Strategy-proof and Efficient Fair Scheduling

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Abstract

In this paper, I study dynamic and sequential fair division problems for players with dichotomous preferences and devise a systematic approach of designing efficient, envy-free, and strategy-proof mechanisms for any generic problem. The mechanisms developed here can accommodate common discount factors to represent players' time preferences between different periods. I also show that the mechanisms proposed in the current research outperform in efficiency the repeated applications of a static strategy-proof mechanism by a factor of the size of set of players in refined problems with unbounded demands. I also contribute a novel comparative statics result on the egalitarian solutions to monotone and concave cooperative games with transferable utilities in characteristic function form. In the process of doing so, I discover a duality-like property of the egalitarian solutions and reconcile the seemingly contradictory process to the objective in the search for them. Finally, I highlight the relative importance of identifying the correct order of priority over choices of payoffs in the pursuit of equality.

1 Introduction

Many interesting assignment and division problems in reality present themselves in a dynamic or sequential manner. A production process may be subject to such procedural constraints that the final product must be produced with several input factors, some of which can be the intermediate output of other cruder materials. Privately supplied public resources may be available for allocation for a certain amount of time, beyond which the supply contract may be renegotiated and the allocation of the resources must be redetermined. The most important distinction between static and dynamic assignment problems manifests itself in scheduling problems. Conference rooms must be scheduled for meetings among different committees; Landing strips must be scheduled for flights flown by various airlines; And cloud computing servers must be scheduled to accept competing computational tasks. In each of these examples, it is fairly unreasonable to require all parties to appear in front of a central distributor with their demand for the resources over the entire time horizon or the full assembly line. Hence, production tasks of rubber are distributed before those for the wheels, welfare electricity supplies to low income families are rationed annually, and library books must be returned and recirculated for each semester. Such problems of dividing resources or tasks with procedural or periodic constraints are the subject of the current research project.

What complicates the matter even more is the fact that each party's demand for the resources is very often her own private information. When workers are about to be assigned to shifts on a common workstation, the manager (the principal) probably has very little idea on the hours the workers (the agents) are available for, although the workers will most likely retain this information from their own calendar[1]. Clearly, the manager would like the workstation to be occupied and running as much as possible hence would prefer to assign shifts only to workers who will actually be available during their assigned shifts, indicating higher efficiency from the workstation. In other

[1] I thank Balázs Szentes for suggesting this example.

words, the agents' private information poses the classical challenge to allocative efficiency. By extension, full efficiency may become achievable for certain only when the principal can incentivize the self-interested agents into honest disclosure of their private information. Moreover, to avoid switching shifts, the principal would most likely prefer to chart the shifts so that no agent is more available during other's shift than during her own, i.e. the shifts should induce no envy among the workers. Therefore, the goal of this paper is to systematically design scheduling or division mechanisms for dynamic or sequential settings that are efficient, strategy-proof, and envy-free[2].

The main contributions of the current study come in two aspects. For applications, I develop a systematic design of scheduling mechanisms satisfying the aforementioned properties for agents endowed with dichotomous preferences[3]. On a theoretical level, I provide a novel comparative statics result for the egalitarian solutions of monotone and concave cooperative games with transferable utilities in characteristic function form. To be expected, the former contribution comes as an extension and implication of the latter. I shall now briefly discuss these results, starting with the second. The set of all feasible payoff profiles is *convex* in a monotone and concave cooperative game. The egalitarian solution of such a game provides the payoff vector that is more equal than any other feasible payoff vector in Lorenz terms: the collective payoff of any number of the worst-off players in the egalitarian solution is higher than the same quantity in any other feasible payoff

[2]It as been widely documented in the assignment literature that envy-free-ness is the strongest notion of fairness, implying various others such as proportionality or fair share, where each player receives at least a $1/n$ fraction of her maximum achievable payoff in a population of size n. See Brams and Taylor (1996) for a standard textbook treatment and introduction. Envy-free-ness was first proposed by Foley (1967). An often cited advantage of adopting envy-free-ness as a fairness measure is that envy-free-ness does not involve comparisons of the same allocation according to different players' preferences. On the exact opposite, envy-free-ness only requires comparing *different* bundles according to a given player's preferences, avoiding the controversy associated with double-standards.

[3]Extensive computer science literature has also studied similar problems in the context of algorithmic game theory and mechanism design. For an excellent introduction and/or collection of known results and challenging open questions, please see Nisan et al. (2007). Problems including scheduling and fair divisions (also known as "cake-cutting") in the dynamic mechanism design context have also been discussed in the computer science literature under the topic of online mechanism design. Finally, dichotomous preferences have also been referred to as piecewise uniform valuation densities in the computer science parlance.

profile. The comparative statics result shows that, when some single player introduces more payoff opportunities to a concave game—effectively generating another concave game, this single player should be given the largest increase in payoff in the egalitarian solution of the new game.

In the process of establishing the comparative statics result, I also discovered an intriguing duality-like feature of the egalitarian solution. The search for the egalitarian solution was first accomplished by Dutta and Ray (1989) algorithmically for *convex* cooperative games and was later converted for concave cooperative games by Chen et al. (2013). The essence of this pair of algorithms are roughly as follows: recursively minimizing the average incremental payoffs among the worst-off of the remaining players leads to the egalitarian solutions in concave games. The most interesting feature in these algorithms is that minimizing poor agents' payoffs runs exactly against the spirit of Lorenz dominance favoring rich-to-poor transfers. Following this route, I will show that recursively maximizing the average leftover payoffs among the best-off of the remaining players also lead to the same unique egalitarian solution. I refer to this process as the "dual" of the aforementioned algorithms for its evident mirroring appearance of the original algorithms.

I reconcile this conflict in the process and result of reaching Lorenz dominance posed by the original and dual algorithms with the following interpretation. More important than the choice of payoffs in achieving egalitarianism is the correct order of priority among the players. In all algorithms, players who end up with lower payoffs also have their joint demands satisfied before players with higher final payoffs do. In the original algorithms in Dutta and Ray (1989) and Chen et al. (2013), players are selected to be awarded their demand while bumping players who have not been chosen yet to lower priority orders. On the other hand, in the dual algorithm presented in this paper, players are selected to yield to players to be selected later. The endogenous tradeoffs between the order of priority and payoffs eliminate any potential Lorenz improvement by the way of rich-to-poor transfers, necessary for the proposed solution concept to be Lorenz dominating any other feasible payoff profile. Furthermore, Dutta and Ray (1989) showed that any other feasible payoff vector can be sequentially Lorenz improved to the egalitarian solution, testifying the latter being the only vector that Lorenz dominates all other payoff vectors. Accompanying the dominant feature of the egalitarian solution is its nature of being the maximizer of separable, increasing, and strictly concave functions of the players payoffs. Several previous work including Dasgupta et al. (1973), Rothschild and Stiglitz (1973), and Shorrocks (1983) have shown the equivalence between the maximizer of such collective welfare functions and the Lorenz dominant income distribution.

To apply the static comparative statics result to the dynamic setting, I consider a carefully chosen adjustment of the egalitarian solutions in period t based on the cumulative payoffs of the players before this period. The correlation between the previous cumulative payoffs and the current deviations from the egalitarian solution is designed to be mildly but strictly negative with a magnitude always strictly less than 1. Hence, higher (lower) previous cumulative payoffs for a certain player become a handicap (benefit) or punishment (reward) for the current allocation. The motivation for such a seemingly insignificant and definitely delicate maneuver is two-fold. First, combined with the collective utility maximizing feature of the egalitarian solution, envy-free-ness directly translates into dynamic settings, the reason for which is equally transparent: were some player, say 2, allocated more in some other player, say 1's, shift than is player 1, it must be the case that player 2 is currently rewarded for having a much lower previous cumulative payoff than has player 1 (since the forenamed correlation is small and negative), which implies that player 1's cumulative payoff at the end of the current period must be strictly greater than player 2's.

Much more importantly, for a class of refined division problems, such negative correlations render truth-telling a *strictly* dominant strategy, making all players telling the truth the *only* equilibrium. The efficiency implication of achieving such equilibrium selection is much more significant than the incentive implication. In particular, when compared to the simple repeated application (a mechanism referred to as $\mathscr{R}_0$) of the static strategy-proof mechanism presented in Chen et al. (2013), the minuscule negative correlation rules out a vast range of weakly dominant strategies that survived $\mathscr{R}_0$. For the measure of efficiency, or lack thereof, in $\mathscr{R}_0$, I adopt the predominant notion of "price of anarchy"[4] from the computer science literature. This measure captures the ratio between the maximum achievable total payoff of a preference profile to the lowest total payoff given the same preference profile in any pure strategy Nash equilibrium and describes, in general, the spread between the maximum and minimum total payoffs in any preference profile and equilibrium combination. Hence, a higher price of anarchy signals a worse efficiency performance. Vis-à-vis $\mathscr{R}_0$, the mechanism proposed in this paper achieves full efficiency with a price of anarchy of 1 while the price of anarchy for $\mathscr{R}_0$ is a considerable n, the number of players.

Aside from being greatly inspired by Dutta and Ray (1989) and Chen et al. (2013), to which the current research owns a tremendous amount, this paper also borrows from several other strands of literature. The first group of literature deals directly with the problem of fair divisions, with existence results dating back to Neyman (1946), Steinhaus (1948), and Weller (1985). These papers address the fundamental issue of the sufficient conditions for the existence of efficient and envy-free divisions and offer generously affirmative answers in groups of domains. More recent work in the same realm include Aziz and Ye (2013), Brams et al. (2006), Brams et al. (2008), Mossel and Tamuz (2010), Procaccia (2013a), and Procaccia (2013b) with an introductory text in Robertson and Webb (1998) and a more methodological work from Barbanel (2005).

