A monotonic core solution for convex TU games

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Abstract

We find that the answer to the open question of whether there is a continuous core solution that satisfies coalitional monotonicity in the class of convex games is yes. We prove that the SD-prenucleolus is the only known continuous core solution that satisfies coalitional monotonicity for convex games, a class of games widely used to model economic situations.

Keywords: Convex games, prenucleolus, core, monotonicity
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1 Introduction

One of the main issues analyzed in the literature of coalitional game theory is how to divide the outcome obtained by agents that cooperate. One approach to dealing with the problem consists of proposing rules or solutions that are used to solve the game. In this approach, the Shapley value (Shapley, 1953) and the prenucleolus (Schmeidler, 1969) stand out as the best-known, most widely analyzed single-valued solutions for coalitional games with transferable utility (TU games). One of the main reasons for the attractiveness of the Shapley value lies in the fact that it respects the principle of monotonicity, i.e. if a new TU game w is obtained from a given TU game v by increasing the worth of a coalition S then the members of S receive a payoff in game w that is no lower than in game v. On the other hand, the prenucleolus respects the core stability principle, i.e. the prenucleolus selects a core allocation whenever the game is balanced. A core allocation provides each coalition with at least the worth of the coalition, the amount that the members of the coalition can obtain by themselves. It seems very attractive to ask for a solution that fulfils both principles, since they share a kind of incentive compatibility principle that can be summarized in the following idea: the higher the worth of a coalition the better for its members. However, in the class of balanced games they are not compatible (Young, 1985) and therefore, the Shapley value does not respect core stability and the prenucleolus fails to satisfy coalitional monotonicity.

There is no such incompatibility if the analysis is restricted to the class of convex games where the Shapley value (Shapley, 1953) satisfies core stability.[1] The prenucleolus and the per capita prenucleolus do not satisfy coalitional monotonicity in this class. These results motivate the question that this paper seeks to solve: Is there any continuous[2] core solution that satisfies

[1] Note also that Dutta and Ray (1989) introduce a solution for convex games that satisfies core stability and coalitional monotonicity.

[2] With the requirement of continuity we avoid core solutions defined as follows. Let ϕ be a solution that coincides with the Shapley value if the game is convex and with the prenucleolus otherwise. Then ϕ satisfies coalitional monotonicity in the class of convex games but is not continuous.

coalitional monotonicity in the class of convex games? The answer is yes: the SD-prenucleolus (the Surplus Distributor prenucleolus).

This solution, introduced by Arin and Katsev (2014), also satisfies aggregate monotonicity and coalitional monotonicity for games with veto players. The SD-prenucleolus is the only known concept that satisfies coalitional monotonicity for convex games while respecting the principle of core stability, that is, it selects a core allocation whenever the game is balanced.

The rest of the paper is organized as follows: Section 2 introduces TU games, solutions and properties. Section 3 provides the definition of the SD-prenucleolus of a game and introduces the SD-reduced game property. Section 4 proves that the SD-prenucleolus satisfies coalitional monotonicity in the class of SD-relevant games. Section 5 proves that convex games are SD-relevant games. Section 6 ends the paper with some concluding remarks.

2 Preliminaries: TU games

A *cooperative n-person game in characteristic function form* is a pair (N, v), where N is a finite set of n elements and $v : 2^N \to \mathbb{R}$ is a real-valued function defined on the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of N is called a *coalition*. A game is *monotonic* if whenever $T \subset S$ then $v(T) \leq v(S)$. The number of players in S is denoted by $|S|$. Given $S \subset N$ we denote by $N \backslash S$ the set of players of N that are not in S. A distribution of $v(N)$ among the players, an *allocation*, is a real-valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i. A distribution satisfying $\sum\limits_{i \in N} x_i = v(N)$ is called an *efficient allocation* and the set of efficient allocations is denoted by $X(N, v)$. We denote $\sum\limits_{i \in S} x_i$ by $x(S)$. The core of a game is the set of efficient allocations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{x \in X(N, v) : x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

A game is balanced when it has a nonempty core (Bondareva (1963) and Shapley (1967)). We say that a game (N, v) is convex if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subseteq N.$$

Let Γ_0 be a nonempty set of games. A solution φ on Γ_0 is a correspondence that associates a set $\varphi(N, v)$ in $\mathbb{R}^N$with each game (N, v) in Γ_0 such that $x(N) \leq v(N)$ for all $x \in \varphi(N, v)$. This solution is *efficient* if the previous inequality holds with equality. We say that a solution satisfies *core stability in* Γ_0 if it selects core allocations whenever the game is balanced and belongs to Γ_0.

The solution is *single-valued* if the set contains a *single* element for each game in the set of games. A *core solution* is a single-valued solution concept that selects a core allocation whenever the game is balanced.

We say that the vector x *weakly lexicographically dominates* the vector y (denoted by $x \succeq_L y$) if either $\widetilde{x} = \widetilde{y}$ or there is k such that $\widetilde{x}_i = \widetilde{y}_i$ for all $i \in \{1, 2, ..., k-1\}$ and $\widetilde{x}_k > \widetilde{y}_k$ where $\widetilde{x}$ and $\widetilde{y}$ are the vectors with the same components as the vectors x, y, but rearranged in a non decreasing order $(i > j \Rightarrow \widetilde{x}_i \geq \widetilde{x}_j)$.

