LOCATION GAME ON THE PLANE

VLADIMIR MAZALOV
Institute of Applied Mathematical Research,
Karelian Research Center of Russian Academy of Sciences,
Pushkinskaya str. 11, Petrozavodsk 185610, Russia,
E-mail: vmazalov@krc.karelia.ru

MINORU SAKAGUCHI
Prof. Emeritus, Osaka University, 3-26-4 Midorigaoka,
Toyonaka, Osaka 560-0002, Japan,
E-mail: smf@mc.kcom.ne.jp

We analyze Hotelling's duopoly model on the plane. There are two players (firms) located in different points inside a circle and the customers are distributed with some density in it. The solution of two game-theoretic problems is derived. The first problem is to find the equilibrium prices for the homogeneous goods, and the second problem is to find the equilibrium allocation of the players inside the circle. The equilibrium in location game is constructed for uniform and non-uniform case.

Key words: Hotelling's duopoly model on the plane, location game, equilibrium prices, equilibrium location. AMS Subject Classification: 91A10, 91A80, 91B24, 91B32.

1. Introduction

Imagine a city in the form of the circle S (see Fig. 1) with population distributed in it with some density function $f(x, y)$. We consider a non-zero-sum location game with two players (Firms) I and II which are located in some different points P_1 and P_2 inside the circle. The objective of the players is to determine the optimal prices for the same goods in dependence of their position on the plane. Let the radius of the circle be equal to 1.

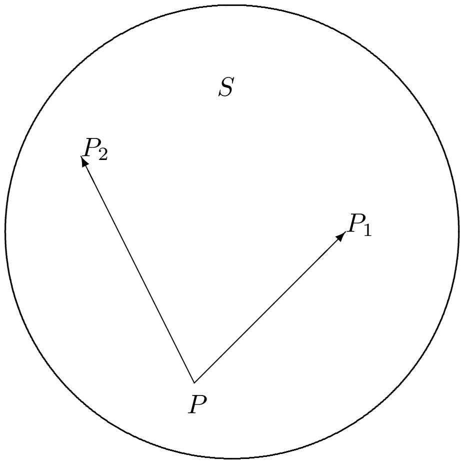


Fig. 1.

Suppose that the firms I and II declare the prices for the goods as p_1 and p_2. We will use here the Hotelling's model to determine which firm will be more attractive for a customer who is located in the point $P \in S$. The customer compares the cost functions $F(p_i, \rho(P, P_i))$ with arguments p_i (price) and $\rho(P, P_i)$ (distance from his location point P to the firm point P_i), $i = 1, 2$, and prefers the firm with minimal value. So, all customers in S are divided for two parts S_1 and S_2 respectively their preferences to firms I and II. So, the gains of the firms I and II can be determined by functions

$$H_1(p_1, p_2) = p_1 \mu(S_1), \quad H_2(p_1, p_2) = p_2 \mu(S_2), \tag{1}$$

where $\mu(S) = \int_S f(x, y) dx dy$ is the probability measure of the set S. Mainly, we analyze here a problem with $F = p + \rho^2$. Our objective is to find the equilibrium prices in this game and the equilibrium allocation of the players on the plane.

The location problem, firstly, installed by Hotelling (1927) as a problem of Nash equilibrium of competitive facilities on a linear market, afterwards was considered in linear variant in the articles of D'Aspremont et al.(1979), Kats (1987), Bester et al. (1996), Zhang et al. (1998), Sakaguchi (2001). Dresner (1982) and Hakimi (1983) consider the location problem as a Stackelberg equilibrium problem on the plane and a network, respectively.

In this paper we analyse Nash equilibrium location problem on the plane. At the begining (Section 2) we find Nash equilibrium in the pricing game model with a population uniformly distributed in the circle. Notice, that in the uniform case (1) will take the following form

$$H_1(p_1, p_2) = p_1 \left[\frac{1}{\pi} \int\limits_{S_1} dxdy \right], \quad H_2(p_1, p_2) = p_2 \left[\frac{1}{\pi} \int\limits_{S_2} dxdy \right].$$

Then, in Section 3 a non-uniform case is analyzed and the equilibrium prices are derived. In Section 4 we consider the problem of the equilibrium points for allocation of players on the plane. Different versions of non-uniform distribution are compared in Section 5. Final remarks and a possibility of another cost functions are presented in the Conclusion.