In the extensive study of the assignment problem, also known as the "one-sided matching problem" in Zhou (1990), Hylland and Zeckhauser (1979) and Bogomolnaia and Moulin (2001) are two phenomenal contributions. As a matter of fact, the static version of the current research, Tian (2015), stands right at the intersection of these two papers and offers arbitrarily asymmetric gener-

[4]The first formal introduction of this notion into the computer science literature is believed to be due to Koutsoupias and Papadimitriou (1999).

alizations of the symmetric static mechanism in Chen et al. (2013) where in Tian (2015), strategy-proof-ness is still maintained. The most recent contribution in assignment problem, Hashimoto et al. (2014), offers two axiomatic characterizations of the probabilistic serial mechanism first proposed in Bogomolnaia and Moulin (2001). Other research in this strand include Alkan et al. (1991), Bogomolnaia and Moulin (2004), Budish et al. (2013), Crès and Moulin (2001), Kojima and Manea (2010), and Mennle and Seuken (2014).

The current application of the equivalence between Lorenz dominance and the maximization of a collective welfare function is first inspired by Quah (2007), which is one of the recent contributions in a seminal line of research in the relationship between complementary, super-modularity, and monotone comparative statics pioneered theoretically by Topkis (1978) and compiled in Topkis (2001), with a series of other stellar results including Antoniadou (2004a), Antoniadou (2004b), Antoniadou (2007) (all on consumer's problem), Athey (2002) (uncertainty), Kukushkin (2011) (objective versus constraint changes), Milgrom and Shannon (1994), Quah and Strulovici (2009) (informativeness), Vives (1990) (equilibrium with strategic complementarities), and Vives (2001) (oligopoly pricing). Finally, an ever growing field of dynamic mechanism design, although all dealing with dynamic auctions or contracting, provides incredible insights with Athey and Segal (2013), Bergemann and Välimäki (2010), Gallien (2006), Gershkov and Moldovanu (2009), Pai and Vohra (2013), and Pavan et al. (2013).

The rest of the paper is organized as follows: section 2 presents the model and introduces notations; section 3 exhibits the comparative statics result of the egalitarian solutions for monotone and concave cooperative games with transferable utilities in characteristic function form; section 4 demonstrates the design of the dynamic mechanisms; section 5 discusses the efficiency implications of the negative correlations between previous cumulative payoffs and current allocations given by the mechanism in section 4; the last section concludes and proposes future research.

2 The Model

A continuum $I_b \equiv [0,b)$ with $0 < b \leq +\infty$ of heterogeneous indivisible public goods or resources must be allocated among a *finite* set of players $N \equiv \{1,2,3,\cdots,n\}$, with $i \in N$ representing a typical player. For convenience, I shall refer to the distributor of the goods or the mechanism designer in discussion as player 0—$N \cup \{0\}$ will be called the **extended set of players**. Let $I_{+\infty} \equiv \mathbb{R}_+$, the set of all non-negative real numbers. Each public good or resource $r \in I_b$ can be assigned to at most one player simultaneously. Therefore, I refer to any division of the interval I_b such that any good is assigned to at most one player among N as a **feasible division**. A feasible division C of I_b is then defined to be a function $C : I_b \to N \cup \{0\}$ where for all $r \in I_b$, $C(r) = i \in N$ means that good r is assigned to player i according to C. $C(r) = 0$ means that the good r remains unassigned or wasted according to C. Clearly, for any feasible division C and $r \in I_b$ where $C(r) = i$, the division C' generated by letting $C'(r) = j \neq i$ and $C(r') = C'(r')$, $\forall r' \neq r$ is still feasible. Let $\mathscr{C}_b$ represent the set of all feasible divisions of I_b.

In this paper, I focus on the sequential or dynamic division of I_b by batches or periods. The sequential division problem describes scenarios where the entire set I_b is available up front for allocation but must be assigned in a discrete number of steps. The dynamic division setting can cover more general scenarios where at any point $r \in I_b$, only a limited interval $[r,r'] \subseteq I_b$ is available for division while the rest will become available at r' at the latest. To this end, let $B_b \equiv \{b_t\}_{t=0}^{T}$ be a *strictly* increasing sequence such that $b_0 = 0$, $0 < b_t \leq b$ and $b_t < +\infty$, $\forall 1 \leq t \leq T$ where $1 \leq T \leq +\infty$. Moreover, when $b < +\infty$, let $T < +\infty$ and $b_T = b$. In the sequential setting, the interval $[b_t, b_{t+1}] \subseteq I_b$ represents the batch of goods or resources to be divided among the players in step $(t+1)$—after the interval $[0,b_t]$ but before the interval $[b_{t+1},b)$ are divided, while in the dynamic setting, $[b_t,b_{t+1}]$ represents the interval to be divided in period t, with time running from period 0 to period $(T-1)$. Equivalently, elements of B mark the end points of the batches of goods to be divided together or the start and end points of the time periods between which the resources must be divided all at once. I require any $B_b = \{b_t\}_{t=0}^{T}$ to be such that

$$\bigcup_{t=0}^{T-1} [b_t, b_{t-1}] = I_b.$$

In other words, the sequence of intervals $\{[b_t, b_{t+1}]\}_{t=0}^{T-1}$ uniquely defines the batches or periods of division, the union of which is the entire set of goods to be assigned. B_b will be referred to as a **segmenting** of the underlying I_b from now on. I shall also omit the subscript b from B_b and simply use the notation B since a segmenting completely characterizes an I_b henceforth where $|B| = T+1$.

Each player i is endowed with a type $\theta_i \subseteq I_b$ that induces some *dichotomous* preferences among all feasible divisions of I_b. Assume all possible types are unions of potentially infinitely many mutually *disconnected closed* intervals within I_b. Given a certain segmenting of I_b, B, let $\Theta^B \subseteq 2^{I_b}$ represent the set of all possible types of any player, where 2^{I_b} is the set of all subsets of I_b. Without loss of generality[5], assume $I_b \in \Theta^B$. Let $\theta \equiv (\theta_1, \theta_2, \cdots, \theta_n) \in \left(\Theta^B\right)^n$ be the vector of the players' types, to which I will refer as the profile of the players' types or preferences. I use the notation $(\theta_i, \theta_{-i}) = \theta$ to single out player i's type from the profile θ, where θ_{-i} represents the profile of all players' types except for i's. For any $[b_t, b_{t+1}]$, define $\Theta_t^B \equiv \left\{\theta \cap [b_t, b_{t+1}] : \theta \in \Theta^B\right\}, \forall 0 \leq t \leq T-1$. I refer to Θ_t^B as the **the restriction of Θ^B to batch $(t+1)$ or period t (or simply "to t")**. The **restrictions of θ_i and θ to t** is similarly defined to be $\theta_{it} \equiv \theta_i \cap [b_t, b_{t+1}]$ and $\vec{\theta}_t \equiv (\theta_{1t}, \theta_{2t}, \cdots, \theta_{nt})$. For tractability, I make the following assumption regarding Θ_t^B:

Assumption 1 (Finite type space)**.** Given any triple (B, T, Θ^B), $\left|\Theta_t^B\right| < +\infty, \forall 0 \leq t \leq (T-1)$.

Assumption 1 is equivalent to stating that $|\Theta^B|$ is finite when T is finite but does not imply so when T is infinite. It simply limits the variety of the players' types to a finite set within each

[5]Were there an $I \subset I_b$ such that $I \not\subseteq \theta_i$ for any $\theta_i \in \Theta^B$, disregard I and consider the problem with $\hat{I}_{\hat{b}} \equiv (I_b \backslash I)$.

batch or period. Moreover, no assumption is implicitly made within Assumption 1 regarding the richness of Θ_t^B as compared to $2^{[b_t, b_{t+1}]}$—Θ_t^B can be arbitrarily large, although the finiteness of $|\Theta_t^B|$ is indispensable. Without loss of generality, also assume that $\theta \in \Theta^B$ and $\theta' \in \Theta^B$ implies that $(\theta \cap \theta') \in \Theta^B$ and $(\theta \cup \theta') \in \Theta^B$. Any Θ^B satisfying Assumption 1 will be referred to as **a type space compatible with** B. It is worth pointing out that there definitely can be multiple type spaces compatible with a segmenting.

I interpret such subset types as the players considering goods within their types attractive or acceptable and those outside unattractive or unacceptable. However, it should be noted that any goods outside the players' types do *not* generate any disutility to the players—the players are simply indifferent between being assigned such goods and otherwise. Specifically, let $\mathbb{1}_i^C : I_b \to \{0,1\}$ be the **indicator function** for player i generated by a feasible division C such that $\mathbb{1}_i^C(r) = 1$ if $C(r) = i$ and $\mathbb{1}_i^C(r) = 0$ otherwise. To accommodate linear substitution between different batches or time preferences between different periods, given any segmenting B of I_b, let the sequence $\beta_i^B \equiv \{\beta_{it}^B\}_{t=0}^{T-1}$ represent the marginal utilities of the goods within the respective batches or the discount factors of respective periods. I will loosely refer to the sequence β_i^B as being **the discount factor compatible with** B **for player** i—please keep in mind that β_i^B also represents the relative value of the different batches for player i in the sequential background. For players' utilities to be well-defined, especially when T is infinite, I make the following assumption:

Assumption 2 (Finite utility)**.** For any triple (B, T, β_i^B) with β_i^B compatible with B for i,

$$\beta_{it}^B \geq 0, \ \forall 0 \leq t \leq T-1, \text{ and } \chi\left(B, \beta^B\right) \equiv \sum_{t=0}^{T-1} \beta_{it}^B \cdot (b_{t+1} - b_t) < +\infty.$$

