Fees versus Royalties: The Case of a Product Improvement

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Abstract

We examine the effect of the chosen licensing method for a product improvement in the downstream market. We analyze four licensing methods: fixed fee, fixed fee with an auction, per-unit royalties and per-unit royalties with an auction. All four methods are analyzed for two cases: when the licensees can produce only the improved product and when the licensees can continue producing the old product as well. It is assumed that in addition to having the right to produce the patented product, the licensee becomes a Stackelberg leader in the downstream market. It was found that in the case of a fixed fee the patent owner sells an exclusive license to a single producer. In contrast, in the case of per-unit royalty the patent owner sells licenses to about half of the producers if the producers are not allowed to produce the old product, and to all of them if they are allowed. The patent owner and the consumer prefer the fixed fee method over the royalty (whether or not the licenses are auctioned).

Introduction

The pricing of a license to use an innovation plays a crucial role in the adoption of new technologies and in the incentivization of innovators. A significant part of the Industrial Organization literature is devoted to this issue. Kamien and Tauman (1986) analyze the case of a process innovation. In an environment of *n* oligopolistic producers and one patent owner, they present a 3-stage game. In the first stage, the patent owner sets the

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price of the license; in the second, each producer decides whether or not to buy the license; and in the final stage all of the producers compete in a Cournot competition setup where the licensees benefit from lower marginal costs. Three main licensing methods are considered: fixed fee, fixed fee with an auction, and per-unit royalty. The first two are shown to dominate the third and only a "drastic" innovation (i.e. a drastic reduction in cost) is licensed to a single producer.

Katz and Shapiro (1985) study the case where the innovator is also a producer. This and the above model assume a linear demand and were extended by Kamien, Muller and Zang (1992) and Sen and Tauman (2012) to a wider class of demand functions. Tauman, Weiss, and Zhao (2015) studied the case of an innovation that reduces the cost of entry into an industry rather than marginal cost. Finally, Katz and Shapiro (1986) examine the optimal licensing strategy for general profit functions without specifying the downstream market structure. They conclude that, auctioning the license yields the inventor higher profit than a fixed fee license.

A parallel strand of literature focused on the same questions for the case of quality improvements. Thus, Stamatopoulos and Tauman (2008) analyze a model with three players: two firms and one innovator, in which there is price competition and quality varies across firms. The innovator has a patented innovation for quality improvement and chooses the licensing method (fees, royalties or a combination of them) and the number of licensees (zero, one or two). The optimal strategy depends on whether or not the market is covered, i.e. whether or not all consumers have to buy the product from one of the firms. In a covered market, the preferred licensing method is via royalties, and both firms become licensees. In an uncovered market the optimal licensing method is a combination of royalties and fixed fees. Bagchi and Mukherjee (2014) show that when a new patented innovation appears, and there is variation in quality among the producers, who have zero opportunity costs, licensing by royalties may be preferable from the point of view of both the innovator and society.

Kamien, Tauman and Zang (1988) also discuss licensing methods for a product innovation. In their model, when an innovation appears, the oligopolistic producers divide into two groups: unlicensed producers that keep producing the old product and licensees that produce the improved product. The 3-stage game described in their model is similar to that in Kamien and Tauman (1986) for the first two stages. In the third stage, the two groups set their quantities simultaneously, where the new and the old product are substitutes from the consumer's point of view. They analyze the case of fixed fee licensing only and found that: (i) when the production cost of the new product is relatively low the innovator sells an exclusive license, which results in monopolization of the market; and (ii) when the production cost is relatively high and the number of firms in the industry is large, the optimal number of licensees is an increasing function of the demand elasticity of the new product with respect to the price of the old product. The first result is similar to that obtained in the current model, where in the case of fixed fees the optimal strategy is to sell an exclusive license. However, in the current model the monopoly result is obtained in the environment of a sequential game in the downstream market where the licensees act as Stackelberg leaders.

The model we analyze considers an innovation, improving upon an existing product, in an oligopolistic market. It differs from previous models in the nature of competition in the product market. Competition is sequential with the patentees moving first. We also introduce a new licensing mechanism and analyze it in addition to the analysis of three other common licensing methods. The three known methods we analyze can be described as follows: In the first one, the patent owner sets a fixed fee, which is independent of the quantities to be produced; the second involves an auction, in which the patent owner sets k, the number of licensees, and accepts the k highest bids for fixed-fee licenses; the third is a per-unit royalty. The new licensing mechanism, involves an auction, similar to that of the second method, whereby the k highest bids for a per-unit royalty are each awarded the right to use the patent.

This model is also innovative in considering technology spillovers from the new product to the old one. The spillovers increase the demand of the overall market. The new product increases the consumers' willing to pay in ε%. The patentees set the new product's quantities and price first. After the production of the new product, the unlicensed firms learn the new product technology, and are able to improve the old product. Hence, the demand for the old product equals the demand for the new product, minus the quantity sold of the new product. Therefore, the sequential environment is an integral part of this model since it allows the learning and the spillover process.

The setup is analyzed as a multi-stage non-cooperative game with $n+1$ players: the patent holder and $\underline{n}$ identical producers. The four-stage game proceeds as follows: In the first stage, the patent holder moves first and sets the payment or, in the case of an auction, the number of licensees. In the second stage, the producers decide whether or not to purchase the license and in the auction scenarios they simultaneously decide on their bid. The third stage takes place in the downstream market, where all the licensees compete via quantities, a standard Cournot model. Once the new product's price has been determined, the consumers who are not willing to pay that price continue to demand the old product. In the final stage, the unlicensed producers compete against one another, again in a standard Cournot setup.

We repeat the above analysis for the case in which the licensees can compete in the final stage, as well, by producing the old product in addition to the new one.

As in Kamien and Tauman (1986), we assume the inverse demand function is: $P=a-Q$, where $a>0$ and Q is the aggregate quantity. We let an improvement of size ε ($\varepsilon>0$) increase the price the consumer is willing to pay by ε percent. Hence, the demand for the new product, following the improvement is given by: $P=(1+\varepsilon)(a-Q)$. The demand for the old product comes from the consumers who decided not to buy the new product, as such it depends on the new product's price as well.

As in Kamien and Tauman (1986), marginal costs are assumed to be constant, and since we focus on changes in the demand side, we normalize them to zero.