2. Solution for Uniform Distribution

Turn over the circle S that points P_1 and P_2 become have the same ordinate y (see Fig. 2). Denote the the coordinates of P_1 and P_2 on the axis O_x as x_1 and x_2 respectively. Without loss of generality suppose that $x_1 \geq x_2$.

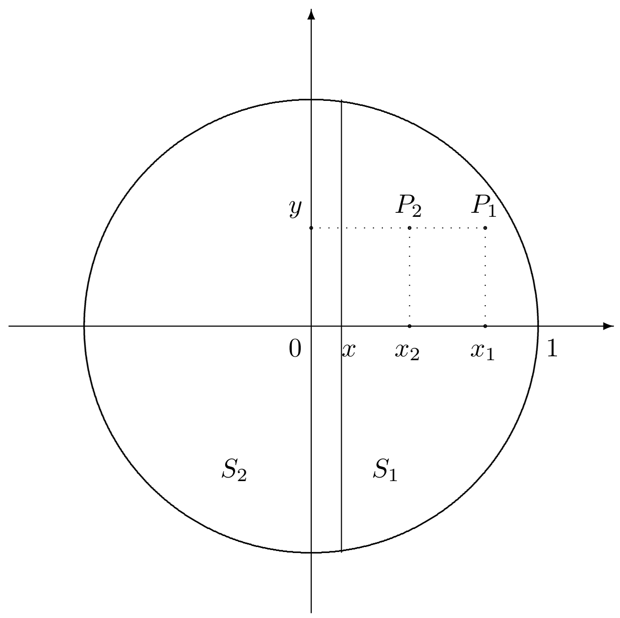


Fig. 2.

According to Hotelling's scheme the regions S_1 and S_2 are distigiuished by the line

$$p_1 + (x - x_1)^2 = p_2 + (x - x_2)^2,$$

which is a parallel for the axis O_y with coordinate

$$x = \frac{1}{2}(x_1 + x_2) + \frac{p_1 - p_2}{2(x_1 - x_2)}. \tag{2}$$

So, according to (1) the payoffs of the players in this game are

$$H_1(p_1, p_2) = p_1\left(\arccos x - x\sqrt{1 - x^2}\right)/\pi \tag{3}$$

$$H_2(p_1, p_2) = p_2\left(\pi - \arccos x + x\sqrt{1 - x^2}\right)/\pi, \tag{4}$$

with x satisfied (2). We find the equilibrium prices from the equation $\frac{\partial H_1}{\partial p_1} = \frac{\partial H_2}{\partial p_2} = 0$.

Find the derivative of (3) in p_1.

$$\pi\frac{\partial H_1}{\partial p_1} = \arccos x - x\sqrt{1 - x^2} + p_1\left[-\frac{1}{\sqrt{1 - x^2}}\frac{1}{2(x_1 - x_2)} - \sqrt{1 - x^2}\frac{1}{2(x_1 - x_2)} + \frac{2x^2}{2\sqrt{1 - x^2}}\frac{1}{2(x_1 - x_2)}\right].$$

Having it equal to zero we obtain

$$p_1 = (x_1 - x_2)\left[\frac{\arccos x}{\sqrt{1 - x^2}} - x\right] \tag{5}$$

Analogously, from $\frac{\partial H_2}{\partial p_2} = 0$ it follows that

$$p_2 = (x_1 - x_2)\left[x + \frac{\pi - \arccos x}{\sqrt{1 - x^2}}\right] \tag{6}$$

Finally, using (2), (5), (6) the equilibrium prices can be represented in the form

$$p_1 = \frac{x_1 - x_2}{2}\left[\frac{\pi}{\sqrt{1 - x^2}} - 2\left(\frac{x_1 + x_2}{2} - x\right)\right] \tag{7}$$

$$p_2 = \frac{x_1 - x_2}{2}\left[\frac{\pi}{\sqrt{1 - x^2}} + 2\left(\frac{x_1 + x_2}{2} - x\right)\right] \tag{8}$$

with

$$x = \frac{x_1 + x_2}{4} - \frac{\pi/2 - \arccos x}{2\sqrt{1 - x^2}}. \tag{9}$$

Remark 2.1. If $x_1 + x_2 = 0$ then $x = 0$ by (2), and, consequently, $p_1 = p_2 = \pi x_1$, by (5)-(6), and $H_1 = H_2 = \pi x_1/2$ by (3)-(4). Hence, the maximal equilibrium prices are reached with $x_1 = 1$ and $x_2 = -1$ and equal to $p_1 = p_2 = \pi$. The optimal payoffs are equal to $H_1 = H_2 = \pi/2 \approx 1.570$. So, if customers locate uniformly in the unit circle, then the two firms should locate as far as possible on a same diameter.