Given a feasible division C and a segmenting B, let $u_{it}^B(C|\theta_i)$ represent player i's period-t utility with her type being θ_i. Therefore,

$$u_{it}^{B}(C|\theta_i) \equiv \int_{\theta_i \cap [b_t, b_{t+1})} \mathbb{1}_i^C(r)\, dr.$$

Clearly, $0 \leq u_{it}^{B}(C) \leq (b_{t+1} - b_t)$. Combined with Assumption 2 and the corresponding β_i^B, let the player i's canonical utility with type θ_i and certain fixed B and C be defined as the discounted sum of period utilities: $u_i^B(C|\theta_i) \equiv \sum_{t=0}^{T-1} \beta_{it}^{B} \cdot u_{it}^{B}(C|\theta_i)$. This canonical utility represents the player's utilities when they have linear preferences among the various batches of acceptable goods in the sequential interpretation and the present discounted amount of resources they receive in a certain feasible division in the dynamic setting. All results in this paper still hold when players' utilities are strictly increasing in their canonical utilities. I may use the symbol $x_{i\bar{t}}^{B}(C|\theta_i)$ to stand for the cumulative utility for player i up to $b_{\bar{t}}$. That is,

$$x_{i\bar{t}}^{B}(C|\theta_i) \equiv \sum_{t=0}^{\bar{t}-1} \left(\beta_{it}^{B} \cdot u_{it}^{B}(C|\theta_i)\right).$$

Without loss of generality, normalize $x_{i0}^{B}(C|\theta_i) = 0$ for all $(i, C, \theta_i) \in (N \times \mathscr{C}_b \times \Theta^B)$. I shall also use the symbol $\vec{u}_t^B(C|\theta)$ to represent the vector of all players' period-t utilities with the type profile θ for a feasible cut C and similarly for $\vec{x}_{\bar{t}}^B(C|\theta)$. I make one more crucial assumption regarding the discount factors serving the emphasis of this paper on allocating the resources to the players who find them acceptable, while shying away from the possibility of player-specific discount factors.

Assumption 3 (Common discount factors)**.** For any pair (B, T), there exists a vector of discount factors β^B such that $\beta_i^B = \beta^B, \forall i \in N$. β^B will be referred to as a **common discount factor** compatible with B for players in N.

Certainly, there can be more than one common discount factor compatible with a segmenting

B, just like multiple B-compatible type spaces are allowed. As restricting as Assumption 3 for a common discount factor may seem, it still entertains various interesting applications. For example, the sequential setting is similar to a sequential auction of multiple objects on which the bidders share common values. The difference between a sequential auction and sequential divisions is that players have common relative values between different batches but maintain acceptance or rejection within a batch in the latter—in a sequential auction, players are generally assumed to receive different signals. On the other hand, in the dynamic setting and especially when $\beta_t^B \leq 1$, $\forall 0 \leq t \leq T-1$, β_t^B can be interpreted as the ex ante (that is, at time 0) probability that the interval $[b_t, b_{t+1}]$ of goods will be available for division at time b_t at the latest.

To summarize the various aforementioned elements, I formally define a **sequential or dynamic division problem** (henceforth abbreviated as "problem") Π to be a quintuple $(b, B, T, \Theta^B, \beta^B)$ where B is a segmenting of I_b, $|B| = T+1$, and both Θ^B and β^B are compatible with B. If T is finite (infinite), Π will be referred to as a(n) **finite (infinite) problem**. Given any sequential or dynamic division problem Π and fixing a type profile $\theta \in (\Theta^B)^n$, a feasible division C of I_b is **Pareto efficient at** θ if and only if there does not exist another feasible division $C' \neq C$ of I_b such that $u_i^B(C'|\theta_i) \geq u_i^B(C|\theta_i), \forall i \in N$, with the inequality being *strict* for at least one player i. A feasible division C of I_b is **envy-free at** θ if and only if

$$u_i^B(C|\theta_i) \geq \sum_{t=0}^{T-1} \left\{ \beta_t^B \cdot \left[\int_{\theta_{it} \cap [b_t, b_{t+1}]} \mathbb{1}_j^C(r)\, dr \right] \right\}, \forall (i,j) \in N \times N.$$

Both definitions are conventional: at any given type profile, a feasible division C is Pareto efficient if and only if there does not exist another feasible division that, at the fixed type profile, strictly benefits some player without hurting any; C is envy-free at the fixed type profile if, for any player i, replacing the goods assigned to i with the goods assigned to j never strictly benefits i.

I shall now very briefly formulate the information structure of the environment to place such division problems into the context of mechanism design. Given any problem Π, all elements of Π—b, B, T, Θ^B, and β^B—are common knowledge among the extended set of players $N \cup \{0\}$. Any player's type is her own private information, although it could potentially be only partially revealed to her batch(es) by batch(es) or period(s) by period(s). In particular, θ_{it} must be revealed to i before $[b_t, b_{t+1}]$ is divided, but no other assumptions are made regarding the revelation of the batch or period restrictions of the types to the players. For example, $\theta_{i(t+t')}$ for $0 < t' \leq (T-t-1)$ can be allowed to be conditional on or correlated with θ_{it}, embodying the situations where players' preferences may be evolving in a way depending on their previous types.

Give a problem Π, a **direct sequential/dynamic division mechanism** (henceforth, "mechanism") is therefore defined to be a function $M : \left(\Theta^B\right)^n \to \mathscr{C}_b$, where recall that $\mathscr{C}_b$ is the set of all feasible divisions of I_b. Let $M(\theta)(r) \equiv C(r)$ if and only if $M(\theta) = C, \forall r \in I_b$. The range of any M being only $\mathscr{C}_b$ entails that no monetary transfer is allowed in such mechanisms. Due to the aforementioned uncertain nature of the future types, I restrict attention to mechanisms that are only history dependent, at least in a payoff-equivalent sense. A mechanism M is **sequentially or dynamically payoff consistent** (henceforth, "consistent") if $\forall \left(\theta, \theta'\right) \in \left(\Theta^B\right)^n \times \left(\Theta^B\right)^n$ and $0 < t \leq \bar{t} \leq T-1$,

$$\left[\theta_t = \theta'_t,\ \forall t \leq \bar{t}\right] \Rightarrow \left[\vec{u}^B_t(M(\theta)|\theta) = \vec{u}^B_t(M(\theta')|\theta'),\ \forall t \leq \bar{t}\right].$$

That is, for any two type profiles that are identical up to $b_{\bar{t}}$, a consistent mechanism must be choosing the same payoff vectors for all periods up to $b_{\bar{t}}$. A consistent mechanism is also consistent in the sense of cumulative payoffs up to $b_{\bar{t}}$ for all $1 \leq \bar{t} \leq T$.

The efficiency and fairness measures of any mechanism are defined by the natural translation of corresponding properties for the feasible divisions. Specifically, given a problem Π, a mechanism M is **Pareto efficient** if and only if $M(\theta)$ is Pareto efficient at θ for all $\theta \in \left(\Theta^B\right)^n$. M is **envy-free** if and only if $M(\theta)$ is envy-free at θ for all $\theta \in \left(\Theta^B\right)^n$. For the strategic property and equilibrium concept, I strive for the strongest form of truthfulness, namely strategy-proof-ness. Given a problem Π, a mechanism M is **strategy-proof** if and only if for all $i \in N$, for all $\left(\theta, \hat{\theta}\right) \in \left(\Theta^B\right)^n \times \left(\Theta^B\right)^n$ such that $\theta_i \neq \hat{\theta}_i$ and $\theta_j = \hat{\theta}_j$ for all $j \neq i$, then

$$u_i^B\left(M(\theta)|\theta_i\right) \geq u_i^B\left(M\left(\hat{\theta}\right)|\theta_i\right).$$

That is, any unilateral pretense of being different from one's true type can never strictly benefit the pretending player—the argument of $M(\cdot)$ represents the types players report to the mechanism while players' payoffs are only conditional on their true types. Implicit in the efficiency and fairness measures of the mechanism is the focus on only when the players are reporting their true types: no such measures are defined for the off dominant strategy equilibrium deviations from truth-telling in a strategy-proof mechanism in the central results in this paper. However, efficiency loss as a result of untruthful reports will be discussed as extensions later.

The goal of this research is to systematically design a class of Pareto efficient, envy-free, and strategy-proof mechanisms for each problem. I would now first take a crucial detour to introduce the egalitarian solutions (see Dutta and Ray (1989)) of concave cooperative games with transferable utilities in characteristic function form (see Chen et al. (2013)) and provide the novel comparative statics results on this solution concept. For almost all of the following results, the distinction between subsets (supersets) and proper subsets (proper supersets) can be significant. Hence, it is worth point out that $\subset$ and $\supset$ represent *proper* subsets and *proper* supersets, respectively, while $\subseteq$ and $\supseteq$ are symbols for (potentially equal) subsets and supersets, respectively.

3 The egalitarian solutions of concave cooperative games

3.1 Preliminaries and previous results

A **cooperative game with transferable utilities in characteristic function form** Γ (with the characteristic function $v(\cdot)$) is defined to be a pair $\Gamma \equiv (N, v(\cdot))$, where N is the finite set of players with $|N| = n < +\infty$ and $v : 2^N \to \mathbb{R}$. I focus on one family of cooperative games that closely fit the context of divisions, namely the monotone and concave ones. First of all, let $v(\emptyset) = 0$. Moreover, Γ is called **monotone** (or "monotonically increasing") if and only if for all $S \subseteq T \subseteq N$, $v(S) \leq v(T)$. The game Γ is called **concave** if and only if

$$v(S \cup T) + v(S \cap T) \leq v(S) + v(T), \forall S, T \subseteq N. \tag{1}$$

Note that if Γ is monotone and concave[6], then $v(S \cup T) \geq \max\{v(S), v(T)\}$ and when $S \cap T = \emptyset$, $v(S \cup T) \leq v(S) + v(T)$[7]. Notice that any monotonic linear transformation of a concave cooperative game Γ yields another concave game Γ'. Specifically, take a concave $\Gamma = (N, v(\cdot))$ and for any $i \in N$, take one fixed $x_i \in \mathbb{R}$. Now, let $\Gamma' \equiv (N, v'(\cdot))$ with the characteristic function $v'(\cdot)$ given by $v'(S) \equiv a \cdot v(S) + \sum_{i \in S} x_i$, $\forall S \subseteq N$ for some $a > 0$. Clearly[8],

[6] Convex cooperative games, with the inequality in (1) reversed, have been studied much more extensively, starting most prominently from Shapley (1951).