Given a game (N, v), a coalition $S \subset N$ and $x \in X(N, v)$, the *satisfaction of S with respect to x* is defined as $f(S, x) = x(S) - v(S)$. Let $\theta(x)$ be the vector of all satisfactions at x arranged in non decreasing order. Schmeidler (1969) introduced the *prenucleolus*[3] of a game v, denoted by $PN(N, v)$, as the unique allocation that lexicographically maximizes the vector of non decreasingly ordered satisfactions over the set of allocations. In formula:

$$PN(N, v) = \{x \in X(N, v) \,|\theta(x) \succeq_L \theta(y) \text{ for all } y \in X(N, v)\}.$$

The *per capita prenucleolus* (Grotte, 1970) is defined analogously by using the concept of per capita satisfaction instead of satisfaction. Given S, $S \neq \emptyset$, and x the *per capita satisfaction* of S at x is

$$f^{pc}(S, x) = \frac{x(S) - v(S)}{|S|}$$

[3]The solution concept was defined using the notion of excess instead of satisfaction. Given a game (N, v) and an allocation x, the *excess of a coalition S with respect to x* in game (N, v) is defined as follows: $e(S, x) := v(S) - x(S)$.

The prenucleolus and the per capita prenucleolus are core solutions. Other *weighted prenucleoli* can be defined in a similar way whenever a weighted satisfaction function is defined.

For two-person games the prenucleolus and the per capita prenucleolus coincide with the *standard solution* that allocates to each player the sum of the worth of his/her individual coalition and half of the surplus of the game (the difference between the worth of the grand coalition and the sum of the worth of individual coalitions).

The following monotonicity properties are defined for single-valued solutions.

- φ satisfies *coalitional monotonicity in* Γ_0 if the following condition holds: If $v, w \in \Gamma_0$, $v(T) < w(T)$ for some $T \subseteq N$ and $v(S) = w(S)$ for all $S \in 2^N \setminus \{T\}$ then $\varphi_i(N, v) \leq \varphi_i(N, w)$ for all $i \in T$.

- φ satisfies *aggregate monotonicity in* Γ_0 if the following condition holds: If $v, w \in \Gamma_0$, $v(N) < w(N)$ and $v(S) = w(S)$ for all $S \in 2^N \setminus \{N\}$ then $\varphi_i(N, w) - \varphi_i(N, v) \geq 0$ for all $i \in N$.

- φ satisfies *strong aggregate monotonicity in* Γ_0 if the following condition holds: If $v, w \in \Gamma_0$, $v(N) < w(N)$ and $v(S) = w(S)$ for all $S \in 2^N \setminus \{N\}$ then $\varphi_i(N, w) - \varphi_i(N, v) = \varphi_j(N, w) - \varphi_j(N, v) \geqslant 0$ for all $i, j \in N$.

Young (1985) proves that no core solution satisfies coalitional monotonicity in the class of balanced games. However, there are core solutions, including the per capita prenucleolus and the SD-prenucleolus (defined in Subsection 3.1), that satisfy strong aggregate monotonicity. The prenucleolus does not satisfy aggregate monotonicity in the class of convex games (Hokari, 2000). The per capita prenucleolus does not satisfy coalitional monotonicity in the class of convex games (see Arin, 2013).

The following notation is widely used in this work. We denote by (N, b_S) the game whose characteristic function is

$$b_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise.} \end{cases}$$

Given two TU games (N, v) and (N, w) with the same set of players we denote by $(N, v + w)$ a new game where $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$. Given a game (N, v) and a real number λ we denote by $(N, \lambda v)$ a new game where $(\lambda v)(S) = \lambda v(S)$ for all $S \subseteq N$.

3 The SD-prenucleolus

3.1 Definition

In 2014, Arin and Katsev introduce the SD-prenucleolus, a single-valued solution concept for TU games. In this section we recall some definitions and results that are needed in the present paper.

The definition of the SD-prenucleolus is based on the concept of satisfaction of a coalition with respect to an allocation. Given a game (N, v) and an allocation $x \in X(N, v)$ we calculate a new satisfaction vector $(F(S, x))_{S \subset N}$. We define the components of this vector recursively by defining an algorithm.

The algorithm has several steps (at most $2^{|N|} - 2$) and at each step we identify the collection of coalitions $\mathcal{H}$ that has obtained the satisfaction. In the first step this collection $\mathcal{H}$ is empty. The algorithm ends when $\mathcal{H} = 2^N \setminus \{N\}$.

For a collection $\mathcal{H} \subset 2^N \setminus \{N\}$ and a function $F : \mathcal{H} \to \mathbb{R}$ we will define the function $F_{\mathcal{H}} : 2^N \setminus \{\mathcal{H} \cup \{N\}\} \to \mathbb{R}$. To this end, we introduce some notation. For $\mathcal{H} \subset 2^N \setminus \{N\}$ and $S \subset N$, we denote

$$\sigma_{\mathcal{H}}(S) = \bigcup_{T \in \mathcal{H}, T \subset S} T.$$

For $S \subset N$ we denote by $f_{\mathcal{H},F}(i, S)$ the satisfaction of player i with respect to a coalition S and a collection $\mathcal{H}$ ($i \in \sigma_{\mathcal{H}}(S)$):

$$f_{\mathcal{H},F}(i, S) = \min_{T : T \in \mathcal{H}, i \in T \subset S} F(T).$$

Now we define a value $F_{\mathcal{H}}(S)$ for all $S \in 2^N \setminus \{\mathcal{H} \cup \{N\}\}$. We consider two cases (since it is evident that $\sigma_{\mathcal{H}}(S) \subseteq S$):

1. Relevant coalitions. $\sigma_{\mathcal{H}}(S) \neq S$. In this case the satisfaction of S is

$$F_{\mathcal{H}}(S) = \frac{x(S) - v(S) - \sum\limits_{i \in \sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i,S)}{|S| - |\sigma_{\mathcal{H}}(S)|}. \tag{1}$$