Some numerical values are presented in the Table 1.

Table 1. The equilibrium prices for various x_1 and x_2.

		$x_2 = 0.6$	0.4	0.2	0	-0.2	-0.4	-0.6	-0.8	-1
$x_1 =$ 0.8	p_1	0.228	0.480	0.755	1.054	1.378	1.729	2.106	2.513	2.949
	p_2	0.416	0.801	1.156	1.481	1.778	2.049	2.293	2.513	2.708
	H_1	0.081	0.179	0.298	0.438	0.601	0.791	1.008	1.256	1.536
	H_2	0.268	0.501	0.699	0.865	1.002	1.111	1.195	1.256	1.296
	x	0.230	0.198	0.165	0.132	0.099	0.066	0.033	0	-0.033
$x_1 =$ 0.6	p_1		0.251	0.527	0.827	1.152	1.504	1.884	2.293	2.732
	p_2		0.385	0.740	1.067	1.366	1.638	1.884	2.106	2.305
	H_1		0.099	0.219	0.360	0.527	0.720	0.942	1.195	1.481
	H_2		0.233	0.432	0.601	0.740	0.853	0.942	1.008	1.054
	x		0.165	0.132	0.099	0.066	0.033	0	-0.033	-0.066

3. Solution in Non-uniform Case

Suppose that customer's distribution is not uniform. Consider here the case when the density be

$$f(r,\theta) = 3(1-r)/\pi, \quad 0 \le r \le 1, 0 \le \theta \le 2\pi. \tag{11}$$

It corresponds to the case with customers which are concentrated closer to the center of the city.

Notice, that it is sufficient to consider here only the case with $x_1 + x_2 \ge 0$, otherwise, we obtain the solution changing the signs of x_1, x_2. Then the expected rewards (1) to the players are

$$H_1(p_1,p_2) = \frac{6}{\pi} p_1 A(x), \quad H_2(p_1,p_2) = p_2\left(1 - \frac{6}{\pi}A(x)\right), \tag{12}$$

where

$$A(x) = \int_x^1 r(1-r)\arccos\left(\frac{x}{r}\right)dr = \frac{1}{6}\left[\arccos x - x\sqrt{1-x^2} - 2x\int_x^1 \sqrt{r^2-x^2}dr\right]$$

$$= \frac{1}{6}\left[\arccos x - 2x\sqrt{1-x^2} - x^3\log x + x^3\log(1+\sqrt{1-x^2})\right]$$

so that

$$\frac{\pi}{6}\frac{\partial H_1}{\partial p_1} = A(x) + p_1 A'(x)\frac{\partial x}{\partial p_1} = A(x) - \frac{p_1}{2(x_1-x_2)}\int_x^1 \sqrt{r^2-x^2}dr,$$

$$\frac{\partial H_2}{\partial p_2} = 1 - \frac{6}{\pi}A(x) - p_2\frac{6}{\pi}A'(x)\frac{\partial x}{\partial p_2} = 1 - \frac{6}{\pi}A(x) - \frac{6}{\pi}\frac{p_2}{2(x_1-x_2)}\int_x^1 \sqrt{r^2-x^2}dr,$$

since

$$A'(x) = -\int_x^1 \frac{r(1-r)}{\sqrt{r^2-x^2}}dr = -\int_x^1 \sqrt{r^2-x^2}dr. \tag{13}$$

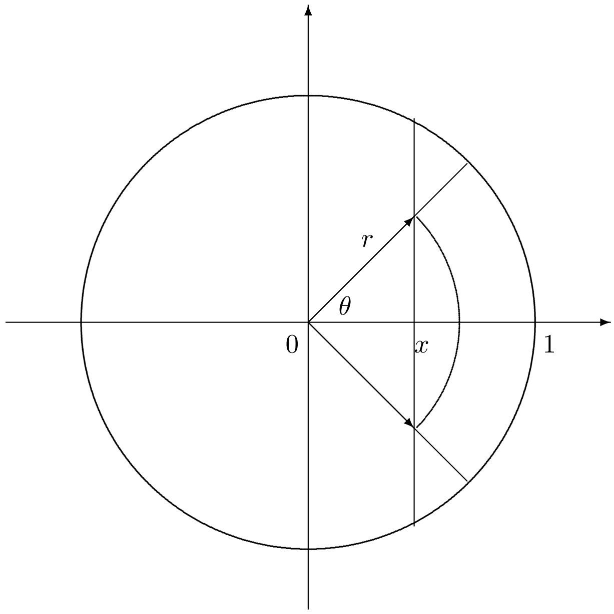


Fig. 3.