[7] Again, the case with the reverse of the inequality here has been predominant in the literature on cooperative games, a property called "super-additivity".

[8] For clarity, keep in mind that for all $T, S \subseteq N$, $T \cup S = (T \backslash S) \cup (S \backslash T) \cup (T \cap S)$ and $T \backslash S = T \backslash (T \cap S)$, implying

$$\sum_{i \in (T \cup S)} x_i + \sum_{i \in (T \cap S)} x_i = \sum_{i \in (T \backslash (T \cap S))} x_i + \sum_{i \in (T \cap S)} x_i + \sum_{i \in (S \backslash (T \cap S))} x_i + \sum_{i \in (T \cap S)} x_i = \sum_{i \in T} x_i + \sum_{i \in S} x_i.$$

$$\begin{aligned}
v'(T \cup S) + v'(T \cap S) =& av(T \cup S) + \sum_{i \in (T \cup S)} x_i + av(T \cap S) + \sum_{i \in (T \cap S)} x_i \\
=& av(T \cup S) + av(T \cap S) + \sum_{i \in T} x_i + \sum_{i \in S} x_i \\
\underbrace{\leq}_{\text{by concavity of } \Gamma} & av(T) + av(S) + \sum_{i \in T} x_i + \sum_{i \in S} x_i = v'(T) + v'(S),
\end{aligned}$$

which shows Γ' is concave as well. For any $S \subseteq N$, $v(S)$ is interpreted as the maximum amount of payoffs that can be shared and divided among the players in S. Dutta and Ray (1989) devised an algorithm solving for the egalitarian solution of any *convex* cooperative game, which was later converted for concave cooperative games by Chen et al. (2013). My first result is an amendment to Chen et al. (2013) with a formal proof that the algorithm provided by Dutta and Ray (1989) does offer the egalitarian solution to monotone and concave cooperative games with transferable utilities in characteristic function form after first formally defining "egalitarian" with Lorenz dominance.

Per Dutta and Ray (1989), take any $y \in \mathbb{R}^n_+$ and let $\mathbf{y} \in \mathbb{R}^n_+$ be the vector given by rearranging the entries of y in a weakly increasing order. y **Lorenz dominates** $y' \in \mathbb{R}^n_+$ if and only if

$$\sum_{l=1}^{n} y_l = \sum_{l=1}^{n} y'_l \text{ and } \sum_{l=1}^{\bar{l}} \mathbf{y}_l \geq \sum_{l=1}^{\bar{l}} \mathbf{y}'_l, \forall 1 \leq \bar{l} \leq n.$$

The natural interpretation of Lorenz dominance is as follows: if y_i represents the income of agent i in a first economy (similarly for y'_i in second economy) and if the two economies have the same average income, the first economy should be considered more egalitarian than the second, in terms of income, if the total income its $\bar{l}$ poorest members is never a lower share of the total income than that in the second. Shorrocks (1983) extended this standard definition with defining generalized Lorenz dominance as follows: y **generalized Lorenz dominates** y' if and only if $\sum_{l=1}^{\bar{l}} \mathbf{y}_l \geq \sum_{l=1}^{\bar{l}} \mathbf{y}'_l, \forall 1 \leq \bar{l} \leq n$. Generalized Lorenz dominance allows for comparisons between two vectors in the Lorenz sense without requiring the two vectors in comparison to be on the same hyperplane of the income space. The motivation for such a definition is straightforward: the first economy should be considered more prosperous and egalitarian if its $\bar{l}$ poorest member take home more total income (not as shares of the total income of the entire first economy) than those in the second. Certainly, implicit in the definition is that the total income in the first economy cannot be lower than that in the second (simply let $\bar{l} = n$), hence the notion of being "more prosperous". I will now introduce an algorithm that locates the (unique) generalized Lorenz dominant vector from a very slight modification of both Theorem 2 in Dutta and Ray (1989) and Mechanism 1 in Chen et al. (2013). First of all, define the following notation: given a monotone and concave cooperative game Γ with transferable utilities in characteristic function form, let

$$\forall S \subset T \subseteq N,\ e(T-S) \equiv \frac{v(T)-v(S)}{|T|-|S|},$$

where, note that T must be a *proper* superset of S. As a special case, $v(S)/|S| = e(S-\emptyset)$ for all $S \neq \emptyset$. Moreover, let $V \equiv \left\{ y \in \mathbb{R}^n_+ : \sum_{i \in S} y_i \leq v(S),\ \forall S \subseteq N \right\}$. V is referred to as the **feasible set** of Γ[9,10]. Some immediate observations of the structure of $e(\cdot)$ are collected in the following lemma.

Lemma 1 (Preliminary observations of $e(\cdot)$)**.** *Given any monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function given by $v(\cdot)$,*

1. $\forall R \subset S \subset T \subseteq N$, $e(T-R) = e(S-R) \Rightarrow e(T-R) = e(S-R) = e(T-S)$;

[9] Not surprisingly, the feasible set V bears an intrinsic characterization of the set of all feasible divisions.

[10] Clearly, V given by any characteristic function $v(\cdot)$ is closed and bounded hence compact.

2. *$\forall R \subset T \subseteq N$ and $R \subset S \subseteq N$, where $R \subset (T \cap S)$ and $T \not\subseteq S$ and $S \not\subseteq T$, such that $e(S-R) = e(T-R)$, then $\min\{e[(T \cup S)-R], e[(T \cap S)-R]\} \leq e(S-R) = e(T-R)$;*

3. *$\forall R \subset T \subseteq N$ and $R \subset S \subseteq N$, where $R = (T \cap S)$ and $T \not\subseteq S$ and $S \not\subseteq T$, such that $e(S-R) = e(T-R)$, then $e[(T \cup S)-R] \leq e(S-R) = e(T-R)$.*

Proof. See Appendix A for the proof of Lemma 1. ∎

Equipped with Lemma 1, now consider the following algorithm, the objective of which is to select a vector $y \in \mathbb{R}^n_+$ that generalized Lorenz dominates any other y' in the feasible set V.

Definition 1 (Algorithm $\mathscr{E}$)**.** Given any monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$, run Algorithm $\mathscr{E}$ through the following steps:

1. Set $S_0 = \emptyset$. In step 1, select S_1 where $\emptyset \subset S_1 \subseteq N$ such that

$$e(S_1 - \emptyset) \leq e(S - \emptyset),\ \forall \emptyset \subset S \subseteq N. \tag{2}$$

   If there are multiple candidates for S_1, select the smallest one: let S_1 be such that, in addition to satisfying (2), for all S such that $\emptyset \subset S \subset S_1$, $e(S-\emptyset) > e(S_1 - \emptyset)$;

   If there are multiple smallest candidates, select any one of them and proceed;

2. Let S_k be subset of N that was selected in step k, for all $k \geq 1$. In step $(k+1)$, select any of the smallest $S_{k+1} \subseteq N$ such that $e(S_{k+1} - S_k) \leq e(S - S_k)$ for all $S \subseteq N$ where $S_k \subset S$;

3. Terminate in step $K \geq 1$ if the only candidate for S_K is N;

4. After termination, let $y^* \in \mathbb{R}^n_+$ be the vector generated by setting $y^*_i = e(S_{k+1} - S_k)$ if $i \in (S_{k+1} \backslash S_k)$; y^* will be referred to as the **egalitarian solution of** Γ.

The sequence of subsets $\{S_k\}_{k=1}^{K}$ selected in the proceedings will be referred to as the **sequence of subsets of** N **generated by Algorithm** $\mathscr{E}$ **for** Γ.

Lemma 1 ensures that Algorithm $\mathscr{E}$ is well-defined in each step and the vector y^* generated after termination is unique. Algorithm $\mathscr{E}$ also terminates in fewer steps than $n = |N|$ since step k involves including at fewest one more player into the new set S_k. An immediate result regarding Algorithm $\mathscr{E}$ is that players selected later receive higher payoffs (that is, the y_i^*'s) than those selected earlier, a result I present in the following lemma which is also included in Chen et al. (2013). An alternative proof to Chen et al. (2013) is deferred to Appendix A.

Lemma 2 (First observation from Algorithm $\mathscr{E}$)**.** *Let $S_1 \subset S_2 \subset \cdots \subset S_K = N$ be the sequence of subsets of N generated by Algorithm $\mathscr{E}$ with $K \leq n = |N|$ for the monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$. Then $\forall 2 \leq k \leq (K-1)$, $e(S_k - S_{k-1}) \leq e(S_{k+1} - S_k)$.*

Proof. See Appendix A for the proof of Lemma 2. ■

In contrast to Dutta and Ray (1989) and Chen et al. (2013), the selection of a smallest S_k in step k turns out to be instrumental for applying this egalitarian solution concept to a sequential or dynamic setting, which will be highlighted in the upcoming sections. Regardless of selecting the largest or the smallest candidate subset in each step, Algorithm $\mathscr{E}$—and the mechanisms in the two aforementioned papers—partitions N into equivalent classes according to the steps where players are included in the selected subsets. When always selecting the largest subset, the equivalence class of any player i is simply the set of players with the same quantities for y_i^* and the resulting partition is the coarsest of N for the egalitarian solution y^*. On the other hand, Algorithm $\mathscr{E}$ partitions N into the finest[11]. The following lemma provides the formal statement for this result, for which I

[11] Given a set N, two partitions $\mathscr{P}$ and $\mathscr{P}'$ of N can be compared according the *partial* order of coarseness with

adopt the following notations: let y^* be the egalitarian solution of $\Gamma = (N, v(\cdot))$, $\forall i \in N$, define $\lhd_i(y^*) \equiv \left\{ j \in N : y_j^* < y_i^* \right\}$, the set of players who receive strictly less payoff than i in y^*.

