# The Supercore for Normal Form Games*

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**Abstract**

We study the supercore of a system derived from a normal form game. For the case of a finite game, we define a sequence of games and show that the supercore coincides with the set of Nash equilibrium strategy profiles of the last game in that sequence. This result is illustrated with the characterization of the supercore for the $n$-person prisoner's dilemma. With regard to the mixed extension of a game we prove that the supercore coincides with the set of NE whenever the game has a finite number of Nash equilibria. This coincidence is not mantained for games with infinite Nash equilibria.

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# 1 Introduction

Stable sets were defined by von Neumann and Morgenstern (1945) as a solution to $n$-person cooperative games. This solution has received a great deal of attention in the literature of games since then (see Lucas (1992).) One of the first criticisms to the von Neumann and Morgenstern (vN&M) stable sets was made by Harsanyi (1974) who argued that this notion is unsatisfactory because it neglects the stabilizing effect of the indirect dominance relation between alternatives. This shortcoming is not shared by the subsolutions developed by Roth (1976), a generalization of the vN&M stable sets which does take into account the indirect dominance relation. Interestingly enough, this notion has not been extensively considered in the game theory literature. This paper shows evidence of the performance for normal form games of the subsolution known as the supercore.

A recent approach to the stable set theory and its connections with other solution concepts in game theory has been developed in the book Theory of Social Situations (TOSS) by Greenberg (1990). In its Chapter 7 it is argued that when modeling social environments, normal form games do not capture the notion of negotiation among players, while an advantage of the approach proposed in TOSS is that the consequences of different types of negotiations among players may be analyzed. One of the negotiations considered by Greenberg is the so called individual contingent threat (ICT) situation, where each single player can object to the prevailing outcome and can threat the others by stating that she will use a different strategy.[1] The ICT situation can be associated to a system where the strategy profiles of a normal form game are the elements of the set and the binary relation defined on that set accounts only for the profitable single deviations.

With respect to the solutions for the system associated to an ICT situation of a finite normal form game, Greenberg shows that the vN&M stable sets always exist in the following two cases: either when there are at most two players or when there are $n$ players, each one with a set of at most two strategies.[2] Unfortunately, however, these theorems cannot be generalized and two counter-

[1] Negotiations where players can jointly object to the prevailing outcome and can threat the others by stating that they will use another strategies, are considered by Greenberg (1989).

[2] Greenberg (1990), pp. 100-1, Theorems 7.4.5 and 7.4.6.

examples are immediately presented (see Greenberg, 1990, pp. 100-1). Even in the case of a game with Nash equilibrium (NE) strategy profiles (Nash (1951)), the existence of a vN&M stable set is not guaranteed.[3] In this line of research, Nakanishi (2001, 2002) has generalized Greenberg's results. He studies the existence of the vN&M stable sets for the prisoner's dilemma and for some other class of normal form games. Arce (1994) and Muto and Okada (1996, 1998) have also studied the vN&M stable sets for different applications using the ICT situation approach.

In this paper, we study the supercore as a solution for the systems associated to an ICT situation of a normal form game and of its mixed extension.

Regarding to the pure strategies case, the supercore of the system associated to an ICT situation contains the NE strategy profiles. The fact that the supercore may include strategy profiles other than the NE ones does not diminish the importance of the supercore as a solution concept. On the contrary, it implies that the supercore is not a reformulation of the NE in a different setting. Moreover, it means that the analysis of the type of strategy profiles contained in the supercore is of interest *per se*. In our study of the supercore we introduce a *procedure* that easily allows the determination of this solution. In particular, given a normal form game we derive a sequence of games and we find that the set of NE strategy profiles of the last game in the sequence exactly coincides with the supercore of the system derived from the original game. This procedure also allows the identification of those games in which the supercore selects exactly the NE strategy profiles. With regard to the content of the supercore, this solution may be interpreted as the outcome of a dynamic model of sequential selection of NE strategy profiles. From this perspective, the supercore is formed by the union of NE and the "NE protected strategy profiles" of each game in the sequence.

We illustrate the previous results with a numerical example and also by characterizing the supercore of the system associated to the $n$-person prisoners' dilemma. As we shall see, the supercore is the unique vN&M stable set and it is formed by the strategy profile where all players choose to defect and by those strategy profiles where the number of players who choose to cooperate is even.

[3]Greenberg (1990), p. 102, Example 7.4.8.

For the case of a system associated to the mixed extension of a normal form game, we show that the supercore of such a system coincides with the set of NE strategy profiles whenever the normal form game has a finite number of NE profiles. A simple example shows that this result is not maintained for the case of infinite NE strategy profiles. The equivalence between the supercore and the set of NE strategy profiles for the mixed extension of a game is obtained by using a weaker dominance relation on the set of strategy profiles. The proofs of such equivalence concludes the paper.

The rest of the paper is organized as follows. Section 2 contains the preliminaries. In Section 3 we introduce the ICT situation associated to a normal form game. We define the sequence of normal form games which allows to determine the supercore of the system associated to a normal form game. We conclude this section with a numerical example and with the characterization of the supercore for the $n$-person prisoner's dilemma. Section 4 contains the study of the supercore associated to the mixed extension of a game.

# 2 Preliminaries

In these preliminaries we introduce the solution concepts for an abstract system used in the paper. We also recall the definitions of a finite normal form game, of its mixed extension and of the Nash equilibrium solution.

An *abstract system* is a pair $(X, R)$, where $X$ is a set of elements and $R$ is an irreflexive binary relation defined on $X$. The relation $R$ is read "dominates" hence, if for two elements $x, x'$ in $X$ we have $xRx'$ then we say that $x$ dominates $x'$.