The conditions $\frac{\partial H_1}{\partial p_1} = \frac{\partial H_2}{\partial p_2} = 0$ yield

$$p_1 = 2(x_1 - x_2)A(x) / \int_x^1 \sqrt{r^2 - x^2} dr, \tag{14}$$

$$p_2 = 2(x_1 - x_2)\left(\frac{\pi}{6} - A(x)\right) / \int_x^1 \sqrt{r^2 - x^2} dr. \tag{15}$$

Substituting these p_1 and p_2 into

$$x = \frac{1}{2}(x_1 + x_2) + \frac{p_1 - p_2}{2(x_1 - x_2)} \tag{2'}$$

we obtain

$$x - \frac{1}{2}(x_1 + x_2) = (2A(x) - \pi/6) / \int_x^1 \sqrt{r^2 - x^2} dr. \tag{16}$$

Remark 3.1. It follows from (13) that $A(x)$ is convex, decreasing with $A(0) = \pi/12$ and $A(1) = 0$. The right side of (16) is negative, hence,

$$x \le (x_1 + x_2)/2.$$

Let us show that the equation (16) has the unique solution. Rewrite it in the form

$$B(x) = -[x - \frac{1}{2}(x_1 + x_2)]A'(x) - (2A(x) - \pi/6) = 0. \tag{17}$$

Derivative of the function $B(x)$ staying at the left side of the equation (17)

$$B'(x) = -3A'(x) - A''(x)(x - \frac{x_1 + x_2}{2})$$

$$= \int_x^1 \left[3\sqrt{r^2 - x^2} + \frac{x}{\sqrt{r^2 - x^2}}(\frac{x_1 + x_2}{2} - x) \right] dr$$

is positive, so $B(x)$ increases in the interval $[0, \frac{x_1+x_2}{2}]$, and $B(0) = -\frac{x_1+x_2}{4} < 0$ and $B(\frac{x_1+x_2}{2}) = \pi/6 - 2A(\frac{x_1+x_2}{2}) \ge 0$.

If $x_1 + x_2 = 0$, then $x = 0$ satisfies the equation (16) and, $p_1 = p_2 = \frac{2}{3}\pi x_1$ by (14)-(15), and $H_1 = H_2 = \frac{1}{3}\pi x_1$, by (12). For $x_1 = 1, x_2 = -1$ we have $p_1 = p_2 = \frac{2}{3}\pi \approx 2.094$ and $H_1 = H_2 = \frac{1}{3}\pi \approx 1.047$. In Table 2 the results of numerical calculations are presented.

Table 2. The equilibrium prices for various x_1 and x_2.

		$x_2 = 0.6$	0.4	0.2	0	-0.2	-0.4	-0.6	-0.8	-1
	p_1	0.131	0.281	0.449	0.639	0.852	1.091	1.362	1.675	2.046
$x_1 =$	p_2	0.383	0.700	0.962	1.175	1.347	1.484	1.591	1.675	1.751
0.8	H_1	0.033	0.080	0.143	0.225	0.330	0.462	0.628	0.837	1.102
	H_2	0.285	0.500	0.655	0.761	0.825	0.855	0.857	0.837	0.807
	x	0.275	0.236	0.198	0.159	0.120	0.080	0.040	0	-0.040
	p_1		0.149	0.319	0.511	0.727	0.972	1.256	1.591	1.979
$x_1 =$	p_2		0.320	0.587	0.808	0.989	1.136	1.256	1.362	1.454
0.6	H_1		0.047	0.112	0.198	0.308	0.448	0.628	0.857	1.140
	H_2		0.218	0.380	0.495	0.570	0.612	0.628	0.628	0.616
	x		0.198	0.159	0.120	0.080	0.040	0	-0.040	-0.080

4. Equilibrium Points for Location

From previous considerations we saw that if the points of player's location P_1 and P_2 are fixed then there are some equilibrium prices p_1 and p_2. So, we have p_1, p_2 as a function of x_1, x_2. Now, we can install the problem: are there equilibrium points x_1^*, x_2^* for the location of players. This problem often appeares when we plan some infrastructure for the socio-economical regional systems. Let us consider this model for the non-uniform case considered in Section 3.