In the case where licensees are restricted to producing only the improved product, the main results are as follows:

- The patent owner prefers a fixed fee over royalties.
- The monopoly producer who purchased the license for a fixed fee prefers that method over royalties.
- The unlicensed producers prefer royalties over a fixed fee.
- Consumers prefer a fixed fee over royalties in the regular pricing case and royalties over a fixed fee when *a* is sufficiently large in the auction case.
- In the fixed fee case, the patent holder sells an exclusive license while in the royalties' case he sells approximately *n*/2 licenses.

In the case where licensees can produce both the old and new products:

- Under fixed fee licensing, the patent owner sells one license when n>3 and two licenses when n=2,3. In the royalties' case, he sells n licenses.
- The licensees produce smaller quantities of the new product and sell it for a higher price relative to the previous results.

The paper proceed as follows: Section 2 discusses the four licensing methods when licensees produce only the improved product. Section 3 discusses the scenario in which the licensees can produce both the old and new products. Section 4 offers conclusions and further discussions. The proofs appear in the appendix.

1. Licensees produce only the improved product

In this section, we discuss the scenario in which the licensees can produce only the improved product, while the unlicensed producers continue producing the old product.

1.1 Licensing by Fixed Fee

The Model

The patent owner sets a license fee for his patent in order to sell it to all or some of the n producers and seeks to maximize his profit. A producer will purchase a license if the profit from producing the new produce less the license fee is greater than the profit from producing the old product. The profit from purchasing a license depends, however, on the total number of licenses sold. We describe the interaction between the patent owner and the n firms in the industry as a 4-stage non-cooperative game, denoted as G1. The first two stages of G1 are as in Kamien and Tauman (1986). Thus, in the first stage, the patent holder chooses for each firm i a fixed license fee α_i. Let $\alpha = (\alpha_1, \ldots\ldots, \alpha_n)$, an n-dimensional vector. In the second stage, all of the producers respond simultaneously and independently in deciding whether or not to purchase the license given α. The set N = {1, . . . , n} of producers is therefore partitioned into two subsets: the set S of k licensees and its complement, the set N/S of n-k unlicensed producers who continue to produce the old product. In the third stage, when S becomes common knowledge, the k licensees compete via quantities as in a standard Cournot setup leading to production levels q_i $(i \in S)$ as a function of α and S. In the final stage, the n-k producers who did not purchase a license compete via quantities as in a standard Cournot setup leading again to production levels q_i $(i \notin S)$ as a function of α and S. The demand for the old product is the sum of demands by consumers that decided not to purchase the new product. Therefore, a strategy of the patent owner is an element of $\mathbb{R}_+^n$ while that of a producer *i* is a pair (τ_i, q_i), where τ_i is a decision rule that determines the threshold α_i^* at which the producer purchases the license at price α iff $\alpha<\alpha_i^*$. Hence, $\tau_i: \mathbb{R}_+^n \to \{0,1\}$ where $\tau_i = 1$ if the producer decides to purchase the license and $\tau_i = 0$ otherwise. $q_i = q_i(\alpha, S)$ is the production level of producer *i* as described before.

The players' profit function can now be described as follows: Each set of strategies defines a market with two products, an old and a new one, which are substitutes. The new product is an improvement over the old one in the sense that consumers are more willing to purchase it. Let $\pi_i = \pi_i(\alpha, (\tau_1, q_1), \dots, (\tau_n, q_n))$ be the profit of the *i*th producer. Thus,

$$\pi_i(\alpha, (\tau_1, q_1), \dots, (\tau_n, q_n)) = \begin{cases} p^{old} \cdot q_i & i \notin S \\ p^{new} \cdot q_i - \alpha_i & i \in S \end{cases}$$

where $p^{new} = (1+\varepsilon)(a - \sum_{i \in S} q_i)$ and $p^{old} = p^{new} - (1+\varepsilon) \sum_{i \notin S} q_i$.

The demand for the old product is the residual demand derived from demand for the new product. Therefore, we obtain the following form for the old product demand: $Q^{old} = \frac{p^{new} - p^{old}}{1+\varepsilon}$, which clearly satisfies the two requirements that when $P^{old} = P^{new}$, the quantity demanded of the old product is zero; and when $P^{old} = 0$, the quantity demanded of the old product is (a – (quantity demanded of the new product)), that is the whole residual demand. Therefore, the inverse demand function is derived as above.

The profit of the patent owner is $\pi_{PH}(\alpha, (\tau_1, q_1), \dots, (\tau_1, q_1)) = \sum_{i \in S} \alpha_i$. To maintain simplicity, we ignore the integer constraint and consider the number of licensees as a continuous variable, unless specified otherwise.

Subgame Perfect Nash Equilibrium analysis

Proposition 1

> *The game G1 has a unique SPE, in which $\alpha_i = \alpha_j$ for any i,j.*

We denote the equilibrium strategies by $\sigma^* = (\alpha^*, (\tau_1{}^*, q_1{}^*), \dots, (\tau_n{}^*, q_n{}^*))$ and S^* is the set of producers who buy the license under σ^*. We let $k^* = |S^*|$.

Proposition 2

$$In\ equilibrium,\ k^* = 1,\ \alpha^*{}_i = a^2(1+\epsilon)\left(\frac{1}{4} - \frac{1}{(1+n)^2}\right) for\ any\ i\ and$$

$$\pi_i{}^* = \begin{cases} \dfrac{a^2(1+\epsilon)}{(1+n)^2} & i \in S^* \\ \dfrac{a^2(1+\epsilon)}{4n^2} & i \notin S \end{cases}$$

$$\pi_{PH}{}^* = k^* \cdot \alpha^*{}_i = a^2(1+\epsilon)\left(\frac{1}{4} - \frac{1}{(1+n)^2}\right)$$

Intuitively, the license fee can be at most the difference between the profit obtained by purchasing a license and the profit from producing the old product. Formally, let $\pi_i(k,n)$ be producer i's profit as a function of n and k. Hence, $\alpha^*{}_i(k,n) = \pi_i(k,n)$ - $\pi_j(k-1,n) \quad i \in S \quad j \notin S$ [2], which depends on k. On the other hand, we can think about k as the reaction of the producers to the size of the license fee. The patent owner maximizes his profit given the tradeoff between the size of the license fee and number of licensees. It is shown that the maximum point is generated when k=1, that is, the patent holder sells an exclusive license. This result is similar to that in Kamien and Tauman (1986) and Kamien, Tauman and Zang (1988) but only in the case of a drastic innovation.[3] In the case of a non-drastic innovation, Kamien and Tauman (1986) conclude that the number of licensees is a decreasing function of the importance of the cost reduction technology and Kamien, Tauman and Zang (1988) calculate the number of licensees only as n goes to ∞.