Note that if collection $\mathcal{H}$ is empty then the current satisfaction of coalition S coincides with its per capita satisfaction:

$$F_{\emptyset}(S) = \frac{x(S) - v(S)}{|S|}.$$

2. Completed coalitions. $\sigma_{\mathcal{H}}(S) = S$. In this case the satisfaction of S is

$$F_{\mathcal{H}}(S) = x(S) - v(S) - \sum_{i \in S} f_{\mathcal{H},F}(i,S) + \max_{i \in S} f_{\mathcal{H},F}(i,S). \tag{2}$$

The algorithm that computes the new satisfaction vector, whose components are denoted by $F(S) = F(S,x)$, is the following:

Algorithm 1 *Let (N,v) be a TU game and $x \in X(N,v)$.*
Step 1: *Set $k = 0$, $\mathcal{H}_0 = \emptyset$ and $F_{\emptyset}(S) = \frac{x(S)-v(S)}{|S|}$.*
Step 2: *Set*

$$\mathcal{H}_{k+1} = \mathcal{H}_k \cup \{S \notin \mathcal{H}_k : F_{\mathcal{H}_k}(S) = \min_{T \notin \mathcal{H}_k} F_{\mathcal{H}_k}(T)\}.$$

Define for each $S \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$:

$$F(S,x) = F_{\mathcal{H}_k}(S).$$

Step 3: *If $\mathcal{H}_{k+1} \neq 2^N \setminus \{N\}$ then let $k = k+1$ and go to Step 2. Otherwise go to Step 4.*
Step 4: *Stop. Return the vector $(F(S,x))_{S \subset N}$.*

We say that a coalition S is *relevant* (*completed*) with respect to allocation x if S was a relevant (completed) coalition at the step where its satisfaction $F(S,x)$ was determined. Given a TU game (N,v), a player $i \in S \subset N$ and $x \in X(N,v)$ we denote by $f_i(S,x) = \min\limits_{T: i \in T \subseteq S} F(T,x)$ and $z_i(S,x) = x_i - f_i(S,x)$ the satisfaction and the coalitional payoff of player i in coalition S at x. If there is no confusion we write $f_i(S)$ and $z_i(S)$ instead of $f_i(S,x)$ and $z_i(S,x)$. Arin and Katsev (2014) prove the following lamé that is used in the proof of Lemma 3.

Lemma 2 *(Arin and Katsev, 2014) Let (N, v) be a TU game and $x \in X(N, v)$. Let function F be the result of Algorithm 1 and let $\{\mathcal{H}_i\}_{i \in \{1,\ldots,k\}}$ be the associated collections of sets. If $S \in \mathcal{H}_i$, $T \notin \mathcal{H}_i$ then $F(T) > F(S)$.*

The lemma below proves that the *surplus of a coalition*[4] $(x(S) - v(S))$ is fully divided among its members if the coalition is relevant. This is not the case with completed coalitions, where the surplus of the coalition is higher than the sum of the surpluses of the members of the coalition.

Lemma 3 *Let (N, v) be a TU game and $x \in X(N, v)$. Then*

1. If $S \subset N$ is relevant then $\sum\limits_{i \in S} f_i(S) = x(S) - v(S)$.

2. If $S \subset N$ is completed then $\sum\limits_{i \in S} f_i(S) < x(S) - v(S)$.

3. If $S \subset T \subset N$ then $f_i(S) \geq f_i(T)$ for all $i \in S$.

Proof. Let k be the step of the Algorithm 1 where the satisfaction of coalition S has been defined.

1. By definition (1),

$$x(S) - v(S) = F_{\mathcal{H}_k}(S)(|S| - |\sigma_{\mathcal{H}_k}(S)|) + \sum_{i \in \sigma_{\mathcal{H}_k}(S)} f_{\mathcal{H}_k,F}(i, S).$$

By Lemma 2, $f_i(S) = F_{\mathcal{H}_k}(S) = F(S)$ for all $i \in S \backslash \sigma_{\mathcal{H}_k}(S)$ and $F_{\mathcal{H}_k}(S) > f_{\mathcal{H}_k,F}(i, S) = f_i(S)$ for all $i \in \sigma_{\mathcal{H}_k}(S)$. Therefore,

$$x(S) - v(S) = \sum_{i \in S \backslash \sigma_{\mathcal{H}_k}(S)} f_i(S) + \sum_{i \in \sigma_{\mathcal{H}_k}(S)} f_i(S) = \sum_{i \in S} f_i(S).$$

2. By definition (2),

$$x(S) - v(S) = \sum_{i \in S} f_{\mathcal{H}_k,F}(i, S) + F_{\mathcal{H}_k}(S) - \max_{i \in S} f_{\mathcal{H}_k,F}(i, S).$$

[4] The name Surplus Distributor prenucleolus reflects this fact. The Algorithm establishes how the surplus of a coalition is divided among its members. In Arin and Katsev (2014), it is discussed the difference between the per capita prenucleolus and the SD-prenucelolus with respect to this fact. Note that the pre capita prenucleolus is based on the idea of equal division of the surplus among the members of the coalition.

By Lemma 2, $f_{\mathcal{H}_k,F}(i,S) = f_i(S)$ for all $i \in S$ and $F_{\mathcal{H}_k}(S) > \max_{i \in S} f_{\mathcal{H}_k,F}(i,S)$. Therefore,

$$\sum_{i \in S} f_i(S) < x(S) - v(S).$$

3. The result is immediately apparent. ■

The SD-prenucleolus is a lexicographic value that selects an optimal element in the set of vectors of satisfactions. We denote the SD-prenucleolus of game (N,v) by $SD(N,v)$.

Given a TU game (N,v) and an allocation $x \in X(N,v)$, let F^x be the vector of all satisfactions at x arranged in non decreasing order. We say that the vector x belongs to the SD-prenucleolus if its satisfaction vector dominates (or weakly dominates) every other satisfaction vector.

