

# Coalitional configurations and value

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## Abstract

In 1977 Owen defined a new value for cooperative games with transferable utility. This value, which is called the coalitional value or Owen value, was defined by generalizing the Shapley value (1953) to a new framework. Owen supposed that due to political, economic or other reasons some players may be more likely to act together than others. More precisely, this author supposed that players are joined in coalitions and form a partition of the set of players, and defined the coalitional value in this framework.

If the players of a game are 1, 2 and 3, Owen considered that the three players can join and form a partition of the set  $\{1, 2, 3\}$ .

In this work, for example, we are going to let player 1 to join up with 2 (and not with 3) and player 3 to join up with player 2 (and not with 1). That is, we are going to let players to join so that to belong to one or more than one coalition. In this case players will not form necessarily a partition of the set of players, but a collection of coalitions ( $\{\{1, 2\}, \{2, 3\}\}$  in the example above), which will be called

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coalitional configuration. If a coalition belongs to a coalitional configuration this will mean that its players are joined up together.

So, in this work we consider a new framework. In this new framework we will define and characterize a value, which will be called configuration value, and this value will be a generalization of the Owen value (1977).

We define the configuration value by employing variations in order to generalize the definition of the Owen value (1977) given by means of orders. This definition gives rise to an explicit formula for the configuration value. The new value will be characterized by adapting the axioms employed by Owen (1977) to characterize his value and by adding a specific axiom for the new framework. Moreover, we give an alternative definition for the configuration value, which will also be a generalization of the original definition of the Owen value (1977). Finally, we prove that the configuration value of a player can be calculated by means of some players' Owen values in a specific game and we calculate the configuration value by employing multilinear extensions.

The definition of our value is as follows.

Let  $N \subset U$  be a finite subset of the universe of players. We will denote by  $G^N$  the space formed by all transferable utility (TU) games on  $N$ .

If  $\mathfrak{B} = \{B_1, B_2, \dots, B_m\}$  is a family of subsets of  $N$  such that  $\bigcup_{i=1}^m B_i = N$ ,  $\mathfrak{B}$  will be called *coalitional configuration* of  $N$ . Notice that a coalitional configuration is not necessarily a partition of  $N$ . We will denote by  $\mathbf{B}^N$  the set of all coalitional configurations of  $N$ .

In the redefinition we will consider variations with repetition of a finite subset  $N \subset U$ , that is, orderings formed by elements of  $N$  where these elements can be repeated. The *order* of a variation is the number of elements of the variation, where each element is counted as many times as it appears in the variation.

Let  $\mathfrak{B} = \{B_1, B_2, \dots, B_m\}$  be a coalitional configuration of  $N$ . For each  $i \in N$  we will denote by  $c_{\mathfrak{B}}(i)$  the number of coalitions of  $\mathfrak{B}$  player  $i$  belongs to.

Given  $\mathfrak{B} \in \mathbf{B}^N$  and  $v \in G^N$ , the configuration value of  $(\mathfrak{B}, v)$  will be defined as follows.

Let us consider players in  $N$  in such a way that player  $i \in N$  is repeated  $c_{\mathfrak{B}}(i)$  times, that is, we have  $\sum_{B_q \in \mathfrak{B}} |B_q|$  elements. Suppose we choose (without replacement) one of these elements (being all the elements equally likely to be picked) and then we add elements successively up to select the  $\sum_{B_q \in \mathfrak{B}} |B_q|$  ones so that coalitions  $B_q$  of  $\mathfrak{B}$  appear successively. When player  $i \in N$  has been chosen

for the  $k^{\text{th}}$  time, in the following step an element will be picked among  $c_{\mathfrak{B}}(i) - k$   $i$ -s and the rest of the elements that have not been selected yet, being all them equally likely to be chosen.

When a player is picked, this player is given his marginal contribution to the set formed by the players that have been chosen before (notice that this contribution can be different from zero only when the player is selected for the first time). When all the elements have been picked each player receives the summation of all his contributions, that is, his contribution when he has been chosen for the first time. The expected value of this contribution will be precisely the configuration value of the player in game  $v \in G^N$  with coalitional configuration  $\mathfrak{B} \in \mathbf{B}^N$ .

Formally, let  $v \in G^N$  and  $\mathfrak{B} \in \mathbf{B}^N$ . Let  $R_{\mathfrak{B}}(N)$  be the set formed by the  $\sum_{B_q \in \mathfrak{B}} |B_q|$  order variations with repetition of subset  $N$  such that each  $i \in N$  is repeated  $c_{\mathfrak{B}}(i)$  times and coalitions of  $\mathfrak{B}$  appear successively.

Let  $R \in R_{\mathfrak{B}}(N)$  and  $i \in N$ . We will denote by  $R[i]$  the set formed by players in  $N$  whose first position in  $R$  is previous to the first position of  $i$  in  $R$ . The *marginal contribution* of player  $i$  in  $R$  will be

$$C_i(v, R) = v(R[i] \cup \{i\}) - v(R[i]).$$

Let  $P$  be the probability distribution on  $R_{\mathfrak{B}}(N)$  all the orders are equally likely in.

**Definition 0.1.** *The configuration value is the mapping  $\phi$  from  $\bigcup_{N \subset U} \mathbf{B}^N \times G^N$  into  $\bigcup_{N \subset U} \mathbb{R}^N$  such that  $\psi(\mathfrak{B}, v) \in \mathbb{R}^N$  if  $(\mathfrak{B}, v) \in \mathbf{B}^N \times G^N$  which is defined as follows for each  $\mathfrak{B} \in \mathbf{B}^N, v \in G^N$  and  $i \in N$ ,*

$$\phi_i(\mathfrak{B}, v) = E_P(C_i(v, \cdot)).$$