STRATEGIES UNDER DISTRIBUTIONAL AND STRATEGIC UNCERTAINTY

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Abstract. I investigate the decision problem which arises in a game of incomplete information under two different types of uncertainty - uncertainty about other players' type distributions and about other players' strategies. I propose a new solution concept which works in two steps. First, I assume common knowledge of rationality and eliminate all strategies which are not best replies. Second, I apply the maximin expected utility criterion. Using this solution concept, one can derive predictions about outcomes and recommendations for players facing uncertainty. A bidder following this solution concept in a first-price auction expects all other bidders to bid their highest rationalizable bid given their valuation. As a consequence, the bidder never expects to win against an equal or higher type and resorts to win against lower types with certainty.

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1. Introduction

1.1. **Motivation.** In many economic settings agents face uncertainty. For example, a seller may be uncertain about the distribution of a buyer's willingness to pay, i.e. face *distributional uncertainty*. Bidders participating in a first-price auction may be uncertain about each other's bidding strategies, i.e. face *strategic uncertainty*. Formally, a player faces uncertainty if the smallest set of distributions and strategies such that the player knows that the other players' true type distribution and strategy is an element of this set, is not a singleton. In the presence of uncertainty I propose a new solution concept which works in two steps: First, I assume common knowledge of rationality and eliminate all actions which are not rationalizable. Afterwards, I apply the maximin expected utility criterion. Using this solution concept, I can derive recommendations for a player facing distributional, strategic or both, distributional and strategic uncertainty. Furthermore, I analyze outcomes under the assumption that every player in the game uses this concept.

The first step is based on the idea that even if an equilibrium exists, in many economic settings a player may be uncertain whether her opponents employ equilibrium strategies, and consequently face strategic uncertainty. As stated by Pearce (1984), "some Nash equilibria are intuitively unreasonable and not all reasonable strategy profiles are Nash equilibria". He argues that if players cannot communicate, then a player will best reply to equilibrium strategies only if she is able to deduce these equilibrium strategies. However, a player may consider more than one strategy of the other players' as possible. For example, this can occur under the existence of multiple equilibria without one equilibrium being focal or salient (Bernheim (1984)). Thus, a Nash equilibrium may not be a suitable solution concept if a player does not observe or does not deduce a unique conjecture about the other players' strategies. Similarly, Renou and Schlag (2010) argue that "common knowledge of conjectures, mutual knowledge of rationality and payoffs, and existence of a common prior" are required in order to justify a Nash equilibrium as a solution concept. Thus, Bernheim and Pearce (and Battigalli and Siniscalchi (2003b) for games of incomplete information) propose to consider strategies which a player can deduce only from common knowledge of rationality.

A player is rational if her action is a best reply given her type and an assumption about the other players' type distribution and strategies. A strategy which a player assumes to be played by another rational player has to be rational as well, i.e. to be a best reply given an assumption about type distributions and strategies. This reasoning continues ad infinitum. Battigalli and Siniscalchi (2003b) show that common knowledge of rationality is equivalent to bidders playing *belief-free rationalizable strategies*. These are strategies which survive the iterated elimination of actions which are not best replies to some distribution of types and to some strategy which consists of actions which have not been eliminated in previous elimination rounds.

In the second step I apply the maximin expected utility criterion due to Gilboa and Schmeidler (1989). A player applying this criterion chooses the action which maximizes her minimum expected utility given her type. The maximin expected utility criterion can be modeled as a simultaneous zero-sum game against an adverse nature. Given the strategy of the adverse nature, the player applying the maximin criterion chooses the action which maximizes her expected utility. The adverse nature's utility is the player's expected utility multiplied by -1. The action space of the adverse nature consists of all distributions and all belief-free rationalizable strategies of the other players which the player considers to be possible. The model as a simultaneous zero-sum game against an adverse nature has the advantage that in equilibrium a player does not only choose an action which maximizes her minimum expected utility but also chooses an optimal action given the adverse nature's strategy. This can be interpreted as maximizing expected utility given some subjective belief, in the following called *subjective maximin belief*, which is determined by the distribution and strategy chosen by nature.

The following two examples illustrate how the proposed solution concept applies under strategic uncertainty and why following a Nash equilibrium might not be a useful recommendation. Afterwards, I will summarize the results for first-price auctions under distributional, strategic and both, distributional and strategic uncertainty.

For the first example consider a sender who has to deposit a package either in places A, B or C. A receiver has to decide to which places she sends one or two drivers in order to pick up the package. If the package is picked up, sender and receiver earn each a payoff of P and zero otherwise. In addition, the receiver faces a cost of c if a driver travels to place A or B and a cost of $\tilde{c}$ if a driver travels to place C. Due to cost savings in administration, the cost of sending two drivers is equal to $2c - \alpha$ where it holds that $-2c + \alpha < -c$. The game is summarized in the following payoff table:

	A	B	C	AB	AC
A	$P; P - c$	$0; -c$	$0; -\tilde{c}$	$P; P - 2c + \alpha$	$P; P - c - \tilde{c} + \alpha$
B	$0; -c$	$P; P - c$	$0; -\tilde{c}$	$P; P - 2c + \alpha$	$0; -c - \tilde{c} + \alpha$
C	$0; -c$	$0; -c$	$P; P - \tilde{c}$	$0; -2c + \alpha$	$0; P - c - \tilde{c} + \alpha$

Assume it is common knowledge that it holds $P - \tilde{c} < -c$ and $P - 2c + \alpha > -c$. The Nash equilibria in this game are $(A; A)$, $(B; B)$ and both players mixing between A and B with probability $\frac{1}{2}$. Although Nash equilibria exist, the players may be uncertain about each other's strategy since there does not exist a particularly salient one. The application of the maximin criterion (as well es the maximin expected utility criterion) leaves both players indifferent between actions A and B. The maximin criterion does not yield to action AB for the receiver since by choosing AB she would face the risk that the sender deposits the package in C, leaving the receiver with the costs of two drivers $-2c + \alpha$. However, the result of the maximin criterion changes after restricting the strategy space to belief-free rationalizable strategies. Excluding actions which are not best replies leads to the elimination of strategies C and AC of the receiver, leading to the elimination of action C for the sender:

	A	B	AB
A	$P; P - c$	$0; -c$	$P; P - 2c + \alpha$
B	$0; -c$	$P; P - c$	$P; P - 2c + \alpha$

Now the maximin criterion leads to action AB for the receiver. In other words, if the receiver anticipates that the sender anticipates that she will never send a driver to C, the application of the maximin criterion leads to the action AB. In this case, the receiver earns a payoff of $P - 2c + \alpha$ with certainty. If she would follow a Nash equilibrium or apply the maximin criterion directly, she would face the risk of getting a payoff of $-c$.

As a second example consider the following payoff table. It illustrates the decision problem of a player who is certain about her opponent's rationality but uncertain about her preferences:

	X	Y	Z
A	*10;10*	0;9	0,0
B	5;1	5;9*	0,0
C	4;1	4; 9*	4;0
D	1;10*	*6;9	0;0

The unique Nash equilibrium in pure strategies, (A, X), is focal in the sense that it is the social optimum and leads to the highest possible payoff for both players. However, the row player may think that her opponent is rational but may have preferences which cause her to choose action Y instead of X. For example, the application of the maximin or the minimax regret criterion would lead to action Y. In other words, the column player may prefer to get a payoff of 9 with certainty instead of aiming for the payoff of 10 and risking to get a payoff of 1. Given this uncertainty about the column player's strategy, the row player may resort to the application of the maximin criterion. This leads to action C which ensures a payoff of 4 for the row player. However, the row player can anticipate that action Z is strictly dominated for the column player. After the elimination of this action, C becomes strictly dominated for the row player which leads to the following payoff table:

	X	Y
A	*10;10*	0;9
B	5;1	5;9*
D	1;10*	*6;9

Now the application of the maximin criterion leads to action B for the row player. That is, after anticipating that the column player will never play Z, the row player can ensure a payoff of 5 instead a payoff of 4.

These examples show how the proposed solution concept provides recommendations under strategic uncertainty. Moreover, they show why players may not expect their opponents to play Nash equilibria and why the application of the maximin utility criterion alone may cause forgone profits. Another example for the failure of the maximin criterion are first-price auctions. If a player faces strategic uncertainty and applies the maximin expected utility criterion, the adverse nature would choose strategies of the other bidders such that they bid arbitrarily high. First, this assumption seems unrealistic. Second, in this case the maximin expected utility criterion does not lead to any recommendation since the bidder would be indifferent between any bid between zero and her valuation. Therefore, it is crucial for a bidder to determine the highest possible bids of her competitors. A first simple restriction of the strategy space would be that bidders never bid above their valuation. However, if a bidder with valuation θ knows that there is positive probability weight on types equal or lower than $\theta' < \theta$ and every bidder bids at most her valuation, she can expect a positive payoff from bidding θ'. Consequently, bidding too close to the own valuation in this case is not rational. Thus, eliminating only bids above valuation still allows for actions which are not rational. The highest belief-free rationalizable bid provides for a bidder exactly the answer to the question which bid of her competitors is the highest possible.

In sections 4-7 I apply my proposed solution concept to first-price auctions. I provide recommendations and analyze outcomes under distributional uncertainty, strategic uncertainty and both, distributional and strategic uncertainty. For the analysis of distributional uncertainty I assume common knowledge of an exogenously given mean μ of bidders' valuations. The latter assumption reflects that in reality bidders often are not able to learn their competitors' value distribution. Although this information is very valuable for the bidders and they go at great lengths in order to learn the value distribution, such learning has its limits and bidders may be able to learn only the range the mean of the value distribution. [1]

Under strategic uncertainty with common knowledge of rationality and common knowledge of a symmetric value distribution for every type there exists a unique highest belief-free rationalizable bid. A bidder applying the proposed solution concept assumes that every other bidder places the highest belief-free rationalizable bid given her type. As a

[1] See Montiero (2009)

consequence, the bidder never expects to win against a bidder with an equal or higher type and therefore bids the highest belief-free rationalizable bid of a lower type in order to win against the lower type with certainty. If every bidder applies this solution concept, then every bidder has the same beliefs about distributions and strategies. Every bidder calculates which highest rationalizable bid of a lower type maximizes her expected payoff. It turns out that due to the symmetry of beliefs about distributions strategies, the higher the type of the bidder, the higher is type whose highest belief-free rationalizable bid maximizes her expected payoff. Therefore, the outcome is efficient.

Under strategic uncertainty with common knowledge of rationality and distributional uncertainty with common knowledge of an exogenously given mean as before, for every type there exists a unique highest belief-free rationalizable bid. A bidder applying the proposed solution concept assumes that every other bidder places the highest belief-free rationalizable bid given her type. Let θ_μ be the lowest valuation which is higher than the mean. The highest belief-free rationalizable bid of a bidder with a valuation lower than θ_μ is her valuation. The subjective maximin belief of a bidder with a valuation $\theta < \theta_\mu$ about the other bidders' value distributions is that the probability weight is distributed between the valuations θ and θ_μ. As a consequence, a bidder with a valuation lower than θ_μ expects a utility of zero and is indifferent between any bid between zero and her valuation. Every bidder with a valuation θ such that $\theta \geq \theta_\mu$ never expects to win against a bidder with the same valuation. Hence, the maximin belief of a bidder about the other bidders' value distribution maximizes the probability weight on θ and makes the bidder indifferent between any highest belief-free rationalizable bid of lower types. As a consequence, the bidder mixes among all highest belief-free rationalizable bids of lower types. Therefore, the outcome is not efficient.

Under distributional uncertainty (without strategic uncertainty) there does not exist an outcome under maximin strategies. That is, an equilibrium in the game with I bidders and an adverse nature where all bidders apply the maximin expected utility criterion, does not exist. In order to gain some intuition, consider the example with two bidders and three valuations given by 0, θ and 1 where $\theta < \mu$. For a bidder with valuation θ the adverse nature will distribute the probability weight in the other bidder's value distribution between types θ and 1. If there exists an equilibrium, then both θ-types bid θ. If the equilibrium is efficient, then both 1-types have to play a mixed strategy on some interval with θ as the lower endpoint. Since the mean μ has to be preserved, the adverse nature minimizes the winning probability (and hence the expected utility) of a bidder with valuation 1 by distributing the probability weight in the other bidder's value distribution between 0 and 1. A best reply to such a distribution would be to bid zero (or an infinitesimally small amount). Hence, an efficient equilibrium does not exist. An inefficient equilibrium can be excluded with similar arguments.

The remainder of the paper is organized as follows. I conclude the introduction with an overview over the related literature. The second section contains the formal model. The third section collects all results for the general model, in particular, it provides sufficient conditions for strategies to be belief-free rationalizable. The fourth section specifies the model for first-price auctions. The fifth, sixth and seventh section characterize the outcomes under the maximin expected utility criterion under distributional, strategic and both, distributional and strategic uncertainty. The appendix contains the proofs not provided in previous sections.

1.2. **Related Literature.** This paper relates to two strand of literature - on the one hand the literature on decision criteria under uncertainty and robustness, on the other hand the literature on rationalizability. Two widely used decision criteria under uncertainty are the maximin utility and the minimax regret criterion. The axiomatization of the maximin expected utility criterion is provided in Gilboa and Schmeidler (1989), the axiomatization of the minimax regret criterion is provided in Stoye (2011). In Bergemann and Schlag (2008) both criteria are applied to a monopoly pricing problem where a seller faces uncertainty about the buyer's value distribution. Since the seller knows that the buyer will obtain the good if the price is equal or lower than her valuation, the seller does not face strategic uncertainty.

The maximin expected utility criterion has particularly been applied to first-price auctions under distributional uncertainty. Lo (1998) derives equilibrium bidding strategies in a first-price auction under the maximin expected utility criterion where it is common knowledge that the true value distribution is an element of a given set of distributions. Salo and Weber (1995) assume that only the range of values is common knowledge and that ambiguity averse bidders use a convex transformation of the uniform distribution as a prior. They find, that the more ambiguity averse a bidder is, the higher is the bid. Chen, Katuščák, and Ozdenoren (2007) analyze first and second-price auction where bidders face one of two possible distributions which can be ordered with respect to FOD. Thus, an ambiguity-averse bidder would assume the stochastically dominating distribution. In their experimental findings they reject the hypothesis that bidders are ambiguity-averse. These three papers use Nash equilibria as a solution concept, that is, agents do not face strategic uncertainty since strategies are observable.

Bose, Ozdenoren, and Pape (2006) derive the optimal auction in a setting where seller and bidders may face different degrees of ambiguity, that is, they may face different sets of possible value distributions. Carrasco, Luz, Kos, Messner, Monteiro, and Moreira (2017) consider a seller facing a single buyer. The set of distributions the seller considers to be possible is determined by a given range and mean. In these two papers strategic uncertainty is not an issue since the seller chooses an incentive compatible mechanism.

Renou and Schlag (2010) analyze strategic uncertainty using the minimax regret criterion. Besides Kasberger and Schlag (2017), I am the only one addressing distributional *and* strategic uncertainty. They use the minimax regret criterion and allow for the possibility that a bidder can impose bounds on the other bidders' bids or value distributions. For example, they consider the case where a bidder can impose a lower bound on the highest bid.

In their literature on robust mechanism design Dirk Bergemann and Stephen Morris consider the problem of a social planner facing uncertainty about the players' actions. In Bergemann and Morris (2005) a social planner can circumvent uncertainty about the players' strategies by choosing ex-post implementable mechanisms. Bergemann and Morris (2013) provides predictions in games independent of the specification of the information structure. In order to do so, they characterize the set of set of Bayes correlated equilibria. An application of this concept to first-price auctions is carried out in Bergemann, Brooks, and Morris (2015). In Carroll (2016) two agents accept or reject a proposed deal where the value for each agent depends on an unknown state. The results provide an upper bound of welfare loss among all information structures.

The concept of belief-free rationalizable strategies has been first introduced by Bernheim (1984) and Pearce (1984) for games with complete information. Battigalli and Siniscalchi

(2003b) extend belief-free rationalizability to games of incomplete information. An application to first-price auctions has been carried out by Dekel and Wolinsky (2001). They apply belief-free rationalizable strategies to a first-price auction with discrete private values and discrete bids. They present a condition on the distribution of types under which the only belief-free rationalizable action is to bid the highest bid below valuation. Battigalli and Siniscalchi (2003a) assume that value distributions in a first-price auction are common knowledge but not the strategies of the bidders. They characterize the set of belief-free rationalizable actions under the assumption of *strategic sophistication*, which implies common knowledge of rationality and of the fact that bidders with positive bids win with positive probability. [2] They find that for a bidder with a given valuation θ all bids in an interval $(0, b^{max}(\theta))$ are belief-free rationalizable where $b^{max}(\theta)$ is higher than the equilibrium bid. Using this result, one can immediately tell that under common knowledge of rationality a bidder applying the maximin expected utility criterion has the subjective maximin belief that every other bidder with valuation θ bids $b^{max}(\theta)$. I replicate this result in section 6 for first-price auctions with discrete values.

To the best of my knowledge I am the first one applying the maximin expected utility criterion to strategic uncertainty and the first one combining rationalizable strategies with a decision criterion under uncertainty.

2. Model

Underlying game of incomplete information. The starting point of the model is a game of incomplete information. Let $\{1, \ldots, I\}$ be the set of players, for every $i \in \{1, \ldots, I\}$ let $A_i \subseteq \mathbb{R}$ be the set of possible actions and $\Theta_i \subseteq \mathbb{R}$ be the set of possible privately known types for player i. Let Θ_0 denote the set of utility-relevant sates of the world such that the true state of the world $\theta_0 \in \Theta_0$ is not known to any of the players. A *pure strategy* for player i is a mapping

$$\beta_i : \Theta_i \to A_i$$

$$\theta_i \mapsto a_i$$

[2]The assumption that it is common knowledge that bidders with positive bids win with positive probability, excludes all weakly dominated bids, including bidding above valuation.

for $\theta_i \in \Theta_i$. The set S_i is the set of all pure strategies of player i. A *mixed strategy* is a mapping

$$\beta_i : \Theta_i \to \Delta A_i$$

$$\theta_i \mapsto a_i$$

where ΔA_i is the set of all probability distributions on A_{-i}. Let

$$u_i : A \times \Theta \to \mathbb{R}$$

$$(a_1, \dots, a_I, \theta) \mapsto u_i(a_1, \dots, a_I, \theta)$$

denote the utility function for player i where

$$(a_1, \dots, a_I) \in A = A_1 \times \dots \times A_I$$

$$\theta = (\theta_0, \theta_1, \dots, \theta_I) \in \Theta = \Theta_0 \times \Theta_1 \times \cdots \times \Theta_I.$$

In the application to first-price auctions in sections 4-7 values are private, i.e. the utility function of player i depends an the actions of all players but only on her own type.