Definition 4 *(Arin and Katsev, 2014) Let (N,v) be a TU game. Then $x \in SD(N,v)$ if and only if for any $y \in X(N,v)$ it holds that $F^x \succeq_L F^y$.*

The SD-prenucleolus is a core solution that satisfies continuity, strong aggregate monotonicity and other interesting properties (see Arin and Katsev, 2014). In this paper, it is also established the equivalent of Kohlberg's theorem (Kohlberg, 1971) for the SD-prenucleolus. The theorem is useful for checking whether an allocation is the SD-prenucleolus of a game or not.

Given a game (N,v), an allocation $x \in X(N,v)$ and a real number α we define the following set of coalitions

$$\mathcal{B}_\alpha(x) = \{S \subset N : F(S,x) \leq \alpha\}.$$

Theorem 5 *(Arin and Katsev, 2014) Let (N,v) be a TU game and $x \in X(N,v)$. Then $x = SD(N,v)$ if and only if the collection of sets $\mathcal{B}_\alpha(x)$ is empty or balanced[5] for all α.*

We now introduce the class of SD-relevant games, a class that includes the class of convex games.

[5]Let N be a finite set of players and let $\mathcal{C}$ be a collection of distinct nonempty subsets of N. Define $\mathcal{C}_i = \{S \in \mathcal{C} : i \in S\}$. $\mathcal{C}$ is said to be a balanced collection of sets over N if there exist positive numbers $(\lambda_S)_{S \in \mathcal{C}}$ such that $\sum_{S \in \mathcal{C}_i} \lambda_S = 1$ for all $i \in N$.

Definition 6 *A TU game (N, v) is SD-relevant if any S, $S \subset N$, is relevant with respect to $SD(N, v)$.*

In the following, we introduce a balanced game that is not SD-relevant. Let (N, v) be a TU game where $N = \{1, 2, 3, 4\}$ and

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \text{ or } S \in \{\{1,2\}, \{3,4\}\} \\ 4 & \text{if } S = N \\ 2 & \text{otherwise.} \end{cases}$$

It can be checked that $SD(N, v) = (1, 1, 1, 1)$, $F(\{1, 2, 3\}) = 1$ and $F(\{1, 3\}) = F(\{2, 3\}) = 0$. Therefore, coalition $\{1, 2, 3\}$ is completed with respect to the allocation $(1, 1, 1, 1)$. Consequently, (N, v) is not SD-relevant.

3.2 SD-relevant games and SD-reduced game property

Like the prenucleolus and other lexicographic values, the SD-prenucleolus satisfies a consistency property called the SD-reduced game property. This property is based on the notion of the SD-reduced game, which we now introduce.

Definition 7 *Let (N, v) be a TU game, $S \subset N$ and $x \in X(N, v)$. A game (S, v^x) is the SD-reduced game with respect to S and x if*

1. $v^x(S) = v(N) - x(N \setminus S)$
2. *for every $T \subset S$*

$$F^{(S,v^x)}(T, (x_i)_{i \in S}) = \min_{U \subseteq N \setminus S} F^{(N,v)}(U \cup T, x)$$

where $F^{(S,v^x)}(T, (x_i)_{i \in S})$ is the satisfaction in game (S, v^x) of coalition T at $(x_i)_{i \in S}$ and $F^{(N,v)}(U \cup T, x)$ is the satisfaction in game (N, v) of coalition $U \cup T$ at x.

As a consequence of Lemma 8, for any game (N, v), any coalition $S \subset N$ and any allocation $x \in X(N, v)$ the SD-reduced game (S, v^x) exists and is unique.

Lemma 8 *Let N be a finite set of players, $x \in \mathbb{R}^N$, let V be a real number and $f \in \mathbb{R}^{2^N \setminus \{N\}}$. Then there is a unique TU game (N, v) such that*

1. $v(N) = V$
2. $(F(S, x))_{S \subset N} = f$.

The proof of the Lemma is immediately apparent and therefore omitted.

We say that a solution ϕ satisfies the *SD-reduced game property* on Γ, SD-RGP, if for every game $(N, v) \in \Gamma$ then $(x_i)_{i \in S} \in \phi(S, v^x)$ for any $S \subset N$ and any $x \in \phi(N, v)$.

The SD-prenucleolus satisfies the SD-reduced game property in the class of all TU games. This type of property[6] plays a determinant role in the characterization of lexicographic values such as the prenucleolus (Sobolev, 1975) and the per capita prenucleolus (Kleppe, 2010). The reduced games associated with the prenucleolus and the per capita prenucleolus can be reformulated explicitly.

If a game is SD-relevant then any SD-reduced game with respect to the SD-prenucleolus of the game is also SD-relevant. Therefore, in the class of SD-relevant games the SD-reduced games with respect to the SD-prenucleolus belong to this class.

Lemma 9 *Let (N, v) be an SD-relevant game, $S \subset N$ and $x = SD(N, v)$. Then (S, v^x) is SD-relevant.*

Proof. Let $y = SD(S, v^x)$ and let P and M be two subsets of S such that $M \cup P \neq S$. Assume, without loss of generality, that $F(M, y) \geq F(P, y)$. We seek to prove that

$$F(M \cup P, y) \leq \max\{F(M, y), F(P, y)\} = F(M, y).$$

Let $F(M, y) = F(M \cup Q, x)$ and $F(P, y) = F(P \cup T, x)$ where $Q, T \subseteq N \setminus S$. Note that $(M \cup Q) \cup (P \cup T) \neq N$. Since all coalitions in the game (N, v) are relevant,

$$F((M \cup Q) \cup (P \cup T), x) \leq$$

[6] Note that the definition of this reduced game depends on the definition of the vector of satisfactions. If the vector of satisfactions considered is $(x(S) - v(S))_{S \subset N}$ then the associated reduced game property is satisfied by the prenucleolus (see Theorem 5.2.7 in Peleg and Sudholter (2007)).