For any $x \in X$, let $\mathcal{D}(x)$ denote the dominion of $x$, *i.e.*, $\mathcal{D}(x) = \{x' \in X : xRx'\}$. Given any subset $A$ of $X$, we define the following sets: $\mathcal{D}(A) = \bigcup_{x \in A} \mathcal{D}(x)$, $\mathcal{U}(A) = X - \mathcal{D}(A)$,[4] and $\mathcal{P}(A) = \mathcal{U}(A) - A$.

A *subsolution* of $(X, R)$ is a subset $A$ of $X$ such that $A \subset \mathcal{U}(A)$,[5] and $A = \mathcal{U}^2(A)$, where $\mathcal{U}^2(A) = \mathcal{U}(\mathcal{U}(A))$. The condition $A \subset \mathcal{U}(A)$ is known as the internal stability condition. With regard to the condition $A = \mathcal{U}^2(A)$, Roth (1976, p. 44) provides the following interpretation:

> ...every point in $\mathcal{U}(A) - A$ is dominated by some other point in the same set and the entire set, thus "overrules" itself leaving only the set as "sound".

In words, if $A$ is a subsolution then $\mathcal{D}(A) \subset \mathcal{D}(\mathcal{P}(A))$.

The intersection of all subsolutions of $(X, R)$ is also a subsolution which is known as *supercore*.

A subset $A \subset X$ is a *vN&M stable set* of $(X, R)$ if $A = \mathcal{U}(A)$. Thus, a vN&M stable set is characterized by the internal stability condition $A \subset \mathcal{U}(A)$, and by $\mathcal{U}(A) \subset A$, known as the external stability condition. Clearly, a vN&M stable set is a subsolution that satisfies $\mathcal{P}(A) = \emptyset$.

A subset $A \subset X$ is the *core* of $(X, R)$ if $A = \mathcal{U}(X)$.

A finite *normal form game* $\Gamma^N$ is a triple $< N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} >$ where $N = \{1, ..., n\}$ is the finite set of players, $S_i$ is the finite set of strategies for player $i$ and $u_i : S = \times_{i \in N} S_i \longrightarrow \mathbb{R}$ is player $i$'s payoff function.

[4]The symbol $-$ stands for the difference binary relation.
[5]The symbol $\subset$ means weakly contained while $\subsetneq$ means strictly contained.

A strategy of player $i$, $\widehat{s}_i$ is a best response to $s_{-i}$ if for all $s_i \in S_i$, $u_i(\widehat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$ where $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$.

Let $s = (s_1, ..., s_n)$ denote a strategy profile. Then, $s^* = (s_1^*, ..., s_n^*)$ is a *Nash equilibrium* in $\Gamma^N$ if $s_i^*$ is a best response to $s_{-i}^*$ for all $i \in N$.

A *mixed extension of the game* $\Gamma^N$ is a triple $< N, \{\Delta S_i\}_{i\in N}, \{U_i\}_{i\in N} >$ where $\Delta S_i$ is the simplex of the mixed strategies for player $i$, and $U_i : \Delta(S) = \times_{i\in N}\Delta S_i \longrightarrow \mathbb{R}$, assigns to $\sigma \in \Delta(S)$ the expected value under $u_i$ of the lottery over $S$ that is induced by $\sigma$ (so that $U_i(\sigma) = \sum\limits_{s\in S}(\prod\limits_{j\in N} \sigma_j(s_j))u_i(s))$.

Let $\sigma$ denote a mixed strategy profile. Then, $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ is a *Nash equilibrium* in the mixed extension of the game $\Gamma^N$ if $\sigma_i^*$ is a best response to $\sigma_{-i}^* = (\sigma_1^*, ..., \sigma_{i-1}^*, \sigma_{i+1}^*, ..., \sigma_n^*)$ for all $i \in N$.

# 3 The Supercore of $(S, \angle)$

This section has 3 subsections. In the first one, we define the system associated to an ICT situation of a normal form game. In the second one we define a sequence of normal form games that allows the determination of the supercore of $(S, \angle)$. We conclude the section with a numerical example that illustrates these results and with the characterization of the supercore of the prisoners dilemma.

## 3.1 A system associated to an ICT situation of a finite normal form game

The application of the approach developed in TOSS to the normal form of a game allows the consideration of different types of negotiation among players, and thus the analysis of their possible consequences. In particular, the negotiation in which each player can deviate from a proposed strategy profile unilaterally is the one that we study in this paper. This notion of negotiation is formalized by the individual contingent threat, ICT, situation.

Let us start with a description of the negotiation procedure:

A strategy profile, say $s$, is proposed to players. If all individuals openly consent to follow $s$, then $s$ will be adopted. If player $i$ objects to $s$, then she has to openly declare that if the remaining players stick to the specified profile

$s$, then she will choose $s'_i$ instead of $s_i$ (contingent threat of player $i$). Thus, each single player can object to the prevailing profile and can threat the others by saying that she will choose another strategy. When player $i$ modifies profile $s$ into profile $s'$ it is said that player $i$ induces $s'$ from $s$. Any player other than player $i$ can then counter the new upcoming strategy profile induced by player $i$. The process just described continues this way. The set of profiles that player $i$ can induce from $s$ is denoted as follows:

$$\gamma_i(s) = \{s' \in S : s'_j = s_j \text{ for all } j \neq i,\ j \in N\}.$$

Thus, $\gamma_i$ determines an inducement correspondence for player $i$ from $S$ into itself. Once we have $\gamma_i$ we can define the ICT situation associated with $\Gamma^N$ as:

$$\Gamma^N_\gamma \equiv (N, S, \{u_i\}_{i \in N}, \{\gamma_i\}_{i \in N}).$$

We are now ready to define the system associated to an ICT situation of a game in normal form.

