \cap T}$. It follows from these inclusions that $\sum_{i \in M_S} a_{i\pi(i)} + \sum_{i \in M_T} a_{i\pi(i)} = \sum_{i \in M_S \cup M_T} a_{i\pi(i)} + \sum_{i \in M_S \cap M_T} a_{i\pi(i)} \leq \sum_{i \in M_{S \cup T}} a_{i\pi(i)} + \sum_{i \in M_{S \cap T}} a_{i\pi(i)}$. From (4.6), this means that $V(S) + V(T) \leq V(S \cup T) + V(S \cap T)$. ■

Condition (4.5) is interpreted as follows: *Each player i has at most one favorite house $\pi(i)$ with positive valuation and has no interest in any other house j. Moreover, each player can obtain his/her favorite house (since π is a permutation).* This seems like a very restrictive sufficient condition. In fact, it is easy to see that it is *not* a necessary condition: the valuation matrix B in Example 3.2 violates (4.5), but permutation game (N, V^C) defined by matrix B is convex (see endnote 13).

Solymosi and Raghavan [15] gave a necessary and sufficient condition for Shapley-Shubik's assignment games to be convex, similar to (4.5), which is also quite restrictive. It would be interesting to identify a necessary and sufficient condition for permutation games to be convex.

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