Possible distributions ans strategies. In order to formalize the maximin expected utility criterion, a new player, denoted by n, is introduced, representing the adverse nature a player i applying the maximin expected utility criterion faces. Players i and n play a simultaneous zero-sum game where utilities are induced by the underlying game of incomplete information. The first step of a formal description of this game is the definition of the adverse nature's action space. It accounts for the residual uncertainty of player i, that is, the adverse nature's action space is the set of all distributions and strategies player i considers to be possible. The following definition formalizes the set of the distributions and strategies a player considers to be possible.

Definition 1. *Let $\Delta\Theta_{-i}$ be the set of all probability distributions on Θ_{-i} and let ΔS_{-i} be the set of all probability distributions on S_{-i}. The set $\Delta_{\Theta_{-i}}$ is the smallest subset of $\Delta\Theta_{-i}$ such that player i knows that the true type distribution is an element in $\Delta_{\Theta_{-i}}$. The set $\Delta_{S_{-i}}$ is the smallest subset of ΔS_{-i} such that player i knows that the strategies of the other $I-1$ players are an element in $\Delta_{S_{-i}}$.*

The following examples illustrate this definition.

Example 1. *Consider a setting where every player i knows the other players' type distribution, denoted by F_{-i} and observes the other players' strategies, denoted by β_{-i}. Then for every player i the set of possible distributions and strategies is equal to*

$$\{F_{-i}\} \times \{\beta_{-i}\}.$$

Thus, if the other players' type distribution and strategies are known to a player, then the set of the distributions and strategies which this player considers to be possible, is a singleton and so is the adverse nature's action space. In other words, in this setting none of the players faces uncertainty. If the given strategies form a Bayes-Nash equilibrium, this equilibrium coincides with the unique outcome under maximin strategies.

Example 2. *Consider a game of incomplete information with two players i and j where player i does not have any knowledge about player j's strategy but knows that player j can have two different types θ_L, θ_H with $\theta_L < \theta_H$ and θ_L can occur either with probability $\frac{1}{3}$ or $\frac{1}{2}$. Then the set of the distributions and strategies which player i considers to be possible is given by*

$$\Delta S_j \times \{F_1, F_2\}$$

where F_1 and F_2 are elements in $\Delta\Theta_j = \{\theta_L, \theta_H\}$ defined by $F_1(\theta_L) = \frac{1}{3}$, $F_1(\theta_H) = 1$ and $F_2(\theta_L) = \frac{1}{2}$, $F_2(\theta_H) = 1$.

This example shows that the more knowledge a player applying the maximin expected utility criterion has about the other players' type distributions and strategies, the smaller is the set distributions and strategies the player considers to be possible, and so is the adverse nature's action space.

The following remarks clarify some aspects of Definition 1. First, as mentioned above, throughout the paper I use the axiomatization of the knowledge operator where the statement that a player knows something implies that it is true. Therefore, for every i the true type distribution is indeed an element in $\Delta_{\Theta_{-i}}$ and the strategies adopted by the other players are an element in $\Delta_{S_{-i}}$.

Second, the assumption that a player knows that the true type distribution (or the true strategy) is an element of some set is w.l.o.g. since it covers any possible knowledge structure. For example, if a bidder i knows only the type spaces of the other bidders but nothing else about the type distribution, then $\Delta_{\Theta_{-i}}$ is equal to $\Delta\Theta_{-i}$, the set of all type distributions on Θ_{-i}. In contrast, if a bidder i faces no distributional uncertainty and knows that the distribution of the other bidders' types is given by a function F_{-i}, then the set $\Delta_{\Theta_{-i}}$ is equal to $\{F_{-i}\}$. In many real-world settings players do not know the exact type distribution of the other players' but invest effort in order to obtain additional knowledge. As a result, players know that the other players' value distribution is an element of some set of value distributions. For example, Carrasco, Luz, Kos, Messner, Monteiro, and Moreira (2017) consider a first-price auction where the seller does not know the bidders' value distributions but knows their mean.

Third, I consider the sets

$$\Delta\Theta_{-i} \text{ and } \Delta S_{-i}$$

and not the sets

$$\Delta\Theta_0 \times \cdots \times \Delta\Theta_{i-1} \times \Delta\Theta_{i+1} \times \cdots \times \Delta\Theta_I \text{ and } \Delta S_1 \times \cdots \times \Delta S_{i-1} \times \Delta S_{i+1} \times \cdots \times \Delta S_I,$$

that is, I allow for correlated types and correlated strategies. [3]

Belief-free rationalizable strategies. As argued in the introduction, in many economic settings players may face uncertainty about the other players' strategies. Even if an equilibrium exists, a player may expect more strategies of the other players to be possible. For example, if there exist multiple equilibria or the equilibrium strategy is not aligned with preferences the other players may have, e.g. maximin or minimax regret preferences. In order to determine the set of strategies a player can expect from rational opponents, I assume common knowledge of rationality. The set of strategies which are compatible with the assumption of common knowledge of rationality are *belief-free rationalizable strategies* as introduced by Battigalli and Siniscalchi (2003b). They consider belief-free rationalizability where players know only type spaces and action spaces. In addition, they introduced the concept of Δ-rationalizability which allows for the fact that players have additional knowledge about possible types and strategies as described in Definition 1.

[3]Nevertheless, Definition 1 is not as general as possible. The most general version would allow for a correlation between strategies and distributions. Since I do not consider the possibility of correlation in the application sections, I do not allow for this type of correlation in the general model for the sake of notation simplicity. For a detailed discussion see Appendix F.

In Battigalli and Siniscalchi (2003b) the following two assumptions hold:

(A1) For every player i the type space Θ_i, the action space A_i, the set of the other players' possible type distributions $\Delta_{\Theta_{-i}}$ and the set of the other players' possible strategies $\Delta_{S_{-i}}$ are common knowledge.

(A2) It is common knowledge that every player is rational, i.e. it is common knowledge that every player i maximizes her expected utility given her type and an assumption about the other players' types in $\Delta_{\Theta_{-i}}$ and an assumption about the other players' strategies in $\Delta_{S_{-i}}$.

These two assumptions lead to the following reasoning. Every player i maximizes her expected utility given her type, an assumption about the other players' type distribution in $\Delta_{\Theta_{-i}}$ and an assumption about the other players' strategies in $\Delta_{S_{-i}}$. The strategy which player i assumes is played by some player $j \neq i$ has also to by compatible with assumptions (A1) and (A2). Therefore, for every possible type of player j, the action prescribed by the strategy assumed by player i maximizes j's expected utility given her type and some assumption about type distributions and strategies which are elements in $\Delta_{\Theta_{-i}}$ and $\Delta_{S_{-i}}$. Again, the strategies assumed by player j have to be compatible with assumptions (A1) and (A2) and therefore for every type of some player $k \neq j$, the action which is prescribed by player j's assumption about player k's strategy has to be a best reply given some assumption about the other players' type distribution and strategies which are elements in $\Delta_{\Theta_{-i}}$ and $\Delta_{S_{-i}}$. This reasoning continues ad infinitum.

Given the type of a player, an action which is compatible with assumptions *(A1)* and *(A2)* is called a *belief-free rationalizable.* Battigalli, Di Tillio, Grillo, and Penta (2011) have shown that it is equivalent to define a belief-free rationalizable action as follows.

Definition 2.

(i) Let $i \in \{1, \ldots, I\}$ be some player and $\theta_i \in \Theta_i$ be some type of player i. The set of belief-free rationalizable actions *for player i is defined as follows. Set $BFR_i^1(\theta_i) := A_i$. Assume that for $k \in \mathbb{N}$ the set $BFR_i^k(\theta_i)$ is already defined. Then the set $BFR_i^{k+1}(\theta_i)$ is defined as the set of all elements a_i in A_i for which there*

exists a type distribution $F_{-i} \in \Delta_{\Theta_{-i}}$ *and a strategy profile of the other players* $\beta_{-i} \in \Delta_{S_{-i}}$ *such that it holds* [4]

(i) $\beta_j(\theta_j)(a_j) > 0 \ \Rightarrow \ a_j \in BFR_j^k(\theta_j)$ *for all* $j \neq i$

(ii) $a_i \in \underset{a'_i \in A_i}{\operatorname{argmax}} \ U_i(\theta_i, a'_i, \beta_{-i}, F_{-i})$

and $BFR_i(\theta_i)$ *is given by*

$$BFR_i(\theta_i) = \bigcap_{k \geq 1} BFR_i^k(\theta_i).$$

(ii) A strategy β_i *of a player* i *is* belief-free rationalizable *if for every* $\theta_i \in \Theta_i$ *every action* a_i *with* $\beta_i(\theta_i)(a_i) > 0$ *is belief-free rationalizable, i.e. an element of* $BFR_i(\theta_i)$*.*

(iii) For a player i *let* BFR_{-i} *be the set of belief-free rationalizable strategies of the other* $I-1$ *players.*

The intuition behind this definition is that an action for a player which is consistent with assumptions $(A1)$ and $(A2)$, i.e. a belief-free rationalizable action, is an action which survives the iterated elimination of actions which are not best replies. An action is a best reply if it maximizes the player's expected utility given her type, an assumption about the other players' type distribution in $\Delta_{\Theta_{-i}}$ and an assumption about the other players' strategies in $\Delta_{S_{-i}}$ which have not been eliminated yet.

The definition of the possible distributions and strategies and of belief-free rationalizable strategies allows for a formal definition of the adverse nature's action space.

Definition 3. *Let* Δ_{-i}^{BFR} *be defined by*

$$\Delta S_{-i}^{BFR} = \Delta_{S_{-i}} \cap BFR_{-i},$$

that is, the set of all strategies of the other $I-1$ *players which player* i *considers to be possible and which are compatible with common knowledge of rationality. Then the action space of the adverse nature faced by a player* i *is given by*

$$\Delta_{\Theta_{-i}} \times \Delta S_{-i}^{BFR}.$$

[4] If $\beta_i(\theta_i)$ is a mixed strategy, then for $a_i \in A_i$, $\beta_i(\theta_i)(a_i)$ denotes the probability with which action a_i is played.

Simultaneous game against adverse nature. Now the formal description of the simultaneous game against the adverse nature can be provided. The game consists of two players: player i who applies the maximin expected utility criterion and the player n representing the adverse nature. Given her type, player i chooses a (possibly mixed) strategy

$$\beta_i : \Theta_i \to \Delta A_i$$

$$\theta_i \mapsto \beta_i(\theta_i).$$

The strategy of the adverse nature β^{n_i} assigns to every possible type of player i a type distribution and a strategy of the other players:

$$\beta^{n_i} : \Theta_i \to \Delta_{\Theta_{-i}} \times \Delta S_{-i}^{BFR}$$

$$\theta_i \mapsto \beta^{n,i}(\theta_i) = (F_{-i}^{n_i,\theta_i}, \beta_{-i}^{n_i,\theta_i}).$$

Here the superscript n_i, θ_i indicates that the other players' type distribution $F_{-i}^{n_i,\theta_i}$ and strategies $\beta_{-i}^{n_i,\theta_i}$ are chosen by the adverse nature faced by player i and depend on her type θ_i.

It should be stressed that the simultaneous game between player i and the adverse nature is a game of complete information, that is, player i's type is common knowledge among player i and the adverse nature.

The utility of player i is determined by the type of player i, the action of player i, the distribution of the other players' types and the other players' strategies chosen by nature. Formally, the utility of player i is given by

$$(1) \quad U_i\left(\theta_i, \beta_i(\theta_i), \beta_{-i}^{n_i,\theta_i}, F^{n_i,\theta_i}\right)$$
$$= \int_{\theta_{-i}} \int_{a_{-i}} u_i(a_1, \ldots, a_i, \ldots, a_I, \theta) \prod_{j=1}^{I} \beta_j^{n_i,\theta_i}(\theta_j)(a_j) d\theta_{-j} dF_{-i}^{n_i,\theta_i}(\theta_{-i}) d\theta_{-i}$$

where the function u_i stems from the underlying game of incomplete information. The utility of nature is given by

$$-U_i\left(\theta_i, \beta_i(\theta_i), \beta_{-i}^{n_i,\theta_i}, F_{-i}^{n_i,\theta_i}\right).$$

If in addition to player i, another player j applies the maximin expected utility criterion, then she also faces an adverse nature represented by the player n. Then n's strategy is given by

$$(\beta^{n_i}, \beta^{n_j}) : \Theta_i \times \Theta_j \to (\Delta_{\Theta_{-i}} \times \Delta S_{-i}^{BFR}) \times \left(\Delta_{\Theta_{-j}} \times \Delta S_{-j}^{BFR}\right)$$
$$(\theta_i, \theta_j) \mapsto (\beta^{n_i}(\theta_i), \beta^{n_j}(\theta_j)) = \left(\left(F_{-i}^{n_i,\theta_i}, \beta_{-i}^{n_i,\theta_i}\right), \left(F_{-j}^{n_j,\theta_j}, \beta_{-j}^{n_j,\theta_j}\right)\right).$$

Since the state of the world the adverse nature chooses for player i is not observed by player j, for every player applying the expected maximin utility criterion the adverse nature faces an independent minimization problem. [5]

The following definition summarizes all components describing a game under uncertainty.

Definition 4. *A* game under uncertainty *consists of players* $1, \ldots, I$*, a subset of players* $\{i_1, \ldots i_k\} \subseteq \{1, \ldots, I\}$ *applying the maximin expected utility criterion, and a player* n*. For every* $i \in \{1, \ldots, I\}$ *a strategy is a mapping from a type space* Θ_i *to an action space* A_i*:*

$$\beta_i : \Theta_i \to A_i.$$

A strategy of n *is a mapping which for every player* $i \in \{i_1, \ldots i_k\}$ *and for every possible type of player* i *assigns a distribution of the other players' values in* $\Delta_{\Theta_{-i}}$ *and a strategy of the other players in* ΔS_{-i}^{BFR} *, that is, a state of the world which player* i *considers to be possible:*

$$(\beta^{n_{i_1}}, \ldots, \beta^{n_{i_k}}) : \Theta_{i_1} \times \ldots \times \Theta_{i_k} \to (\Delta_{\Theta_{-i_1}} \times \Delta S_{-i_1}^{BFR}) \times \cdots \times \left(\Delta_{\Theta_{-i_k}} \times \Delta S_{-i_k}^{BFR}\right).$$

The utility of player i *is given by*

$$U_i\left(\theta_i, \beta_i(\theta_i), \beta_{-i}^{n_i,\theta_i}, F^{n_i,\theta_i}\right)$$

which is defined as in (1) and depends on the utility function of player i *in the underlying game of incomplete information, denoted by* u_i*:*

$$u_i : \Theta \times A \to \mathbb{R}.$$

$$(a_1, \ldots, a_I, \theta) \mapsto u_i(a_1, \ldots, a_I, \theta)$$

[5]Equivalently, one could introduce an additional adverse nature for every player applying the minimax expected utility criterion.

The utility of player nature is given by

$$-\sum_{j=1}^{k} U_i\left(\theta_i, \beta_i(\theta_i), \beta_{-i}^{n_i,\theta_i}, F^{n_i,\theta_i}\right).$$

Throughout the remainder of the paper it will be assumed that a game of uncertainty is given without explicitly stating all its ingredients.

The term uncertainty can include distributional uncertainty or strategic uncertainty or both. If only one type of uncertainty is present, I will refer to this case as *pure distributional* or *pure distributional uncertainty.*

Now it is possible to define a maximin strategy in a game under distributional and strategic uncertainty.

Definition 5. *In a game under uncertainty for a player i a strategy*

$$\beta_i : \Theta_i \to \Delta A_i$$

is a maximin strategy *if there exists an equilibrium in the simultaneous game between nature and player i such that β_i is player i's equilibrium strategy.*

As described above, such a maximin strategy has two properties. First, if a player would not choose an action according to a maximin strategy given her type, then there would exist a state of the world (i.e. a distribution of the other players' types and a strategy of the other players) which the player considers to be possible under which the player's expected utility is lower than under the action prescribed by a maximin strategy. Second, the distribution and the strategy chosen by nature can be interpreted as the player's subjective belief about the state of the world against which she maximizes her expected utility given her type. The second property is formalized in the following definition.

Definition 6. *In a game under uncertainty let β^{n_i} be the adverse nature's equilibrium strategy projected on the $i'th$ component. A* subjective maximin belief *of player i with valuation θ_i is defined as*

$$\beta^{n_i}(\theta_i) = \left(F_{-i}^{n_i,\theta_i}, \beta_{n_i}^{n_i,\theta_i}\right),$$

that is, the adverse nature's equilibrium strategy evaluated at θ_i.

Note that the subjective maximin belief of player is not necessarily unique. However, every best reply of a player i to any subjective maximin belief induces the same expected utility for player i.

3. Outcomes under distributional and strategic uncertainty

So far, I have characterized the set of strategies of a player which are obtained if this particular player applies the maximin expected utility criterion. In addition to the derivation of maximin strategies for particular players, one can analyze what happens if *all* players adopt maximin strategies. Since under strategic uncertainty players do not observe each other's strategies, I do not use the term *equilibrium*, but the term *outcome*.

Definition 7. *In a game under uncertainty an* outcome under maximin strategies *is a strategy profile* $(\beta_1, \dots, \beta_I)$ *such that every player's strategy is a maximin strategy.*

Note that every strategy which is played in an outcome under maximin strategies can nevertheless be seen as an equilibrium strategy since it is a strategy played in an equilibrium in the simultaneous game against the adverse nature.

One commonly known example for outcomes under maximin strategies are Bayes-Nash equilibria which are formally defined in the following definition.

Definition 8. *In a game of incomplete information a strategy profile* $(\beta_1, \dots \beta_I)$ *together with a profile of type distributions* $(F_1, \dots, F_I)$ *is a* Bayes-Nash equilibrium with a common prior *if for every* $i \in \{1, \dots, I\}$ *it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i(\theta_i, a_i', \beta_{-i}, F_{-i}).$$

That is, every player maximizes her expected utility given the other players' strategies and the other players' commonly known type distributions.

Example 3. *Consider a game under uncertainty with a commonly known strategy profile* $(\beta_1, \dots, \beta_I)$ *and a commonly known profile of beliefs* $(F_1, \dots, F_I)$*, i.e. for every* $i \in \{1, \dots I\}$ *the sets* $\Delta_{\Theta_{-i}}$ *and* ΔS_{-i}^{BFR} *are given by*

$$\Delta_{\Theta_{-i}} = \{F_{-i}\} \quad \text{and} \quad \Delta S_{-i}^{BFR} = \{\beta_{-i}\}.$$

Then the set of Bayes-Nash equilibria with a common prior equals to the set of outcomes under maximin strategies.

The remainder of this section collects results concerning outcomes under maximin strategies. The first Proposition follows directly from the definition of belief-free rationalizable strategies and of an outcome under maximin strategies.