Lemma 3 (Finest partition of N)**.** *Let y^* be the egalitarian solution given by Algorithm $\mathscr{E}$ that terminates in K steps for the monotone and concave game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$. Let $\mathscr{P}^*$ be the partition of N implied by Algorithm $\mathscr{E}$, namely $\mathscr{P}^* \equiv \{(S_k \backslash S_{k-1})\}_{k=1}^K$.*

*A partition $\mathscr{P}$ of N is y^*-**compatible** if and only if*

1. *For all $i \in N$ and $j \in N$, $i \in S \in \mathscr{P}$ and $j \in S \in \mathscr{P}$ implies that $y_i^* = y_j^*$;*
2. *And for any $S \in \mathscr{P}$, $\sum_{j \in (\lhd_i(y^*) \cup S)} y_j^* = v\left(\lhd_i(y^*) \cup S\right)$, $\forall i \in S$.*

$\mathscr{P}^*$ is the finest among all y^*-compatible partitions of N. That is, let $\mathscr{P}$ be a y^*-compatible partition of N. Then any element of $\mathscr{P}$ is a union of some elements of $\mathscr{P}^*$.

Proof. See Appendix A for the proof of Lemma 3. ■

From now on, I shall call the element of $\mathscr{P}^*$ that contains i the **clique of i at** y^* with the symbol $Q_i(y^*)$. It should be noted that even given the same y^*, the finest partition $\mathscr{P}^*$ hence the cliques $Q_i(y^*)$ may be different for different game Γ's, although in the scope of the current paper, such multiplicities and abuse of notation will not cause any confusion and the same clarity also applies to the definition of $\lhd_i(y^*)$. All results will be stated in a self-containing manner with designations clear in the contexts of the corresponding results. The arbitrary selection of one of the smallest candidates in Definition 1 may also result in multiple sequences of $\{S_k\}_{k=1}^K$. However, the resulting partition $\mathscr{P}^*$ of N from these potentially different sequences are all the same, highlighting

respect to the following definition: $\mathscr{P}$ is coarser than $\mathscr{P}'$ (or equivalently, $\mathscr{P}'$ is finer than $\mathscr{P}$) if and only if every element of $\mathscr{P}$ is the union of some elements of $\mathscr{P}'$. Notice that arbitrary selections that are neither the largest nor the smallest among the candidate sets such as those made in Chen et al. (2013) lack such features, which may prove inconvenient in some cases especially because coarseness is a partial ordering.

the importance of such a finest partition in illustrating the mechanics of Algorithm $\mathscr{E}$. Lemma 3 also implies that, given Γ, Algorithm $\mathscr{E}$ always terminates in the same number of steps, regardless of the order of the subsets being selected. Of course, same argument can be made for the selection of the largest candidate adopted in Dutta and Ray (1989).

On the other hand, Lemma 1 does dictate the solution y^* to be independent of the selection among subsets of N within each step with the same values of $e(\cdot)$, which makes selecting the smallest candidate even more subtle and significant—the resulting vector being the same, only selecting smallest candidates renders the egalitarian solutions extendable to dynamic settings.

As alluded before, the y^* obtained from Algorithm $\mathscr{E}$ generalized Lorenz dominates any other vector y in the feasible set V given by the characteristic function $v(\cdot)$. I record this result, along with the fact that $y^* \in V$, with following lemma. Both results are included Dutta and Ray (1989) and Chen et al. (2013), either under different contexts or missing formal proofs[12].

Lemma 4 (Feasibility and dominance of y^*)**.** *Given any monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$ with V being the feasible set given by $v(\cdot)$, the egalitarian solution y^* defined by Algorithm $\mathscr{E}$ generalized Lorenz dominates any $y \in V$ and $y^* \in V$.*

Proof. See Appendix A for the proof of Lemma 4. ■

A quick calculation reveals that for all $1 \leq k \leq K \leq N$, $\sum_{i \in S_k} y_i^* = v(S_k)$, because

$$\sum_{i \in S_k} y_i = \sum_{i \in S_1} y_i + \cdots + \sum_{i \in (S_k \setminus S_{k-1})} y_i = v(S_1) + v(S_2) - v(S_1) + \cdots + v(S_k) - v(S_{k-1}) = v(S_k), \quad (3)$$

[12] Dutta and Ray (1989) first developed Algorithm $\mathscr{E}$ but for convex cooperative games and adopting Lorenz (instead of generalized Lorenz) dominance and Chen et al. (2013) relied on Dutta and Ray (1989) for both feasibility and Lorenz dominance. So the current work is either amending their results or providing alternative justifications for some previously known facts.

which proves handy in the proof of Lemma 4. Lemma 4 directly leads to the following characterization of the egalitarian solution. This following lemma follows directly from Theorem 2 in Shorrocks (1983)—the proof is thus entirely omitted.

Lemma 5 (Nash collective utility maximizer)**.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be strictly increasing, strictly concave, and continuous on $\mathbb{R}_+$. Define $F : \mathbb{R}^n_+ \to \mathbb{R}$ as $F(y) = \sum_{i=1}^n f(y_i)$ where $y = (y_1, y_2, \cdots, y_n)$. For any two vectors y and y' in $\mathbb{R}^n_+$, y generalized Lorenz dominating y' implies that $F(y) \geq F(y')$.*

In particular, given a monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form where $|N| = n$ and V is the feasible set given by the characteristic function $v(\cdot)$, then

$$y^* = \arg\max_{y \in V} F(y),$$

where y^* is the egalitarian solution of Γ given by Algorithm $\mathscr{E}$.

From now on, I shall refer to the combination of strictly increasing, strictly concave, and continuous on $\mathbb{R}_+$ (and additive separability) as the **regularity conditions** for $f(\cdot)$ (and for $F(\cdot)$). y^* is actually the unique maximizer of $F(\cdot)$ on V[13]. With y_i representing player i's payoff, $F(\cdot)$ has been referred to as the (symmetric) **Nash collective utility function**—see, for example, Moulin (2004)—the motivation of which should be quite obvious: y^* is the Nash bargaining solution when the feasible set of the bargaining problem is V[14].

Lemma 5 provides a convenient criterion of checking whether a feasible division satisfies envy-free-ness in applying the egalitarian solutions to the sequential or dynamic scenarios, in addition to an alternative motivation for the solution concept in the first place. In other words, any feasible allocation that is not envy-free can be easily rejected by the egalitarian solution as a candidate maximizer of a Nash collective utility function, guiding toward a direction of designing

[13] See, for example, Rockafellar (1997).
[14] See, for example, Osborne and Rubinstein (1990).

sequential/dynamic envy-free division mechanisms by deploying the maximization of some Nash collective utility functions batch-by-batch or period-by-period.

As a matter of fact, the additively separable structure of $F(\cdot)$ is adopted in the current paper almost entirely for the ease of checking envy-free-ness. Shorrocks (1983) proves that generalized Lorenz dominance is equivalent to the maximizer of a wider range of collective utilities functions, which includes the current form of $F(\cdot)$. Without indulging in details, the main result (Theorem 2) in Shorrocks (1983) roughly states the following: if a society prefers more income, and given any vector of income, weakly prefers the average of the permutations of the income vector to the income vector itself, then an income vector generalized Lorenz dominating another is equivalent to the first resulting in a higher social welfare than the second. The weak preference for the average of the permutations over any single one of the permutations is called "$\mathscr{S}$-concave" in both Dasgupta et al. (1973) and Rothschild and Stiglitz (1973), neither of which imposes preferences for higher incomes. Hence, in Dasgupta et al. (1973) and Rothschild and Stiglitz (1973), the equivalence was between Lorenz (*not* generalized Lorenz) dominance and $\mathscr{S}$-concavity of social welfare functions.

The egalitarian solution is directly Pareto efficient among all $y \in V$ simply because $F(\cdot)$ is increasing in any y_i, holding every other y_j's constant. Algorithm $\mathscr{E}$ is also reminiscent of the serial dictatorship mechanism for assignment problems, with the additional flexible feature that the order of the players being the dictators are endogenous instead of being chosen by some preset exogenous randomizing mechanism. Moreover, Algorithm $\mathscr{E}$ allows groups of players to share the same priority hence the same payoff. In the light of collective welfare, the group of players with the lowest average demand represents the group that will yield the highest marginal collective welfare for the society, were their constraints to be relaxed first. Hence, players with lower average demand jointly as a group are given higher order of priority and whose demands are satisfied first.

On the other hand, there exists a tradeoff between any player's order of priority and her payoff—higher order of priority means lower payoff in comparisons between the players. Furthermore, players with lower order of priority, hence higher payoff, must sacrifice payoffs if they want to be taken with a higher priority, indicating that Algorithm $\mathscr{E}$ favors smaller groups of agents with smaller average demand. I convey these intuitions with the following lemma, part of which was alluded to in Chen et al. (2013) without formal proofs.