Assume that the player II selects the point for location $x_2 < 0$. The objective of player I is to find the point x_1 which maximizes his reward $H_1(p_1, p_2)$. Let us find the solution of the equation $\frac{\partial H_1}{\partial x_1} = 0$. By (12)

$$\frac{\pi}{6}\frac{\partial H_1}{\partial x_1} = \frac{\partial p_1}{\partial x_1}A(x) + p_1 A'(x)\frac{\partial x}{\partial x_1} = 0. \tag{18}$$

Differentiating (14) and (17) in x_1 we obtain

$$\frac{1}{2}\frac{\partial p_1}{\partial x_1} = -\frac{A(x)}{A'(x)} - (x_1 - x_2)\left[1 - \frac{A'(x)A''(x)}{[A'(x)]^2}\right]\frac{\partial x}{\partial x_1}, \tag{19}$$

and

$$-\left(\frac{\partial x}{\partial x_1} - \frac{1}{2}\right)A'(x) - [x - \frac{1}{2}(x_1 + x_2)]A''(x)\frac{\partial x}{\partial x_1} - 2A'(x)\frac{\partial x}{\partial x_1} = 0,$$

and, consequently,

$$\frac{\partial x}{\partial x_1} = A'(x)\left[6A'(x) + 2(x - \frac{x_1 + x_2}{2})A''(x)\right]^{-1}. \tag{20}$$

The equation (18)-(20) can be used to find the optimal response x_1 of player I.

Notice, that from symmetry of the problem it follows that if the equilibrium exists then it must be among collections of the form $(x_1, x_2 = -x_1)$. In this case $x = 0$, $A(0) = \pi/12$, $A'(0) = -1/2$, $A''(0) = 0$. From (20) we receive

$$\frac{\partial x}{\partial x_1} = (-1/2)/(-3 + 0) = 1/6,$$

and from (19)

$$\frac{\partial p_1}{\partial x_1} = \frac{\pi}{3} - \frac{2}{3}x_1.$$

Substituting it into (18) we obtain

$$\left(\frac{\pi}{3} - \frac{2}{3}x_1\right)\frac{\pi}{12} + \left(\frac{2}{3}\pi x_1\right)\cdot(-\frac{1}{2})\cdot\frac{1}{6} = 0,$$

and, finally,

$$x_1^* = \frac{\pi}{4}.$$

So, the optimal points for location of the players are $x_1^* = \pi/4, x_2^* = -\pi/4$, with equilibrium prices $p_1 = p_2 = \pi^2/6$ and rewards $H_1 = H_2 = \pi^2/12$.

Remark 4.1. Revisiting to the uniform distribution case (Section 2), the similar calculation made in Section 4 proceeds as follows:

From (3), (7) and (9), we get

$$\pi\frac{\partial H_1}{\partial x_1} = \frac{\partial p_1}{\partial x_1}\left(\arccos x - x\sqrt{1 - x^2}\right) - 2p_1\sqrt{1 - x^2}\frac{\partial x}{\partial x_1},$$

$$\frac{\partial p_1}{\partial x_1} = \frac{\pi}{2\sqrt{1 - x^2}} + x - x_1 + \frac{x_1 - x_2}{2}\left(2 + \pi x(1 - x^2)^{-3/2}\right)\frac{\partial x}{\partial x_1},$$

$$\frac{\partial x}{\partial x_1} = \frac{1}{4}\left[1 + \frac{1}{2(1 - x^2)} + \frac{x}{2(1 - x^2)^{3/2}}\left(\frac{\pi}{2} - \arccos x\right)\right]^{-1},$$

and hence

$$\pi\left[\frac{\partial H_1}{\partial x_1}\right]_{x=0} = \left[\frac{\partial p_1}{\partial x_1}\right]_{x=0}\frac{\pi}{2} - 2[p_1]_{x=0}\left[\frac{\partial x}{\partial x_1}\right]_{x=0}$$

$$= \left(\frac{\pi}{2} - \frac{2x_1}{3}\right)\frac{\pi}{2} - 2\pi x_1\frac{1}{6} = \frac{\pi}{4}\left(\pi - \frac{8}{3}x_1\right) > 0, \quad \forall x_1 \in (0,1).$$