**Definition 1** *An individual dominance system associated to an ICT situation of a game $\Gamma^N$ is a pair $(S, \angle)$, where $\angle$ is the binary relation defined on $S$ as follows:*

$$s' \angle s \text{ if there exists } i \in N \textit{ such that } s' \in \gamma_i(s) \text{ and } u_i(s') > u_i(s).$$

This means that $s'$ dominates $s$ if $s'$ is derived from $s$ via a deviation of a player $i$ who is better off under $s'$ than under $s$.

(Hereafter, if there is no confusion the individual dominance system will be called simply as the system.)

## 3.2 A procedure to compute the supercore of $(S, \angle)$

Let us consider a game $\Gamma^N$ with at least one NE strategy profile. This assumption is not restrictive since if the game $\Gamma^N$ has not any NE strategy profile, then the supercore of $(S, \angle)$ is the empty set (Roth, (1976)).

Let $S^*$ be the set of NE strategy profiles of the game $\Gamma^N$ and let $s^* \in S^*$. Starting from $s^*$, consider the set of strategy profiles obtained by a deviation of a player who gets a lower payoff than the payoff provided by $s^*$. This set is the dominion of $s^*$, that is, $\mathcal{D}(s^*) = \{s \in S : s^* \in \gamma_i(s)$ and $u_i(s^*) > u_i(s)$ for some $i \in N\}$. Hence, it is clear that moving from $s$ into $s^*$ is always profitable for player $i$. Then, the dominion of $S^*$ is $\mathcal{D}(S^*) = \bigcup_{s^* \in S^*} \mathcal{D}(s^*)$.

Let $\upsilon_i(\Gamma^N)$ be the lowest payoff for player $i$ in the game $\Gamma^N$. That is, $\upsilon_i(\Gamma^N) = \min\{u_i(s)\colon\ s \in S\}$. Denote by $\upsilon(\Gamma^N) = (\upsilon_1(\Gamma^N), ..., \upsilon_n(\Gamma^N))$ the vector of lowest payoffs.

In what follows we give a procedure to determine the supercore of $(S, \angle)$, but first let us consider an intuitive description of this procedure.

Starting from the game $\Gamma^N$, we define a new game $\Gamma_1^N$ with the same set of players and strategies for every player and with the players' payoffs modified in the following way: The payoff for each player at every profile in $\mathcal{D}(S^*)$ is equal to his lowest payoff in the game $\Gamma^N$, while the payoffs corresponding to the remaining strategy profiles are maintained. The idea behind this modification is to take any power away from the strategy profiles dominated by the NE strategy profiles. By assigning them the lowest payoffs of the game, these strategy profiles cannot dominate any profile.

With this modification at hand, we may verify whether or not game $\Gamma_1^N$ has additional NE strategy profiles than those that $\Gamma^N$ has. If $\Gamma_1^N$ has a new NE strategy profiles then a game $\Gamma_2^N$ can be defined, and the procedure may continue iteratively in this manner.

This procedure generates a sequence of games $\left\langle \Gamma_t^N \right\rangle_{t=0}^{\infty}$ and a sequence of systems $\langle (S, \angle_t) \rangle_{t=0}^{\infty}$ defined inductively as follows:

(i) $\Gamma_0^N = \Gamma^N$ and $(S, \angle_0) = (S, \angle)$.

(ii) For $t \geq 1$, $\Gamma_t^N = < N, \{S_i\}_{i \in N}, \{u_i^t\}_{i \in N} >$, where

$$u_i^t(s) = \begin{cases} \upsilon_i(\Gamma^N) & \text{if } s \in \mathcal{D}(S_{t-1}^*) \text{ in } (S, \angle_{t-1}) \\ u_i^{t-1}(s) & \text{otherwise,} \end{cases}$$

and $(S, \angle_t)$ is the associated system to $\Gamma_t^N$ where $\angle_t$ is the binary relation on $S$ given by

$s' \angle_t s$ if there is a player $i \in N$ such that $s' \in \gamma_i(s)$ and $u_i^t(s') > u_i^t(s)$.

($S^*_{t-1}$ denotes the set of NE strategy profiles of $\Gamma^N_{t-1}$.)

Formally, the procedure described above can be summarized as follows:

*Step 0:* Let $\Gamma^N_0 = \Gamma^N$. Compute $S^*_0$ and determine $\mathcal{D}(S^*_0)$ in $(S, \angle_0)$. Game $\Gamma^N_1$ is generated according to the player's payoff function $\left\{u^1_i\right\}_{i\in N}$, and the system $(S, \angle_1)$ is generated by relation $\angle_1$.

*Step t:* Let be the game $\Gamma^N_t$. Compute $S^*_t$.

If $S^*_t = S^*_{t-1}$, then we conclude the procedure.

If $S^*_{t-1} \subsetneq S^*_t$, determine $\mathcal{D}(S^*_t)$ in $(S, \angle_t)$. Game $\Gamma^N_{t+1}$ is generated according to the player's payoff function $\left\{u^{t+1}_i\right\}_{i\in N}$, and the system $(S, \angle_{t+1})$ is generated by the relation $\angle_{t+1}$.

Given that $S$ is finite, there exists a $k \in \mathbb{N}$

such that $S^*_t \neq S^*_{t+1}$ for all $t = 0, ...k-2$ and $S^*_k = S^*_{k-1}$.

Now, we can establish the following theorem:

**Theorem 1** *Let $S^*_k$ be the set of NE strategy profiles of the game $\Gamma^N_k$. Then $S^*_k$ is the supercore of $(S, \angle)$.*

**Proof.** We will prove that the following two conditions hold:

(i) $S^*_k$ is a subsolution of $(S, \angle)$.

(ii) Any other subsolution $\overline{S}$ of $(S, \angle)$ contains $S^*_k$.