Lemma 6 (Second observation of Algorithm $\mathscr{E}$)**.** *Let $S_1 \subset S_2 \subset \cdots \subset S_K = N$ be the sequence of subsets of N generated by Algorithm $\mathscr{E}$ with $K \leq n = |N|$ for the monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$. Then $\forall 0 \leq k \leq (K-1)$, $[S \supset S_k$ and $|S| > |S_{k+1}|] \Rightarrow e(S - \emptyset) \geq e(S_{k+1} - \emptyset)$.*

Proof. See Appendix A for the proof of Lemma 6. ■

In the next subsection, I will establish a key novel observation regarding y^*, leading to a critical comparative statics result as the cornerstone for strategy-proof-ness in a class of sequential/dynamic division mechanisms.

3.2 Comparative statics of the egalitarian solution

Consider the following thought experiment: if we consider the characteristic functions as delineating the boundaries of the payoff possibilities of the cooperative entities and suppose that one single player suddenly "brings more to the table", which results in a relaxation of these boundaries. Now, in this new cooperative game, who should be rewarded with more payoffs, as compared to before in the original cooperative game, within this new set of boundaries? Should the player who brings more to the table be given the largest increase in payoff in the egalitarian solution?

More specifically, for such relaxations, it is reasonable to assume that no player i should be able to increase the maximum amount of payoff for a group of players that excludes i. Moreover, any player i's influence on a larger group (or more precisely, a superset) of players (that includes i, of course) should be less than that on a smaller group (a subset that, again, includes i)—otherwise, the new cooperative game after player i brings more to the table may no longer be concave. The main comparative statics result in this paper shows that in the egalitarian solution, such relaxations will result in the largest increase in payoff given to player i. Along with player i, there may be a set of players who see their payoffs increased as well while some other players, who will have lower *priority* than player i in the egalitarian solution of the new game, will see their payoffs (weakly) decreased. I now formally present this main result with the following theorem.

Theorem 1 (Maximal increase in payoffs)**.** *Let $\Gamma = (N, v(\cdot))$ and $\hat{\Gamma} = (N, \hat{v}(\cdot))$ be two monotone and concave cooperative games with transferable utilities in characteristic function form such that there exists a unique $i \in N$ such that, for all $S \subseteq N$,*

1. $i \notin S \Rightarrow v(S) = \hat{v}(S)$; 2. $i \in S \Rightarrow v(S) \leq \hat{v}(S)$; 3. $i \in S \subset T \Rightarrow \hat{v}(T) - v(T) \leq \hat{v}(S) - v(S)$.

Let y^* and $\hat{y}^*$ be the egalitarian solutions of Γ and $\hat{\Gamma}$, respectively. Then, for all $j \in N$,

$$\hat{y}_i^* - y_i^* \geq \hat{y}_j^* - y_j^*. \tag{4}$$

Proof. See Appendix A for the proof of Theorem 1 after the proof of Lemma 7. ■

It can be quickly confirmed that if the games Γ and $\hat{\Gamma}$ are linearly transformed to Γ' and $\hat{\Gamma}'$ by

$$v'(S) = av(S) + \sum_{j \in S} x_j \text{ and } \hat{v}'(S) = a\hat{v}(S) + \sum_{j \in S} x_j$$

for some positive factor a and fixed vector $\vec{x}$, then the three conditions above still hold. Hence, the comparative statics result in Theorem 1 is robust with respect to monotone linear transformations of the games in discussion. Theorem 1 is established from the following series of observations:

Players with strictly lower payoffs than i at y^* will have the same payoff at $\hat{y}^*$ as at y^*; Players with the same payoff as i at y^* but not in i's clique will have the same payoffs at $\hat{y}^*$ as at y^*; Player i's payoff will be weakly higher at $\hat{y}^*$ than at y^*; Players with higher payoffs than i at $\hat{y}^*$ will have weakly lower payoffs than they did at y^*. The focus of proving Theorem 1 will be on establishing the last statement: the first two statements above are immediate observations from the definition of Algorithm $\mathscr{E}$—these players can be arranged to be selected before i is in both Γ and $\hat{\Gamma}$ according to Algorithm $\mathscr{E}$ and since the values of the characteristic functions $v(\cdot)$ and $\hat{v}(\cdot)$ for any subset excluding i are the same, these players' payoffs will be exactly the same at y^* and at $\hat{y}^*$.

The third statement follows directly from Proposition 3 in Tian (2015), which actually offers a bit more than merely ensuring i's payoff never decreasing from y^* to $\hat{y}^*$. First of all, it applies to maximizers of *any* Nash collective utility functions, not just the symmetric ones. It also postulates that some *group* of players, rather than just i, will have weakly higher payoffs at $\hat{y}^*$ than at y^*. In the current context of the egalitarian solutions, the Proposition implies that all players in i's *clique* at y^* will have weakly higher payoffs at $\hat{y}^*$. However, it does not confirm that i's clique, along with any other players included in the first two statements above, are the only players with such properties—the Proposition simply guarantees the existence of a subset of players who will *never* see their payoffs decreased from y^* to $\hat{y}^*$, although there may well be many such groups.

Forward looking to the practice of designing division mechanisms, the appeal of such simultaneous and unidirectional movement in a group of players' payoffs caused by a single player's relaxation of the boundaries are mostly for the objective toward strategy-proof-ness. The concurrent increase in players' payoffs are often studied through the optimization of some objective functions over a lattice structure in the literature[15]. Essentially, I wish to build in a moderate degree of complementarity among the players' payoffs to use other players' payoff increase to limit that of the

[15]See, for example, Topkis (1978) and Topkis (2001) for classical treatment or the more recent Quah (2007).

potentially deviating player[16]. The aforementioned papers established that super-modularity of the objective function may be the most direct answer for such quests, in a very loose sense necessitating the additively separable structure of the Nash collective utility function while also maintaining considerable flexibility for the mechanism designers—separability easily promises both super- and sub-modularity. I will now turn to proving that any player with strictly higher payoff than i has at $\hat{y}^*$ will have weakly lower payoff than they did at y^*. The proof is largely driven by the following observation regarding Algorithm $\mathscr{E}$.

Lemma 7 (Third observation of Algorithm $\mathscr{E}$)**.** *Given a monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form and let $\{S_k\}_{k=1}^K$ be any sequence of subsets generated by Algorithm $\mathscr{E}$. Then*

$$\forall 1 \leq k \leq K,\ e(S_k - S_{k-1}) = \max_{S \subset S_k} e(S_k - S).$$

Proof. See Appendix A for the proof of Lemma 7. ■

Moreover, the mechanics of Algorithm $\mathscr{E}$ also provides the following observation in the mechanism design context: suppose one player disturbs the choice of the egalitarian solution by lying about her type, resulting in a relaxation of the feasible set as described in Theorem 1, will her own payoff necessarily change? The following lemma gives an affirmative answer.

Lemma 8 (Unilateral disturbance)**.** *Let $\Gamma = (N, v(\cdot))$ and $\hat{\Gamma} = (N, \hat{v}(\cdot))$ be two monotone and concave cooperative games with transferable utilities in characteristic function form such that there exists a unique $i \in N$ such that, for all $S \subseteq N$,*

1. $i \notin S \Rightarrow v(S) = \hat{v}(S)$; 2. $i \in S \Rightarrow v(S) \leq \hat{v}(S)$; 3. $i \in S \subset T \Rightarrow \hat{v}(T) - v(T) \leq \hat{v}(S) - v(S)$.

[16] Bear in mind that such complementarity can only demonstrate itself in the solutions, which makes choosing the "right" objective function the caveat in designing a well-behaved mechanism.

Let y^* and $\hat{y}^*$ be the egalitarian solutions and $\mathscr{P}^*$ and $\hat{\mathscr{P}}^*$ generated by Algorithm $\mathscr{E}$ for Γ and $\hat{\Gamma}$, respectively. Then $y_i^* = \hat{y}_i^* \Rightarrow \left[y^* = \hat{y}^* \text{ and } \mathscr{P}^* = \hat{\mathscr{P}}^*\right]$. Otherwise,

$$\left[\hat{y}_j^* \geq \hat{y}_i^* \text{ and } j \notin Q_i\left(\hat{y}^*\right)\right] \Rightarrow \hat{y}_j^* \leq y_j^*.$$

Proof. See Appendix A for the proof of Lemma 8 after the proof of Theorem 1. ■

On the other hand, the scope of Lemma 7 extends beyond proving the crucial comparative statics result in Theorem 1. It suggests that Algorithm $\mathscr{E}$ maintains a duality-like property, which I formalize with the following results. I first present a lemma similar to Lemma 1.

Lemma 9 (Secondary observations of $e(\cdot)$)**.** *Given any monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function being given by $v(\cdot)$, then*

1. *$\forall S \subset T \subset R \subseteq N$, $e(R-T) = e(R-S) \Rightarrow e(R-S) = e(R-T) = e(T-S)$;*
2. *$\forall T \subset R \subseteq N$ and $S \subset R \subseteq N$ such that $e(R-S) = e(R-T)$, then*

$$\max\left\{e\left[R-(T\cup S)\right], e\left[R-(T\cap S)\right]\right\} \geq e(R-S) = e(R-T).$$

Proof. See Appendix A for the proof of Lemma 9. ■

Now, consider the following "dual" algorithm to Algorithm $\mathscr{E}$.

