DECOMPOSITION PRINCIPLE IN COST GAMES

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ABSTRACT. In this work we consider cost games in characteristic function form where the players are not homogenous. It is supposed that each player is formed by a number of indistinguishable agents. We determined tariffs for the agents in such a way the amount is gathered coincide with the cost of give service to the gran coalition. The tariffs are determined supposing additivity and the principle of decomposition using the canonical dual base of the space of cooperative games.

1. Overview

By a cost game we mean a triplet (N, α, c) where $N = \{1, \ldots, n\}$ is a finite set of players, $\alpha \in \mathbb{R}_+^N$ is a vector of real no negative numbers indexed by the players in N and $c : 2^N \to \mathbb{R}_+$ is a cost function such that $c(\emptyset) = 0$. We suppose that the players in N need to hire a service, the player $i \in N$ consists of α_i indistinguishable agents and $c(S)$ is the cost that players in S need to pay in order to get the service. Alternatively, each number α_i can be thought as the intensity with which the player i uses the service; each partial cost assigned to the player i is divided by this number. We are looking for a fair tariffs for the agents in order to cover the cost of the service for all the players.

Suppose the cost function could be decompose in a sequence of basic costs $\{\delta_{T_1}, \ldots, \delta_{T_m}\}$ in such a way each player in T_j is equally responsible for the basic cost δ_{T_j}. Without loss of generality $\{T_1, \ldots, T_m\} \equiv 2^N$, because if some $T \in 2^N$ is missing, it could be include with zero basic cost and when a coalition is repeated, we can join them in only one basic cost. In such a case, we could express c in the following form,

$$(1.1) \qquad c(S) = \sum_{\{T \mid T \cap S \neq \emptyset\}} \delta_T.$$

This expression of c suggests to divide equally each δ_T among the players in T, in which case we get the Shapley value. However, we have heterogeneous players, so we suggest to divide equally each δ_T among the agents in T, what takes us to tariffs for the agents. It is worth mentioning that this tariff are not equivalent to the weighted Shapley value.

We will proof that for every cost function $c : 2^N \to \mathbb{R}_+$ such that $c(\emptyset) = 0$, there exists a unique set of real numbers $\{\delta_T\}_{T \in 2^N}$ such that c satisfies (1.1). We will say that the game (N, α, c) is decomposable when all the real numbers $\{\delta_T\}_{T \in 2^N}$

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in (1.1) are no negative and we say that $\{\delta_T\}_{T \in 2^N}$ are the basic costs. Let G be the set of all decomposable games.

Definition 1. *By a tariffs rule we mean a function*

$$p : G \to \mathbb{R}^N$$

such that $\sum_{i \in N} \alpha_i p_i(N, \alpha, c) = c(N)$.

Given c, $d \in G$ and $\lambda \in R_+$ we define $c + d$ and λc as usual, i.e. , $(c + d)(S) = c(S) + d(S)$ and $(\lambda c)(S) = \lambda c(S)$. Clearly, G is closed under these sum and under positive linear combination. Let

$$a_T(S) = \begin{cases} 1 & \text{si } S \cap T \neq \emptyset \\ 0 & \text{de otra forma} \end{cases}.$$

be a cost function for every $T \subseteq N, T \neq \emptyset$. The set of cost functions $\{a_T | T \subseteq N, T \neq \emptyset\}$ is known as the dual base.

Axiom (additivity). We will say that the tariffs rule $p : G \to \mathbb{R}^N$ is additive if and only if $p(c + d) = p(c) + p(d)$ for every $c, d \in G$.

Axiom (decomposition principle). Every basic cost δ_T is equally share among the agents in T, i.e. ,

(1.2) $$p_k(\delta_T a_T) = \begin{cases} \frac{\delta_T}{\alpha(T)} & \text{if } k \in T \\ 0 & \text{otherwise} \end{cases}$$

The additive axiom says that the tariff of an agent is formed adding what corresponds him from each basic cost and the decomposition principle axiom request that each basic cost is divided equally among the agents involved it.

Lemma 1. *For every S, $T \subseteq N$ we have that*

$$\sum_{\{R | T \subseteq R \subseteq S\}} (-1)^{r+t} = \begin{cases} 1 & \text{si } T = S \\ 0 & \text{de otra forma} \end{cases}$$

Lemma 2. *Let*

(1.3) $$\delta_T = \sum_{\{R | N \setminus T \subseteq R\}} (-1)^{n+r+t+1} c(R)$$

be for every $T \subseteq N$, then

    a) $\sum_{T \subseteq N \setminus S} \delta_T = -c(S)$
    b) $\sum_{T \subseteq N} \delta_T = 0$
    c) $\delta_\emptyset = -c(N)$
    d) $\sum_{\{T | T \cap S \neq \emptyset\}} \delta_T = c(S)$

Theorem 1. *There exist a unique tariffs rule $p : G \to \mathbb{R}^N$ such that satisfies the axioms of additivity and decomposition principle. Moreover, it is given by*

(1.4) $$p_i(c) = \sum_{\{S : i \notin S\}} \sum_{\{T : T \subseteq S\}} \frac{(-1)^{s+t}}{\alpha(N \setminus T)} (c(S \cup \{i\}) - c(S))$$

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