Proposition 1. *In a game under uncertainty let $(\beta_1, \dots, \beta_I)$ be an outcome under maximin strategies. Then for every $i \in \{1, \dots, I\}$ it holds that β_i is a belief-free rationalizable strategy for player i.*

Proof. Every player maximizes her expected utility given a distribution of the other players' types and a belief-free rationalizable strategy of the other players chosen by nature. Formally, it is to show that for every player i and for every type θ_i an action a_i which is played with positive probability is an element of $BFR_i^k(\theta_i)$ for every $k \geq 1$. Let $(\beta_1, \dots, \beta_I)$ be the outcome under maximin strategies. The proof works by induction. It is true that for every $i \in \{1, \dots, I\}$ any action $a_i \in A_i$ is an element in $BFR_i^1(\theta_i)$ since it holds by definition that $BFR_i^1(\theta_i) = A_i$. Assume it is already shown for every $i \in \{1, \dots, I\}$ that a_i with $\beta_i(a_i) > 0$ is an element of $BFR_i^k(\theta_i)$. Since n can choose only among belief-free rationalizable strategies, it holds for every $j \neq i$ that $F_{-i}^{n_i,\theta_i}$ and $\beta_j^{n_i,\theta_i}$ fulfill the following two properties:

(i) $\beta_j^{n_i,\theta_i}(\theta_j)(a_j) > 0 \Rightarrow a_j \in {BFR_j}^k(\theta_j)$ for all $j \neq i$

(ii) $\beta_i(\theta_i)(a_i) > 0 \Rightarrow a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}}\ U_i\left(\theta_i, a_i', \beta_{-i}^{n_i,\theta_i}, F_{-i}^{n_i,\theta_i}\right)$.

Hence, it holds that a_i is an element in $BFR_i^{k+1}(\theta_i)$ and it follows by induction that a_i is an element in $BFR_i^k(\theta_i)$ for every $k \geq 1$. □

The following conclusions can be derived from this proposition. First, this proposition shows that the maximin expected utility criterion is consistent with common knowledge of rationality. That is, every action resulting from the application of the maximin utility criterion is belief-free rationalizable. Second, it provides a sufficient condition for a strategy to be belief-free rationalizable. Third, the proof also shows that an action which is a best reply to a belief-free rationalizable strategy is again belief-free rationalizable. The last statement is formalized in the following corollary.

Corollary 1. *In a game under uncertainty let $i \in \{1, \ldots, I\}$ be a player with valuation θ_i and for $j \in \{1, \ldots, I\} \backslash \{i\}$ let β_j be a belief-free rationalizable strategy for player j. Let $a_i \in A_i$ be a best reply to β_{-i} and $F_{-i} \in \Delta\Theta_{-i}$, i.e. it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}}\ U_i(\theta_i, a_i', \beta_{-i}, F_{-i}),$$

then $a_i \in BFR_i(\theta_i)$, that is, a_i is a belief-free rationalizable action for player i with valuation θ_i..

I will now provide another simple condition which is sufficient for an action to be belief-free rationalizable and therefore facilitates to derive maximin strategies. In order to do so, the following definition is needed.

Definition 9. *For a game under uncertainty a profile of strategies $(\beta_1, \ldots, \beta_n) \in \Delta S_1 \times \cdots \times \Delta_I$ together with a profile of subjective beliefs about the other players' type distributions $(F^1_{-1}, \ldots, F^I_{-I}) \in \Delta_{\Theta_{-1}} \times \cdots \times \Delta_{\Theta_{-I}}$ is called* subjective-belief equilibrium with given strategies *if every player acts optimally given her belief and the other players' strategies, i.e. for every $i \in \{1, \ldots, n\}$ it holds that*

$$\beta_i(\theta_i)(a_i) > 0 \ \Rightarrow\ a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}}\ U_i(\theta_i, a_i', \beta_{-i}, F^i_{-i}).$$

That is, in a subjective-belief equilibrium players observe each other's strategies but do not know each other's type distributions. Every player forms a subjective belief about the other players' type distributions and acts optimally given this subjective belief and the other players' strategies which are observable. An example for a subjective-belief equilibrium is a Bayes-Nash equilibrium with a common prior.

Example 4. *Let the strategy profile $(\beta_1, \ldots \beta_I)$ together with the profile of beliefs $(F_{-1}, \ldots, F_{-I})$ be a Bayes-Nash equilibrium with a common prior. Then $(\beta_1, \ldots \beta_I)$ together with $(F_{-1}, \ldots, F_{-I})$ constitutes a subjective-belief equilibrium.*

The following proposition states that a strategy which is played in a subjective-belief equilibrium is belief-free rationalizable.

Proposition 2. *In a game under uncertainty an action $a_i \in A_i$ is belief-free rationalizable for a player i with valuation θ_i if there exists a subjective-belief equilibrium with strategies $(\beta_1, \ldots, \beta_n)$ such that $\beta_i(\theta_i)(a_i) > 0$.*

Proof. I show by induction that for every $j \in \{1, \ldots, I\}$, for every $k \geq 1$ and for all $\theta_j \in \Theta_j$ it holds that

$$\beta_j(\theta_j)(a_j) > 0 \Rightarrow a_j \in BFR_j^k(\theta_j).$$

Then it follows that $a_j \in BFR_j(\theta_j)$ and one can conclude $a_i \in BFR_i(\theta_i)$ because $\beta_i(\theta_i)(a_i) > 0$. It holds for all $j \in \{1, \ldots, I\}$ that

$$\beta_j(\theta_j)(a_j) > 0 \Rightarrow a_j \in BFR_j^1(\theta_j) \text{ for all } \theta_j \in \Theta_j$$

since $BFR_j^1(\theta_j) = A_j$ by definition. Assume it is already shown for $k \in \mathbb{N}$ that for all $j \in \{1, \ldots, I\}$ it holds that

$$\beta_j(\theta_j)(a_i) > 0 \Rightarrow a_j \in BFR_{\theta_j}^k \text{ for all } \theta_j \in \Theta_j.$$

Let j be some player with type θ_j and subjective belief $F_{-j}^j = (F_1^j, \ldots, F_{j-1}^j, F_{j+1}^j, \ldots, F_I^j)$. Then F_{-j}^j and β_{-j} fulfill the properties

(i) $\beta_l(\theta_l)(a_l) > 0 \Rightarrow a_l \in {BFR_l}^k(\theta_l)$ for all $l \neq j$
(ii) $\beta_j(\theta_j)(a_j) > 0 \Rightarrow a_j \in \underset{a_j' \in A_j}{\operatorname{argmax}} \; U_j(\theta_j, a_j', \beta_{-j}, F_{-j}^j)$.

The first property follows from the induction hypothesis and the second property follows form the definition of a subjective-belief equilibrium with given strategies. By definition of a belief-free rationalizable action, it follows that $\beta_j(\theta_j) \in BFR_j^{k+1}$. Hence, it is shown that $\beta_j(\theta_j)(a_j) > 0 \Rightarrow a_j \in BFR_j(\theta_j)$. □

As will be seen in the applications, for the derivation of maximin strategies it is mostly sufficient to consider the class of belief-free rationalizable strategies which occur in subjective-belief equilibria, in particular Bayes-Nash equilibria. Therefore, the following corollary will be useful in the proofs concerning the applications. It states that every strategy played in a Nash equilibrium is belief-free rationalizable and that best replies to Bayes-Nash equilibria are belief-free rationalizable,

Corollary 2. *Let $(\beta_1, \ldots, \beta_I)$ together with the common prior $(F_1, \ldots, F_I)$ constitute a Bayes-Nash equilibrium of a game of incomplete information. Then the following holds true:*

(i) For every $i \in \{1, \ldots, I\}$ the strategy β_i is belief-free rationalizable.

(ii) Let $i \in \{1, \dots I\}$ be a player with valuation θ_i and let $a_i \in A_i$ be a best reply to β_{-i} and some distribution of the other players' types $F'_{-i} \in \Delta_{\Theta_{-i}}$, i.e. it holds that

$$a_i \in \underset{a'_i \in A_i}{\operatorname{argmax}} U_i(\theta_i, a'_i, \beta_{-i}, F'_{-i}),$$

then $a_i \in BFR_i(\theta_i)$, that is, a_i is a belief-free rationalizable action for player i with valuation θ_i.

Proof. As stated in Example 4, every Bayes-Nash equilibrium is a subjective-belief equilibrium. It follows from Proposition 2 that every strategy played in a Bayes-Nash equilibrium is belief-free rationalizable. Hence, every strategy played in a Bayes-Nash equilibrium is belief-free rationalizable which proves the first part. Corollary 1 states that best replies to belief-free rationalizable strategies are belief-free rationalizable. Therefore, a best reply to a strategy which is played in a Bayes-Nash equilibrium is belief-free rationalizable which shows the second part. ☐

4. Applications To First-Price Auctions: Setup

The following sections investigate maximin strategies and outcomes under maximin strategies in first-price auctions under distributional and strategic uncertainty. This section specifies the general model in the setting of first-price auctions and provides an overview over the results in subsection 4.2.

4.1. Model.

Underlying game of incomplete information. As in the general model, the model description starts with the specification of the underlying game of incomplete information.

There are I risk-neutral bidders competing in a first-price sealed-bid auction for one indivisible good. Before the auction starts, each bidder i privately observes her valuation (type) $\theta_i \in \Theta = \{0 = \theta^1, \theta^2, \dots, \theta^{m-1}, 1 = \theta^m\}$ with $i \in \{1, ..., I\}$. A *(mixed) strategy* β_i of a bidder i maps the valuation (type) of a bidder to a distribution of bids:

$$\beta_i : \Theta \to \Delta\mathscr{B}$$

$$\theta_i \mapsto \beta_i(\theta_i)$$

where $\mathscr{B}$ is a finite (arbitrarily fine) grid of bids on $[0,1]$ with $\Theta \subseteq \mathscr{B}$ and $\Delta\mathscr{B}$ is the set of all probability distributions over $\mathscr{B}$. [6] For every $0 < b < 1$ there exists a predecessor in $\mathscr{B}$ denoted by

$$b^- = \max_{b' \in \mathscr{B}} b' < b$$

and a successor in $\mathscr{B}$ denoted by

$$b^+ = \min_{b' \in \mathscr{B}} b' > b.$$

I assume that the bid grid is sufficiently fine, that is, if a bidder i believes that some bidder j with valuation $\theta^k \in \Theta$ bids b with strictly positive probability and there is strictly positive probability weight on type θ^k of bidder j, then b^+ induces a higher expected utility for bidder i than bid b. [7]

A *pure strategy* for bidder i with valuation θ_i is a probability distribution which puts probability weight 1 on one bid. In the auction the bidders submit bids, the bidder with the highest bid wins the object and pays her bid. In addition, it holds an efficient tie-breaking rule. [8] Thus, the utility of bidder i with valuation θ_i and bid b_i given that the other bids are b_{-i} is denoted by

$$u_i(\theta_i, b_i, b_{-i}) = \begin{cases} \theta_i - b_i & \text{if } b_i > \max\limits_{j \neq i} b_j \\ \theta_i - b_i & \text{if } b_i = \max\limits_{j \neq i} b_i \text{ and } \theta_i > \max\limits_{j \neq i} \{\theta_j \mid b_j = b_i\} \\ \frac{1}{k}(\theta_i - b_i) & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i = \theta_{j_1} = \cdots = \theta_{j_{k-1}} = \max\limits_{j \neq i} \theta_j \\ 0 & \text{if } b_i < \max\limits_{j \neq i} b_j \end{cases}$$

where θ_j denotes the valuation of bidder j with bid b_j for $j \in \{1, ..., n\}$ and $k := \#\{max\{\theta_j \mid \beta_j(\theta_j) = \max\limits_{l \neq j} \beta_l(\theta_l)\}\}$.

[6] A finite grid is used for the set of all possible bids instead of the interval $[0,1]$ because of the following reason. Consider two bidders 1 and 2 with the same valuation θ. If bidder 1 bids some amount $b < \theta$, one has to identify the smallest bid which is strictly higher than b since this would be the unique best reply of bidder 2. This allows a more formal analysis than using expressions like "bidding an arbitrarily small amount more than b". The grid is assumed to be finite in order to ensure that any subset of the bid grid is compact. Since the grid can be arbitrarily fine, I assume for simplicity that $\Theta \subseteq B$.

[7] One can always find a sufficiently fine bid grid. Assume it is required that $b^+ < b + \epsilon$ for an $\epsilon > 0$. Then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < n$. A suitable bid grid is given by $\{\frac{k}{n} \ : \ k \in \{1, \dots, n\}\}$.

[8] The core statements in the results do not depend on the choice of the tie-breaking rule.

It is assumed that all components of the underlying game of incomplete information are common knowledge among all bidders.

Possible distributions and strategies. As in the general model, the action space of player n depends on the possible distributions and strategies the players applying the maximin expected utility criterion consider to be possible. For the possible distributions I will consider the following two assumptions:

(D1) The bidders' value distributions are common knowledge, i.e. for every $i \in \{1, \dots, I\}$ it holds that $\Delta_{\Theta_{-i}}$ is a singleton.

(D2) It is common knowledge that every bidder's value distribution has range $[0, 1]$ and an exogenously given mean μ. Formally, let

$$\mathscr{F}_{\mu}^{n-1} = \left\{ F_1 \times \cdots \times F_{n-1} \in \Delta^{n-1}(\Theta) \mid \sum_{i=1}^{m} \theta_i Pr(\theta^i) = \mu \right\},$$

the set of all distributions of independently drawn values for $n-1$ bidders with mean μ. Then it holds for every $i \in \{1, \dots, I\}$ that

$$\Delta_{\Theta_{-i}} = \mathscr{F}_{\mu}^{n-1}.$$

The first assumption is a standard assumption in auction literature, the second assumption evolves from recent auction literature. Since Wilson (1987) the assumption that in an auction bidders know each other's value distribution has been dropped in recent economic literature. However, it is possible that bidders exert effort in order to obtain information about the other bidders' values and in the end learn the range and the mean of the other bidders' value distribution, as assumed for example in Carrasco, Luz, Kos, Messner, Monteiro, and Moreira (2017). Moreover, without any restriction of the possible distributions, the adverse nature would always choose the distribution which puts all the probability weight on the highest possible type 1. Hence, the maximin expected utility criterion would not be a useful decision criterion since every bidder with a type lower than 1 would never expect to win the auction.

I will consider two possible sets of possible strategies which are defined by the following two assumptions:

(S1) The bidders' strategies are common knowledge, i.e. for every $i \in \{1, \ldots, I\}$ it holds that $\Delta_{S_{-i}}$ is a singleton.

(S2) It is common knowledge that all bidders are rational, that is, they play belief-free rationalizable strategies. Formally, it holds for every $i \in \{1, \ldots, I\}$ that

$$\Delta S_{-i}^{BFR} = BFR_{-i}.$$

Again, the first assumption is a standard assumption while the second assumption reflects the case where bidders have no knowledge about each other's strategies besides the fact that they are rational.

The rest of this section analyzes the outcomes in a first-price auction under the different combination of the assumptions (D1), (D2), (S1) and (S2). The outcomes under assumptions (D1) and (S1) coincide with the set of Bayes-Nash equilibria and therefore do not require any further analysis. Before the formal analysis I provide a rather informal preview of the results.

4.2. **Preview of results.**

Pure distributional uncertainty. Under pure distributional uncertainty there does not exist an outcome under maximin strategies. That is, an equilibrium in the game with I bidders and an adverse nature where all bidders apply the maximin expected utility criterion, does not exist.

Pure strategic uncertainty.

(i) Under strategic uncertainty with common knowledge of rationality and common knowledge of valuations, the bidder with the highest valuation bids the second-highest valuation and every other bidder is indifferent between any bid between zero and her valuation. If at least two bidders have the highest valuation, then every bidder is indifferent between zero and her valuation.

(ii) Under strategic uncertainty with common knowledge of rationality and common knowledge of a symmetric value distribution, an outcome under maximin strategies always exists. The bidders' strategies are equal in every outcome and every outcome is efficient.

For every type there exists a unique highest belief-free rationalizable bid. For every bidder and every type the adverse nature chooses as the strategy of the other bidders that every bidder places the highest belief-free rationalizable bid given her type. As a consequence, a bidder never expects to win against a bidder with an equal or higher type and therefore bids the highest belief-free rationalizable bid of a lower type.

Distributional and strategic uncertainty. Under strategic uncertainty with common knowledge of rationality and distributional uncertainty with common knowledge of an exogenously given mean of the value distributions, an outcome under maximin strategies always exists. If there exists types $\theta^k, \theta^{k'}, \theta^{k''} \in \Theta$ such that $0 < \theta^k \leq \mu < \theta^{k'} < \theta^{k''}$, then every outcome is inefficient.

For every type there exists a unique highest belief-free rationalizable bid. For every bidder and every type the adverse nature chooses as the strategy of the other bidders that every bidder places the highest belief-free rationalizable bid given her type.

Let θ_μ be the lowest valuation which is higher than the mean. The highest belief-free rationalizable bid of a bidder with a valuation lower than θ_μ is her valuation. The subjective maximin belief of a bidder with valuation lower than θ^μ about the other bidders' value distributions is that the probability weight is distributed between the valuations $\theta^{\mu-1}$ and θ_μ. As a consequence a bidder with a valuation lower than μ expects a utility of zero and is indifferent between any bid between zero and her valuation.

Every bidder with a valuation θ^k such that $\theta^k \geq \theta_\mu$ never expects to win against a bidder with the same valuation. Hence, the maximin belief of a bidder about the other bidders' value distribution maximizes the probability weight on θ^k and makes the bidder indifferent between any highest belief-free rationalizable bid of lower types. As a consequence, the bidder mixes among all highest belief-free rationalizable bids of lower types. Therefore, if types $\theta^k, \theta^{k'}, \theta^{k''} \in \Theta$ such that $0 < \theta^k \leq \mu < \theta^{k'} < \theta^{k''}$ exist, then with positive probability type $\theta^{k''}$ bids zero and type $\theta^{k'}$ bids the highest belief-free rationalizable bid of type θ^k which is θ^k. Conclusively, the outcome is not efficient.

4.3. **Notation and definitions.** For the formal analysis it is useful to have an overview over the notation and definitions which will be used in the remainder of this section.

- For $\theta^k \in \Theta$ let $\overline{b}^{\theta^k}$ be the highest belief-free rationalizable bid for a bidder with valuation θ^k.
- For $\theta^k, \theta^l \in \Theta$ let $x_{\theta^l}^{\theta^k}$ denote the probability that a bidder with type θ^l occurs in in the subjective maximin belief of a bidder with valuation θ^k.
- Let $G_i^{\theta^j}$ denote the bid distribution of bidder i with valuation θ^j.

Definition 10. *An* auction mechanism *is a double (x, p) of an allocation function x and a payment function p. The allocation function*

$$x : b \to (x_1, \dots, x_n) \quad \text{with } x_i \in [0, 1], \ \sum x_i \leq 1$$

determines for each participant the probability of receiving the item and the payment function

$$p : b \to (p_1, \dots, p_n) \quad \text{with } p_i \in \mathbb{R}^+$$

determines each participant's payment.