$$\max\{F(M \cup Q, x), F(P \cup T, x)\} = F(M \cup Q, x).$$

Note that $(M \cup Q) \cup (P \cup T) = (M \cup P) \cup (Q \cup T)$. Therefore,

$$F(M \cup P, y) \leq F((M \cup Q) \cup (P \cup T), x) \leq$$

$$\leq F(M \cup Q, x) = F(M, y).$$

Consequently, (S, v^x) is SD-relevant. ■

Lemma 9 allows for a different interpretation of the SD-reduced game of an SD-relevant game. SD-reduced games with respect to the SD-prenucleolus of the game can be easily computed according to the result established by the following lemma.

Lemma 10 *Let (N, v) be an SD-relevant game, $S \subset N$ and $x = SD(N, v)$. Consider the SD-reduced game (S, v^x) and $T \subset S$. Then*

$$v^x(T) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} z_i(T \cup (N \setminus S)) = \sum_{i \in T} z_i(T \cup (N \setminus S)).$$

Proof. By Lemma 9, (S, v^x) is SD-relevant. For game (S, v^x) we denote by $f^{x,S}$ the analog of function f in Algorithm 1. By definition of $f_i^{x,S}(T)$ it holds that

$$f_i^{x,S}(T) = \min_{i \in U \subseteq T} F^{(S,v^x)}(U) = \min_{i \in U \subseteq T} \min_{R \subseteq N \setminus S} F(U \cup R) =$$

$$= \min_{i \in M \subseteq T \cup (N \setminus S)} F(M) = f_i(T \cup (N \setminus S)).$$

Hence,

$$v^x(T) = x(T) - \sum_{i \in T} f_i^{x,S}(T) = x(T) - \sum_{i \in T} f_i(T \cup (N \setminus S)) =$$

$$= \sum_{i \in T} z_i(T \cup (N \setminus S)).$$

Since coalition $T \cup (N \setminus S)$ is relevant in the game (N, v), by Lemma 3 $v(T \cup (N \setminus S)) = \sum_{i \in T \cup (N \setminus S)} z_i(T \cup (N \setminus S))$. Therefore,

$$\sum_{i \in T} z_i(T \cup (N \setminus S)) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} z_i(T \cup (N \setminus S)).$$

■

The corollary below presents a simple formula for computing some SD-reduced games. This result is used in the proof of the main theorem.

Let (N,v) be a TU game and $x \in X(N,v)$. We denote by $\mathcal{B}(x)$ the set of coalitions with minimal satisfaction with respect to x.

Corollary 11 *Let (N,v) be an SD-relevant game, $x = SD(N,v)$ and $S \in \mathcal{B}(x)$. Consider the SD-reduced game $(N \backslash S, v^x)$ and coalition $T \subset N \backslash S$. Then $v^x(T) = v(T \cup S) - v(S)$.*

Proof. Since $S \in \mathcal{B}(x)$, it holds that $f_i(S) = \frac{x(S)-v(S)}{|S|}$ and for any $i \in T$ such that $S \subset T$ it holds that $f_i(T) = f_i(S)$. By applying Lemma 10,

$$v(T \cup S) - v^x(T) = \sum_{i \in S} z_i(T \cup S) = \sum_{i \in S} x_i - \sum_{i \in S} f_i(T \cup S) =$$

$$\sum_{i \in S} (x_i - f_i(S)) = x(S) - |S| \frac{x(S) - v(S)}{|S|} = v(S).$$

■

3.3 Antipartitions and SD-equivalent games

The notion of antipartition (Arin and Inarra, 1998) also plays a central role in the proof of the main result of this paper. A collection of sets $\mathcal{C} = \{S : S \subset N\}$ is called *antipartition* if the collection of sets $\{N \backslash S : S \in \mathcal{C}\}$ is a partition of N. An antipartition is a balanced collection of sets. In order to balance an antipartition $\mathcal{Q}$ each coalition receives the same weight, i.e. $\frac{1}{|\mathcal{C}|-1}$. Given a TU game (N,v) the *satisfaction of an antipartition* $\mathcal{C}$ is defined by

$$\mathcal{F}(\mathcal{C}, v) = \frac{v(N) - \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}|-1} v(S)}{|N|}. \tag{3}$$

If there is no confusion we write $\mathcal{F}(\mathcal{C})$ instead of $\mathcal{F}(\mathcal{C}, v)$. Given a TU game (N,v) and $x \in X(N,v)$ we denote by $\mathcal{B}(x)$ the set of coalitions with minimal satisfaction at x.

Lemma 12 *Let (N,v) be a TU game and $x \in X(N,v)$. Let $\mathcal{C}$ be an antipartition contained in $\mathcal{B}(x)$. Then $F(S) = \mathcal{F}(\mathcal{C})$ for all $S \in \mathcal{B}(x)$.*

Proof. It is emmediately apparent that for $S \in \mathcal{C}$,

$$F(S) = \frac{x(S) - v(S)}{|S|} = \alpha.$$

Since $\mathcal{C}$ is balanced, $\sum_{S \in \mathcal{C}} \lambda_s x(S) = \sum_{i \in N} x_i = v(N)$. By definition in (3), $|N| \mathcal{F}(\mathcal{C}) = v(N) - \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}|-1} v(S)$. Since $\mathcal{C}$ is balanced,

$$v(N) - \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}|-1} v(S) = \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}|-1} (x(S) - v(S)) =$$

$$\sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}|-1} |S| F(S) = \alpha \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}|-1} |S| = \alpha |N|.$$

This last equality is a consequence of the fact that each player is present in all coalitions of the antipartition but one. ■

Note that if the set of coalitions with minimal satisfaction with respect to the SD-prenucleolus of the game contains an antipartition then the satisfaction of those coalitions only depends on the characteristic function of the game.