(i) We show that $S^*_k$ satisfies in $(S, \angle)$ the conditions $S^*_k \subset \mathcal{U}(S^*_k)$ and $S^*_k = \mathcal{U}^2(S^*_k)$.

By the construction of the sequence of games $< \Gamma^N_0,...,\Gamma^N_k >$, we can write that for each $s \in S$ and for all $i \in N$

$$u^k_i(s) = \begin{cases} \upsilon_i(\Gamma^N) & \text{if } s \in \mathcal{D}(S^*_k) \text{ in } (S, \angle) \\ u_i(s) & \text{otherwise.} \end{cases} \tag{1}$$

Clearly, the NE strategy profiles of $\Gamma^N_k$, in the system $(S, \angle)$, cannot dominate each other and can only be dominated by the strategy profiles belonging to $\mathcal{D}(S^*_k)$. Hence, $S^*_k \subset \mathcal{U}(S^*_k)$ and $S^*_k \subset \mathcal{U}(\mathcal{U}(S^*_k))$. Now, let us assume that there is a strategy profile $s \in \mathcal{U}(\mathcal{U}(S^*_k))$ such that $s \notin S^*_k$ then $s$ will be dominated in

$(S, \angle_k)$ by some $s' \in S$. So $s' \angle_k s$, and from (1), it follows that $s' \notin \mathcal{D}(S_k^*)$ in $(S, \angle)$. Since $s, s' \notin \mathcal{D}(S_k^*)$, the players' payoffs corresponding to the profiles $s$ and $s'$ are the same in the games $\Gamma_k^N$ and $\Gamma^N$, it follows that $s' \angle s$. Therefore, $s \in \mathcal{D}(\mathcal{U}(S_k^*))$, which contradicts that $s \in \mathcal{U}(\mathcal{U}(S_k^*))$. Consequently, $S_k^* = \mathcal{U}(\mathcal{U}(S_k^*))$ and (i) has been proved.

(ii) We argue by contradiction. Suppose that for some subsolution $\overline{S}$ of $(S, \angle)$, $S_k^* \not\subset \overline{S}$. Now, let us consider $S_0^* \subset ... \subset S_k^*$ and define $l = \min\{t : S_t^* \not\subset \overline{S},\, t = 0, ..., k\}$. Since $S_0^*$ is the core of $(S, \angle)$, it is included in any subsolution, therefore $l \neq 0$.

Let $s \in S_l^*$ such that $s \notin \overline{S}$. Then, either $s \in \mathcal{D}(\overline{S})$ or $s \in \mathcal{P}(\overline{S})$ in $(S, \angle)$. Given that $s$ is a Nash equilibrium in $\Gamma_l^N$, then it can only be dominated by some strategy profiles in $\mathcal{D}(S_{l-1}^*)$ and, by the definition of $l$ we have that $\overline{S}$ is a subsolution such that $S_{l-1}^* \subset \overline{S}$ so that $s \notin \mathcal{D}(\overline{S})$. Hence, $s \in \mathcal{P}(\overline{S})$. However, given that $s' \in \mathcal{D}(S_{l-1}^*)$ and $S_{l-1}^* \subset \overline{S}$ then $s' \notin \mathcal{P}(\overline{S})$ and we have arrived to a contradiction. ■

**Corollary 1** *The core of* $(S, \angle)$ *coincides with the supercore of* $(S, \angle)$ *if and only if* $S_0^* = S_1^*$.

**Proof.** Given that $S_0^*$ is the core of $(S, \angle)$ the result follows directly from Theorem 1. ■

One question that immediately arises is the type of profiles included in the supercore. The content of this set for $(S, \angle)$ may be interpreted as the result of a dynamic model of sequential selection of strategy profiles.[6]

Starting from $\Gamma_0^N$ and $S_0^*$, assume that the strategy profiles in $\mathcal{D}(S_0^*)$ "lose power," so that the current payoffs of the players are replaced by their corresponding lowest payoffs in the game. By taking into account these lowered payoffs, we determine $\Gamma_1^N$, a game where the profiles in $\mathcal{D}(S_0^*)$ cannot dominate any profile. Thus, starting from $\Gamma_1^N$, we determine the set of NE profiles $S_1^*$. Clearly the strategy profiles belonging to $S_1^* - S_0^*$ are dominated only by some profiles in $\mathcal{D}(S_0^*)$ and we call them "*NE protected profiles*".

In general, given the game $\Gamma_t^N$, $t = 1, ..., k$, $S_t^*$ is formed by the set $S_{t-1}^*$ and by those NE profiles protected by $S_{t-1}^*$. Therefore, we can establish that the

[6]See the dynamic model presented by Roth (1978).

supercore is formed by $S_0^*$ and the NE protected profiles of each game in the rest of the sequence.

### 3.3 Two examples

In what follows we first present a simple example illustrating some previous results. In order to do it we apply the procedure described above.

**Example 1** Consider the following game $\Gamma^N$

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| $a_1$ | 6,6 | 5,5 | 1,3 | 2,2 |
| $a_2$ | 3,4 | 4,4 | 7,2 | 1,3 |
| $a_3$ | 6,2 | 2,3 | 8,8 | 6,2 |
| $a_4$ | 2,3 | 2,5 | 9,4 | 2,5 |

*Step 0*: Let $\Gamma_0^N = \Gamma^N$. The vector of lowest payoffs is $(\upsilon_1(\Gamma^N), \upsilon_2(\Gamma^N)) = (1, 2)$. The set of NE strategy profiles of $\Gamma_0^N$ is $S_0^* = \{(a_1, b_1)\}$ and the dominion of $S_0^*$ is $\mathcal{D}(S_0^*)$=$\{(a_1, b_2), (a_1, b_3), (a_1, b_4), (a_2, b_1), (a_4, b_1)\}$. Substituting the payoffs' profiles in $\mathcal{D}(S_0^*)$ by $(1, 2)$ game $\Gamma_1^N$ is obtained.