Definition 11. *A bidder i with valuation θ_i* overbids *a bidder j with valuation θ_j if for every b, b' such that $\beta_i(\theta_i)(b) > 0$ and $\beta_j(\theta_j)(b') > 0$ it holds that $b \geq b'$ if $\theta_i > \theta_j$ and $b > b'$ if $\theta_i \leq \theta_j$.*

Note that due to the efficient tie-breaking rule, a bidder who overbids every other bidder wins in any auction mechanism where the highest bid wins.

In order to evaluate outcomes in terms of social surplus and revenue, I introduce the following definitions.

Definition 12. *Let $(\beta_1, \dots, \beta_I)$ be an outcome under maximin strategies of an auction mechanism. Let $(b_1, \dots, b_I)$ be a vector of bids which is played with positive probability in this outcome, that is, for all $b_j \in \{b_1, \dots, b_I\}$ there exists a valuation $\theta_j \in \Theta$ such that $\beta_j(\theta_j)(b_j) > 0$. The outcome $(\beta_1, \dots, \beta_I)$ is* efficient *if for all such bid vectors $(b_1, \dots, b_I)$ which are played with positive probability in this outcome it holds that*

$$x_i(b_1, \dots, b_I) > 0 \ \Leftrightarrow \theta_i = \max_{j \neq i} \theta_j.$$

That is, the good is allocated with probability one to a group of bidders who have the highest valuation.

Definition 13. *Let* (x^1, p^1) *and* (x^2, p^2) *be two auction mechanisms. Let bidders' valuations be distributed on an interval* $[0, \overline{v}]$ *according to distribution functions* $F_1, \ldots, F_I$*. The auction mechanism* (x^1, p^1) dominates *the auction mechanism* (x^2, p^2) in terms of expected revenue *if for every Bayes-Nash equilibrium* $(\beta_1^1, \ldots, \beta_I^1)$ *of* (x^1, p^1) *and every Bayes-Nash equilibrium* $(\beta_1^2, \ldots, \beta_I^2)$ *of* (x^2, p^2) *it holds that*

$$\sum_{i=1}^{I} \int_{\theta \in [0,\overline{v}]^I} \int_{b=(b_1,\ldots,b_I)} p_i^1(b_1, \ldots, b_I) \prod_{j=1}^{I} \beta_j^1(\theta_j)(b_j) db \; dF_1(\theta_1) \cdots dF_I(\theta_I) d\theta$$

$$> \sum_{i=1}^{I} \int_{\theta \in [0,\overline{v}]^I} \int_{b=(b_1,\ldots,b_I)} p_i^2(b_1, \ldots, b_I) \prod_{j=1}^{I} \beta_j^2(\theta_j)(b_j) db \; dF_1(\theta_1) \cdots dF_I(\theta_I) d\theta$$

i.e. the expected revenue from auction (x^1, p^1) *exceeds the expected revenue from auction* (x^2, p^2)*.*

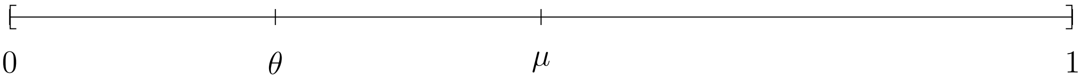
5. Outcomes under pure distributional uncertainty

I begin with the setting under pure distributional uncertainty, i.e. assumptions (D2) and (S1) hold. In this case an outcome under maximin strategies, i.e. an equilibrium in the game with I bidders and an adverse nature where all bidders apply the maximin expected utility criterion, does not exist if there are at least three different valuations with at least one valuation above the mean.

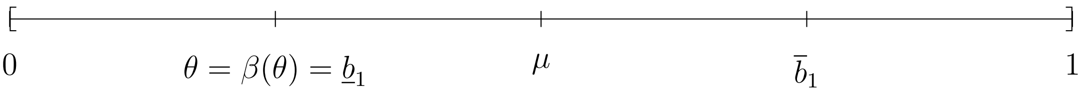
Proposition 3. *Consider a first-price auction under uncertainty where assumptions (D2) and (S1) hold. Moreover, it holds that* $m \geq 3$ *and there exists at least one valuation* $\theta > \mu$*. Then there does not exist an outcome under maximin strategies.*

Proof. The formal proof is relegated to the appendix. □

If there are only two valuations, then the constraints, that probabilities have to add up to one and the mean has to be preserved, already determine the value distribution. Hence, the adverse nature has only one possible action and an equilibrium exists. To give an intuition why an equilibrium does not exist if there are at least three different valuations with at least one valuation $\theta > \mu$, consider the case with two bidders 1 and 2 and three possible valuations 0, θ and 1 with mean $\mu > \theta$.



In this case an outcome does not exist independent of the particular choice of θ and μ. Since the existence of inefficient or asymmetric equilibria can be ruled out similarly to the existence of symmetric efficient equilibria, assume that if an equilibrium exists, it has to be symmetric and efficient. Then for every possible type $\theta_k \in \{0, \theta, 1\}$ there exists a bidding interval $[\underline{b}_{\theta_k}, \overline{b}_{\theta_k}]$. Since a bidder with valuation 0 always bids zero, it holds that $\underline{b}_0 = \overline{b}_0 = 0$. The value distribution which minimizes the expected utility of a bidder with valuation θ puts zero probability weight on the 0-type. Then the unique equilibrium equilibrium strategy for the θ-types of both bidders is to bid θ. As a consequence, a bidder with valuation θ earns an expected utility of zero and therefore indeed the distribution which puts zero probability weight on the 0-type, is player n's strategy for the θ-types. Hence, it holds that $\underline{b}_\theta = \overline{b}_\theta = \theta$. Since gaps between the bidding intervals are not possible, it holds for the lower endpoint of the 1-type's bidding interval that $\overline{b}_1 = \underline{b}_\theta = \theta$.



Given the strategy of the 1-type, the value distribution which the adverse nature chooses for the 1-type minimizes the sum of the probability weights of the 0- and the θ-type. Since the mean μ has to be preserved, this results in a distribution which puts zero probability weight on the θ-type. But then both 1-types have an incentive to deviate to bidding zero. Hence, a symmetric efficient equilibrium does not exist. [9]

6. Outcomes under pure strategic uncertainty

Now I consider the setting under pure strategic uncertainty, i.e. assumptions (D1) and (S2) hold. I start with the special case where the value distributions put all probability weight on one type, or in other words, the bidders now each other's valuations.

[9]From the non-existence of an equilibrium in the game against the adverse nature it follows that equilibrium existence criteria as in Reny (1999) do not apply. It may help to gain some intuition to understand why the game is not better-reply secure. The reason is that there exists a discontinuity in the utility function of the adverse nature. Consider the case where both θ-types believe that there is no θ-type and bid zero. If one θ-type bids an arbitrarily small positive amount, she wins against the zero- and the θ-type which reduces the adverse nature's utility by as strictly positive amount.

6.1. **Common knowledge of valuations and strategic uncertainty.**

Proposition 4. *Consider a first-price auction under uncertainty where every bidder's valuation is common knowledge and (S2) holds. Then the following holds true for an outcome under maximin strategies:*

(i) If $\theta_k > \max_{j \neq i} \theta_j$, i.e. there exists a unique bidder k with the highest valuation, then bidder k bids $\theta_{k'} = \max_{\theta_j \in \Theta} \theta_j < \theta_k$, i.e. the bidder with the highest valuation bids the second-highest valuation and every bidder $i \neq k$ is indifferent between any bid between zero and her valuation.

(ii) If it holds that $\theta_k = \theta_l = \max_{j \in \{1,\ldots,I\}} \theta_j$, i.e. there exist at least two bidders k and l with the highest valuation, then every bidder is indifferent between any bid between zero and her valuation.

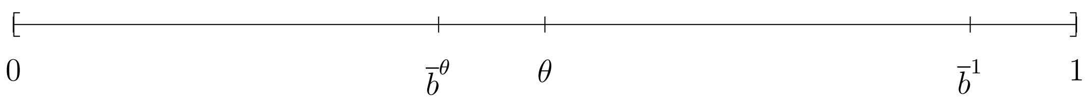
Proof. The formal proof is relegated to the appendix. □

The intuition behind part (i) is that one can show that the second-highest valuation $\theta_{k'}$ is the highest belief-free rationalizable bid for bidder k with the highest valuation θ_k. If the adverse nature chooses for all other bidders the subjective maximin belief that bidder k bids $\theta_{k'}$, this induces a payoff of zero for any other bidder. Hence, any strategy of the adverse nature has to induce an expected utility of at most zero for all bidders besides k. That is, the subjective belief of a bidder $i \neq k$ with valuation θ_i is that at least one bidder bids an amount which is equal or greater than θ_i. As a consequence, all bidders are indifferent between zero and their valuation. The adverse nature chooses the belief for bidder k such that at least one bidder bids the second-highest valuation $\theta_{k'}$. Hence, it is a best reply for bidder k to bid $\theta_{k'}$. Similar arguments apply to part (ii).

Note that while the unique Nash equilibrium in this setting is belief-free rationalizable, there are much more belief-free rationalizable actions than the Nash equilibrium. In particular, in the case of two bidders who have the same valuation v all actions in the interval $[0, v]$ are belief-free rationalizable. This leaves room for more outcomes than the unique Nash-equilibrium which is weakly dominated.

6.2. **Known distribution and strategic uncertainty.** Now I consider the case where not the bidders' valuations but the distribution of the valuations is common knowledge. In this case for every type there exists a unique highest belief-free rationalizable bid. For every bidder and every type it is a weakly dominant strategy for the adverse nature to choose a strategy of the other bidders such that every other bidder will bid the highest belief-free rationalizable bid given her type. As a consequence, it is never a best reply for a bidder to overbid bidders with the same type. Hence, every bidder overbids only lower types and it depends on the commonly known value distribution which types are overbid. Since the strategy chosen by the adverse nature is the same for every bidder and every type, this results in an efficient outcome. This is illustrated by the following example.

Example 5. *Consider a first-price auction under strategic uncertainty (i.e. assumption (S2) holds) with two bidders 1 and 2 and three possible valuations* $0, \theta$ *and* 1 *which are identically and independently distributed according to a commonly known distribution function* $F \in \Delta\{0, \theta, 1\}$*. For every type* $\theta^k \in \{0, \theta, 1\}$ *there exists a highest belief-free rationalizable bid* $\overline{b}^{\theta^k}$*. For every bidder and every type the adverse nature chooses a strategy of the other bidder such that every other bidder bids the highest belief-free rationalizable bid. That is, every bidder with every type has the subjective belief that the 0-type bids zero, the* θ*-type bids* $\overline{b}^{\theta}$ *and the 1-type bids* $\overline{b}^{1}$*.*



Hence, bidder 1 with type θ *never expects to win against bidder 2 with type* θ *(and vice versa) and therefore bids 0. Bidder 1 with type 1 never expects to win against bidder 2 with type 1 and has to decide between bidding* 0 *and bidding* $\overline{b}^{\theta}$ *(and vice versa). In any case the outcome is efficient.*

The insights from this example are formalized in the following proposition.

Proposition 5. *In a first-price auction under uncertainty such that (S2) holds and where all bidders' valuations are distributed independently and identically according to a commonly known distribution function, there exists an outcome under maximin strategies. Every outcome is efficient.*

Proof. The formal proof is relegated to the appendix.

□

I will show the existence of an efficient outcome by construction for which I proceed in the steps listed below. Then I will show that every strategy of the adverse nature in an outcome under maximin strategies induces the same bidding strategies and therefore every outcome has to be efficient.

(I) Show that for every type $\theta^k \in \Theta$ there exists a unique highest belief-free rationalizable bid $\overline{b}^{\theta^k}$.
(II) Show that for every type zero is a belief-free rationalizable bid.
(III) Show that for every type $\theta^k \in \Theta$ every bid in the interval $[0, \overline{b}^{\theta^k}]$ is belief-free rationalizable.
(IV) Calculate for every type $\theta^k \in \Theta$ the highest belief-free rationalizable bid $\overline{b}^{\theta^k}$.

The first step follows from the fact that $\mathscr{B}$ is compact and well-ordered with respect to $\leq$. For a proof sketch of step (II) consider a proof by induction with respect to the valuations in Θ. Assume it has been shown that for every bidder with a type θ^j such that $j < k+1$ bidding zero is a belief-free rationalizable action. Assume that a bidder with valuation θ^{k+1} believes that all lower types bid zero. Due to step (I), for every type there exists a highest belief-free rationalizable bid. Assume further, that the bidder with valuation θ^{k+1} believes that all higher types bid their highest belief-free rationalizable bid, then it is a best reply of this bidder to bid zero. As stated in Corollary 1, a best reply to a belief-free rationalizable strategy profile is belief-free rationalizable and therefore zero is a belief-free rationalizable action for a bidder with valuation θ^{k+1}.

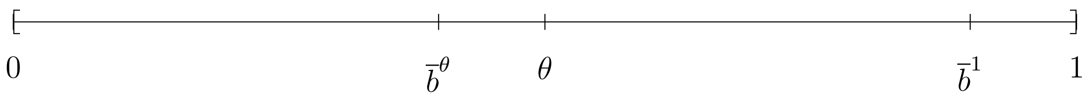
For an intuition of step (III) consider the bid 0^+. Since bidding zero is a belief-free rationalizable action for every bidder and every type, it is straight-forward that for a sufficiently fine bid grid bidding 0^+ is a belief-free rationalizable action for every bidder and every type besides zero. Because if a bidder believes that all bidders bid zero, than she could win the auction with probability 1 by bidding 0^+. The same holds for $(0^+)^+$ and so on. This process reaches some bid b such that for type θ^2 it is more profitable to bid zero and win against the zero-type than to bid b^+ even if all other bidders with a type higher than zero bid b. Then b is the highest belief-free rationalizable bid for type θ^2 and all bids in the interval $[0, b]$ are belief-free rationalizable for a bidder with valuation θ^2. The analogue reasoning applies to every higher type. Since the bids in $\mathscr{B}$ are well-ordered with respect to $\leq$, one can show the result by double induction with respect to the types and the bids.

Given step (III), one can calculate the highest belief-free rationalizable bid for every type. The highest belief-free rationalizable bid $\overline{b}^{\theta^k}$ for some bidder i with valuation θ^k is induced by the belief about the other bidders' strategies such that

(i) All bidders with a lower type bid their highest belief-free rationalizable bid.
(ii) All bidders with an equal or higher type bid $\left(\overline{b}^{\theta^k}\right)^-$.

This is, $\overline{b}^{\theta^k}$ is a best reply to the belief which maximizes the expected utility of bidding $\overline{b}^{\theta^k}$. The strategies in (i) are belief-free rationalizable by definition and it follows from step (III) that the strategies described in (ii) are belief-free rationalizable. Hence, the highest belief-free rationalizable bid $\overline{b}^{\theta^k}$ for type θ^k makes this type indifferent between winning with probability 1 by bidding $\overline{b}^{\theta^k}$ and the most profitable overbidding of a lower type given that all lower types bid their highest belief-free rationalizable bid. The following example continues with Example 5 and illustrates the steps above.

Example 6. *Consider again the case with two bidders 1 and 2 and three possible valuations 0,θ and 1 which are identically and independently distributed according to a commonly known distribution function $F \in \Delta\{0, \theta, 1\}$.*



The highest belief-free rationalizable bid for type zero is zero. The highest belief-free rationalizable bid for type θ is given by the bid $\overline{b}^{\theta}$ which makes her indifferent between winning with probability 1 by bidding $\overline{b}^{\theta}$ and just overbidding type zero:

$$\theta - \overline{b}^{\theta} = F(0)\,(\theta - 0)$$

$$\Leftrightarrow \overline{b}^{\theta} = \theta(1 - F(0)) + F(0).$$

The highest belief-free rationalizable bid for type 1 is given by the bid $\overline{b}^{1}$ which makes her indifferent between winning with probability 1 by bidding $\overline{b}^{1}$ and the most profitable overbidding of a lower type. That is, type 1 has to be indifferent between bidding $\overline{b}^{1}$ and *the maximum utility of bidding either* $0 = \overline{b}^0$ *or* $\overline{b}^\theta$*:*

$$1 - \overline{b}^1 = \max \left\{ F(0)\,(1-0)\,, F(\theta)\left(1 - \overline{b}^\theta\right) \right\}.$$

For a numerical example consider the parameters $\theta = \frac{1}{2}$, $F(0) = \frac{1}{3}$, $F(\theta) = \frac{2}{3}$ *and* $F(1) = 1$. *Then it holds that*

$$\overline{b}^\theta = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{3}$$

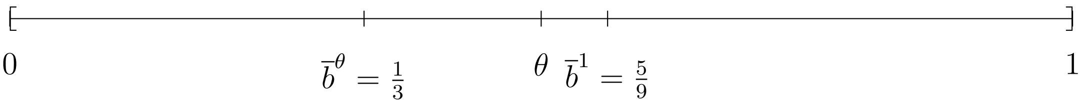
and

$$\max \left\{ F(0), F(\theta)\left(1 - \overline{b}^\theta\right) \right\} = \max \left\{ \frac{1}{3}, \frac{2}{3}\left(1 - \frac{1}{3}\right) \right\} = \frac{4}{9}$$

from which follows that

$$\overline{b}^1 = 1 - \frac{4}{9} = \frac{5}{9}$$

which is illustrated below:



After computing the highest belief-free rationalizable bids one can compute bidding strategies in an outcome under maximin strategies. Type zero bid zero. Since type θ *of bidder 1 believes that type* θ *of bidder 2 bids* $\overline{b}^\theta$, *type* θ *of bidder 1 bids 0 (and analogously for type* θ *of bidder 2). Type 1 of bidder 1 does not expect to win against type 1 of bidder 2 and therefore has to decide whether to overbid type 0 or type* θ *of bidder 2. In any case the outcome is efficient.* [10] *Bidding zero gives an expected utility of*

$$F(0) = \frac{1}{3}$$

and bidding $\overline{b}^\theta = \frac{1}{3}$ *gives an expected utility of*

$$F(\theta)(1 - \overline{b}^\theta) = \frac{2}{3}\left(1 - \frac{1}{3}\right) = \frac{4}{9}.$$

Hence, type 1 of bidder 1 will bid $\overline{b}^\theta$ *(and analogously for type* θ *of bidder 2).*

[10] Due to the efficient tie-breaking rule the outcome is efficient even if different types submit equal bids. However, efficiency does not depend on thy choice of the tie-breaking rule. If the tie-breaking rule would be to allocate the good randomly among the bidders with the highest bid, then type 1 would just decide between the bide 0^+ and $\left(\overline{b}^\theta\right)^+$.

Applying the same procedure, one can compute the highest belief-free rationalizable bids for every number of types and every choice of parameters and then compute the bids under maximin strategies. The following two graphs show the highest belief-free rationalizable bids for m equidistant types with a uniform distribution for $m = 10$ and $m = 20$.