So we conclude that the expected reward attains at its maximum at $x_1^* = -x_2^* = 1$. And this location gives, from (3) and (7),

$$p_i^* = \pi \approx 3.1415 \text{ and } H_i^* = \pi/2 \approx 1.5708, \quad i = 1, 2. \tag{21}$$

5. Another Non-uniform Case

Suppose that the customer's distribution has the density

$$f(r,\theta) = a + \frac{3}{2}\left(\pi^{-1} - a\right)r, \quad (0 \le r \le 1, 0 \le \theta \le 2\pi) \tag{22}$$

where $a > 0$ is chosen such that $f(r,\theta) \ge 0, \forall r$ and θ. The adequate choice of a will be explicitely given later. Also, equation (22) reduces to the uniform case if $a = \pi^{-1}$.

The expected rewards to the players for the price-pair (p_1, p_2) are

$$H_1(p_1,p_2) = p_1 J(x), \tag{23}$$

$$H_2(p_1,p_2) = p_2(1 - J(x)), \tag{24}$$

where x satisfies the equation (2), and

$$J(x) \equiv a(\arccos x - x\sqrt{1-x^2}) + \frac{3}{\pi}(1 - \pi a)\int_x^1 r^2 \arccos\frac{x}{r}dr. \tag{25}$$

Note that

$$J'(x) = 3(\pi^{-1} - a)\int_x^1 \sqrt{r^2 - x^2}dr - (3\pi^{-1} - a)\sqrt{1-x^2},$$

$$J''(x) = x\left[(3\pi^{-1} - a)(1-x^2)^{-1/2} - 3(\pi^{-1} - a)\int_x^1 \frac{dr}{\sqrt{r^2 - x^2}}\right],$$

and hence

	$J(x)$	$J'(x)$	$J''(x)$
$x = 0$	$1/2$	$-\frac{1}{2}(3\pi^{-1} + a)$	0
$x = 1$	0	0	$+\infty$

which will be used later.

The condition $\frac{\partial H_1}{\partial p_1} = \frac{\partial H_2}{\partial p_2} = 0$ yields

$$p_1 = 2(x_1 - x_2)J(x)/(-J'(x)), \tag{26}$$

$$p_2 = 2(x_1 - x_2)(1 - J(x))/(-J'(x)). \tag{27}$$

Substituting these prices into (2), we obtain

$$\frac{1}{2}(x_1 + x_2) - x = \frac{2J(x) - 1}{J'(x)}. \tag{28}$$

The right side of (28) for $x \geq 0$ is an increasing and non-negative function with $\frac{2J(0)-1}{J'(0)} = 0$ and $\frac{2J(1)-1}{J'(1)} = \infty$. Therefore, the solution of the equation (28) exists and is unique, and satisfies $x \leq \frac{x_1+x_2}{2}$.

If $x_1 + x_2 = 0$, then $x = 0$ satisfies (28), and, from (26)–(27),

$$p_1 = p_2 = 4x_1/(3\pi^{-1} + a) \tag{29}$$

$$H_1 = H_2 = 2x_1/(3\pi^{-1} + a). \tag{30}$$

Now we repeat the procedure employed in Section 4. From (23), (26) and (28), we get

$$\frac{\partial H_1}{\partial x_1} = \frac{\partial p_1}{\partial x_1} J(x) + p_1 J'(x) \frac{\partial x}{\partial x_1},$$

$$-\frac{1}{2}\frac{\partial p_1}{\partial x_1} = \frac{J(x)}{J'(x)} + (x_1 - x_2)\left\{1 - J(x)J''(x)(J'(x))^{-2}\right\}\frac{\partial x}{\partial x_1},$$

$$\frac{\partial x}{\partial x_1} = \frac{1/2}{3 + (1 - 2J(x))J''(x)(J'(x))^{-2}},$$

and hence, using the values of $J(0), J'(0)$ and $J''(0)$,

$$\left[\frac{\partial H_1}{\partial x_1}\right]_{x=0} = \frac{1}{2}\left[\frac{\partial p_1}{\partial x_1}\right]_{x=0} - \frac{1}{2}(3\pi^{-1} + a)[p_1]_{x=0} \cdot \left[\frac{\partial x}{\partial x_1}\right]_{x=0}$$

$$= (3\pi^{-1} + a)^{-1} - \frac{2}{3}x_1,$$

since

$$\left[\frac{\partial x}{\partial x_1}\right]_{x=0} = \frac{1}{6}$$

$$\left[\frac{\partial p_1}{\partial x_1}\right]_{x=0} = 2(3\pi^{-1} + a)^{-1} - \frac{2}{3}x_1$$