There are three main differences between this model and Kamien, Tauman and Zang (1988). The first is in the way the innovation is modeled. In the current model, the innovation is taken to be demand increasing whereas in Kamien, Tauman and Zang (1988) the innovation is a reduction in the marginal cost of the new product. The second is in the timing of actions, in my model I consider a sequential environment in contrast to the simultaneously competition between the new and the old producers in Kamien, Tauman and Zang (1988). The last difference is the spillover effect I introduce from the technology of the new product to the old product. In Kamien, Tauman and Zang (1988) the old technology did not improve because of the new product appearance.

[2] See Kamien and Tauman (1986) for further details.

[3] An innovation is drastic if the monopoly price of the product, using the new technology, is lower than the marginal cost of production in the old technology.

These three substantial differences lead to different results in the equilibrium outcomes of the two models.

Proposition 3

i. *The profit of all players is increasing in ε.*
ii. *The profit of the monopoly licensee is always higher than that of the other producers.*
iii. *The patent owner's profit is increasing in n while the producers' profit is decreasing in n.*

Notice that in Kamien and Tauman (1986) in the case of a non-drastic innovation the patent owner's profit increases as the innovation improves similar to (i), whereas the profit of the unlicensed producers decreases in improvement size in contrast to (i). Note also that in Kamien and Tauman (1986) the patent owner profit maximizing level of n depends on the innovation size, whereas in the current setting for any innovation size the profit of the patent owner always increases in n.

The following proposition compares the market outcomes before and after the innovation:

Proposition 4

i. *The price of the new product is higher than that of the old product prior to the innovation.*
ii. *When $\varepsilon > \frac{n-1}{n+1}$, the improvement increases the old product's price.*
iii. *Consumers benefit from the improvement.*
iv. *The patent owner benefits from the improvement.*
v. *When $\varepsilon > \frac{(n-1)(3n+1)}{(1+n)^2}$, all the producers benefit from the improvement.*

Remarks:

1. The intuition underlying Claims (ii) and (v) is as follows: When an improvement has been introduced, and the improved product is produced in a monopolistic market, there are two opposing effects on the demand for the old product. On the one hand, some consumers are willing to buy the improved product in the monopoly market, thus reducing demand for the old product; on

the other hand, the demand of all consumers has increased by ε. The second effect will dominate for large enough ε.

2. In Kamien and Tauman (1986), for a non-drastic innovation, each producer is worse off following the innovation, and the consumers are better off.

Auction

Another method of selling licenses is by auction. Denote the non-cooperative game between the patent owner and the *n* producers in this case as G2. In this game, the patent owner again makes the first move by setting the number of potential licensees (at *k*) in the first stage. In the second stage, the producers respond by determining their bids and the *k* highest bidders win the auction, hence obtain a license. The rest of the game's description is similar to that of G1.

Proposition 5

Proposition 1 holds for G2 and, in equilibrium, $k^* = 1$, $\alpha^*{}_i = \frac{a^2(n^2-1)(1+\epsilon)}{4n^2}$ *for any* i, *and*

$$\pi_i{}^* = \frac{a^2(1+\epsilon)}{4n^2} \ \ for\ any\ i$$

$$\pi_{PH}{}^* = k^* \cdot \alpha^*{}_i = \frac{a^2(n^2-1)(1+\epsilon)}{4n^2}$$

By choosing the auction method, the patent owner guarantees himself the entire monopoly surplus. As in Kamien and Tauman (1986), a producer makes a bid when he knows for certain that one bid will win, and the profit if he loses the auction is less than that if he purchases the license in G1. The reason for this is that $\pi_j(0,n) > \pi_j(1,n)$ for any $j \notin S$ and therefore the alternative opportunity costs are lower and fees are higher in the case of an auction (As Katz and Shapiro [1986] have shown). We can see that in equilibrium producers are indifferent between producing the new and the old product. The equilibrium results in the downstream market are similar to those in G1 (proposition 4 excluding part (iv) also holds in G2).

1.2 Licensing by Royalties

We now analyze the case in which the patent owner awards licenses in exchange for per-unit royalties.

The model

We denote the game in this case as G3, which is identical to G1 except that in G3 the fee charged for a license involves per-unit royalties while in G1 there was a fixed license fee. The patent owner chooses the royalty level *r*. We view the royalty as a constant marginal cost. The producers choose whether to produce the new product and pay a per-unit royalty or to continue producing the old product. In order to create positive demand for the licenses, the patent owner should set *r* to be in the interval [0, (1+ε)a]. As a result, the set N = {1, . . . , *n*} of producers is partitioned into two subsets: the set S of *k* licensees and its complement N/S consisting of *n*-*k* unlicensed producers who continue to produce the old product. In the final stage, having observed *r* and *k* the licensees determine q_i, $i \in S$ and the other producers having observed $P_{new} = a - \sum_{i \in S} q_i$ and *k* determine q_i, $i \in N/S$ as in the game G1.

The payoffs are defined by:

$$\pi_i = \begin{cases} p^{old} \cdot q_i, & i \notin S \\ (p^{new} - r) \cdot q_i & i \in S \end{cases}$$

where $p^{new} = (1+\varepsilon)(a - \sum_{i \in S} q_i)$ and $p^{old} = p^{new} - (1+\varepsilon)\sum_{i \notin S} q_i$.

The profit of the patent owner is $\pi_{PH} = r \cdot \sum_{i \in S} q_i$.

We denote by r* and k* the level of the royalty payment and the equilibrium number of licensees in G3, respectively.