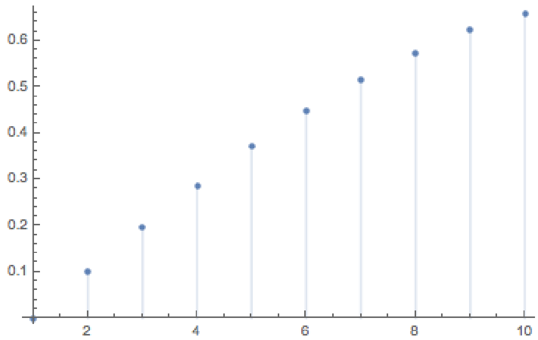


FIGURE 1. Highest belief-free rationalizable bids for $m = 10$

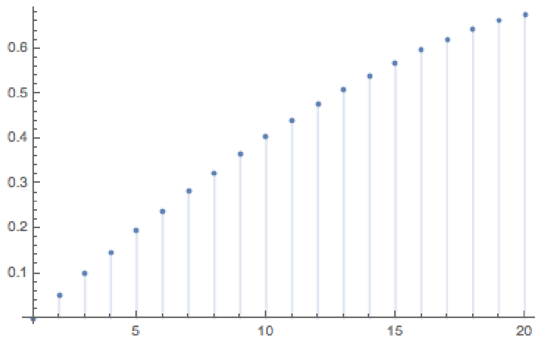


FIGURE 2. Highest belief-free rationalizable bids for $m = 20$

The following two graphs show the bids in an outcome under maximin strategies for m equidistant types with a uniform distribution for $m = 10$ and $m = 20$.

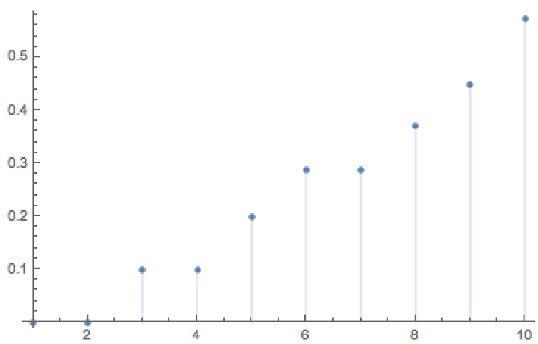


FIGURE 3. Bids in an outcome under maximin strategies for $m = 10$

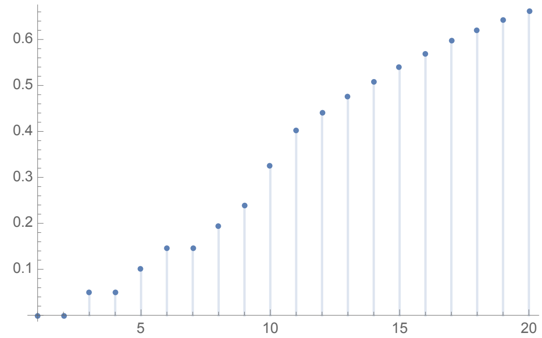


FIGURE 4. Bids in an outcome under maximin strategies for $m = 20$

Figures 3 and 4 show that the outcome under maximin strategies is efficient since the bidder with the highest valuation wins the auction with probability 1. However, it is possible that different types submit equal bids. Whenever a bidder is not indifferent between two bids, her bidding strategy is unique which is the case in figures 3 and 4.

The strategy of the adverse nature is not necessarily unique. Assume that it is a best reply of a bidder i with valuation $\theta^k \in \Theta$ to bid $\overline{b}^{\theta^l}$ for $l < k$. Then in the belief of bidder i with valuation θ^k the adverse nature could decrease the bid of some bidder with type $\theta^{l'}$ for $l \neq l'$ without changing the best reply of bidder i and hence without changing her expected utility. Since all possible strategies of the adverse nature induce the same bidding strategies, the non-uniqueness of the adverse nature's strategy does not affect efficiency.

The recursive computation of highest belief-free rationalizable bids for all types as described in step (IV) and the example, is formalized in the following proposition.

Proposition 6. *In a first-price auction under uncertainty such that (S2) holds and where all bidders' valuations are distributed independently and identically according to a commonly known distribution function the highest belief-free rationalizable bids are obtained as follows. The highest belief-free rationalizable bid for type zero is zero. Assume that for every type θ^j with $j < k$ the highest belief-free rationalizable bid $\overline{b}^{\theta^j}$ has been already computed. Then the highest belief-free rationalizable bid is determined by the equality*

$$\theta^k - \overline{b}^{\theta^k} = \max_{\theta^j < \theta^k} F^{I-1}\left(\theta^j\right)\left(\theta^k - \overline{b}^{\theta^j}\right). \tag{2}$$

Proof. The formal proof is relegated to the appendix. □

Proposition 6 states that the highest belief-free rationalizable bid of a bidder with valuation θ^k makes this bidder indifferent between winning the auction with probability 1 by bidding $\overline{b}^{\theta^k}$ and the most profitable overbidding of some lower type given that all lower types bid their highest belief-free rationalizable bid.

The following proposition provides the strategies of the adverse nature and the bidders.

Proposition 7. *In a first-price auction under uncertainty such that (S2) holds and where all bidders' valuations are distributed independently and identically according to a commonly known distribution function the following holds true:*

(i) For every bidder and every type the adverse nature chooses as the other bidders' strategies that every bidder places the highest belief-free rationalizable bid given her type.

(ii) A bidder with valuation θ^k bids

$$\max_{\theta^j<\theta^k} F^{I-1}\left(\theta^j\right)\left(\theta^k-\overline{b}^{\theta^j}\right).$$

This proposition states that every bidder chooses the most profitable overbidding of a lower type. Moreover, it shows that a bidder with a given type does not need to know higher types but only lower types. This stems from the fact that a bidder with a given type does not expect to win against bidders with the same or a higher type.

Similarly as in the case where bidders' valuations are known, the concept of belief-free rationalizable actions allows for more actions than the Bayes-Nash equilibrium as formalized in the following proposition.

Proposition 8. *Consider a first-price auction under uncertainty such that (S2) holds and where all bidders' valuations are distributed independently and identically according to a commonly known distribution function. Let $\overline{b}_*^{\theta^k}$ be the highest bid which is played with positive probability by a bidder with type θ^k in a Bayes-Nash equilibrium. Then for all $k \neq 1$ it holds that $\overline{b}_*^{\theta^k} < \overline{b}^{\theta^k}$.*

Proof. The formal proof is relegated to the appendix. □

Proposition 6 provides the explanation for this result. Since the Bayes-Nash equilibrium is efficient, the highest bid in the Bayes-Nash equilibrium is induced if a bidder overbids all bidders with an equal or lower type. In contrast, the highest belief-free rationalizable bid is induced if a bidder overbids all other bidders.

7. Distributional and strategic uncertainty

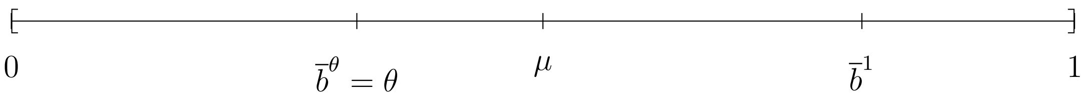
Finally, I analyze outcomes under distributional and strategic uncertainty. That is, it is common knowledge that every bidder's value distribution has range $[0, 1]$ and an exogenously given mean μ and it is common knowledge that bidders are rational (assumptions (D2) and (S2)). As in the case of pure distributional uncertainty, the subjective maximin belief of every bidder is that all other bidders place the highest belief-free rationalizable bid given their valuation. In addition to the subjective maximin belief about the other bidders' strategies, every bidder forms a subjective maximin belief about the other bidders' value distributions. The derivation of subjective maximin beliefs and bidding strategies is illustrated in the following example.

Example 7. *Consider a first-price auction under distributional and strategic uncertainty (i.e. assumptions (D2) and (S2) hold) with two bidders 1 and 2 and three possible valuations* $0, \theta$ *and* 1 *which are identically and independently distributed with a commonly known mean* μ*. Assume that it holds* $\theta < \mu$*. The first step is to calculate the highest belief-free rationalizable bid for every valuation.*

The highest belief-free rationalizable bid for a bidder with valuation zero is zero. Assume that bidder 1 and bidder 2 have the subjective belief that the other bidder's value distribution distributes the probability weight between types θ *and 1, i.e. there is zero probability weight on type 0. Given that bidder 1 and bidder 2 have this subjective belief, the following strategies constitute a Bayes-Nash equilibrium:*

- *(i) Type* θ *of bidder 1 and bidder 2 bids* θ
- *(ii) Type 1 of bidder 1 and bidder 2 plays a mixed strategy on the interval* $[\theta, b^1]$ *for* $\theta < b^1 < 1$*.*

Thus, it is part of a subjective-belief equilibrium that a bidder with valuation θ *bids* θ*. It follows from Proposition 2 that bidding* θ *is a belief-free rationalizable action for a bidder* *with valuation θ. Since bidding above valuation cannot be belief-free rationalizable, the highest belief-free rationalizable action for a bidder with valuation θ is to bid θ.* [11]



Let $\overline{b}^1$ denote the highest belief-free rationalizable bid of a bidder with valuation 1. In order to compute $\overline{b}^1$, consider the subjective belief of bidder with valuation 1 that the strategy of the other bidder is such that

(iii) *Type zero bids zero,*
(iv) *Type θ bids θ,*
(vi) *Type 1 bids $\left(\overline{b}^1\right)^-$.*

It has been already shown that (iv) is belief-free rationalizable and similarly as before, one can show that (vi) is belief-free rationalizable. It follows from Corollary 1 that a best reply to the strategy described in $(iii) - (vi)$ is belief-free rationalizable. Thus, this is the belief-free rationalizable strategy which maximizes the expected utility of bidding $\overline{b}^1$ and therefore induces the highest belief-free rationalizable bid of a bidder with valuation 1, i.e. bidding $\overline{b}^1$ is a best reply to this strategy.

A belief-free rationalizable bid is a best reply to a strategy of the other bidders and to a distribution of their valuations. Hence, in a addition to the strategy inducing $\overline{b}^1$, one has to derive the value distribution inducing $\overline{b}^1$. Let $(x_0^1, x_\theta^1, x_1^1)$ denote the probability weights in this distribution. It must hold that

$$1 - \overline{b}^1 \geq x_1^0$$

$$1 - \overline{b}^1 \geq \left(x_1^0 + x_1^\theta\right)(1-\theta)$$

which is equivalent to

$$1 - \overline{b}^1 \geq \max\left\{x_1^0, \left(x_1^0 + x_1^\theta\right)(1-\theta)\right\}.$$

[11]It turns out that the subjective belief in (i) and (ii) coincides with the subjective maximin belief a bidder with valuation θ. However, belief-free rationalizable strategies can be induced by a subjective-belief equilibrium where the subjective belief does not coincide with the subjective maximin belief.

Since $\overline{b}^1$ is the highest bid for which this condition is fulfilled, it holds that

$$\overline{b}^1 = 1 - \min\max\left\{x_1^0, \left(x_1^0 + x_1^\theta\right)(1-\theta)\right\}$$

which is equivalent to

$$\overline{b}^1 = 1 - x_1^0 = 1 - \left(x_1^0 + x_1^\theta\right)(1-\theta)\,. \tag{3}$$

Since probabilities have to add up to zero and the mean has to be preserved, the vector $(x_0^1, x_\theta^1, x_1^1)$ is the unique solution to the following system of linear equations

$$x_0^1 + x_\theta^1 + x_1^1 = 1$$

$$x_0^1 0 + x_\theta^1 \theta + x_1^1 1 = \mu$$

$$x_1^0 = \left(x_1^0 + x_1^\theta\right)(1-\theta).$$

After obtaining the solution

$$x_0^1 = \frac{1-\mu}{1+\theta}, \quad x_\theta^1 = \frac{\theta(1-\mu)}{1-\theta^2}, \quad x_1^1 = \frac{\mu-\theta^2}{1-\theta^2},$$

one can compute $\overline{b}^1$ using equation (3), i.e. it holds

$$\overline{b}^1 = 1 - x_1^0 = 1 - \left(x_1^0 + x_1^\theta\right)(1-\theta).$$

After deriving the highest belief-free rationalizable bids for every type, the second step is to derive the adverse nature's strategy. In the setting of strategic and distributional uncertainty the adverse nature's strategy determines for every bidder and every type a strategy and a value distribution of the other bidder. For every bidder and every type the adverse nature chooses as the strategy of the other bidder to place the highest belief-free rationalizable bid given her valuation.

The value distribution for a bidder with valuation zero is irrelevant since such a bidder always earns a utility of zero. For a bidder with valuation θ the adverse nature chooses a distribution of the other bidder's values which puts zero probability weight on type zero. Since type θ bids θ, this induces an expected utility of zero for a bidder with valuation θ. A bidder with valuation 1 never expects to win against a bidder with valuation 1. Therefore, *a bidder with valuation 1 has to decide between bidding zero and bidding θ. Hence, the adverse nature has to choose a value distribution $\left(\tilde{x}_1^0, \tilde{x}_1^\theta, \tilde{x}_1^1\right)$ such that it holds*

$$\min\max\left\{\tilde{x}_1^0, \left(\tilde{x}_1^0+\tilde{x}_1^\theta\right)(1-\theta)\right\}.$$

Since probabilities have to add up to one and the mean has to be preserved, the vector $\left(\tilde{x}_1^0, \tilde{x}_1^\theta, \tilde{x}_1^1\right)$ is the unique solution of the same system of linear equations as the vector $\left(x_1^0, x_1^\theta, x_1^1\right)$. Therefore, it holds that

$$\left(\tilde{x}_1^0, \tilde{x}_1^\theta, \tilde{x}_1^1\right) = \left(x_1^0, x_1^\theta, x_1^1\right).$$

In the final step, for every bidder and every type one has to find the set of best replies to the adverse nature's strategy. Moreover, one has to identify the best replies such that the adverse nature does not have an incentive to deviate from her strategy derived in the second step. It is a weakly dominant strategy for the adverse nature to choose for every bidder and every type the strategy of the other bidder which places the highest rationalizable bids. Hence, it is sufficient to check whether the adverse nature has an incentive to deviate from the chosen distributions.

A bidder with valuation zero bids zero. A bidder with valuation θ expects a utility of zero and is indifferent between any bid in the interval $[0,\theta]$. Hence, the adverse nature does not have an incentive to deviate. A bidder with valuation 1 is indifferent between bidding 0 and θ. In an equilibrium in the game against the adverse nature, a bidder with valuation 1 mixes between 0 and θ in a way such that the adverse nature is indifferent among any value distribution which fulfills the constraints that probabilities add up to one and the mean μ is preserved. Therefore, the adverse nature does not have an incentive to deviate.

Note that the distribution of the other bidder's values which the adverse nature chooses for a type is the same distribution which induces the highest belief-free rationalizable bid for this type. That is, a bidder i with a given type assumes that her opponent j has the same assumption about i's value distributions as i's assumption about j's value distributions. But bidder i assumes that j has a different belief about i's strategy than i's belief about j's strategy.

The insight from the example about bidders' strategies is generalized in the following Proposition.

Proposition 9. *Consider a first-price auction under distributional and strategic uncertainty such that assumptions (D2) and (S2) hold. There exists an outcome under maximin strategies. In every outcome the bidding strategies are characterized as follows:*

(i) Every bidder with valuation θ^k such that $\theta^k \leq \mu$ is indifferent between any bid in the interval $[0, \theta^k]$.

(ii) Every bidder with valuation θ^k such that $\theta^k \leq \mu$ mixes among the bids $\{\bar{b}^{\theta^j} \mid j < k\}$, that is, among the set of all highest belief-free rationalizable bids of lower types.

A direct implication of this proposition is the inefficiency of the outcome.

Corollary 3. *Consider a first-price auction under distributional and strategic uncertainty such that assumptions (D2) and (S2) hold. If there exist types $\theta^k, \theta^{k'}, \theta^{k''} \in \Theta$ such that $0 < \theta^k \leq \mu < \theta^{k'} < \theta^{k''}$, then the outcome is not efficient.*

The inefficiency stems from the fact that every type above μ mixes between all highest belief-free rationalizable bids of all lower types. With positive probability type $\theta^{k''}$ bids zero and type $\theta^{k'}$ bids the highest belief-free rationalizable bid of type θ^k which is θ^k. Conclusively, the outcome is not efficient.

Similarly as under pure strategic uncertainty, I will show the existence of an outcome under maximin strategies by construction for which I proceed in the following steps:

(I) Show that for every type $\theta^k \in \Theta$ there exists a unique highest belief-free rationalizable bid $\bar{b}^{\theta^k}$.

(II) Show that for every type zero is a belief-free rationalizable bid.

(III) Show that for every type $\theta^k \in \Theta$ every bid in the interval $[0, \bar{b}^{\theta^k}]$ is belief-free rationalizable.

(IV) Calculate for every type $\theta^k \in \Theta$ the other bidders' value distribution which induces the highest belief-free rationalizable bid.

The explanation for steps (I)-(III) works analogously as for steps (I)-(III) in the case of pure strategic uncertainty. For the calculation of the highest belief-free rationalizable bids, first, consider valuations equal or below μ. Analogously as in the example, on can show that the highest belief-free rationalizable bid for a bidder with valuation θ^k such that $\theta^k \leq \mu$ is θ^k. This bid is induced by the subjective belief equilibrium where the probability weight is distributed between types θ^k and θ_μ and all bidders with valuation θ^k bid θ^k, where θ_μ is the smallest valuation strictly higher than μ.

The calculation of the highest belief-free rationalizable bids for higher types works by recursion. Assume that $\theta^k \geq \theta_\mu$ and that for all $j < k$ the highest belief-free rationalizable bids has been already computed. Then $\overline{b}^{\theta^k}$ is a best reply to the other bidders' strategies such that:

(i) Every bidder with valuation θ^j such that $\theta^j < \theta^k$ bids her highest belief-free rationalizable bid.
(ii) Every bidder with valuation θ^k bids $\left(\overline{b}^{\theta^k}\right)^-$.

For a bidder i with valuation θ^k the value distribution of the other bidders which induces bidder i to bid $\overline{b}^{\theta^k}$ has to minimize the incentive to bid another bid. In addition, probabilities have to add up to zero and the mean μ has to be preserved. Let $\left(x^{\theta^k}_{\theta^1}, \ldots, x^{\theta^k}_{\theta^m}\right)$ be a vector of probabilities such that according to the value distribution inducing $\overline{b}^{\theta^k}$, type θ^l of some bidder $j \neq i$ occurs with probability $x^{\theta^k}_{\theta^l}$.

Hence, the vector $\left(x^{\theta^k}_{\theta^1}, \ldots, x^{\theta^k}_{\theta^m}\right)$ is the solution to the following minimization problem

$$\min\max\left\{\left(x^{\theta^k}_{\theta^1}\right)^{I-1}\theta^k, \left(x^{\theta^k}_{\theta^1}+x^{\theta^k}_{\theta^2}\right)^{I-1}\left(\theta^k-\overline{b}^{\theta^2}\right), ..., \left(x^{\theta^k}_{\theta^1}+\cdots+x^{\theta^k}_{\theta^{k-1}}\right)^{I-1}\left(\theta^k-\overline{b}^{\theta^{k-1}}\right)\right\}$$

$$s.t.\;\; x^{\theta^k}_{\theta^1}+\cdots+x^{\theta^k}_{\theta^m}=1$$
$$x^{\theta^k}_{\theta^1}\theta^1+\cdots+x^{\theta^k}_{\theta^m}\theta^m=\mu.$$

In the solution of this minimization problem all terms of the form

$$\left(\sum_{i=1}^{j} x^{\theta^k}_{\theta^i}\right)^{I-1}\left(\theta^k-\overline{b}^{\theta^j}\right) \quad \text{for } 1 < j < k$$

have to be equal.