$$\left[p_1\right]_{x=0} = 4(3\pi^{-1} + a)^{-1}x_1.$$

Therefore, the condition $\left[\frac{\partial H_1}{\partial x_1}\right]_{x=0} = 0$ gives

$$x_1^* = \frac{3\pi}{2(3 + \pi a)}. \tag{31}$$

If we choose $a > \frac{3}{2}(1 - 2\pi^{-1}) \approx 0.5451$, then $x_1^* \in (0, 1)$. So, if we choose a such that

$$\frac{3}{2}(1 - 2\pi^{-1}) \approx 0.5451 < a < 3\pi^{-1} \approx 0.9549,$$

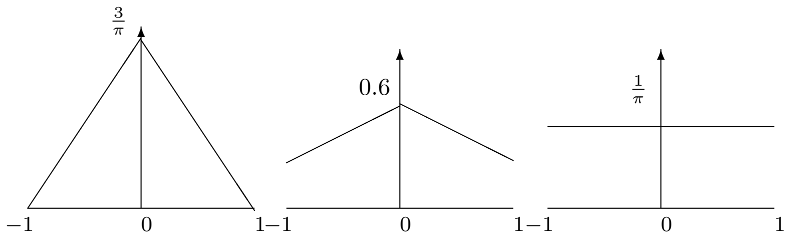
then $x_1^* = \frac{3\pi}{2(3+\pi a)} \in (0, 1)$, and $f(r, \theta) = a + \frac{3}{2}(\pi^{-1} - a)r > 0, \forall r : 0 \leq r \leq 1$.

For example, for $a = 0.6$

$$x_1^* = 0.9647, \quad \text{and } f(r, \theta) = 0.6 - 0.4225r. \tag{32}$$

Remark 5.1. Comparing the three types of customer's distribution, we obtain Table 3, and the result is consistent with our reasonable intuition.

Table 3. Solution for three types of customer's distribution.

	"Center"	"Center and Seaside"	"Uniform"
			
x_1^*	$\pi/4 \approx 0.7854$	0.9647	1
p_1^*	$\pi^2/6 \approx 1.6450$	2.4815	$\pi \approx 3.1415$
H_1^*	$\pi^2/12 \approx 0.8225$	1.2408	$\pi/2 \approx 1.5708$
Based on	Section 3	Eq. (32)	Eq. (21) in Remark 4.1

We observe that the optimal location approaches the center of the circle, as much as the customer's distribution concentrates at the center. It is interesting to notice that the equilibrium prices and the optimal rewards for the players become smaller. So, "small profits and quick returns" is true, even in the competition situation.

6. Conclusion

We considered above the location model with the cost function of the form $F_1 = p + \rho^2$. The same approach can be used for the analysis of the game with another cost functions, for instance, $F_2 = p^2 + \rho^2$ and $F_3 = p + \rho$.

Consider, for example, the location model with $F = p^2 + \rho^2$ where the roles of the price and the distance for the customer are equal. In this case, the regions S_1 and S_2 are determined by the line

$$p_1^2 + (x - x_1)^2 = p_2^2 + (x - x_2)^2,$$

which is a parallel for the axis O_y with coordinate

$$x = \frac{x_1 + x_2}{2} + \frac{p_1^2 - p_2^2}{2(x_1 - x_2)}, \tag{33}$$

and the payoffs of the players will have the same form (3)-(4). Demanding that the prices (p_1, p_2) satisfy the equilibrium conditions we obtain the formulas:

$$p_1^2 = \frac{x_1 - x_2}{2}\left[\frac{\arccos x}{\sqrt{1-x^2}} - x\right] \tag{34}$$

and

$$p_2^2 = \frac{x_1 - x_2}{2}\left[x + \frac{\pi - \arccos x}{\sqrt{1-x^2}}\right]. \tag{35}$$

With (33) we can represent it in the form

$$p_1 = (x_1 - x_2)^{1/2}\left[\frac{\pi}{4\sqrt{1-x^2}} - \left(\frac{x_1+x_2}{2} - x\right)\right]^{1/2}$$

$$p_2 = (x_1 - x_2)^{1/2}\left[\frac{\pi}{4\sqrt{1-x^2}} + \left(\frac{x_1+x_2}{2} - x\right)\right]^{1/2}$$

with

$$x = \frac{x_1 + x_2}{3} - \frac{\pi/2 - \arccos x}{3\sqrt{1-x^2}}. \tag{36}$$

The results of numerical calculations are presented in the Table 4.