Proposition 6

G3 has a unique symmetric equilibrium in which r and k* satisfy the following equations:*

$$(1)\quad \frac{a\left(1+k\big(2-k(2+n)\big)\right)(1+\epsilon)}{\big(-1+k(2+n)\big)^2} = -\frac{r(a-r+a\epsilon)}{k(1+k)(a-2r+a\epsilon)}$$

$$(2)\ r = \frac{a(1+k^2-k(1+n))(1+\epsilon)}{1-k(2+n)}$$

Since $q_i = q_j$ for any $i, j \epsilon S$ we can denote $q = q_i$ for any $i \in S$. The intuition underlying this result is that the equilibrium point (k*,r*) is a tangency point between the reaction function of the producers (how many firms will purchase the license as a function of *r*) and the isoprofit curve $r \cdot \sum_{i=1}^{k} q_i\, , i \in S$. Since the reaction function is concave and the profit is convex, in the relevant region, there is a unique tangency point. As before, each producer is willing to pay for the license as long as his profit with the license exceeds his profit without it. Equation (1) represents the tangency condition while equation (2) represents the reaction function constraint.

The solution of (1) and (2) is given in the appendix.

Proposition 7

i. *k* is independent of both a and ε, and r is proportional to both a and (1+ε).*
ii. *k* is "slightly above" n/2.*
iii. *The patent owner prefers a fixed fee over royalties (as in Kamien and Tauman (1986)).*
iv. *The monopoly producer who purchases the license also prefers a fixed fee over royalties (as in Kamien and Tauman (1986))*
v. *The unlicensed producers prefer royalties over a fixed fee. (In Kamien and Tauman (1986), all the producers purchase a license in the case of royalties.)*

Remark:

The equations for comparing consumer surplus between the licensing methods are highly complicated and therefore numeric examples were used to identify the situations in which consumers prefer a fixed fee over royalties (yielding results similar to Kamien and Tauman (1986)). The numerical examples can be found in the appendix.

Auction

As in the case of a fixed license fee, we can also introduce an auction in the case of royalties. We define the game G4 as follows: In the first stage, the patent owner sets *k*, the number of winning bids. In the second stage, each producer decides on his bid for the license, in terms of a per-unit royalty. The *k* highest bidders are awarded a license. The third and fourth stages are similar to those in G1.

Proposition 8

There is unique equilibrium in G4 which satisfies: $R = \frac{an(1+\epsilon)}{2(1+n)}, k = \frac{n}{2}$.

The intuition is similar to that of proposition 6. Thus, equilibrium is obtained at a tangency point between the producers' reaction function (the bid as a function of k) and the isoprofit curve $r \cdot \sum_{i=1}^{k} q_i, i \in S$. The change from the previous case is that producers will be willing to pay at most the difference between the level of profit with a license when |S|=*k* and the level of profit without a license when |S|=*k*. In the previous case, producers were willing to pay at most the difference between the profit with a license when |S|=*k* and the level of profit without a license when |S|=*k*-1. In this case, the two equations that need to be solved are: $-\frac{k(1+k)(a-2r+a\epsilon)}{r(R-a(1+\epsilon))} = \frac{1+n}{a(1+\epsilon)}$ (tangency) and $k = n - \frac{r(1+n)}{a(1+a)}$ (reaction function).

Therefore, we arrive at the following proposition:

Proposition 9

i. For $n \geq 2$, the patent owner prefers auctioning the licenses using a fixed fee (in which case *k*=1) rather than royalties (in which case *k*=n/2)
ii. For *a*>3, consumer surplus is greater in the case of auctioning the licenses with royalties than in the case of auctioning the licenses with a fixed fee.
iii. For $n \geq 3$, the patent owner's profit is greater in the case of auctioning with a fixed fee than in the case of auctioning with royalties. The opposite is true in the case of *n*=1,2.
iv. The unlicensed producers are better off in the case of auctioning with royalties than in the case of auctioning with a fixed fee.

2. Licensees produce both the old and the new product

In this section, we discuss the case in which licensees can produce both the old and the new product. This captures the often observed scenario where manufacturers sell products based on more than one technology level (cellular phones, personal computers …). In this case, the licensees internalize the effect of the new technology on the manufacturers of the old product. As mentioned before, the new technology has two main effects on them. The first is that they become Stackelberg followers and supply only the residual demand, the second is the increase in demand of the overall market.

2.1 Licensing by Fixed Fee

The Model

In the case of a fixed fee, we describe the interaction between the patent owner and the producers as a game, G5, which is similar to G1 in the first three stages. In stage four, instead of *n-k* producers in Cournot competition we consider *n* producers, since all producers have the option of producing the old product. We define α, N, S, *k*, $\tau_i$as in G1 while q_i is now defined as a pair $q_i = (q_i^{new}, q_i^{old})$ where q_i^{new} is the quantity set in stage 3 and q_i^{old} is the quantity set in stage 4. Notice that $q_i^{new} = 0$ for any $i \notin S$. We again let $\pi_i = \pi_i(\alpha, (\tau_1, q_1), \dots, (\tau_n, q_n))$ be the profit of the i[th] producer. Hence,

$$\pi_i(\alpha, (\tau_1, q_1), \dots, (\tau_1, q_1)) = \begin{cases} p^{old} \cdot q_i^{old} & i \notin S \\ p^{new} \cdot q_i^{new} + p^{old} \cdot q_i^{old} - \alpha_i & i \in S \end{cases}$$

where $p^{new} = (1+\varepsilon)(a - \sum_i q_i^{new})$ and $p^{old} = p^{new} - (1+\varepsilon)\sum_i q_i^{old}$.

The profit of the patent owner is $\pi_{PH}(\alpha, (\tau_1, q_1), \dots, (\tau_1, q_1)) = \sum_{i \in S} \alpha_i$.

Proposition 10

The game G5 has a unique subgame perfect Nash equilibrium in which αi=αj for all i,j.

We denote the equilibrium strategies by $\sigma^* = (\alpha^*, (\tau_1{}^*, q_1{}^*), \dots, (\tau_1{}^*, q_1{}^*))$, and by S^* the set of producers who purchase the license under σ^* , with $k^* = |S^*|$. The next proposition provides the explicit expressions for these strategies.