Definition 2 (Algorithm $\bar{\mathscr{E}}$)**.** Given any monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$, run Algorithm $\bar{\mathscr{E}}$ through the following steps:

1. Set $T_0 = N$. In step 1, select T_1 where $T_1 \subset N$ such that

$$e(N - T_1) \geq e(N - T),\ \forall T \subset N. \tag{5}$$

If there are multiple candidates for T_1, select the largest one: let T_1 be such that, in addition to satisfying (5), for all T such that $T \supset T_1$, $e(N - T) < e(N - T_1)$;

If there are multiple largest candidates, select any one and proceed;

2. Let T_k be subset of N that was selected in step k, for all $k \geq 1$. In step $(k+1)$, select any of the largest $T_{k+1} \subset T_k$ such that $e\,(T_k - T_{k+1}) \geq e(T_k - T)$ for all $T \subset T_k \subseteq N$;

3. Terminate in step $K \geq 1$ if the only candidate for T_K is $\emptyset$;

4. After termination, let $\bar{y}^* \in \mathbb{R}^n_+$ be the vector generated by setting $\bar{y}^*_i = e\,(T_k - T_{k+1})$ if $i \in (T_k \backslash T_{k+1})$; The sequence of subsets $\{T_k\}_{k=1}^K$ selected in the proceedings and the resulting vector $\bar{y}^*$ will be referred to as the **sequence of subsets generated by Algorithm $\bar{\mathscr{E}}$** and the **dual egalitarian solution of** Γ, respectively.

As with Algorithm $\mathscr{E}$, Lemma 9 ensures that Algorithm $\bar{\mathscr{E}}$ is well-defined. By the way Algorithm $\bar{\mathscr{E}}$ is phrased, the following result should not come as a surprise.

Theorem 2 ("Duality")**.** *Let $\Gamma = (N, v(\cdot))$ be a monotone and concave cooperative game with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$. Then*

1. *Both Algorithm $\mathscr{E}$ and Algorithm $\bar{\mathscr{E}}$ terminate in the same number $K \leq N$ of steps;*

2. *For any sequence of subsets of N, $\{S_k\}_{k=1}^K$, generated by Algorithm $\mathscr{E}$, there exists a sequence of subsets of N, $\{T_k\}_{k=1}^K$, generated by Algorithm $\bar{\mathscr{E}}$, such that $T_k = S_{K-k}$, $\forall 0 \leq k \leq K$;*

3. *Moreover, the egalitarian solution y^* is equal to the dual egalitarian solution $\bar{y}^*$;*

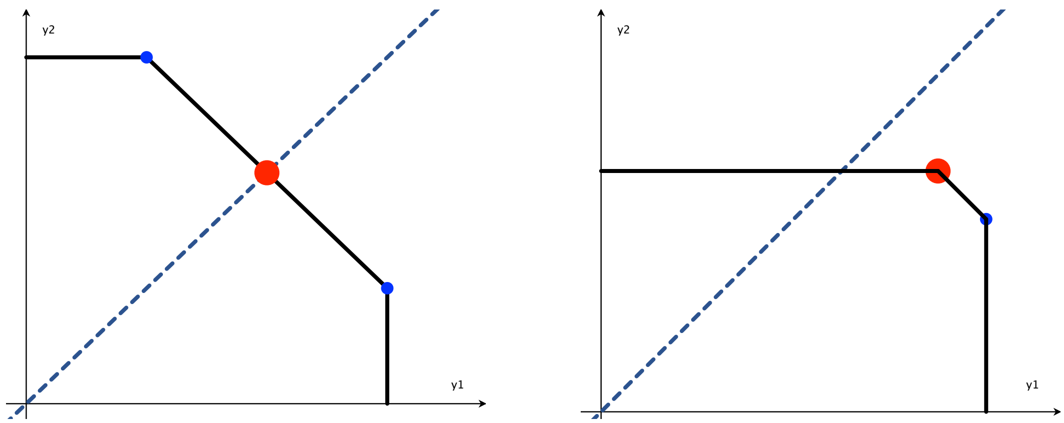
4. *Furthermore,* $\{(T_k \backslash T_{k+1})\}_{k=0}^{K-1}$ *generated by Algorithm* $\bar{\mathscr{E}}$ *coincides with* $\mathscr{P}^*$ *generated by Algorithm* $\mathscr{E}$ *as defined in Lemma 3.*

Proof. See Appendix A for the proof of Theorem 2. ∎

Theorem 2 appears very counter-intuitive at first sight since it states that recursively maximizing the average payoff of the best-off among the remaining players will result in a generalized Lorenz dominant payoff distribution, which goes right against the spirit of Lorenz dominance (generalized or not) favoring transfers from the rich agents to the poor. As a matter of fact, the same question can be raised for Algorithm $\mathscr{E}$—how is minimizing the incomes of the poorest agents supposed to lead to egalitarianism? However, a closer look reveals how Theorem 2 accurately locates the egalitarian solution. First, it correctly identities the players who contribute more payoff possibilities to the cooperative game and subsequently allocates higher payoffs to them. To achieve generalized Lorenz dominance—maximal amount of total payoff and equality, simultaneously—players who face more relaxed boundaries should receive higher payoffs so as not to compete with players who face more restraining constraints. Second, although players selected earlier in Algorithm $\bar{\mathscr{E}}$ walk away with higher payoffs, they are assigned the left-overs from the players selected later since their payoffs are the average incremental from the immediately succeeding players.

Essentially, selecting the egalitarian solution boils down to selecting the set of payoff constraints that lead to Lorenz dominance. Both Algorithm $\mathscr{E}$ and Algorithm $\bar{\mathscr{E}}$ set the players in the correct order of priority—the difference being Algorithm $\mathscr{E}$ selects the players with minimum payoffs first while Algorithm $\bar{\mathscr{E}}$ selects those with the maximum. Consider, on the contrary to Algorithm $\mathscr{E}$ and Algorithm $\bar{\mathscr{E}}$, a "naive" proposal that *minimizes* the average income of the best-off players. The example demonstrated in Figure 1 shows how this proposal identifies the wrong constraints to bind. In both panels, the red dot is the egalitarian solution while the blue dots are those selected by the "naive" proposal.

Figure 1: Egalitarian solutions and outcome of the "naive" proposal



Example 1. The characteristic functions are, respectively

$$v(\{1\}) = v(\{2\}) = 7.5,\ v(\{1,2\}) = 10 \text{ and } v'(\{1\}) = 8,\ v'(\{2\}) = 5,\ v'(\{1,2\}) = 12.$$

Not surprisingly, the payoff vectors selected by the "naive" proposal is (generalized) Lorenz dominated by the egalitarian solutions. Notice also that in the left panel, there are two distinct "naive" solutions, although they are equivalent in payoff distributions in the Lorenz sense. The Lorenz comparisons between the egalitarian solutions and the "naive" solutions are as follows when the payoffs of the players are sorted in a weakly increasing order:

$$\mathbf{y} = (2.5, 7.5),\ \mathbf{y}^* = (5,5) \text{ and } \mathbf{y}' = (4,8),\ \mathbf{y}'^* = (5,7).$$

In both panels, a transfer of payoff from the better-off player to the worse-off in the "naive" solutions results in the Lorenz improvement to the egalitarian solutions, highlighting the mistake by the "naive" proposal in placing a higher priority on the payoff to a potentially better-off player (player 1, in this case) than on that to the other player. Of course, this feature is more evident in the right-panel where the Lorenz improvement from the "naive" to the egalitarian solution does not change the identities of the better- and worse-off players.

Disguised under the two-player examples are the more fatal issue of feasibility. To better understand this, consider the "dual" version of the "naive" proposal. A first group of players are selected according to the *maximum* average demand, then a second group is selected according to the maximum average incremental demand. As mentioned before, this algorithm was adopted first by Dutta and Ray (1989) for *convex* cooperative games with transferable utilities. An immediate concern with such selections when applied to concave games is that the constraints for subsets of the first group of players may easily be violated. Granted, the focus of egalitarian solutions in convex cooperative games is to achieve individual rationality (i.e. respecting participation constraints—hence the considerable attention paid to defining the concept of Lorenz *core* in Dutta and Ray (1989)) and maximum equality among the players' payoffs simultaneously. On the other hand, feasibility (or more mechanically and more fundamentally, concavity) is the constraint holding the egalitarian solution together in concave games.

I conclude this section with two quick notes on the relationship between the current work, especially Theorem 2, and the previous work in Dutta and Ray (1989) and Chen et al. (2013). It is quite obvious that the current work extended Chen et al. (2013) by offering the insight on the two different sequences of subsets of N to achieve egalitarianism with Theorem 2. In the light of this new interpretation, it should be a simple routine to show that the "naive" proposal will become the "dual" of the algorithm developed by Dutta and Ray (1989) for convex cooperative games. Finally, the "naive" proposal (or the algorithm in Dutta and Ray (1989)) cannot be directly applied to concave games because it quickly runs into the issue of multiple candidate solutions even in instances as simple as Example 1.

4 Sequential/dynamic fair division mechanisms

In this section, I demonstrate how to construct a Pareto efficient, strategy-proof, and envy-free sequential or dynamic division mechanism for any problem, given a set of players N with $|N| = n < +\infty$. Recall that, given a set of players N, a problem is defined to be a quintuple $\Pi \equiv \left(b, B, T, \Theta^B, \beta^B\right)$ where the sequence B segments the interval $I_b \equiv [0, b)$ into to a partition of size T, β^B being the sequence of discount factors for the intervals in the partition of I_b, and Θ^B the space of the players' preferences. For the brevity of discussion, I will fix a problem Π from now on and show how to construct a desired mechanism from the quintuple of parameters of Π, hence devising a systematic strategy of constructing mechanisms for any generic problem.