The recursive calculation of the highest belief-free rationalizable bids and the distributions inducing them, is formalized in the following Proposition.

Proposition 10. *Consider a first-price auction under distributional and strategic uncertainty such that assumptions (D2) and (S2) hold. Let* $\theta_\mu = \min\{\theta^k \in \Theta \mid \theta^k > \mu\}$. *For* $\theta^k < \theta_\mu$

the vector of probability weights denoted by $x^{\theta^k} = (x^{\theta^k}_{\theta^1}, \ldots, x^{\theta^k}_{\theta^m})$*, is defined by*

$$x^{\theta^k}_{\theta^{\theta^k}} = \frac{\theta_\mu - \mu}{\theta_\mu - \theta^k}, \quad x^{\theta^k}_{\theta_\mu} = \frac{\mu - \theta^k}{\theta_\mu - \theta^k} \quad \text{and} \quad x^{\theta^k}_{\theta^j} = 0 \quad \text{for } \theta^j \neq \theta^k, \theta_\mu,$$

i.e. the probability weight is distributed between types θ^k *and* θ_μ *such that the mean* μ *is preserved. For* $\theta^k \leq \theta_\mu$ *the highest belief-free rationalizable bid* $\bar{b}^{\theta}$ *is equal to* θ^k*.*

Assume that for all $j < k$*, the highest belief-free rationalizable bid* $\bar{b}^{\theta^j}$ *has been already defined and it holds* $k > z$*. Then for the vector* $x^{\theta^k} = (x^{\theta^k}_{\theta^1}, \ldots, x^{\theta^k}_{\theta^k})$ *it holds that* $x^{\theta^k}_{\theta^j} = 0$ *for* $j > k$ *and the vector* $(x^{\theta^k}_{\theta^1}, \ldots, x^{\theta^k}_{\theta^k})$ *is the unique solution of the following system of* k *linear equations given by*

$$\sum_{i=1}^{k} x^{\theta^k}_{\theta^i} = 1$$

$$\sum_{i=1}^{k} x^{\theta^k}_{\theta^i} \theta^i = \mu$$

$$\left(x^{\theta^k}_{\theta^1}\right)^{I-1} \theta^k = \left(\sum_{i=1}^{j} x^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \bar{b}^{\theta^j}\right) \quad \text{for } 1 < j < k.$$

The highest belief-free rationalizable bid $\bar{b}^{\theta^k}$ *is obtained by the equation*

$$\bar{b}^{\theta^k} = \theta^k - \left(x^{\theta^k}_{\theta^1}\right)^{I-1} \theta^k.$$

Finally, the following Proposition specifies the adverse nature's strategy.

Proposition 11. *Consider a first-price auction under distributional and strategic uncertainty such that assumptions (D2) and (S2) hold. It holds that*

(i) For bidder i *and every type* θ^k *the adverse nature chooses as the strategy of the other bidders that every bidder places the highest belief-free rationalizable bid given her type.*

(ii) For every bidder i *and every type* θ^k *the adverse nature chooses as the distribution of the other bidders' values the value distribution defined by the vector* $x^{\theta^k} = (x^{\theta^k}_{\theta^1}, \ldots, x^{\theta^k}_{\theta^m})$ *as specified in Proposition 10.*

8. Discussion

Choice of decision criterion. The decision criterion under uncertainty used in this paper is the maximin expected utility criterion. The analogous analysis could be conducted with other criteria such as the minimax expected regret criterion.

Possible distributions and strategies. In this paper I restricted the set of possible strategies by assuming common knowledge of rationality and the set of possible distributions by assuming common knowledge of a mean. This restriction is crucial for the application of the maximin expected utility criterion. Otherwise there would exist a distribution or strategy inducing an expected utility of zero for a player independent of her action. However, there exist other possibilities to restrict the set of possible distributions and strategies. For instance, one could investigate outcomes under distributional uncertainty under the assumption that further moment conditions of the type distribution are common knowledge.

Cognitive complexity. Formally, the derivation of the set of belief-free rationalizable actions for an agent with a given type requires an infinite intersection of sets. However, the proofs use a finite number of recursion steps. In the model under strategic uncertainty and in the model under distributional uncertainty the bid a bidder with type θ^k is obtained after at most k recursion steps. One could argue that a sufficiently rational player can conduct the necessary calculations. But one could also argue that for some players these calculations may be too difficult. Therefore, similarly as in level-k models, one could define the concept of k-rationalizability. That is, a player i could know that her opponent can compute the set BFR_j^k for all players j and for $k \in \mathbb{N}$, but cannot compute the sets $BFR_j^{k'}$ for $k' > k$. Depending on the parameters, this knowledge could influence player i's maximin strategy.

Robustness. In addition to the maximin expected utility criterion, one could introduce an additional robustness criterion in the following sense: does the maximin strategy of an agent change if the adverse nature deviates from her strategy to another strategy in an ϵ-neighborhood? If there is a change, does the strategy and the resulting expected utility change continuously?

As an example, consider a first-price auction under pure strategic uncertainty with a commonly known distribution function, two bidders and three valuations 0, θ and 1. Bidder 1 with valuation 1 has the subjective maximin belief that bidder 2 with valuation 1 bids $\overline{b}^1$. Hence, bidder 1 with valuation 1 bids either θ or zero. However, all bids in the interval $[0, \overline{b}^1]$ are belief-free rationalizable for a bidder with valuation 1. Hence, (if the bid grid is sufficiently small) an ϵ-neighborhood of $\overline{b}^1$ and its intersection with the set of belief-free rationalizable actions contains bids lower than $\overline{b}^1$. If bidder 1 with valuation 1 has the subjective belief that bidder 2 with valuation 1 bids lower than $\overline{b}^1$, e.g. $\left(\overline{b}^1\right)^-$, then $\overline{b}^1$ becomes a best reply for bidder 1 with valuation 1. This constitutes a discontinuity in her best reply.

As a second example, consider a first-price auction under pure strategic uncertainty with two bidders and a commonly known common value v. To bid v is the highest belief-free rationalizable action for both bidders. Therefore, bidder 1 has the subjective maximin belief that bidder 2 bids v. As a consequence, bidder 1 is indifferent between any bid in $[0, v]$. Assume that bidder 1 chooses the action v (or v^-). As any other bid, this leads to a utility of zero given the subjective maximin belief that bidder 2 bids v. An ϵ-neighborhood of v and its intersection with the set of belief-free rationalizable actions contains only bids below v, e.g. it contains the bids v,v^- and $(v^-)^-$. The best replies to these bids are in an ϵ-neighborhood of v (or v^-) and the induced utility are in an ϵ-neighborhood of zero. Hence, bidding v (or v^-) fulfills the robustness property that an ϵ-deviation of the subjective maximin belief induces an ϵ-deviation of the best replies and expected utility.

Appendices

Appendix A. Proof of Proposition 3

Proof. The proof works by contradiction. Assume there exists an outcome under maximin strategies. Let θ_μ be the smallest type which is strictly greater than μ. I will consider three different cases of possible strategy profiles of all bidders. For every possible case and for some arbitrary bidder i I will calculate the distribution of the other bidders' value distributions which the adverse nature chooses for bidder i with valuation θ_μ given the strategy profile considered in the particular case. Then I will show that given this value distribution the action of bidder i with valuation θ_μ which is prescribed by the strategy profile is not optimal. As a result, the assumption that an outcome under maximin strategies exists, leads to a contradiction.

In order to show these steps, the following lemmas are needed:

Lemma 1. *Every bid which is played with a strictly positive probability by a bidder with a type strictly lower than θ_μ is smaller or equal than θ^{z-1},* [12] *i.e. for every bidder $i \in \{1, \ldots, I\}$ and every type θ^j with $j < z$ it holds for all b with $\beta_i(\theta^j)(b) > 0$ that $b \leq \theta^{z-1}$.*

Proof. Assume the Lemma is not true. Let bidder i with type θ^j such that $j < z$ be the bidder who plays the highest bid with positive probability among the bidders with a type lower than θ_μ, i.e. it holds

$$\bar{b}_i^{\theta^j} = \max_{k \in \{1, \ldots I\}, l < z} \{\bar{b}_k^{\theta^l}\}.$$

Since $\theta_\mu > \mu$, there must be positive probability weight on at least one type smaller than θ_μ. Hence, with positive probability bidder i wins the auction and pays more than her own valuation which leads to a contradiction. □

Lemma 2. *Bidder i with valuation θ_μ loses the auction with positive probability, i.e. there exists a bidder k, a valuation θ^j and a bid b such that bidder k with valuation θ^j bids b with positive probability and it holds that $\underline{b}_i^{\theta_\mu} < b$.*

[12] Although bidding above the own valuation is a weakly dominated strategy, I still show this claim in order to state Proposition 3 as general as possible and therefore not to impose assumptions on weak-dominance rationality

Proof. Assume the statement in Lemma 2 is not true and for every bidder $k \neq i$ and every $\theta^j \in \Theta$ it holds that $\underline{b}_i^{\theta_\mu} \geq \overline{b}_k^{\theta^j}$. Then it must hold that $\underline{b}_i^{\theta_\mu} = \overline{b}_i^{\theta_\mu}$ and there exists a bidder k with valuation θ^j who bids either $\underline{b}_i^{\theta_\mu}$ or $\left(\underline{b}_i^{\theta_\mu}\right)^-$ with positive probability because otherwise bidder i with valuation θ_μ could bid strictly less while preserving her winning probability. Given these bids, the adverse nature would put strictly positive probability weight on type θ_μ of bidder i in the belief of bidder k with valuation θ^j since this minimizes her winning probability. It follows that bidder k with valuation θ^j has an incentive to bid $\left(\underline{b}_i^{\theta_\mu}\right)^+$ which leads to a contradiction. $\square$

Let $i \in \{1, \dots I\}$ be some bidder. I consider the following cases.

Case 1: For every bidder $k \neq i$ it holds that bidder i with type θ_μ overbids only the 0-type of bidder k.

The winning probability of bidder i is given by $\prod_{k \neq i} x_{i,\theta_\mu}^{k,\theta^1}$. Hence, the adverse nature will choose distributions of the other bidders' valuations for bidder i with valuation θ_μ such that the probability weight on type zero is zero, for example by distributing the probability weight between types θ^{z-1} and θ_μ in the value distribution of every bidder k with $k \neq i$. Hence, the expected utility of bidder i with type θ_μ is zero. Since θ_μ is strictly greater than μ, in the value distribution of an arbitrary bidder k there must be positive probability weight on some type θ^j with $j < z$. If bidder i with valuation θ_μ deviates to bidding θ^j, it follows from Lemma 1 that she wins against type θ^j of bidder k with positive probability. Since there is positive probability weight on type θ^j of bidder k and $\theta^j < \theta_\mu$, bidder i would earn a strictly positive expected utility by deviating to θ^j. Hence, the assumption that bidder i overbids only the 0-type of every bidder leads to a contradiction.

Case 2: There exist bidders $k_1, \dots k_l$ such that bidder i with valuation θ_μ overbids at least two types of those bidders and bidder i with valuation θ_μ bids zero. Consider a bidder $k \in \{k_1, \dots k_l\}$ and the two following subcases.

Case 2.1: For every $j \in \{1, \dots z-1\}$ it holds that bidder i with valuation θ_μ overbids bidder k with valuation θ^j.

Since bidder i with valuation θ_μ bids zero, it holds for all $j \in \{1, \dots z-1\}$ that bidder k

with valuation θ^j bids zero. For every $j \in \{1, \dots z-1\}$ it holds that bidder i with valuation θ^j believes that there is positive probability weight on some type $\theta^{j'}$ of bidder k with $1 \leq j' < z$. Therefore, bidder i with valuation θ^j will bid 0^+. Since this holds for every θ^j with $1 \leq h < z$, it follows that type θ^{z-1} of bidder k will be overbid by bidder i with probability 1. Since bidder k with valuation θ_μ does not expect to earn a positive utility and there has to be positive probability weight on one type θ^j of bidder i with $1 \leq j < z$, it holds that bidder k with valuation θ^{z-1} never expects to win against a type θ^j of bidder i with valuation $1 \leq j < z$. Due to Lemma 1, it holds that type θ^{z-1} of bidder i bids θ^{z-1}. Hence, type θ^{z-1} of bidder i does not expect to earn a positive utility. By bidding zero bidder i with valuation θ^{z-1} could win with positive probability against all types θ^j of bidder k. Hence, the given strategy profile cannot constitute an outcome under maximin strategies.

Case 2.2: There exists a $j \in \{1, \dots z-1\}$ such that that bidder i with valuation θ_μ does not overbid bidder k with valuation θ^j.
This case works similarly to case 2 since the adverse nature will distribute the probability weight on type θ^j and some other type θ^l with $l \geq z$ which is not overbid by bidder i with valuation θ_μ. Since bidder i with valuation θ_μ bids zero, such a bidder and type exist. The lowest possible bid for type $\theta^m = 1$ is zero and due to the efficient tie-breaking rule, bidder i with valuation θ_μ always loses against type $\theta^m = 1$ unless $z < m$. If $z = m$, then in the belief of every bidder there has to be positive probability on type θ_μ. Since bidder i with type θ_μ bids zero, all other bidders have an incentive to bid 0^+. Conclusively, the adverse nature can choose value distributions of bidder k such that bidder i with valuation θ_μ expects a utility of zero. Bidding θ^{z-1} is deviation where she earns a strictly positive expected utility and therefore the considered strategy profile cannot constitute an outcome under maximin strategies.

Case 3: There exist bidders $k_1, \dots k_l$ such that bidder i with valuation θ_μ overbids at least two types of those bidders and bidder i with valuation θ_μ bids a strictly positive amount. Let k be an element in $\{k_1, \dots k_l\}$ and let $\theta^{j_1} \leq \dots \leq, \theta^{j_h}$ be the types of bidder k which are overbid by bidder i with valuation θ_μ. It follows from Lemma 2 that there exists a valuation θ^j with $j > j_h$ such that bidder k with valuation θ^j is not overbid by bidder i

with valuation θ_μ. Then the adverse nature will choose a value distribution of bidder k such that the probability weight is distributed between the types θ^1 and θ^j. Hence, in the belief of bidder i with valuation θ_μ she can win only against the 0-type and therefore bids 0. This a contradiction to the assumption that bidder i with valuation θ_μ bids a strictly positive amount.

Since for every possible case the assumption that an outcome under maximin strategies exists, leads to contradiction, an outcome under maximin strategies under assumptions (D2) and (S1) does not exist.

□

Appendix B. Proof of Proposition 4

Proof. (i) At first, I consider the case where there exists a unique bidder k and show that her highest belief-free rationalizable bid is the second-highest valuation, denoted by θ'_k. In order to do so, I will show by induction that for every bidder $i \neq k$ the bids in the interval $(\theta'_k, 1]$ are not belief-free rationalizable. Let i be an arbitrary bidder which is not bidder k. Hence, bidder i's valuation is strictly lower than 1, The induction steps are descending and start with 1. Since 1 is the highest possible bid, bidder i wins with strictly positive probability if she bids 1 which cannot be belief-free rationalizable since she would earn a negative utility with positive probability. For the induction step assume that it has been shown that all bids equal or higher than b with $b \in (\theta'_k, 1]$ are not belief-free rationalizable for all bidders $i \neq k$. It is to show that for an arbitrary bidder $i \neq k$ the bid b^- is not belief-free rationalizable if $b^- > \theta'_k$. Since all bids strictly higher than b^- are not belief-free rationalizable for all bidders $i \neq k$, it is never a best reply for bidder k to bid strictly higher than b^-. Therefore, bidder i wins with strictly positive probability if she bids b^-. Since b^- is strictly higher than her valuation, this cannot be optimal. This completes the induction step from which follows that for all bidders $i \neq k$ the bids in the interval $(\theta'_k, 1]$ are not belief-free rationalizable. It follows that for bidder k all bids in the interval $(\theta'_k, 1]$ are not belief-free rationalizable. In every Nash equilibrium the highest bidder bids the second-highest valuation $\theta_{k'}$. Since according to part (i) of Corollary 2 a strategy played in a Bayes-Nash equilibrium is belief-free rationalizable, the bid $\theta_{k'}$ is belief-free rationalizable for bidder k. It follows that $\theta_{k'}$ is the highest belief-free rationalizable bid for bidder k.

If the adverse nature chooses for all bidders $i \neq k$ as the action of bidder k to bid θ'_k , i.e. $\beta_k^{n_i,\theta_i}(\theta_k) = \theta_{k'}$, every bidder $i \neq k$ with valuation θ_i expects a utility of zero independent of her action. Therefore, any other strategy of the adverse nature which is played in equilibrium, has to induce an expected utility of zero for every bidder $i \neq k$ independent of bidder i's action. As a result, every bidder $i \neq k$ is indifferent between all bids in the interval $[0, \theta_i]$. It is left to show that a bidder $i \neq k$ does not bid above her valuation. Assume there exists a bidder i with valuation θ_i who bids $b > \theta_i$. Since for all bidders $j \neq k$ bidding zero is belief-free rationalizable, it is belief-free rationalizable for bidder k to bid zero. Given that bidder i bids b, the adverse nature chooses as the strategy of the other bidders to bid zero, i.e. for every $j \neq i$ it holds that $\beta_j^{n_i,\theta_i}(\theta_j) = 0$. As a result, bidder i wins with probability 1 and earns a negative utility which cannot be part of an equilibrium in the game against the adverse nature. Hence, non of the bidders places bids strictly higher than her valuation in equilibrium.

In order to minimize the expected utility of bidder k, the adverse nature chooses as the strategy of the second-highest bidder to bid her valuation i.e. $\beta_k^{n_i,\theta_i}(\theta_k) = \theta_{k'}$. This is the highest belief-free rationalizable bid which can be placed by a bidder who is not bidder k. As a consequence, bidder k bids $\theta_{k'}$.

(ii) Finally, I consider the case where at least two bidders have the highest valuation θ_k. Analogously as before, one can show by induction that for every bidder the bids in the interval $(\theta_k, 1]$ are not belief-free rationalizable. In every Nash equilibrium every highest bidder bids her valuation θ_k. Therefore, it holds due Corollary 2 that the bid θ_k is belief-free rationalizable for every highest bidder. It follows that θ_k is the highest belief-free rationalizable bid and therefore is the action which the adverse nature chooses as the action of a highest bidder k for a bidder $i \neq k$, i.e. $\beta_k^{n_i,\theta_i}(\theta_k) = \theta_k$. This implies that every bidder does not expect to earn a positive utility and therefore is indifferent between any bid between zero and her valuation. Bid strictly higher than the own valuation can be excluded analogously as above. $\square$

Appendix C. Proof of Propositions 5 and 6

In order to prove Propositions 5 and 6, I will show the following lemmas which formalize steps (I) -(III).