Proposition 11

i. *When n>3, in equilibrium,* $k^* = 1$, ${\alpha^*}_i = \frac{a^2(-1+n(2+n))^2(1+\epsilon)}{4n(1+n)^2(2+n)}$ *for any i,* *and*

$$\pi_i{}^* = \begin{cases} \frac{a^2(1+n)^2(1+\epsilon)}{4n(2+n)} - {\alpha^*}_i & i \in S^* \\ \frac{a^2(1+n)^2(1+\epsilon)}{4n^2(2+n)^2} & i \notin S \end{cases}$$

$$\pi_{PH}{}^* = k^* \cdot {\alpha^*}_i = \frac{a^2(-1+n(2+n))^2(1+\epsilon)}{4n(1+n)^2(2+n)}$$

ii. *When n=2,3, in equilibrium,* $k^* = 2$, ${\alpha^*}_i = \frac{a^2(-1+n+3n^2+n^3)^2(-1+4n(2+n))(1+\epsilon)}{2n^2(2+n)^2(1-3n(2+n))^2}$ *for any i*

and

$$\pi_i{}^* = \begin{cases} \frac{a^2n(1+n)^2(2+n)(1+\epsilon)}{(1-3n(2+n))^2} - {\alpha^*}_i & i \in S^* \\ \frac{a^2(1+n)^2(1+\epsilon)}{(1-3n(2+n))^2} & i \notin S \end{cases}$$

$$\pi_{PH}{}^* = k^* \cdot {\alpha^*}_i = \frac{a^2(-1+n+3n^2+n^3)^2(-1+4n(2+n))(1+\epsilon)}{2n^2(2+n)^2(1-3n(2+n))^2}$$

When the patent holder sets the license fee, he faces a tradeoff between the fee size and the number of licensees. When the number of producers is low, and any producer in stage 4 (even if he has no license) can sell the old product, the monopoly power enjoyed by a single produce that bought a license in stage 2 is weaker, and therefore the license fee is lower. In this case, the patent holder prefers to set lower fees, and sell to two producers instead of one.

Next, we compare the outcomes for the two cases depending on whether the licensees can choose whether or not to produce the old product.

Proposition 12

a. *For any n*

i. *The licensees produce smaller quantities of the new product and sell it for a higher price in the equilibrium of G5 than in the equilibrium of G1.*

ii. *The profit of the unlicensed producers in G5 in smaller than in G1.*

iii. *The total quantity produced of the old product is higher in G5 than in G1 (although the quantity per producer is lower) and the price is lower.*

b. *For n>3,*

The profit of the monopoly producer in G5 equals his profit in G1.

c. *For* $n \leq 3$

The profit of the monopoly producer in G5 in lower than his profit in G1.

Note that as proposition 12.a.ii implies, if the licensee could have committed not to participate in the market for the old product he would benefit from it. By participating, the licensee lowers the price of the old product, which depresses the demand for his new product in the earlier stage, as consumers anticipate lower prices for the old product.

Auction

The interaction between the patent owner and the n producers in the case that licenses are auctioned is described by the game G6, which is similar to G5 in the downstream market. Thus, in the first stage, the patent owner sets k, the number of potential licensees, and in the second stage the producers decide on their bids, knowing that the k highest bidders will win.

Proposition 13

Proposition 9 holds for G6 and, in equilibrium, $k^* = 1$, $\alpha^*{}_i = \frac{a^2(1+n)^2(-1+n(2+n))(1+\epsilon)}{4n^2(2+n)^2}$ $for\ any\ i,\ \pi_i{}^* = \frac{a^2(1+\epsilon)}{4n^2}$ $for\ any\ i$ *and* $\pi_{PH}{}^* = k^* \cdot \alpha^*{}_i = \frac{a^2(1+n)^2(-1+n(2+n))(1+\epsilon)}{4n^2(2+n)^2}$

Proposition 14

i. *The licensees produce smaller quantities of the new product and sell it for a higher price in the equilibrium of G6 than in the equilibrium of G2.*

ii. *The profit of the monopoly producer in G6 is smaller than in G2.*

iii. *The profit of the unlicensed producer in G6 in smaller than in G2.*

iv. *The total quantity produced of the old product is higher in G6 than in G2 (although the quantity per producer is lower) and the price is lower.*

2.2 Licensing by Royalties

In the case of per-unit royalties, the interaction between the patent owner and the *n* producers is described by the game G7 which is similar to G3 in the first three stages. In stage four, instead of *n-k* producers, we consider *n* producers since all producers have the option of producing the old product. We define α, N, S, k, $\tau_i$as in G3 while q_i is now defined as a pair $q_i = (q_i^{new}, q_i^{old})$ where q_i^{new} are the quantities set in stage 3 and q_i^{old} are the quantities set in stage 4. Notice that $q_i^{new} = 0$ for any $i \notin S$. We again let $\pi_i = \pi_i(\alpha, (\tau_1, q_1), \dots, (\tau_n, q_n))$ be the profit of the $i^{\underline{th}}$ producer. Thus,

$$\pi_i(\alpha, (\tau_1, q_1), \dots, (\tau_1, q_1)) = \begin{cases} p^{old} \cdot q_i^{old} & i \notin S \\ (p^{new} - r) \cdot q_i^{new} + p^{old} \cdot q_i^{old} & i \in S \end{cases}$$

where $p^{new} = (1+\varepsilon)(a - \sum_i q_i^{new})$ and $p^{old} = p^{new} - (1+\varepsilon)\sum_i q_i^{old}$

The profit of the patent owner is $\pi_{PH}(\alpha, (\tau_1, q_1), \dots, (\tau_1, q_1)) = r \cdot \sum_{i \in S} q_i^{new}$.

We denote by r* and k* the royalty level and the number of licensees in the equilibrium of G7, respectively.

Proposition 15

In the equilibrium of G7, k=n and r*=* $\frac{a(-1+n(2+n))(1+\epsilon)}{2(1+n)^2}$.

To see the intuition underlying Proposition 15, we first assume that k*=*n* and then show that in this case the best choice of the patent holder is to set r*= $\frac{a(-1+n(2+n))(1+\epsilon)}{2(1+n)^2}$. Finally we show that under this r*, the best response of the producers is k*=*n*.

Auction

Holding an auction on the royalties leads to zero payoff for the innovator. To see that, assume first that *k*=*n*. In this case, producers win the auction with any bid. Therefore, they bid r=0 in equilibrium and the innovator's profit is zero. When k<n, winning or losing in the auction has no effect on the payoff in stage 4, since the number of licensees is fixed. Therefore, in the auction the royalty each producer bids must equal the equilibrium price of the product given there are k licensees. This in turn implies each licensee produces zero and the innovator's profit is zero as well.

3. Appendix- Proofs

Proof of Proposition 1

The proof of proposition 1 is obtained from the proof of proposition 2 below.