Arin and Inarra (1998) prove that, given a convex game, the collection of coalitions with minimal satisfaction with respect to the prenucleolus of the game contains either a partition or an antipartition. In the case of the SD-prenucleolus of SD-relevant games only antipartitions should be considered, as the following lemma shows.

Lemma 13 *Let (N,v) be an SD-relevant game. Then $\mathcal{B}(SD(N,v))$ contains an antipartition.*

Proof. Let $x = SD(N,v)$ and let S be a maximal coalition in $\mathcal{B}(x)$, that is, there is no coalition T in $\mathcal{B}(x)$ such that $S \subset T$. Since $\mathcal{B}(x)$ is balanced (Theorem 5), for each $i \in S$ there exists a coalition, T^i, such that $i \notin T^i$ and $T^i \in \mathcal{B}(x)$. Since (N,v) is SD-relevant, the maximality of S implies that $N \backslash S \subset T^i$. Let $\{T^i : i \in S\}$ be the set of maximal coalitions for each i in S

((it may occur that $T^i = T^j$ for players i, j). Then $\{T^i : i \in S\} \cup \{S\}$ is an antipartition. It is immediately apparent that $(N \backslash T^i) \cap (N \backslash S)$ is empty. If for any $i, j \in S$ it holds that $(N \backslash T^i) \cap (N \backslash T^j)$ is nonempty then $T^i \cup T^j \neq N$ which contradicts the maximality of T^i and T^j since the fact that (N, v) is SD-relevant implies that $T^i \cup T^j$ is in $\mathcal{B}(x)$. ■

Note that Lemma 13 also holds for the per capita prenucleolus since the set of coalitions with minimal satisfaction with respect to the SD-prenucleolus of the game and the set of coalitions with minimal satisfaction with respect to the per capita prenucleolus of the game coincide. We now introduce the notion of SD-equivalent games that is also used in the proof of the main results of the paper.

Definition 14 *We say that TU games (N, v) and (N, w) are SD-equivalent if $\mathcal{B}(SD(N, v)) \cap \mathcal{B}(SD(N, w))$ contains an antipartition.*

The next lemma allows us to consider only SD-equivalent games while analyzing the coalitional monotonicity of the SD-prenucleolus in the class of SD-relevant games.

Lemma 15 *Let (N, v) be a TU game such that for some $S \subset N$ and all $\gamma \in [0, \alpha]$ the game $(N, v + \gamma b_S)$ is SD-relevant. Then there exists $\beta \in (0, \alpha]$ such that:*

1 (N, v) and $(N, v + \beta b_S)$ are SD-equivalent games.

2 $(N, v + \beta b_S)$ and $(N, v + \alpha b_S)$ are SD-equivalent games.

Proof. Let $y = SD(N, v+\alpha b_S) = SD(N, w)$ and $x = SD(N, v)$. Let $\mathcal{Q}$ be an antipartition contained in $\mathcal{B}(x)$. If $\mathcal{Q}$ is contained in $\mathcal{B}(y)$ then with $\alpha = \beta$ the statement is true. If $\mathcal{Q} \nsubseteq \mathcal{B}(y)$ then $S \in \mathcal{B}(y)$ and $\mathcal{F}(\mathcal{Q}, v) > F(S, y, w)$. Furthermore, any antipartition in $\mathcal{B}(y)$ must include coalition S. Let $\mathcal{M}$ be an antipartition in $\mathcal{B}(y)$. Since $\mathcal{M} \nsubseteq \mathcal{B}(x)$,

$$\mathcal{F}(\mathcal{M}, v) > \mathcal{F}(\mathcal{Q}, v) \geq \mathcal{F}(\mathcal{Q}, w) > \mathcal{F}(\mathcal{M}, w) = \mathcal{F}(\mathcal{M}, v) - \frac{\alpha}{|N| (|\mathcal{M}| - 1)}.$$

Therefore, there exists $\beta \in (0, \alpha)$ such that

$$\mathcal{F}(\mathcal{M}, q) = \mathcal{F}(\mathcal{Q}, v) = \mathcal{F}(\mathcal{Q}, q) > \mathcal{F}(\mathcal{M}, w)$$

where $(N, q) = (N, v + \beta b_S)$. ■

4 Monotonicity of the SD-prenucleolus

In the following we analyze the coalitional monotonicity of the SD-prenucleolus in the class of SD-relevant games. The proof of the main result uses the following facts:

1 Only SD-equivalent games need to be considered.

2 Given an SD-relevant game, the set of coalitions with minimal satisfaction with respect to the SD-prenucelolus of the game contains an antipartition. The satisfaction of the coalitions included in the antipartition only depends on the characteristic function of the game.

3 The SD-reduced games with respect to the SD-prenucleolus of the game are SD-relevant and can be easily computed.

Now we are in a position to present the theorem.

Theorem 16 *Let (N, v) be a TU game such that for some $S \subset N$ and all $\gamma \in [0, \alpha]$ the game $(N, v + \gamma b_S)$ is SD-relevant. Then $SD_i(N, v + \alpha b_S) \geq SD_i(N, v)$ for any $l \in S$.*

Proof. Since the SD-prenucleolus and the standard solution coincide, the statement holds if $|N| \leq 2$. Assume that the statement is true for games with no more than k players. We seek to prove that it also holds for games with $k + 1$ players.