*Step 1*: Let be the game $\Gamma_1^N$

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| $a_1$ | 6,6 | 1,2 | 1,2 | 1,2 |
| $a_2$ | 1,2 | 4,4 | 7,2 | 1,3 |
| $a_3$ | 6,2 | 2,3 | 8,8 | 6,2 |
| $a_4$ | 1,2 | 2,5 | 9,4 | 2,5 |

Here, we have $S_1^* = \{(a_1, b_1), (a_2, b_2)\}$, and $\mathcal{D}(\{(a_1, b_1), (a_2, b_2)\}) = \mathcal{D}(\{(a_1, b_1)\}) \cup \{(a_2, b_3), (a_2, b_4), (a_3, b_2), (a_4, b_2)\}$. Substituting the payoffs' profiles in $\mathcal{D}(S_1^*)$ by $(1, 2)$ game $\Gamma_2^N$ is obtained.

*Step 2:* Let be the game $\Gamma_2^N$

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| $a_1$ | 6,6 | 1,2 | 1,2 | 1,2 |
| $a_2$ | 1,2 | 4,4 | 1,2 | 1,2 |
| $a_3$ | 6,2 | 1,2 | 8,8 | 6,2 |
| $a_4$ | 1,2 | 1,2 | 9,4 | 2,5 |

The set of NE strategy profiles of $\Gamma_2^N$ is $S_2^* = \{(a_1, b_1), (a_2, b_2)\}$. Since $S_2^* = S_1^*$ the procedure ends.

This procedure generates the sequence of games $\langle \Gamma_0^N, \Gamma_1^N, \Gamma_2^N \rangle$. The set of NE strategy profiles of the game $\Gamma_2^N$ is the supercore for $(S, \angle)$. The two vN&M stable sets of the system $(S, \angle)$ are $\{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}$ and $\{(a_1, b_1), (a_2, b_2), (a_4, b_3), (a_3, b_4)\}$.

**Example 2** *The supercore for the Prisoners Dilemma*:[7]

The $n$-person prisoner's dilemma represents situations where the cooperative outcome, all players selecting cooperation, cannot be attained as a NE. Let us formally define this game.[8] Let $N$ be the set of players. Assume that every player has two actions $C$ (cooperation) and $D$ (defection). The payoff of player $i$ is given by

$$f_i(a\,|r\,),\; a = C, D, \text{ and } r = 0, ..., n-1,$$

where $a$ is player $i$'s action and $r$ is the number of other players who select action $C$.

The following three assumptions on the payoff function define an $n$-person prisoner's dilemma:

- A.1 Every player is better off by choosing $D$ than by choosing $C$. That is, for all $i \in N$: $f_i(C\,|r\,) < f_i(D\,|r\,)$ for all $r = 0, ..., n-1$.

- A.2 If all players choose $D$, then the outcome of the game is worse to all of them than the outcome which arises if they all choose $C$. That is, for all $i \in N$: $f_i(C\,|n-1\,) > f_i(D\,|0\,)$.

- A.3 The payoff of player $i$, given her action, increases as the number of other players that select $C$ increases; that is, $f_i(C\,|r\,)$ and $f_i(D\,|r\,)$ are increasing functions of $r$.

[7] Arce (1994) studies the vN&M stable set for a 3-person prisoner's dilemma. Using sets of continuous strategies for all players Nakanishi (2001) shows that a vN&M stable set always exists for an $n$-person prisoner's dilemma.

[8] Here, we follow Nishihara (1997) formulation of the $n$-person prisoner's dilemma. See also Okada (1993).

It is straightforward that under these assumptions the unique NE strategy profile is that all players select $D$.

Let $(S_{pd}, \angle)$ denote the system associated to the $n$-person prisoner's dilemma.

**Proposition 2** *The supercore of the n-prisoners dilemma is the unique vN&M stable set of* $(S_{pd}, \angle)$*, and it is formed by* $(D, ..., D)$ *and by those strategy profiles such that the number of players who choose* $C$ *is even.*

**Proof.** Using the procedure described above we have a sequence of games $\langle \Gamma_0^N, ..., \Gamma_k^N \rangle$ derived in the following way. *Step 0*: Let $\Gamma_0^N = \Gamma^N$. The set of NE strategy profiles is $S_0^* = \{(D, ..., D)\}$ and by A.1 the dominion of $S_0^*$ is $\mathcal{D}(S_0^*) = \{s \in S_{pd} : s_i = C$ and $s_j = D$, for all $j \neq i.\}$. *Step t*, $(t \geq 1)$: Let be the game $\Gamma_t^N$. The set of NE strategy profiles is $S_t^* = S_{t-1}^* \cup \widetilde{S}_t$ where $\widetilde{S}_t$ is the set of profiles of the game $\Gamma_t^N$ such that exactly $2t$ players choose $C$. The dominion of $S_t^*$ is $\mathcal{D}(S_t^*) = \mathcal{D}(S_{t-1}^*) \cup \mathcal{D}(\widetilde{S}_t)$ where $\mathcal{D}(\widetilde{S}_t)$ is the set of strategy profiles such that exactly $(2t+1)$ players choose $C$. Step $k$: Since $S_{k-1}^* = S_k^*$, it must happen that $k = \frac{n}{2} + 1$ if $n$ is even, and $k =$ integer part of $\frac{n}{2}$ plus 1 if $n$ is odd. It is clear that $S_{pd} = S_k^* \cup \mathcal{D}(S_k^*)$, so we may conclude that the supercore of $(S_{pd}, \angle)$ is a vN&M stable set and it is obviously unique. ■

Lastly, we conclude this section with two comments:

C.1) A drawback of the supercore is that, in general, it does not include the efficient strategy profiles (a strategy profile is efficient if there is no other strategy profile where all players are strictly better off.) However, in the prisoner's dilemma game the inclusion in the supercore of some efficient strategy profiles is guaranteed. Notice that: (i) if the number of players in the game is even then the strategy profile $(C, ..., C)$ is in the supercore and (ii) if the number of players is odd then all those profiles with exactly one player choosing $D$ are in the supercore. It is easy to see that A.1, A.2 and A.3 guarantee that $(C, ..., C)$ and the where exactly one player chooses $D$ are efficient profiles. Example 1 illustrates the case where the supercore does not include any of the two efficient profiles, $(a_3, b_3)$ and $(a_4, b_3)$.