Proof of Proposition 2

In stage 4, as a result of the standard equilibrium outcome in Cournot competition, we obtain: $p^{old} = \frac{p^{new}}{n-k+1}$, $q_i = \frac{p^{new}}{(1+\epsilon)(n-k+1)}$ for $i \notin S$ and the profit of each unlicensed producer is $\frac{(\frac{a}{k+1}(1+\epsilon))^2}{(n-k+1)^2(1+\epsilon)}$. In stage 3, we obtain: $p^{new} = \frac{a}{k+1}(1+\epsilon)$, $\mathrm{q}_i = \frac{a}{k+1}$ for $i \in S$ and the profit of each licensee is $a^2 \frac{(1+\epsilon)}{(k+1)^2}$. The license fee as a function of k is obtained by finding the indifference point for the kth producer between purchasing the license and not purchasing it. Let $\pi_i(k,n)$ be producer i's profit as a function of n and k. Then, $\alpha^*{}_i(k,\mathrm{n}) = \pi_i(k,n)$ -$\pi_j(k-1,n)$ $\quad i \in S \quad j \notin S$ and we get $\alpha_i = \frac{a^2(1+\epsilon)}{(1+k)^2} - \frac{a^2(1+\epsilon)}{k^2(2-k+n)^2}$. This equation also represents the inverse reaction function of producers to the fee level set by the patent owner. The patent owner maximizes $\sum_{i \in S} \alpha_i$ subject to the given reaction function. And since $\underset{n \geq k \in \mathbb{N}^+}{\operatorname{argmax}}\, k \left(\frac{a^2(1+\epsilon)}{(1+k)^2} - \frac{a^2(1+\epsilon)}{k^2(2-k+n)^2}\right) = 1$, the result is a single monopolistic licensee. By substitution, we obtain the profit of each player as in proposition 2

Proof of Proposition 3

Claims i-iii can be easily derived from proposition 2.

Proof of Proposition 4

The price of the product before the innovation is $\frac{a}{n+1}$, and consumer surplus is $\frac{an(a-\frac{a}{1+n})}{2(1+n)}$. The aggregate consumer surplus from both the old and the new product is: $\frac{a^2(1+2(-1+n)n)(1+\epsilon)}{8n^2}$. The producers' profit before the innovation is $\frac{a^2}{(1+n)^2}$. By substitution, we obtain proposition 4.

Proof of Proposition 5

The equilibrium of G2 in stages 3 and 4 is the same as in G1. In stage 2, we obtain the following reaction function for producers: $\alpha^*{}_i(k,\mathrm{n}) = \pi_i(k,n)$ -$\pi_j(k,n)$ $\quad i \in S \quad j \notin S$ since for any *k* the profit of an unlicensed producer is $\pi_j(k,n)$. This is due to the fact that there are *k* licensees regardless of the producer's decision. Hence, the reaction function (i.e. the bid level as a function of *k*) in the case of an auction is $\alpha^*{}_i = \frac{a^2(n^2-1)(1+\epsilon)}{4n^2}$ for any i. In stage 1, the patent owner maximizes his expected profit, taking into account the reaction function. Hence, we obtain $\underset{n\geq k\in\mathbb{N}^+}{\operatorname{argmax}} k\left(\frac{a^2(n^2-1)(1+\epsilon)}{4n^2}\right) = 1$. Substituting $k^*, \alpha^*{}_i$ and the equilibrium of stages 3,4 in the profit function of G2 yields proposition 5.

Proof of Proposition 6

In the equilibrium in stage 4 of G3, we have: $q_i = \frac{p^{new}}{(1+\epsilon)(n-k+1)}$, $\pi_i = \frac{((1+\epsilon)a-r)^2}{(k+1)^2}(\frac{1}{1+\epsilon})$ for any $i \notin S$ and $p^{old} = \frac{p^{new}}{(n-k+1)}$ and in stage 3: $q_i = \frac{(1+\epsilon)a-r}{(1+\epsilon)(k+1)}$, $\pi_i = \frac{p^{new^2}}{(n-k+1)^2(1+\epsilon)}$ for any $i \in S$ and $p^{new} = \frac{(1+\epsilon)a+k*r}{(k+1)}$. Taking into account the same considerations as before, we obtain the reaction function of the producers (the number of licensees as a function of *r*) given as an implicit function: $\pi_i(k,n) - \pi_j(k-1,n) = 0$ for $i \in S$, $j \notin S$. Therefore, we obtain the following equation as the reaction function: $\frac{\frac{(a-r+a\epsilon)^2}{(1+k)^2} - \frac{(a+(-1+k)r+a\epsilon)^2}{k^2(2-k+n)^2}}{1+\epsilon} = 0$. This equation can also be written in the domain $D = \{0 \leq k \leq n, 0 \leq r \leq a(1+\epsilon)\}$ as $r = -\frac{a(1+k^2-k(1+n))(1+\epsilon)}{-1+k(2+n)}$. We can see that the last term is monotonically decreasing and strictly concave in *k* within the domain D. The iso-profit curve of the patent owner is given by: $\pi_{PH} = r \cdot \sum_{i\in S} q_i = \frac{kr(a-r+a\epsilon)}{(1+k)(1+\epsilon)}$ which is convex and has a minimum point in $r = \frac{1}{2}a(1+\varepsilon)$. The patent owner's maximum profit occurs at a tangency point between the iso-profit curve and the reaction function, in the range given by $E = \{0 \leq k \leq n, 0 \leq r \leq \frac{a(1+\epsilon)}{2}\}$. The slope of the iso-profit curve with respect to k in E is: $-\frac{r(a-r+a\epsilon)}{k(1+k)(a-2r+a\epsilon)}$ while the slope of the reaction function with respect to k is $\frac{a\left(1+k\left(2-k(2+n)\right)\right)(1+\epsilon)}{\left(-1+k(2+n)\right)^2}$. Therefore, the optimal point (k*, r*) must solve the equations (1) and (2) appearing in the proposition. The exact solution of (1) and, (2) is given by (I "apologize" for the "long" expressions, I verified these yield the solution in E for several numerical examples):