Lemma 3. *For every bidder i and every valuation $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\overline{b}_i^{\theta_i}$.*

Proof. For every bidder i and every valuation θ_i the set of belief-free rationalizable actions $BFR_i(\theta_i)$ is a finite set in a metric space and therefore compact. Since every compact set contains its supremum, there exists a maximum element of the set $BFR_i(\theta_i)$. Since this is a subset of $\mathscr{B}$ and by definition, $\mathscr{B}$ is well-ordered with respect to the relation $\leq$, the maximum element of $BFR_i(\theta_i)$ has to be unique. $\square$

Lemma 4. *For every type zero is a belief-free rationalizable bid.*

Proof. The proof works by induction with respect to the types in Θ. The induction starts with $\theta^1 = 0$. Montiero (2009) shows that with a given commonly known distribution there exists a Bayes-Nash equilibrium in the first-price auction with discrete values where type zero bids zero. It follows from part (i) of Corollary 2 that zero is a belief-free rationalizable action for type zero.

For the induction step assume that it has been already shown for all types θ^j with $j \leq k$ that zero is a belief-free rationalizable action for type θ^j. Consider a bidder i with valuation θ^{k+1} who believes that all other bidders with type θ^j such that $j < k+1$ bid zero which is belief-free rationalizable by assumption. According to Lemma 3 for every bidder and every type there exists a highest belief-free rationalizable bid. Let the belief of bidder i with valuation θ^{k+1} be such that every other bidder with type θ^j such that $j \geq k+1$ bids her highest belief-free rationalizable bid. Then it is a best reply of bidder i with valuation θ^{k+1} to bid zero. As stated in Corollary 1, a best reply to a belief-free rationalizable strategy profile is belief-free rationalizable and therefore zero is a belief-free rationalizable action for bidder i with type θ^{k+1}. This completes the induction step and hence one can conclude that for every bidder and every type zero is a belief-free rationalizable action. $\square$

Lemma 5. *For every type $\theta^k \in \Theta$ it holds that every bid in $[0, \overline{b}^{\theta^k}]$ is belief-free rationalizable.*

Proof. The proof works by showing a slightly stronger statement by induction with respect to the types in Θ. The statement is that for every type $\theta^k \in \Theta$ it holds that every bid in the interval $[0, \overline{b}^{\theta^k}]$ is belief-free rationalizable for every type θ^j such that $j \geq k$.

The induction starts with $\theta^1 = 0$. The highest belief-free rationalizable bid for type θ^1 is zero and it follows from Lemma 4 that zero is a belief-free rationalizable bid for every type.

For the induction step assume that it has been already shown that for all $l \leq k$ it holds that every bid in the interval $[0, \overline{b}^{\theta^l}]$ is belief-free rationalizable for every type θ^j such that $j \geq l$. By using induction with respect to the bids, I will show that the same statement holds for type θ^{k+1}. The induction starts with the bid zero. It follows from Lemma 4 that zero is belief-free rationalizable for every type. For the induction step assume that it has been already shown that every bid in the interval $[0, b]$ with $b < \overline{b}^{\theta^{k+1}}$ is belief-free rationalizable for every type θ^j with $j \geq k+1$. In order to show that b^+ is belief-free rationalizable for every type θ^j with $j \geq k+1$ consider a bidder i with valuation θ^{k+1} and strategies of the other bidders such that

(i) Every type θ^l with $l < k+1$ bids her highest belief-free rationalizable bid

(ii) Every type θ^j with $j \geq k+1$ bids b

The strategies in (i) are belief-free rationalizable by definition and the strategies in (ii) are belief-free rationalizable by the assumption in the induction step. Given this belief about the other bidders' strategies it is optimal for bidder i with valuation θ^{k+1} to bid b^+. If it is not optimal, then it would hold that b is the highest belief-free rationalizable bid for θ^{k+1} because any other belief about strategies than in (i) and (ii) makes bidding b^+ less profitable. Any change in part (i) would imply that a type θ^l with $l < k+1$ bids some bid $b^{\theta^l} < \overline{b}^{\theta^l}$ instead of $\overline{b}^{\theta^l}$ which makes overbidding this type more profitable. Formally, instead of the inequality

$$\theta^{k+1} - b^+ \geq F\left(\theta^{l+1}\right)\left(\theta^{k+1} - \overline{b}^{\theta^l}\right)$$

the inequality

$$\theta^{k+1} - b^+ \geq F\left(\theta^{l+1}\right)\left(\theta^{k+1} - b^{\theta^l}\right)$$

has to hold. [13] Any deviation from (ii) implies that a type θ^j with $j \geq k+1$ either bids higher or lower. If some type θ^j bids lower, then the same reasoning as above applies. If a type θ^j deviates to a higher bid, then by bidding b^+ bidder i with valuation θ^{k+1} does not overbid type θ^j anymore. Hence, for every $l < k+1$ instead of

$$\theta^{k+1} - b^+ \geq F\left(\theta^{l+1}\right)\left(\theta^{k+1} - \overline{b}^{\theta^l}\right)$$

the inequality

$$\left(1 - \left(F\left(\theta^j\right) - F\left(\theta^{j-1}\right)\right)\right)\left(\theta^{k+1} - b^+\right) \geq F\left(\theta^{l+1}\right)\left(\theta^{k+1} - \overline{b}^{\theta^l}\right)$$

has to hold.

Conclusively, any deviation form the strategies in (i) and (ii) makes bidding b^+ less profitable. Therefore, if bidding b^+ is not a best reply to the beliefs in (i) and (ii), then b is the highest belief-free rationalizable bid for type θ^{k+1}. This shows that any bid in the interval $[0, \overline{b}^{\theta^{k+1}}]$ is belief-free rationalizable for type θ^{k+1}. Analogously one can show that any bid in the interval $[0, \overline{b}^{\theta^{k+1}}]$ is belief-free rationalizable for type θ^j with $j \geq k+1$. This completes the induction step of the first induction. Therefore, it has been shown that for every type $\theta^k \in \Theta$ it holds that every bid in the interval $[0, \overline{b}^{\theta^k}]$ is belief-free rationalizable for every type θ^j such that $j \geq k$. □

Proof of Proposition 6

Proof. As shown in the proof of Lemma 5, for every type the belief as described in (i) and (ii) induces the highest belief-free rationalizable bid for this type, that is the highest belief-free rationalizable bid is a best reply to this belief. Given this belief, the expected utility of a bidder i with some type $\theta^k \in \Theta$ who bids $\overline{b}^{\theta^k}$ is given by

$$\theta^k - \overline{b}^{\theta^k}.$$

This utility has to be higher than the utility induced by any other bid. A bid can be a best reply only if with this bid bidder i just overbids some other bidder. Formally, a bid b can be best reply only if there exists a bidder $j \neq i$ and a valuation θ^l such that bidder j with valuation θ^l bids b^-. Hence, the only potential candidates for best replies besides $\overline{b}^{\theta^k}$

[13]For a simpler notation I write down the inequalities for the 2-bidder case.

are bids $\overline{b}^{\theta^j}$ with $j < k$. Hence, equation 2 ensures that bidding $\overline{b}^{\theta^k}$ induces at least the same expected utility than ony other bid which can be a best reply.

□

Proof of Proposition 5

Proof. I show the existence of an equilibrium by construction. According to Lemma 3 for every type there exist a unique highest belief-free rationalizable bid. For every type and every player the adverse nature chooses the other bidders' strategies such that every bidder places her highest belief-free rationalizable bid, i.e. for every bidder $i \in \{1, \dots, I\}$ it holds that $\beta_k^{n_i,\theta_i}(\theta_k) = \overline{b}^{\theta^k}$ for all $k \neq i$. Independent of the bidders' strategies there does not exist another strategy of the adverse nature which induces a lower expected utility for any of the bidders. Thus, the adverse nature does not have an incentive to deviate from this strategy. Every bidder plays a best reply given her type and the adverse nature's strategy. Due to the compactness of $\mathscr{B}$, such a best reply always exist and is unique. I will show that the outcome defined by these best replies is efficient.

For a bidder with type $\theta^k \in \Theta$ the best reply is given by the most profitable overbidding of a lower type, that is, by

$$\underset{\overline{b}^{\theta^j}, j<k}{\operatorname{argmax}} \; F(\theta^j)\left(\theta^k - \overline{b}^{\theta^j}\right).$$

Let $\overline{b}^{\theta^l}$ be a best reply of a bidder with valuation θ^k. Then it holds for all $j \in \{1, \dots, l-1\}$ that

$$F(\theta^l)\left(\theta^k - \overline{b}^{\theta^l}\right) \geq F(\theta^j)\left(\theta^k - \overline{b}^{\theta^j}\right) \tag{4}$$

$$\Leftrightarrow \theta^k\left(F(\theta^l) - F(\theta^j)\right) - F(\theta^l)\overline{b}^{\theta^l} + F(\theta^j)\overline{b}^{\theta^j} \geq 0. \tag{5}$$

Since $F(\theta^l) - F(\theta^j) > 0$, it follows from (5) that for all $\theta^{k'}$ such that $\theta^{k'} > \theta^k$ and for all $l \in \{1, \dots l-1\}$ it holds that

$$\theta^{k'}\left(F(\theta^l) - F(\theta^j)\right) - F(\theta^l)\overline{b}^{\theta^l} + F(\theta^j)\overline{b}^{\theta^j} \geq 0$$

$$\Leftrightarrow F(\theta^l)\left(\theta^{k'} - \overline{b}^{\theta^l}\right) \geq F(\theta^j)\left(\theta^{k'} - \overline{b}^{\theta^j}\right). \tag{6}$$

First, consider the case where for every $j \in \{1, \dots, l-1\}$ the inequality in 4 is strict. Then for a bidder with valuation θ^k there exists a unique best reply, denoted by $\overline{b}^{\theta^l}$. Hence, in order to show efficiency, it is sufficient to show that every best reply of a bidder with valuation $\theta^{k'}$ with $\theta^{k'} > \theta^k$ is equal or greater than $\overline{b}^{\theta^l}$.

It holds that for every $j \in \{1, \dots, l-1\}$ that the inequality in (6) is strict. It follows that none of the bids $\overline{b}^{\theta^j}$ for $j \in \{1, \dots, l-1\}$ can be a best reply for a bidder with valuation $\theta^{k'}$. Thus, a best reply of a bidder with valuation $\theta^{k'}$ with $\theta^{k'} > \theta^k$ is equal or greater than $\overline{b}^{\theta^l}$.

Second, consider the case where for at least one $j \in \{1, \dots, m\}$ the expression in 4 holds with equality. Let $j_1, \dots j_h$ be all indices for which it holds that the expression in 4 holds with equality. Then for all $j \in \{j_1, \dots, j_h\}$ it must hold that $F(\theta^l) > F(\theta^j)$. Thus, for all $j \in \{j_1, \dots, j_h\}$ expression in 6 holds with strict inequality. For all $j \in \{1, \dots, I\} \backslash \{j_1, \dots, j_h\}$ the expression in 4 holds with strict inequality and therefore also the expression in 6. Therefore, it holds for all $j \in \{1, \dots, j_h\}$ that the inequality in 6 is strict. Analogously to the first case, this implies that the best reply of a bidder with valuation $\theta^{k'}$ with $\theta^{k'} > \theta^k$ is equal or higher than the highest best reply of a bidder with valuation θ^k. Therefore, the outcome is efficient.

So far, I have shown by construction that an outcome under maximin expected utilities exists and that this outcome is efficient. It remains to show that every outcome is efficient. Let β^n denote the strategy of the adverse nature which chooses for every bidder i and every type the strategy of the other bidders' such that every bidder places her highest belief-free rationalizable bid, i.e. for every bidder $i \in \{1, \dots, I\}$ it holds that $\beta_k^{n_i, \theta_i}(\theta_k) = \overline{b}^{\theta^k}$ for all $k \neq i$. As shown above, there exists a unique best reply of the bidders to this strategy and the outcome defined by these best replies is efficient. A different outcome is possible only if the adverse nature plays another strategy, denoted by $\beta^{n'}$. Assume the outcome defined by the best replies to $\beta^{n'}$ is not efficient. Then the best replies to $\beta^{n'}$ have to differ from the best replies to β^n and there exists a bidder i with valuation θ_i such that $B^{n'} \backslash B^n$ is non-empty, where B^n is the set of best replies of bidder i with valuation θ_i to β^n and $B^{n'}$ is the set of best replies to $\beta^{n'}$. Let b be an element in B^n and b' be an element in $B^{n'} \backslash B^n$. Since β^n is a weakly dominant strategy for the adverse nature, the expected utilitys of bidder i of bidding b given β^n and of bidding b' given $\beta^{n'}$ must be equal, let Π denote this expected utility. Since $\beta^{n'}$ is not an element of B^n, it must hold that there exists a bidder k with valuation θ^l such that the subjective maximin belief of bidder i about bidder k's strategy evaluated at θ^l is not equal to $\overline{b}^{\theta^l}$ but is either b' or $(b')^-$ (depending on whether $\theta_i > \theta^l$ or $\theta_i \leq \theta^l$), i.e. $\beta_k^{n_i,\theta_i}(\theta^l) \in \{b', (b')^-\}$. Since $\overline{b}^{\theta^l}$ is the highest belief-free rationalizable bid, it must hold that the subjective maximin belief of bidder i with valuation θ^i about bidder k's strategy at θ^l is strictly lower than $\overline{b}^{\theta^l}$, i.e. $\beta_k^{n_i,\theta_i}(\theta^l) < \overline{b}^{\theta^l}$. Given that bidder i with valuation θ_i plays b' with positive probability, the adverse nature has an incentive to deviate and to change the subjective maximin belief of bidder i with valuation θ^i about bidder k's strategy at θ^l to $\overline{b}^{\theta^l}$. This deviation induces a strictly lower utility than Π because otherwise b' would be an element in B^n. □

Appendix D. Proof of Propositions 8

Proof. The proof works by induction with respect to the type. Since $\overline{b}^{\theta^1} = \overline{b}_*^{\theta^1}$, the induction starts with θ^2. The highest belief-free rationalizable bid for type θ^2 is obtained by the equation

$$\theta^2 - \overline{b}^{\theta^2} = F^{I-1}(0)\theta^2$$
$$\Leftrightarrow \overline{b}^{\theta^2} = \theta^2 \left(1 - F^{I-1}(0)\right).$$

The highest bid which is played with positive probability by a bidder with valuation θ^2 in a Bayes-Nash equilibrium is obtained by the equation

$$F^{I-1}\left(\theta^2\right)\left(\theta^2 - \overline{b}_*^{\theta^2}\right) = F^{I-1}(0)\theta^2$$
$$\Leftrightarrow \overline{b}_*^{\theta^2} = \frac{\theta^2\left(F^{I-1}\left(\theta^2\right) - F^{I-1}(0)\right)}{F^{I-1}\left(\theta^2\right)}.$$

Since $F^{I-1}\left(\theta^2\right) < 1$ it holds that

$$F^{I-1}\left(\theta^2\right) F^{I-1}(0) < F^{I-1}(0)$$
$$\Leftrightarrow F^{I-1}\left(\theta^2\right) - F^{I-1}\left(\theta^2\right) F^{I-1}(0) > F^{I-1}\left(\theta^2\right) - F^{I-1}(0)$$
$$1 - F^{I-1}(0) > \frac{F^{I-1}\left(\theta^2\right) - F^{I-1}(0)}{F^{I-1}\left(\theta^2\right)}$$
$$\overline{b}^{\theta^2} > \overline{b}_*^{\theta^2}.$$

For the induction step assume that it has been already shown that $\overline{b}^{\theta^j} > \overline{b}_*^{\theta^j}$ for all $j \leq k$. It has to be shown that

$$\overline{b}^{\theta^{k+1}} > \overline{b}_*^{\theta^{k+1}}.$$

As stated in Proposition 6, it holds that

$$\theta^{k+1} - \overline{b}^{\theta^{k+1}} = \max_{\theta^j < \theta^{k+1}} F^{I-1}\left(\theta^j\right)\left(\theta^{k+1} - \overline{b}^{\theta^j}\right).$$

Let

$$F^{I-1}\left(\theta^l\right)\left(\theta^{k+1} - \overline{b}^{\theta^l}\right) = \max_{\theta^j < \theta^{k+1}} F^{I-1}\left(\theta^j\right)\left(\theta^{k+1} - \overline{b}^{\theta^j}\right).$$

Since $\overline{b}_*^{\theta^{k+1}}$ is a best reply, it must induce an expected utility which is greater or equal than the expected utility induced by any other bid. Hence, it holds that

$$F\left(\theta^{k+1}\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^{k+1}}\right) \geq F\left(\theta^l\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^l}\right).$$

Due to the induction assumption it holds that $\overline{b}_*^{\theta^l} < \overline{b}^{\theta^l}$ from which follows that

$$\theta^{k+1} - \overline{b}^{\theta^{k+1}} = F^{I-1}\left(\theta^l\right)\left(\theta^{k+1} - \overline{b}^{\theta^l}\right) < F\left(\theta^l\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^l}\right) \leq F\left(\theta^{k+1}\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^{k+1}}\right)$$

and therefore it holds that

$$\theta^{k+1} - \overline{b}^{\theta^{k+1}} < F\left(\theta^{k+1}\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^{k+1}}\right).$$

$$\Leftrightarrow \overline{b}_*^{\theta^{k+1}} < \frac{\overline{b}^{\theta^{k+1}} - \theta^{k+1}\left(1 - F\left(\theta^{k+1}\right)\right)}{F\left(\theta^{k+1}\right)}. \tag{7}$$

It holds that

$$\begin{aligned}
\theta^{k+1} - \overline{b}^{\theta^{k+1}} &\geq 0 \\
\Leftrightarrow \theta^{k+1}\left(1 - F\left(\theta^{k+1}\right)\right) - \overline{b}^{\theta^{k+1}}\left(1 - F\left(\theta^{k+1}\right)\right) &\geq 0 \\
\Leftrightarrow \overline{b}^{\theta^{k+1}} - \theta^{k+1}\left(1 - F\left(\theta^{k+1}\right)\right) &\leq F\left(\theta^{k+1}\right)\overline{b}^{\theta^{k+1}} \\
\Leftrightarrow \frac{\overline{b}^{\theta^{k+1}} - \theta^{k+1}\left(1 - F\left(\theta^{k+1}\right)\right)}{F\left(\theta^{k+1}\right)} &\leq \overline{b}^{\theta^{k+1}}.
\end{aligned}$$

Due to equation (7), it follows that

$$\overline{b}^{\theta^{k+1}} > \overline{b}_*^{\theta^{k+1}}.$$

This completes the induction step and the proof.