C.2) The selected strategy profiles by the supercore may not be form by rationalizable strategies. (A strategy is rationalizable if it survives the iterated removal of strategies that are never best response, see Bernheim (1984) and Pearce (1984).) For example, the supercore for the 2-person prisoner's dilemma is the set $\{(D, D),(C, C)\}$, while only $D$ is a rationalizable.

# 4 The Supercore of $(\Delta(S), \angle)$

In this section we study the supercore of a system associated to the mixed extension of a normal form game. We find that the supercore of such a system coincides with the set of NE of the mixed extension of the game whenever the number of NE profiles is finite. A simple example shows that this is not longer maintained for the case of infinite Nash equilibrium strategy profiles.

An *ICT situation of the mixed extension of a game* $\Gamma^N$ is a 4-tuple $\langle N, \Delta(S), (U_i)_{i\in N}, (\gamma_i)_{i\in N}\rangle$ where $\gamma_i$ is the correspondence from $\Delta(S)$ into itself defined by

$$\gamma_i(\sigma) = \{\sigma' \in \Delta(S) : \sigma'_j = \sigma_j \text{ for all } j \neq i,\, j \in N\}.$$

Thus, $\gamma_i(\sigma)$ is the set of profiles which may be induced by player $i$ from $\sigma$.

**Definition 2** *An individual dominance system associated to an ICT situation of the mixed extension of a game* $\Gamma^N$ is a pair $(\Delta(S), \angle)$ where $\angle$ is the binary relation defined on $\Delta(S)$ such that

$$\sigma' \angle \sigma \text{ if there exits } i \in N \text{ such that } \sigma' \in \gamma_i(\sigma) \text{ and } U_i(\sigma') > U_i(\sigma).$$

Let $\Sigma^*$ be the set of NE strategy profiles of the mixed extension of the game $\Gamma^N$ and let $\sigma^* \in \Sigma^*$. The dominion of $\sigma^*$ is $\mathcal{D}(\sigma^*) = \{\sigma \in \Delta(S) : \sigma^* \in \gamma_i(\sigma)$ and $U_i(\sigma^*) > U_i(\sigma)$ for some $i \in N\}$. Then, the dominion of $\Sigma^*$ will be $\mathcal{D}(\Sigma^*) = \bigcup_{\sigma^*\in\Sigma^*} \mathcal{D}(\sigma^*)$.

**Theorem 3** *If* $\Sigma^*$ *is finite then* $\Sigma^*$ *is the supercore of* $(\Delta(S), \angle)$.

**Proof.** We first prove that $\Sigma^*$ is a subsolution of $(\Delta(S), \angle)$. That is, $\Sigma^* \subset \mathcal{U}(\Sigma^*)$ and $\Sigma^* = \mathcal{U}^2(\Sigma^*)$.

Given that $\Sigma^* \subset \mathcal{U}(\Sigma^*)$, if $\Sigma^* = \mathcal{U}(\Sigma^*)$ then $\Sigma^* = \mathcal{U}^2(\Sigma^*)$ and $\Sigma^*$ is a subsolution. If $\Sigma^* \neq \mathcal{U}(\Sigma^*)$ we have to show that $\mathcal{P}(\Sigma^*) \subset \mathcal{D}(\mathcal{P}(\Sigma^*))$, which is equivalent to the condition $\Sigma^* = \mathcal{U}^2(\Sigma^*)$ given that $\Sigma^* \subset \mathcal{U}(\Sigma)$.

Let $\sigma \in \mathcal{P}(\Sigma^*)$. We will show that $\sigma \in \mathcal{D}(\mathcal{P}(\Sigma^*))$.

Since $\sigma \notin \Sigma^*$ then $\sigma_i$ will not be the best response to $\sigma_{-i}$ for some player $i$. Therefore there exists a profile $\sigma' \in \gamma_i(\sigma)$ such that $U_i(\sigma') > U_i(\sigma)$. Now, set $\sigma_\lambda = \lambda\sigma + (1-\lambda)\sigma'$ for all $\lambda \in [0,1)$. By the linearity of $U_i$ we have $U_i(\sigma_\lambda) > U_i(\sigma)$, and since $\sigma_\lambda \in \gamma_i(\sigma)$, it follows that $\sigma_\lambda \angle \sigma$ for all $\lambda \in [0,1)$. So $\sigma_\lambda$ dominates $\sigma$, and $\sigma_\lambda \notin \Sigma^*$ given that $\sigma \in \mathcal{P}(\Sigma^*)$.