$$
\begin{aligned}
R = & \\
& -\Big(a \Big(-18 n + 3 n^2 + 6 n^3 + n^4 - 6 n \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{1/3} - \\
& \quad 2 n^2 \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{1/3} + \\
& \quad \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{2/3}\Big) \\
& \Big(-18 n + 3 n^2 + 6 n^3 + n^4 + 6 \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{1/3} + \\
& \quad 3 n \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{1/3} + \\
& \quad n^2 \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{1/3} + \\
& \quad \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{2/3}\Big) (1 + \epsilon)\Big) \Big/ \\
& \Big(36 (2 + n)^2 \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{2/3}\Big),
\end{aligned}
$$

$$
\begin{aligned}
k = & \\
& \frac{1}{6 (2 + n)} \\
& \Big(6 + 3 n + n^2 + (n (3 + n) (-6 + n (3 + n)))\Big) \Big/ \\
& \quad \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{1/3} + \\
& \quad \left(324 + 108 n - 81 n^2 - 27 n^3 + 18 n^4 + 9 n^5 + n^6 + 3 (2 + n) (3 + n) \sqrt{324 - 3 (-3 + n) n (-36 + (-3 + n) n)}\right)^{1/3}\Big)
\end{aligned}
$$

Proof of Proposition 7

Claims (i) and (ii) can be shown directly from the previous equations. Substituting these equations into the equation for the patent owner's profit, i.e. $r \cdot k \cdot q$, we can obtain claim (iii). The profit of a licensee under the royalties and fixed fee regimes can be expressed as $\frac{((1+\epsilon)a-r)^2}{(k+1)^2}(\frac{1}{1+\epsilon})$ and $a^2 \frac{(1+\epsilon)}{4}$ respectively (when k=1 in the fixed fee regime). Substituting k* and r* as above, we obtain claim (iv). The profit of an unlicensed producer under the royalties and fixed fee regimes can be expressed as $\frac{(a+kr+a\epsilon)^2}{(1+k)^2(1-k+n)^2(1+\epsilon)}$ and $\frac{a^2(1+\epsilon)}{4n^2}$ respectively. Substituting k* and r* as above, we obtain claim (v).

Table 3.1 presents numerical examples for consumer surplus in the cases of fixed fee and royalty licensing. Recal that in the case of a fixed fee there is only one licensee, we denote by k the (integer) number of licensees in the case of royaslties:

Table 3.1: numeric examples for consumer surplus in the cases of fixed fee and royalty licensing:

Consumer surplus (fixed fee)	Consumer surplus (royalties)	Patent owner's profit (fixed fee)	Patent owner's profit (royalties)	R (royalties)	k (royalties)	a	ε	n
0.902	0.741	0.977	0.79	0.88	3	2	0.1	5
14.43	11.86	15.64	12.67	3.52	3	8	0.1	5
1.558	1.28	1.688	1.36	1.52	3	2	0.9	5
24.92	20.48	27.02	21.88	6.08	3	8	0.9	5
1.029	0.96	1.08	0.97	1.026	8	2	0.1	15
16.46	15.368	17.32	15.57	4.1	8	8	0.1	15
1.777	1.659	1.87	1.68	1.77	8	2	0.9	15
28.44	26.54	29.92	26.9	7.09	8	8	0.9	15
1.055	1.012	1.092	1.015	1.096	12	2	0.1	24
16.88	16.19	17.487	16.24	4.38	12	8	0.1	24
1.82	1.74	1.887	1.75	1.89	12	2	0.9	24
29.159	27.97	30.2	28.06	7.57	12	8	0.9	24

We can see that the consumer surplus is higher in the case of fixed fee with compere to the royalty case.

Proof of Proposition 8

In equilibrium in stage 4 of G4, we have: $q_i = \frac{p^{new}}{(1+\epsilon)(n-k+1)}$, $\pi_i = \frac{((1+\epsilon)a-r)^2}{(k+1)^2}(\frac{1}{1+\epsilon})$ for any $i \notin S$ and $p^{old} = \frac{p^{new}}{(n-k+1)}$. In equilibrium in stage 3, we have: $q_i = \frac{(1+\epsilon)a-r}{(1+\epsilon)(k+1)}$, , $\pi_i = \frac{p^{new^2}}{(n-k+1)^2(1+\epsilon)}$ for any $i \in S$ and $p^{new} = \frac{(1+\epsilon)a+k*r}{(k+1)}$. Taking into account the same considerations as before, we obtain the reaction function of the producers (i.e. the number of licensees as a function of r) as an implicit function: $\pi_i(k,n) - \pi_j(k,n) = 0$ for $i \in S$, $j \notin S$. Therefore, we obtain the following equation as the reaction function: $\frac{(a-r+a\epsilon)^2 - \frac{(a+kr+a\epsilon)^2}{(1-k+n)^2}}{(1+k)^2(1+\epsilon)} = 0$. This term can be written in the domain D as: $k = n - \frac{r(1+n)}{a(1+\epsilon)}$. This function can be interpreted as the producer's reaction function, which is monotonically decreasing and linear. The slope of the iso-profit curve with respect to r in the domain E is: $\frac{k(a-2r+a\epsilon)}{(1+k)(1+\epsilon)}$ while the slope of the reaction function with respect to r is: $\frac{(1+n)}{a(1+\epsilon)}$. Therefore, the optimal point (k*,r*) must solve the equations $\frac{(1+n)}{a(1+\epsilon)} = \frac{k(a-2r+a\epsilon)}{(1+k)(1+\epsilon)}$, $k = n - \frac{r(1+n)}{a(1+\epsilon)}$. The solution is given by: $r = \frac{an(1+\epsilon)}{2(1+n)}, k = \frac{n}{2}$.

Proof of Proposition 9

The patent owner's payoff in the case of a fixed fee is $\frac{a^2(n^2-1)(1+\epsilon)}{4n^2}$ and in the case of royalties is $\frac{a^2n^2(1+\epsilon)}{4(1+n)^2}$. Comparing these values, we can easily obtain claim (i). The consumer surplus in the case of a fixed fee is $\frac{1}{8}\left(\frac{1-2n}{n^2} + 2a(1+\epsilon)\right)$ and in the case of royalties is: $\frac{a^2n^2(1+\epsilon)}{4(1+n)^2}$. Comparing those values, we obtain claim (ii). The licensees' profit in the case of a fixed fee is $\frac{a^2(-4+n(4+n))^2(1+\epsilon)}{4(2+n)^4}$ and in the case of royalties is: $\frac{a^2(1+\epsilon)}{(1+n)^2}$. Comparing those values, we can obtain claim (iii). The unlicensed producers' profit in the case of a fixed fee is: $\frac{a^2(1+\epsilon)}{4n^2}$ and in the case of royalties is: $\frac{a^2(1+\epsilon)}{(1+n)^2}$. Comparing these values, we obtain claim (iv).