Table 4. The equilibrium prices for various x_1 and x_2.

		$x_2 = 0.6$	0.4	0.2	0	-0.2	-0.4	-0.6	-0.8	-1
$x_1 =$ 0.8	p_1	0.309	0.454	0.578	0.692	0.802	0.909	1.015	1.121	1.227
	p_2	0.488	0.671	0.798	0.895	0.972	1.033	1.082	1.121	1.151
	H_1	0.088	0.143	0.199	0.259	0.325	0.397	0.475	0.560	0.652
	H_2	0.348	0.460	0.523	0.560	0.578	0.582	0.575	0.560	0.538
	x	0.342	0.295	0.247	0.198	0.149	0.099	0.049	0	-0.049
$x_1 =$ 0.6	p_1		0.334	0.489	0.621	0.742	0.858	0.970	1.082	1.190
	p_2		0.461	0.633	0.752	0.843	0.914	0.970	1.015	1.050
	H_1		0.114	0.183	0.251	0.324	0.401	0.485	0.575	0.672
	H_2		0.302	0.396	0.447	0.475	0.486	0.485	0.475	0.458
	x		0.247	0.198	0.149	0.099	0.049	0	-0.049	-0.099

The optimal location of the players can be found in the same way as in Sections 4 and 5. By (34), (36)

$$2p_1\frac{\partial p_1}{\partial x_1} = \frac{1}{2}\left[\frac{\arccos x}{\sqrt{1-x^2}} - x\right] + \frac{x_1 - x_2}{2}\left[-\frac{1}{1-x^2} + \frac{x\arccos x}{(1-x^2)^{3/2}} - 1\right]\frac{\partial x}{\partial x_1},$$

$$\frac{\partial x}{\partial x_1} = \left[3 + (1-x^2)^{-1} + x(\pi/2 - \arccos x)(1-x^2)^{-3/2}\right]^{-1},$$

and, consequently, (see, also, Remark 4.1)

$$\pi\left[\frac{\partial H_1}{\partial x_1}\right]_{x=0} = \frac{\pi}{2}\left[\frac{\partial p_1}{\partial x_1}\right]_{x=0} - 2[p_1]_{x=0}\cdot\left[\frac{\partial x}{\partial x_1}\right]_{x=0}$$

$$= \frac{\pi}{2}\frac{\pi - 2x_1}{4\sqrt{2\pi x_1}} - 2\sqrt{\pi x_1/2}\frac{1}{4}.$$

Therefore, the condition $\left[\frac{\partial H_1}{\partial x_1}\right]_{x=0} = 0$ gives us the optimal location for this case

$$x_1^* = \frac{\pi}{6} \approx 0.5253.$$

Comparing with the uniform case considered in Section 2, where the optimal location was $x_1^* = 1$, we can conclude that the choice of the cost function influences strongly on the solution.

Acknowledgements

The research was supported by the Russian Fund for Basic Research (projects 01-01-00126, 01-01-00113) and Japan Society for the Promotion of Science (grant L 01530).

References

d'Aspremont, C., Gabszewicz and J. F. Thisse (1979). "On Hotelling's stability in competition". *Econometrica*, Vol. 47, 1245-1150.

Bester, H., A. De Palma, W. Leininger, J. Thomas and E.-L. Von Tadden (1996). "A non-cooperative analysis of Hotelling's location game". *Games and Economics Behavior*, Vol. 12, 165-186.

Dresner, Z. (1982). "Competitive location strategies for two facilities". *Regional Science and Urban Economics*, Vol. 12, 485-493.

Hakimi, S.L. (1983). "On locating new facilities in a competitive environment". *European Juornal of Operational Research*, Vol. 12, 29-35.

Hotelling, H. (1929). "Stability in competition". *Economic Journal*, Vol. 39, 41-57.

Kats, A. (1987). "Location-price equilibria in a spatial model of discriminatory pricing". *Economic Letters*, Vol. 25, 105-109.

Sakaguchi, M. (2001). "Pure strategy equilibrium in a location game with discriminatory pricing". *Game Theory and Applications*, Vol. 6, 132-140.

Zhang, Y. and Y. Teraoka (1998). "A location game of spatial competition". *Math. Japonica*, Vol. 48, 187-190.