□

Appendix E. Proof of Propositions 9,10 and 11

First, I prove Proposition 10 which formalizes the recursive calculation of the highest belief-free rationalizable bids for every type. This calculation is crucial for the proofs of Propositions 9 and 11. In order to prove Proposition 10, I state the following three lemmas which formalize steps (I)-(III) in section 7. The proofs work analogously as for lemmas 3, 4 and 5 in section 6.

Lemma 6. *For every bidder i and every valuation $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\overline{b}_i^{\theta_i}$.*

Lemma 7. *For every type zero is a belief-free rationalizable bid.*

Lemma 8. *For every type $\theta^k \in \Theta$ it holds that every bid in $[0, \overline{b}^{\theta^k}]$ is belief-free rationalizable.*

Proof of Proposition 10

Proof. First, I examine the highest belief-free rationalizable bids of a bidder with valuation θ^k such that θ^k is lower or equal than μ. Consider a subjective belief equilibrium where every bidder has the subjective belief that the other bidders' value distribution distributes the probability weight between types θ^k and θ^μ. Formally, the distribution of the other bidders' valuation is defined by the vector $x^{\theta^k} = (x^{\theta^k}_{\theta^1}, \ldots, x^{\theta^k}_{\theta^m})$ where for all $j \in \{1, \ldots, \}$ it holds that $x^{\theta^k}_{\theta^j}$ denotes the probability with which type θ^j occurs. This vector is defined by

$$x^{\theta^k}_{\theta^{\theta^k}} = \frac{\theta_\mu - \mu}{\theta_\mu - \theta^k}, \quad x^{\theta^k}_{\theta_\mu} = \frac{\mu - \theta^k}{\theta_\mu - \theta^k} \quad \text{and} \quad x^{\theta^k}_{\theta^j} = 0 \quad \text{for } \theta^j \neq \theta^k, \theta_\mu.$$

Given this subjective belief, in every subjective-belief equilibrium every bidder with valuation θ^k bids θ^k. It follows from Proposition 2 that bidding θ^k is a belief-free rationalizable action for a bidder with valuation θ^k. Since it is not belief-free rationalizable to bid above valuation, θ^k is the highest belief-free rationalizable bid for a bidder with valuation θ^k.

Now I examine the highest belief-free rationalizable bids of a bidder with valuation θ^k such that θ^k is strictly greater than μ. Analogously as in the proof Proposition 6, the highest belief-free rationalizable bid of a bidder with valuation θ^k is induced by the strategy of the other bidders' such that

(i) All bidders with a lower type bid their highest belief-free rationalizable bid.
(ii) All bidders with an equal or higher type bid $\left(\overline{b}^{\theta^k}\right)^-$.

The strategies in (i) are belief-free rationalizable by definition and the strategies in (ii) are belief-free rationalizable due to Lemma 8. It follows from Corollary 1 that a best reply to these strategies is belief-free rationalizable. The highest belief-free rationalizable bid for a bidder with valuation is a best reply to the strategies in (i) and (ii) and to distribution of the other bidders' values. Let the vector $x^{\theta^k} = (x^{\theta^k}_{\theta^1}, \ldots, x^{\theta^k}_{\theta^m})$ define this distribution, i.e. for all $j \in \{1, \ldots, \}$ it holds that $x^{\theta^k}_{\theta^j}$ denotes the probability with which type θ^j occurs. For all $j \in \{1, \ldots, m\}$ it must hold that

$$\theta^k - \overline{b}^{\theta^k} \geq \left(\sum_{i=1}^{j} x^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right).$$

Hence, the vector $\left(x^{\theta^k}_{\theta^1}, \ldots, x^{\theta^k}_{\theta^m}\right)$ is the solution to the following minimization problem

$$\min \max_{l<k} \left\{ \left(\sum_{i=1}^{l} x^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^l}\right) \right\}$$

$$s.t. \;\; x^{\theta^k}_{\theta^1} + \cdots + x^{\theta^k}_{\theta^m} = 1$$

$$x^{\theta^k}_{\theta^1}\theta^1 + \cdots + x^{\theta^k}_{\theta^m}\theta^m = \mu,$$

which I denote by M^{θ^k} since the solution of this minimization problem is the belief which induces the highest belief-free rationalizable bid of a bidder with valuation θ^k. It is to show that for the solution of this minimization problem it holds that

$$\left(\sum_{i=1}^{j} x^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) = \left(\sum_{i=1}^{j'} x^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^{j'}}\right) \quad \text{for all } j, j' < k.$$

Assume there exist $j, j' < k$ such that

$$\left(\sum_{i=1}^{j} x^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) > \left(\sum_{i=1}^{j'} x^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^{j'}}\right).$$

Then it also holds that

$$\max_{l<k}\left\{\left(\sum_{i=1}^{l} x_{\theta^i}^{\theta^k}\right)^{I-1}\left(\theta^k-\overline{b}^{\theta^l}\right)\right\} > \left(\sum_{i=1}^{j'} x_{\theta^i}^{\theta^k}\right)^{I-1}\left(\theta^k-\overline{b}^{\theta^{j'}}\right).$$

Thus one can assume w.l.o.g. that

$$\left(\sum_{i=1}^{j} x_{\theta^i}^{\theta^k}\right)^{I-1}\left(\theta^k-\overline{b}^{\theta^j}\right) = \max_{l<k}\left\{\left(\sum_{i=1}^{l} x_{\theta^i}^{\theta^k}\right)^{I-1}\left(\theta^k-\overline{b}^{\theta^l}\right)\right\}.$$

If

I will consider two cases: $j < j'$ and $j > J'$. In both cases I will construct a vector $\tilde{x}^{\theta^k} = \left(\tilde{x}_{\theta^1}^{\theta^k}, \ldots, \tilde{x}_{\theta^m}^{\theta^k}\right)$ which fulfills all constraints of the minimization problem but leads to a lower value of the objective function. Hence, the vector $x^{\theta^k} = \left(x_{\theta^1}^{\theta^k}, \ldots, x_{\theta^m}^{\theta^k}\right)$ cannot be the solution of minimization problem M^{θ^k}.

Case 1: $j < j'$.
Let $\delta > 0$ be defined by

$$\left(\sum_{i=1}^{j} x_{\theta^i}^{\theta^k}\right)^{I-1}\left(\theta^k-\overline{b}^{\theta^j}\right) = \left(\sum_{i=1}^{j'} x_{\theta^i}^{\theta^k}\right)^{I-1}\left(\theta^k-\overline{b}^{\theta^{j'}}\right)+\delta.$$

(8) $$\left(\sum_{i=1}^{j} x_{\theta^i}^{\theta^k}\right)\sqrt[I-1]{\left(\theta^k-\overline{b}^{\theta^j}\right)} = \left(\sum_{i=1}^{j'} x_{\theta^i}^{\theta^k}\right)\sqrt[I-1]{\left(\theta^k-\overline{b}^{\theta^{j'}}\right)}+\delta.$$

Let the vector $\tilde{x}^{\theta^k} = \left(\tilde{x}_{\theta^1}^{\theta^k}, \ldots, \tilde{x}_{\theta^m}^{\theta^k}\right)$ be defined by

$$\left(\tilde{x}_{\theta^1}^{\theta^k}, \ldots, \tilde{x}_{\theta^{j+1}}^{\theta^k}, \ldots, \tilde{x}_{\theta^m}^{\theta^k}\right) = \left(x_{\theta^1}^{\theta^k}+\epsilon_1, \ldots, x_{\theta^{j+1}}^{\theta^k}+\epsilon_{j+1}, \ldots, x_{\theta^m}^{\theta^k}+\epsilon_m\right)$$

where ϵ_1, ϵ_{j+1} and ϵ_m fulfill the conditions

$$\epsilon_1, \epsilon_{j+1}, \epsilon_m > 0$$

(9) $$-\epsilon_1+\epsilon_{j+1}-\epsilon_m$$

(10) $$\epsilon_{j+1}\theta^{j+1}-\epsilon_m = 0$$

$$\epsilon_{j+1} < \frac{\delta}{\left(1-\theta^{j+1}\right)\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} + \theta^{j+1}\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)}}. \tag{11}$$

Equation (10) is equivalent to

$$\epsilon_m = \epsilon_{j+1}\theta^{j+1}$$

from which follows that if $\epsilon_{j+1} > 0$, then also ϵ_m. Plugging this into equation (9) gives

$$-\epsilon_1 + \epsilon_{j+1} - \epsilon_{j+1}\theta^{j+1} = 0$$

$$\Leftrightarrow \epsilon_1 = \epsilon_{j+1}\left(1-\theta^{j+1}\right)$$

which shows that if $\epsilon_{j+1} > 0$, then also ϵ_1. Conclusively, $\epsilon_1, \epsilon_{j+1}, \epsilon_m$ fulfilling the conditions above indeed exist if ϵ_{j+1} is chosen sufficiently small.

Inequality (11) is equivalent to

$$\delta > \epsilon_{j+1}\left(\left(1-\theta^{j+1}\right)\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} + \theta^{j+1}\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)}\right)$$

$$\Leftrightarrow \delta - \epsilon_{j+1}\left(1-\theta^{j+1}\right)\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} > \left(-\epsilon_{j+1}\left(1-\theta^{j+1}\right)+\epsilon_{j+1}\right)\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)}$$

$$\delta - \epsilon_1\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} > \left(-\epsilon_1 + \epsilon_{j+1}\right)\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)}$$

Adding equation (8) gives

$$\delta + \left(x^{\theta^k}_{\theta^1} - \epsilon_1, \ldots, x^{\theta^k}_{\theta^j}\right)\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} >$$
$$\left(x^{\theta^k}_{\theta^1} - \epsilon_1, \ldots, x^{\theta^k}_{\theta^j}, x^{\theta^k}_{\theta^{j+1}} + \epsilon_{j+1}, \ldots, x^{\theta^k}_{\theta^{j'}}\right)\sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} + \delta$$

$$\Leftrightarrow \left(\sum_{i=1}^{j}\tilde{x}^{\theta^k}_{\theta^i}\right)^{I-1}\left(\theta^k - \overline{b}^{\theta^j}\right) > \left(\sum_{i=1}^{j'}\tilde{x}^{\theta^k}_{\theta^i}\right)^{I-1}\left(\theta^k - \overline{b}^{\theta^{j'}}\right).$$

Thus, it holds that

$$\left(\sum_{i=1}^{j}\tilde{x}^{\theta^k}_{\theta^i}\right)^{I-1}\left(\theta^k - \overline{b}^{\theta^j}\right) = \max_{l<k}\left\{\left(\sum_{i=1}^{l}\tilde{x}^{\theta^k}_{\theta^i}\right)^{I-1}\left(\theta^k - \overline{b}^{\theta^l}\right)\right\}$$

and

$$\left(\sum_{i=1}^{j}\tilde{x}^{\theta^k}_{\theta^i}\right)^{I-1}\left(\theta^k - \overline{b}^{\theta^j}\right) < \left(\sum_{i=1}^{j}x^{\theta^k}_{\theta^i}\right)^{I-1}\left(\theta^k - \overline{b}^{\theta^j}\right).$$

Hence, the vector $\tilde{x}^{\theta^k} = \left(\tilde{x}^{\theta^k}_{\theta^1}, \ldots, \tilde{x}^{\theta^k}_{\theta^m}\right)$ fulfills all constraints of the minimization problem but leads to a lower value of the objective function which leads to a contradiction.

Case 2: $j > j'$.
Let $\delta > 0$ be defined by

$$\left(\sum_{i=1}^{j} x^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) = \left(\sum_{i=1}^{j'} x^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^{j'}}\right) + \delta.$$

$$\left(\sum_{i=1}^{j} x^{\theta^k}_{\theta^i}\right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} = \left(\sum_{i=1}^{j'} x^{\theta^k}_{\theta^i}\right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)} + \delta. \tag{12}$$

Let the vector $\tilde{x}^{\theta^k} = \left(\tilde{x}^{\theta^k}_{\theta^1}, \ldots, \tilde{x}^{\theta^k}_{\theta^m}\right)$ be defined by

$$\left(\tilde{x}^{\theta^k}_{\theta^1}, \ldots, \tilde{x}^{\theta^k}_{\theta^j}, \ldots, \tilde{x}^{\theta^k}_{\theta^m}\right) = \left(x^{\theta^k}_{\theta^1} + \epsilon_1, \ldots, x^{\theta^k}_{\theta^j} - \epsilon_j, \ldots, x^{\theta^k}_{\theta^m} + \epsilon_m\right)$$

where ϵ_1, ϵ_j and ϵ_m fulfill the conditions

$$\epsilon_1, \epsilon_j, \epsilon_m > 0$$

$$\epsilon_1 - \epsilon_j + \epsilon_m = 0 \tag{13}$$

$$-\epsilon_j \theta^j + \epsilon_m = 0 \tag{14}$$

$$\epsilon_j < \frac{\delta}{\theta^j \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} + (1 - \theta^j) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)}}. \tag{15}$$

Equation (14) is equivalent to

$$\epsilon_m = \epsilon_j \theta^j$$

from which follows that if $\epsilon_j > 0$, then also ϵ_m. Plugging this into equation (13) gives

$$\epsilon_1 - \epsilon_j + \epsilon_j \theta^j = 0$$

$$\Leftrightarrow \epsilon_1 = \epsilon_j \left(1 - \theta^j\right) \tag{16}$$

which shows that if $\epsilon_j > 0$, then also ϵ_1. Conclusively, $\epsilon_1, \epsilon_j, \epsilon_m$ fulfilling the conditions above indeed exist if ϵ_j is chosen sufficiently small.

Inequality (15) is equivalent to

$$\delta > \epsilon_j \left(\theta^j \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} + (1-\theta^j) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)} \right)$$

$$\Leftrightarrow \delta - \epsilon_j \theta^j \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} > \epsilon_j \left(1-\theta^j\right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)}$$

$$\Leftrightarrow \delta + \left(\epsilon_j(1-\theta^j) - \epsilon_j\right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} > \epsilon_j \left(1-\theta^j\right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)}$$

$$\Leftrightarrow \delta + \left(\epsilon_1 - \epsilon_j\right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^j}\right)} > \epsilon_1 \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^{j'}}\right)}.$$

Thus, it holds that

$$\left(\sum_{i=1}^{j} \tilde{x}_{\theta^i}^{\theta^k}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) = \max_{l<k} \left\{ \left(\sum_{i=1}^{l} \tilde{x}_{\theta^i}^{\theta^k}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^l}\right) \right\}.$$

It follows from equation (16) that $\epsilon_j > \epsilon_1$ and therefore it holds that

$$\left(\sum_{i=1}^{j} \tilde{x}_{\theta^i}^{\theta^k}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) < \left(\sum_{i=1}^{j} x_{\theta^i}^{\theta^k}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right).$$

Hence, the vector $\tilde{x}^{\theta^k} = \left(\tilde{x}_{\theta^1}^{\theta^k}, \ldots, \tilde{x}_{\theta^m}^{\theta^k}\right)$ fulfills all constraints of the minimization problem but leads to a lower value of the objective function which leads to a contradiction.

□

Appendix F. Additional explanation to Definition 1

The most general version of Definition 1 can be formalized such that a player i knows that the possible distributions of types and actions are described by some subset Δ_{Θ_j, A_j} of the set of all distributions on $\Theta_{-i} \times A_{-i}$ denoted by $\Delta(\Theta_{-i} \times A_{-i})$. Definition 1 does not account for the type of correlation where distributions and strategies could be correlated. As an example, consider the case with two players i and j. The type space of bidder j is given by $\Theta_j = \{\theta_1, \theta_2\}$ and the action space by $A_j = \{a_1, a_2\}$ and there is no unknown utility type θ_0. If a strategy can depend on the distribution, then it is possible that if player

j has the value distribution $(\frac{1}{2},\frac{1}{2})$, she chooses action a_1 given valuation θ_1 and action a_2 given valuation θ_2, but if player j has the value distribution $(\frac{1}{4},\frac{3}{4})$, she chooses action a_2 given any of the valuations. Hence, it is possible that the set $\Delta_{\Theta_{-i},A_{-i}} \subseteq \Delta(\Theta_j \times A_j)$ consists only of the two elements

$$\left(\frac{1}{2}(\theta_1,a_1),\frac{1}{2}(\theta_2,a_2)\right) \quad \text{and} \quad \left(\frac{1}{4}(\theta_1,a_2),\frac{3}{4}(\theta_2,a_2)\right).$$

In contrast, if these distributions are elements of the set $\Delta\Theta_j \times \Delta S_j$, then also

$$\left(\frac{1}{2}(\theta_1,a_2),\frac{1}{2}(\theta_2,a_2)\right) \quad \text{and} \quad \left(\frac{1}{4}(\theta_1,a_1),\frac{3}{4}(\theta_2,a_2)\right)$$

have to be elements. Formally, the set $\Delta\Theta_j \times \Delta S_j$ is not a subset of $\Delta(\Theta_j \times A_j)$. However, there exists an injective function $i : \Delta\Theta_j \times \Delta S_j \hookrightarrow \Delta(\Theta_j \times A_j)$ and therefore $\Delta\Theta_j \times \Delta S_j$ can be viewed as a subset of $\Delta(\Theta_j \times A_j)$. A tuple of distributions $(F_j^{\Theta_j}, F_j^{A_j}) \in \Delta\Theta_j \times \Delta S_j$ is mapped to the distribution where the tuple (θ_j, a_j) occurs with probability $F_j^{\Theta_j}(\theta_j)\sum_{\beta_j \in S_j:\beta_j(v_j)=a_j} F_j^{A_j}(\beta_j)$. As an example consider the tuple of the value distribution where θ_1 and θ_2 occur with probability $\frac{1}{2}$ and the distribution of strategies where the strategy $(\theta_1 \mapsto a_1, \theta_2 \mapsto a_2)$ occurs with probability $\frac{1}{3}$ and the strategy $(\theta_1 \mapsto a_2, \theta_2 \mapsto a_2)$ occurs with probability 1, denoted by $\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{3}\left(\theta_1 \mapsto a_1, \theta_2 \mapsto a_2\right),\frac{2}{3}\left(\theta_1 \mapsto a_2, \theta_2 \mapsto a_2\right)\right)\right)$. This element is mapped to $\left(\frac{1}{6}(v_1,a_1),\frac{1}{3}(v_1,a_2),\frac{1}{2}(\theta_2,a_2)\right)$. I do not consider this type of correlation because players know their type and have an ex-interim rather than an ex-ante perspective. Therefore, given that players know their type, they may not have an assumption about their own distribution. Moreover, the separation between distributions and strategies allows for a formalization of distributional and strategic uncertainty such that that case of pure distributional or pure strategic uncertainty can be nested in the general model. If there is no distributional uncertainty, then the set $\Delta_{\Theta_{-i}}$ is a singleton and if there is no strategic uncertainty, then the set $\Delta_{S_{-i}}$ is a singleton. Since I do not consider the possibility of correlation in the application sections, I do not allow for this type of correlation in the general model for the sake of notation simplicity.

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