It remains to prove that $\sigma_\lambda \in \mathcal{P}(\Sigma^*)$ for some $\lambda$: If $\sigma_\lambda \in \mathcal{D}(\Sigma^*)$ for all $\lambda$, then there exists $\sigma^*_\lambda \in \Sigma^*$ such that $\sigma^*_\lambda \in \gamma_j(\sigma_\lambda)$ for some player $j$, and $U_j(\sigma^*_\lambda) > U_j(\sigma_\lambda)$. If $j = i$, we have $U_i(\sigma^*_\lambda) > U_i(\sigma_\lambda) > U_i(\sigma)$. Hence, $\sigma^*_\lambda$ will dominate $\sigma$ which implies that $\sigma \in \mathcal{D}(\Sigma^*)$. Otherwise the subset $\{\sigma^*_\lambda \colon \lambda \in [0,1)\}$ of $\Sigma^*$ will be infinite, which contradicts the fact that $\Sigma^*$ is finite. Therefore, $\sigma_\lambda \in \mathcal{P}(\Sigma^*)$ for some $\lambda$, and since $\sigma_\lambda \angle \sigma$ it follows that $\sigma \in \mathcal{D}(\mathcal{P}(\Sigma^*))$.

Lastly, since the supercore is the intersection of all subsolutions and any subsolution contains to $\Sigma^*$ we can conclude the result. ∎

The result above is not maintained when the mixed extension of the game $\Gamma^N$ has infinite Nash equilibrium strategy profiles. The following example illustrates that non Nash equilibrium strategy profiles may belong to the supercore of $(\Delta(S), \angle)$.

**Example 3** Consider the mixed extension of the following game:

| | $b_1$ | $b_2$ |
|---|---|---|
| $a_1$ | 1,0 | 1,1 |
| $a_2$ | -1,1 | 1,0 |

Let $p$ be the probability that player 1 chooses $a_1$ and let $q$ be the probability that player 2 chooses $b_1$. It is easy to check that the set of Nash equilibria is $\Sigma^* = \{(p,\, 1-p,\, 0,\, 1) : \frac{1}{2} \le p \le 1\}$. The dominion of the set of Nash equilibria is $\mathcal{D}(\Sigma^*) = \{(p,\, 1-p,\, q,\, 1-q) : \frac{1}{2} < p \le 1,\, 0 < q \le 1\}$ and the set of profiles undominated by the set of Nash equilibria excluded them, is $\mathcal{P}(\Sigma^*) = \{(p,\, 1-p,\, q,\, 1-q) : 0 \le p \le \frac{1}{2},\, 0 < q \le 1\} \cup \{(p, 1-p, 0, 1) : 0 \le p < \frac{1}{2}\}$. It is straightforward to show that the supercore of $(\Delta(S), \angle)$ is $\Sigma^* \cup \{(\frac{1}{2},\, \frac{1}{2},\, q,\, 1-q) : 0 < q \le 1\} \cup \{(p, 1-p, 0, 1) : 0 \le p < \frac{1}{2}\}$.

In view of this example the equivalence of the supercore and the set of NE strategy profiles for the mixed extension of a game requires the definition of a weaker dominance relation on the set of strategy profiles. In what follows it is proved that the set of NE strategy profiles for the mixed extension of the normal form game coincides with the supercore of the system associated considering a somewhat different dominance relation.

Let us define the new dominance relation.

**Definition 3** Let $(\Delta(S), <<)$ be the weakly individual dominance system associated to the mixed extension of the game $\Gamma^N$, where $<<$ is the binary relation defined on $\Delta(S)$ as follows: $\sigma' << \sigma$ if there exists a player $i \in N$ such that $\sigma' \in \gamma_i(\sigma)$ and either $U_i(\sigma') > U_i(\sigma)$ or $U_i(\sigma') = U_i(\sigma)$ whenever $\sigma' \in \Sigma^*$ and $\sigma \notin \Sigma^*$.

With this definition at hand we can establish the following two lemmas.

**Lemma 1** $\Sigma^*$ is a compact subset of $\Delta(S)$.

**proof** See the Appendix.

**Lemma 2**$\mathcal{D}(\Sigma^*) \cup \Sigma^*$ in $(\Delta(S), <<)$ is a closed subset of $\Delta(S)$.

**proof** See the Appendix.

Finally, we will show the equivalence between the set of NE strategy profiles and the supercore of $(\Delta(S), <<)$.

**Theorem 4** $\Sigma^*$ *is the supercore of* $(\Delta(S), <<)$.

**Proof.** See the Appendix. ■

To conclude, let us relate the results of this section with the ones presented by Kalai and Schmeidler (1977). These authors study the admissible set in various bargaining situations. In particular, they present different binary relations to study the equivalence between the admissible set and the NE strategy profiles for the mixed extension of a game. They find that under the binary relation $\angle$, the admissible set may be "too large". For instance, in the 2-person matching pennies game the admissible set coincides with the entire space of mixed strategies. However, since in this example the set of NE profiles is finite, we have that the supercore coincides with the unique NE strategy profile. In addition, Kalai and Schmeidler show that the coincidence of the supercore with the set of NE strategy profiles holds under a rather technical dominance relation definition. This is also our case.

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# 5 Appendix

Proofs omitted from the text are provided below.

**Proof of Lemma 1**

We prove first that $\Sigma^*$ is closed.

Let us consider a sequence $\{\sigma_n^*\}_{n\in\mathbb{N}} \subset \Sigma^*$ such that $lim_{n\to\infty}\sigma_n^* = \sigma^*$. We will see that $\sigma^* \in \Sigma^*$.

Since $\sigma_n^* \in \Sigma^*$ then $(\sigma_n^*)_i$ is a best response to $(\sigma_n^*)_{-i}$ for each $i \in N$. That is,

$$U_i((\sigma_n^*)_i, (\sigma_n^*)_{-i}) \geq U_i(\sigma_i, (\sigma_n^*)_{-i}) \text{ for all } \sigma_i \in \Delta(S_i).$$

Taking the limit to each side of the last expression we have:

$$lim_{n\to\infty}U_i((\sigma_n^*)_i, (\sigma_n^*)_{-i}) \geq lim_{n\to\infty}U_i(\sigma_i, (\sigma_n^*)_{-i}) \text{ for all } \sigma_i \in \Delta(S_i).$$

Since $lim_{n\to\infty}\sigma_n^* = \sigma^*$, and $U_i$ is a continuous function it follows that

$$U_i((\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*) \text{ for all } \sigma_i \in \Delta(S_i).$$

Therefore, $\sigma_i^*$ is player's $i$ best response to $\sigma_{-i}^*$ for every $i \in N$. In other words, $\sigma^*$ is a NE strategy profile.