Consider game (N, v) where $|N| = k + 1$ and let $(N, w) \equiv (N, v + \alpha b_S)$ for some $S \subset N$ and $\alpha > 0$. Let $x = SD(N, v)$ and $y = SD(N, w)$. We show that for each $i \in S$ it holds that $x_i \leq y_i$. Assume that (N, w) and (N, v) are SD-equivalent games. Then there exists an antipartition, say Q, contained in $\mathcal{B}(x) \cap \mathcal{B}(y)$. Let $T \in Q$. We seek to compare the SD-prenucleolus of the two SD-reduced games $(N \backslash T, v^x)$ and $(N \backslash T, w^y)$. We distinguish 3 cases:

1. $S \notin Q$ and $T \nsubseteq S$. By applying Corollary 11, the two SD-reduced games coincide. Therefore, players in $S \cap (N \backslash T)$ receive the same payoff in both games. Since the SD-prenucleolus satisfies the SD-reduced game property, it must be the case that in games (N, v) and (N, w) players in $S \cap (N \backslash T)$ also receive the same payoff.

2. $S \in Q$. Since $S, T \in Q$, $N \backslash T \subset S$. Since $S, T \in Q$ and $v(S) < w(S)$, $\mathcal{F}(Q, w) < \mathcal{F}(Q, v)$. Since $v(T) = w(T)$,

$$y(T) = w(T) + |T| \mathcal{F}(\mathcal{Q}, w) < v(T) + |T| \mathcal{F}(\mathcal{Q}, v) = x(T).$$

Hence, $y(N \backslash T) > x(N \backslash T)$. By applying Corollary 11, the two SD-reduced games assign the same worth to coalitions that are subsets of S. Therefore,

$$w^y(U) = \begin{cases} y(N \backslash T) & \text{if } U = N \backslash T \\ v^x(U) & \text{otherwise} \end{cases}$$

This means that (by strong aggregate monotonicity of the SD-prenucleolus) for $i \in (N \backslash T)$ it holds that $SD_i(N \backslash T, w^y) > SD_i(N \backslash T, v^x)$. Since the SD-prenucleolus satisfies $SD\text{-}RGP$, $y_i > x_i$ for all $i \in N \backslash T$.

3. $S \notin Q$ and $T \subset S$. By applying Corollary 11,

$$w^y(U) = \begin{cases} v^x(U) + \alpha & \text{if } U = S \backslash T \\ v^x(U) & \text{otherwise.} \end{cases}$$

In this case the TU game $(N \backslash T, w^y)$ can be written as $(N \backslash T, v^x + \alpha b_{S \backslash T})$. By Lemma 9 the SD-reduced games $(N \backslash T, v^x)$ and $(N \backslash T, v^x + \alpha b_{S \backslash T})$ are SD-relevant. Note also that for any $\gamma \in [0, \alpha]$ it also holds that $(N \backslash T, v^x + \gamma b_{S \backslash T})$ is an SD-relevant game.[7] Since $|N \backslash T| \leq k$, it is true that for $i \in (N \backslash T) \cap S$

$$y_i = SD_i(N \backslash T, w^y) \geq SD_i(N \backslash T, v^x) = x_i$$

It is thus proved that for any player i in $S \cap (N \backslash T)$ it holds that $y_i \geq x_i$. Since this is true for any coalition T in Q, it must be concluded that for any player i in S it holds that $y_i \geq x_i$

Assume that (N, w) and (N, v) are not SD-equivalent. By Lemma 15 there exists β, $\beta < \alpha$, such that (N, v) and $(N, v + \beta b_S)$ are SD-equivalent and (N, w) and $(N, v + \beta b_S)$ are SD-equivalent. Hence, the above arguments can be used to conclude that for any player i in S it holds that $x_i \leq SD_i(N, v +$

[7] This is so because $(N \backslash T, v^x + \gamma b_{S \backslash T})$ is the SD-reduced game of $(N, v + \gamma b_S)$ with respect to the SD-prenucleolus of $(N, v + \gamma b_S)$. Recall that $(N, v + \gamma b_S)$ is by assumption SD-relevant.

βb_S). Similarly, it must be concluded that for any player i in S it holds that $SD_i(N, w) \geq SD_i(N, v + \beta b_S)$. Therefore, $x_i \leq y_i$ for all $i \in S$. ■

The following example illustrates that the theorem is not necessarily true if the games are not SD-relevant. Consider the game (N, v) introduced after Definition 6 and the game $(N, w) = (N, v + 2b_{\{1,2,3\}})$. It can be checked that $SD(N, v) = x = (1, 1, 1, 1)$ and $SD(N, w) = y = (2, 2, 0, 0)$. Since $\{\{1, 3\}, \{2, 4\}\} = \mathcal{B}(x) \cap \mathcal{B}(y)$, (N, v) and (N, w) are SD-equivalent games. However the two games are not SD-relevant. Note that

$$F(\{1, 2, 4\}, x, v) = 1 > \max(F(\{1, 4\}, x, v), F(\{2, 4\}, x, v)) = 0$$

and similarly,

$$F(\{1, 2, 4\}, y, w) = 2 > \max(F(\{1, 4\}, y, w), F(\{2, 4\}, y, w)) = 0.$$

Coalition $\{1, 3\}$ belongs to the antipartition included in $\mathcal{B}(x) \cap \mathcal{B}(y)$. Consider the SD-reduced games $(\{1, 3\}, v^x)$ and $(\{1, 3\}, w^y)$. Then

$$v^x(\{1\}) = 1 \neq v(\{1, 2, 4\}) - v(\{2, 4\}) = 0$$

and

$$w^y(\{1\}) = 2 \neq w(\{1, 2, 4\}) - w(\{2, 4\}) = v(\{1, 2, 4\}) - v(\{2, 4\}) = 0.$$

The worth of coalitions in games $(\{1, 3\}, v^x)$ and $(\{1, 3\}, w^y)$ depends on x and y and not only on the characteristic functions v and w.