Proof of Proposition 10

The proof of this proposition is obtained from the proof of proposition 11 below.

Proof of Proposition 11

In stage 4, we have: $p^{old} = \frac{p^{new}}{n+1}$, ${q_i}^{old} = \frac{p^{new}}{(1+\epsilon)(n+1)}$ for any i, and the profit of a producer in stage 4 is: (*) $\frac{{p^{new}}^2}{(1+n)^2(1+\epsilon)}$. In stage 3, the licensees maximize their aggregate profit in Stages 3 and 4. Their objective function is: (**) $\frac{{p^{new}}^2}{(1+n)^2(1+\epsilon)} + q_i^{new} \cdot p^{new}$ where $p^{new} = (1+\varepsilon)(a - \sum_i q_i^{new})$. The solution is: $p^{new} =$

$\frac{a(-1+n(2+n))}{(1+n)^2+k(-1+n(2+n))}$, $q_i^{new} = \frac{a(-1+n(2+n))}{(1+n)^2+k(-1+n(2+n))}$ for any $i \in S$. The license fee as a function of k is obtained by finding the indifference point for the *k*th producer between purchasing the license and not purchasing it. As before, $\alpha^*{}_i(k, \mathrm{n}) = \pi_i(k,n)$ - $\pi_j(k-1,n) \quad i \in S \quad j \notin S$. Hence, we obtain:

$$\alpha_i{}^* = a^2(1+n)^2(-\frac{1}{(2+k(-1+n(2+n)))^2} + \frac{n(2+n)}{((1+n)^2+k(-1+n(2+n)))^2})(1+\epsilon).$$

This equation also represents the inverse reaction function of the producer to the fee set by the patent owner. The patent owner maximizes $\sum_{i \in S} \alpha_i$ subject to the given reaction function. And since for *n*>3

$$\underset{n \geq k \in \mathbb{N}^+}{\operatorname{argmax}} k \left(a^2(1+n)^2(-\frac{1}{(2+k(-1+n(2+n)))^2} + \frac{n(2+n)}{((1+n)^2+k(-1+n(2+n)))^2})(1+\epsilon). \right) = 1,$$

by substituting *k*=1 in (*),(**) we obtain claim (i). For *n*=2,3 we obtain $\underset{n \geq k \in \mathbb{N}^+}{\operatorname{argmax}} k \left(a^2(1+n)^2(-\frac{1}{(2+k(-1+n(2+n)))^2} + \frac{n(2+n)}{((1+n)^2+k(-1+n(2+n)))^2})(1+\epsilon). \right) = 2$ and substituting *k*=2 in (*),(**) we obtain claim (ii).

Proof of Proposition 12

All the claims in Proposition 12 can easily be shown by substituting the relevant values in the profit, quantity and price functions.

Proof of Proposition 13

The results for equilibrium in Stages 3 and 4 are similar to those for G5. The relationship between k and α_i is given by: $\alpha^*{}_i(k, \mathrm{n}) = \pi_i(k,n)$ -$\pi_j(k,n) \quad i \in S \quad j \notin S$. Hence, we obtain

$\alpha_i{}^* = \frac{a^2(1+n)^2(-1+n(2+n))(1+\epsilon)}{((1+n)^2+k(-1+n(2+n)))^2}$. Since $\underset{n \geq k \in \mathbb{N}^+}{\operatorname{argmax}} k \frac{a^2(1+n)^2(-1+n(2+n))(1+\epsilon)}{((1+n)^2+k(-1+n(2+n)))^2} = 1$, we obtain $k^* = 1$.

Proof of Proposition 14

The proof of this proposition is similar to that of Proposition 12, except for claim (ii) since the difference between the two propositions is the license fee. Note also that this proposition holds for any *n*>1 and not only for *n*>3, as in Proposition 12. We prove claim (ii) by substituting $\alpha_i{}^* = \frac{a^2(1+n)^2(-1+n(2+n))(1+\epsilon)}{((1+n)^2+k(-1+n(2+n)))^2}$ for $\alpha_i{}^* = a^2(1+n)^2(-\frac{1}{(2+k(-1+n(2+n)))^2} + \frac{n(2+n)}{((1+n)^2+k(-1+n(2+n)))^2})(1+\epsilon)$, as in Proposition 12.

Proof of Proposition 15

The results for equilibrium in stage 4 of G7 are similar to those for G5. In Stage 3, the licensee maximizes his profit in period 3 and 4. His target function is: (***) $\frac{p^{new\,2}}{(1+n)^2(1+\epsilon)} + q_i^{new} \cdot (p^{new} - r)$ where $p^{new} = (1+\varepsilon)(a - \sum_i q_i^{new})$. The maximization solution is: $p^{new} = \frac{(1+n)^2(a+kr+a\epsilon)}{(1+n)^2+k(-1+n(2+n))}, q_i^{new} = q^{new} = \frac{-(1+n)^2r+a(-1+n(2+n))(1+\epsilon)}{((1+n)^2+k(-1+n(2+n)))(1+\epsilon)}$ for any $i \in S$.

Assume k*=*n*. The patent owner's profit function is then: $\pi_{PH} = r \cdot k \cdot q^{new}$. Under the previous assumption, we obtain $r^* = \frac{a(-1+n(2+n))(1+\epsilon)}{2(1+n)^2}$. Now assume that $r^* = \frac{a(-1+n(2+n))(1+\epsilon)}{2(1+n)^2}$. Then, a licensee’s profit is: $\frac{a^2(1+n(12+n(28+n(4+n)(13+2n(4+n)))))(1+\epsilon)}{4(1+n)^2(1+n+3n^2+n^3)^2}$ where the profit of an unlicensed producer is: $\frac{a^2(2+n(1+n)(3+n))^2(1+\epsilon)}{4(1+n)^2(1+n+3n^2+n^3)^2}$. Hence, we obtain that for any *n*>1, *a*>0, ε>0, the profit of a licensee is higher than that of a non-licensed producer and therefore all producers will purchase the license, i.e. k*=*n*.

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