Lastly, given that $\Sigma^*$ is a closed subset of the compact set $\Delta(S)$ we conclude that $\Sigma^*$ is compact.■

**Proof of Lemma 2**

By Lemma 1, $\Sigma^*$ is closed. Hence it is sufficient to prove that the closure of $\mathcal{D}(\Sigma^*)$ is contained in $\mathcal{D}(\Sigma^*) \cup \Sigma^*$.

Let us consider a sequence $\{\sigma_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\Sigma^*)$ such that $\lim_{n\to\infty}\sigma_n = \sigma$. We will see that $\sigma \in \mathcal{D}(\Sigma^*) \cup \Sigma^*$.

Since $\sigma_n \in \mathcal{D}(\Sigma^*)$, there is a NE strategy profile $\sigma_n^*$ such that for some player $i \in N$, $\sigma_n^* \in \gamma_i(\sigma_n)$ and $U_i(\sigma_n^*) \geq U_i(\sigma_n)$. Taking into account that the set $\Sigma^*$ is compact (Lemma 1) and that $\{\sigma_n^*\}_{n\in\mathbb{N}} \subset \Sigma^*$, we can assume without loss of generality the existence of a profile $\sigma^* \in \Sigma^*$ such that $\lim_{n\to\infty}\sigma_n^* = \sigma^*$. (If this is not the case then we substitute that sequence by the appropiate subsequence).

Now, set $\mathbb{N}(i) = \{n \in \mathbb{N} : \sigma_n^* \in \gamma_i(\sigma_n)\}$ for each $i \in N$. It is clear that for some $j \in N$ the set $\mathbb{N}(j)$ is numerable. Hence, we can choose the subsequences

$\{\sigma'_n\}_{n\in\mathbb{N}}$ of $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{\sigma^{*\prime}_n\}_{n\in\mathbb{N}}$ of $\{\sigma^*_n\}_{n\in\mathbb{N}}$ such that $(\sigma^*)'_n \in \gamma_j(\sigma'_n)$ and $U_j((\sigma^*)'_n) \geq U_j(\sigma'_n)$ for all $n \in \mathbb{N}$. Therefore, taking the limit to each side in the last expression we have

$$lim_{n\to\infty}U_j((\sigma^*)'_n) \geq lim_{n\to\infty}U_j(\sigma'_n).$$

Since $lim_{n\to\infty}((\sigma^*)'_n) = \sigma^*$, $lim_{n\to\infty}(\sigma'_n) = \sigma$, and $U_j$ is a continuous function, we have $U_j(\sigma^*) \geq U_j(\sigma)$. Given that $\sigma^* \in \gamma_i(\sigma)$ it follows that if $\sigma \notin \Sigma^*$ then $\sigma^* << \sigma$, so either $\sigma \in \mathcal{D}(\Sigma^*)$ or $\sigma \in \Sigma^*$ and Lemma 5 follows. ■

**Proof of Theorem 4**

Given that any subsolution of $(\Delta(S), <<)$ contains $\Sigma^*$, it is sufficient to prove that $\Sigma^*$ is a subsolution. That is, $\Sigma^* \subset \mathcal{U}(\Sigma^*)$ and $\Sigma^* = \mathcal{U}^2(\Sigma^*)$.

Clearly, $\Sigma^* \subset \mathcal{U}(\Sigma^*)$. If $\Sigma^* = \mathcal{U}(\Sigma^*)$ then $\Sigma^*$ is a vN&M stable set, and thus $\Sigma^*$ is a subsolution. So, let us assume that $\mathcal{P}(\Sigma^*) \neq \emptyset$. We must prove that $\Sigma^* = \mathcal{U}^2(\Sigma^*)$ or equivalently that $\mathcal{P}(\Sigma^*) \subset \mathcal{D}(\mathcal{P}(\Sigma^*))$.

Let $\sigma \in \mathcal{P}(\Sigma^*)$. Since $\sigma \notin \Sigma^*$, $\sigma_i$ is not the best response to $\sigma_{-i}$ for some player $i$. Hence, there is a $\sigma' \in \gamma_i(\sigma)$ such that $U_i(\sigma') > U_i(\sigma)$.

Now, if $\sigma' \in \mathcal{P}(\Sigma^*)$ then we are done. If this is not the case then, set $\sigma_\lambda = \lambda\sigma + (1-\lambda)\sigma'$ for all $\lambda \in [0,1)$. By the linearity of $U_i$ we have that $U_i(\sigma_\lambda) > U_i(\sigma)$, and since $\sigma_\lambda \in \gamma_i(\sigma)$, it follows that $\sigma_\lambda << \sigma$.

By Lemma 2 we know that $\mathcal{D}(\Sigma^*) \cup \Sigma^*$ is a closed subset of $\Delta(S)$. Therefore, $\mathcal{P}(\Sigma^*)$ is an open subset of $\Delta(S)$. This implies that there exists an $\varepsilon > 0$ such that the open ball $B(\sigma,\varepsilon) \subset \mathcal{P}(\Sigma^*)$ . By choosing a $\lambda \in (0,1)$ such that $\sigma_\lambda \in B(\sigma,\varepsilon)$ we have that $\sigma_\lambda \in \mathcal{P}(\Sigma^*)$. Since $\sigma_\lambda << \sigma$, we conclude that $\sigma \in \mathcal{D}(\mathcal{P}(\Sigma^*))$ and Theorem 4 yields.■