5 Convex games

In the class of convex games (Shapley, 1971) core stability and coalitional monotonicity are compatible since in this class the Shapley value satisfies both properties. The Shapley value is not a core solution. Therefore, the issue of whether there is a core solution that satisfies coalitional monotonicity in the class of convex games has hitherto been an open question. The following theorem answers the question in the affirmative.

Theorem 17 *In the class of convex games the SD-prenucleolus satisfies coalitional monotonicity.*

The proof of this theorem results immediately from the facts that convex games are SD-relevant games (see lemma below) and the fact that if (N, v) and $(N, v + \alpha b_S)$ are convex then $(N, v + \gamma b_S)$ is convex for any $\gamma \in [0, \alpha]$.

Lemma 18 *Let (N, v) be a convex game and $x \in X(N, v)$. Then all coalitions are relevant with respect to x.*

Proof. The lemma is obviously true for coalitions included in $\mathcal{B}(x)$. We seek to prove that given any two relevant coalitions, S and T, $S \cup T$ is relevant. Assume that there exist two relevant coalitions S and T such that $S \cup T \neq N$ and $S \cup T$ is completed. By convexity,

$$x(S \cup T) - v(S \cup T) + x(S \cap T) - v(S \cap T) \leq x(S) - v(S) + x(T) - v(T).$$

Since S and T are relevant, by Lemma 3, $x(S) - v(S) = \sum_{i \in S} f_i(S)$ and $x(T) - v(T) = \sum_{i \in T} f_i(T)$. Since $S \cup T$ is completed, by Lemma 3 $x(S \cup T) - v(S \cup T) > \sum_{i \in S \cup T} f_i(S \cup T)$. Therefore,

$$\sum_{i \in S \cup T} f_i(S \cup T) + x(S \cap T) - v(S \cap T) < \sum_{i \in S} f_i(S) + \sum_{i \in T} f_i(T).$$

Let $\mathcal{D} = \{Q \subset S \cup T : Q \nsubseteq S, Q \nsubseteq T \text{ and } F(Q) < \max(F(S), F(T))\}$. We consider two cases.

a) $\mathcal{D} = \emptyset$. In this case,

$$\sum_{i \in S \cup T} f_i(S \cup T) = \sum_{i \in S \setminus T} f_i(S) + \sum_{i \in T \setminus S} f_i(T) + \sum_{i \in T \cap S} \min(f_i(T), f_i(S)).$$

Therefore,

$$x(S \cap T) - v(S \cap T) < \sum_{i \in T \cap S} \max(f_i(T), f_i(S)). \quad (4)$$

By Lemma 3, $x(S \cap T) - v(S \cap T) \geq \sum_{i \in T \cap S} f_i(S \cap T)$. Since $(S \cap T) \subset S$ and $(S \cap T) \subset T$, by part 3 of Lemma 3 $f_i(S) \leq f_i(S \cap T)$ and $f_i(T) \leq f_i(S \cap T)$ for all $i \in S \cap T$. Consequently, $f_i(S \cap T) \geq \max(f_i(T), f_i(S))$ for all $i \in S \cap T$. Therefore,

$$x(S \cap T) - v(S \cap T) \geq \sum_{i \in T \cap S} \max(f_i(T), f_i(S)),$$

which contradicts inequality (4).

b) $\mathcal{D} \neq \emptyset$. Let $Q \in \underset{R \in \mathcal{D}}{\arg\min} F(R)$. Let coalitions S^1 and T^1 be:

$$S^1 = \begin{cases} S & \text{if } F(Q) \geq F(S) \\ S \cup Q & \text{if } F(Q) < F(S) \end{cases} \text{ and } T^1 = \begin{cases} T & \text{if } F(Q) \geq F(T) \\ T \cup Q & \text{if } F(Q) < F(T). \end{cases}$$

We consider two cases:

b1) Coalitions S^1 and T^1 are relevant. Using coalitions S^1 and T^1, the arguments used for coalitions S and T can repeat. Since Q has been chosen with minimal satisfaction, for these two coalitions ($S^1$and T^1), case b) does not occur. Hence, $S^1 \cup T^1 = S \cup T$ is relevant.

b2) Assume, without loss of generality, that S^1 is completed. Note that $S^1 \subset S \cup T$. We repeat the proof with coalitions S and Q. This ends up either in a contradiction (cases a) and b1)) or in case b2) with two coalitions S and P (or Q and P) such that $S \cup P$ (or $Q \cup P$) is completed and $P \in \underset{R \in \mathcal{D}^2}{\arg\min} F(R)$ where

$$\mathcal{D}^2 = \{R \subset S \cup Q : R \nsubseteq S, R \nsubseteq Q \text{ and } F(R) < \max(F(S), F(Q))\}.$$

Now repeat the proof taking coalitions S and P (or Q and P). If the proof ends in case a) or b1) the contradiction is found. If not, repeat the proof with another two coalitions. Note that at the end two coalitions need to be found for which case b2) does not occur since the number of coalitions is finite and the satisfaction of the new coalition is lower than the satisfaction of the removed coalition. ■

6 Concluding remarks

This paper follows up the research started by Arin and Katsev in 2014. Considering the results included in the two papers the SD-prenucleolus stands out as the only known core solution that satisfies coalitional monotonicity in the class of convex games and in the class of veto balanced games[8]. Convex games and games with veto players have been widely used to model many different economic situations. In both classes the compatibility between core stability and coalitional monotonicity was known. However the existence of a continuous core solution satisfying coalitional monotonicity in those two classes was an open question that has been answered in the positive way: the SD-prenucleolus is a continuous core solution that satisfies coalitional monotonicity in both classes of games.

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