I start with linking the egalitarian solutions to the division problem. Consider the restriction $\vec{\theta}_t \in \left(\Theta_t^B\right)^n$ of a preference profile to $[b_t, b_{t+1}]$ for some $0 \leq t \leq (T-1)$. For any set of players S,

$$v(S|\ \vec{\theta}_t) \equiv \int_{\bigcup_{i \in S} \theta_{it}} 1\, dr.$$

Lemma 5 in Tian (2015) showed that $v(\cdot|\vec{\theta}_t)$ so defined generates the characteristic functions for a monotone and concave game for any $\vec{\theta}_t$, i.e. $v(S \cup T|\vec{\theta}_t) + v(S \cap T|\vec{\theta}_t) \leq v(S|\vec{\theta}_t) + v(T|\vec{\theta}_t)$ and $S \subset T \subseteq N \Rightarrow v(S|\vec{\theta}_t) \leq v(T|\vec{\theta}_t)$, $\forall (S, T, \vec{\theta}_t) \in \left(N \times N \times \left(\Theta_t^B\right)^n\right)$. Hence, let $y^*(\vec{\theta}_t)$ and $\mathscr{P}^*(\vec{\theta}_t)$ represent the egalitarian solutions and the partition of N generated by Algorithm $\mathscr{E}$, respectively, for the monotone and concave $\Gamma(\vec{\theta}_t) \equiv \left(N, v(\cdot|\vec{\theta}_t)\right)$. Similarly, I can extend the definition of the function $e(\cdot)$ into the context with $\vec{\theta}_t$ as

$$e(T - S|\vec{\theta}_t) \equiv \frac{v(T|\vec{\theta}_t) - v(S|\vec{\theta}_t)}{|T| - |S|},\ \forall (S, T, \vec{\theta}_t) \in \left(2^N, 2^N, \left(\Theta_t^B\right)^n\right) \text{ where } S \subset T.$$

In addition, the concept of feasible set can also be easily extended to include $\vec{\theta}_t$:

$$V(\vec{\theta}_t) \equiv \left\{ y \in \mathbb{R}^n_+ : \sum_{i \in S} y_i \leq v(S|\vec{\theta}_t), \ \forall S \subseteq N \right\}.$$

Definition 3. For any problem $\Pi = \left(b, B, T, \Theta^B, \beta^B\right)$, define the following quantities:

$$E_t \equiv \left\{ e(T - S|\vec{\theta}_t) : (S, T, \vec{\theta}_t) \in \left(2^N, 2^N, \left(\Theta^B_t\right)^n\right) \text{ where } S \subset T \subseteq N \right\} \cup \{0\},$$

which gives rise to the following quantity

$$0 < \delta_t \equiv \min \left\{ \beta^B_t \cdot |z_1 - z_2| : (z_1, z_2) \in E^2_t \text{ and } z_1 - z_2 \neq 0 \right\}.$$

Moreover, for $1 \leq t \leq T$, let

$$\chi_t \equiv \left[\sum_{s=1}^{t} \beta^B_{s-1} \cdot (b_s - b_{s-1}) \right] > 0.$$

Finally, recursively select the entries of two sequences $\{\rho_t\}_{t=1}^{T-1}$ and $\{\Delta_t\}_{t=1}^{T}$ as follows:

1. Set $\Delta_1 = \delta_0$; Select any ρ_1 from the interval $\left(0, \min\left\{\frac{\Delta_1}{\chi_1 + \Delta_1}, \frac{\delta_1}{2\chi_1}\right\}\right)$;

   Set $\Delta_2 = \min\{(1 - \rho_1)\Delta_1 - \rho_1\chi_1, \delta_1 - \rho_1\chi_1\}$;

2. Given Δ_t, select any ρ_t from the interval $\left(0, \min\left\{\frac{\Delta_t}{\chi_t + \Delta_t}, \frac{\delta_t}{2\chi_t}\right\}\right)$;

   Set $\Delta_{t+1} = \min\{(1 - \rho_t)\Delta_t - \rho_t\chi_t, \delta_t - \rho_t\chi_t\}$.

Remark 1. Notice that since Θ^B_t is a finite set, E_t is a finite set, which means that δ_t is well-defined for all t. Secondly, χ_t represents the maximum cumulative payoff any player can receive before any goods in $[b_t, b)$ is assigned. Last but not least, it can be easily confirmed that, by definition, all entries in the sequence $\{\Delta_t\}_{t=1}^{T}$ is strictly positive while all entries in the sequence $\{\rho_t\}_{t=1}^{T-1}$ is strictly between 0 and 1. In terms of the mechanics, the sequence $\{\Delta_t\}_{t=1}^{T}$ is generated to define the sequence $\{\rho_t\}_{t=1}^{T-1}$. Now, consider the following (class of) sequential/dynamic division mechanism.

Definition 4 (Mechanism $\mathscr{R}$)**.** Given any problem $\Pi = \left(b, B, T, \Theta^B, \beta^B\right)$ with players N, let $\{\rho_t\}_{t=1}^{T-1}$ be any sequence selected in Definition 3. Define Mechanism $\mathscr{R}$ with the following steps:
Given any preference profile θ,

1. In step 1, select any feasible division $C_1 \in \mathscr{C}_b$ such that

$$\vec{u}_0^B(C_1|\theta) = \arg\max_{y\in V(\vec{\theta}_0)} \sum_{i\in N} W_1(y_i) \text{ and } r \notin \theta_i \Rightarrow C_1(r) \neq i, \forall r \in I_b$$

   for any strictly increasing and strictly concave $W_1 : \mathbb{R}_+ \to \mathbb{R}$ continuous on $\mathbb{R}_+$;

2. In step $(t+1)$, given C_t with $1 \leq t \leq (T-1)$, select any C_{t+1} such that $C_{t+1}(r) = C_t(r)$ for all $r \leq b_t$ and

$$\vec{u}_t^B(C_{t+1}|\theta) = \arg\max_{y\in V(\vec{\theta}_t)} \sum_{i\in N} W_{t+1}\left(\rho_t \cdot x_{it}^B(C_t|\theta_i) + \beta_t^B y_i\right) \equiv \vec{y}_{t+1}\left(\vec{\theta}_t\right)$$

   and $r \notin \theta_i \Rightarrow C_{t+1}(r) \neq i, \forall r \in I_b$, for any strictly increasing and strictly concave function $W_{t+1} : \mathbb{R}_+ \to \mathbb{R}$ continuous on $\mathbb{R}_+$;

3. Define $\mathscr{R}(\theta)$ to be the pointwise limit of the sequence of feasible divisions $\{C_t\}_{t=1}^{T}$.

Remark 2*.* The pointwise limit of the sequence of functions $\{C_t\}_{t=1}^{T}$ exists for any θ since for any given $r \in I_b$, there exists a $b_{\bar{t}+1} > r$ (recall that the collection of batches/periods $[b_t, b_{t+1}]$ partitions I_b), which means that sequence $\{C_t(r)\}_{t=1}^{T}$ will simply be constant at $C_{\bar{t}}(r)$ for $t \geq \bar{t}$. Of course, if T is finite, then $\mathscr{R}(\theta)$ will simply be equal to C_T. The range for all the entries in the sequence

Table 1: Period and cumulative payoffs given by Mechanism $\mathscr{R}$ for preference profile θ

$i \backslash t$	0	1	2	3	4	5	6	u
$1y$	1	0	0	0	0.5003102	0.5000000	0	NA
$1x$	0	1	1	1	1	1.5003102	2.0003101	2.0003101
$2y$	0	1	0	1/2	0.4996898	0	0.5000000	NA
$2x$	0	0	1	1	3/2	1.9996898	1.9996898	2.4996898
$3y$	0	0	1	1/2	0	0.5000000	0.5000000	NA
$3x$	0	0	0	1	3/2	3/2	2.0000000	2.5000000

of functions $\{C_t\}_{t=1}^{T}$ is N, which implies so must be the range for the pointwise limit function $\mathscr{R}(\theta)$, which, in turn, implies that the division chosen by Mechanism $\mathscr{R}$ is feasible. Therefore, $u_i^B(\mathscr{R}(\theta)|\theta_i)$, with a finite upper bound for $u_i^B(\cdot|\cdot)$ of *any* feasible division, is well-defined for all players. Furthermore, in each step $(t+1)$, $V(\vec{\theta}_t)$ completely characterizes the set of all feasible divisions of $[b_t, b_{t+1}]$, a fact proven by Proposition 1 in Tian (2015). Mechanism $\mathscr{R}$ is well-defined.

The choice of the function W_t in step t can be completely arbitrary as long as it satisfies the conditions stated in Definition 4 due to Lemma 5—any admissible choice of such functions will not alter the vector of period payoffs chosen, which seemingly makes Lemma 5 a trivial result. However, as mentioned before, Lemma 5 provides a direct and tractable proof for the fairness property of Mechanism $\mathscr{R}$. I will now present the main result regarding Mechanism $\mathscr{R}$.

Proposition 1 (C-PE-EF-SP)**.** *For any problem Π for a given set of player N, Mechanism $\mathscr{R}$ as defined in Definition 4 is consistent, Pareto efficient, envy-free, and strategy-proof.*

Proof. See Appendix B for the proof of Proposition 1. ■

Mechanism $\mathscr{R}$ is clearly consistent simply by the way it is defined. If two preferences profile are identical up to some $b_{\bar{t}}$, so will the players' period payoffs up to $b_{\bar{t}}$ since the maximizers in any step up to $\bar{t}$ is unique according to Lemma 5 and the fact that $W_t(\cdot)$ is *strictly* concave for any $t \leq \bar{t}$. For Pareto efficiency, a moment of reflection confirms that it is implied by the following two conditions jointly: a feasible division C is Pareto efficient at a preference profile θ if

1. For any $r \in I_b$, $r \notin \theta_i \Rightarrow C(r) \neq i$;

2. And for any $r \in I_b$, if there exists an $i \in N$ such that $r \in \theta_i$, then $C(r) \neq 0$.

The first condition states that no good or resource is ever assigned to some player who does not need it and the second one states that as long as there is some player needing a good or resource, this good or resource cannot remain unassigned. Obviously, both conditions are satisfied by $\mathscr{R}(\theta)$ for any θ simply by design. Hence, the proof of Proposition 1 will solely focus on envy-free-ness and strategy-proof-ness. I will now demonstrate the mechanics and properties of Mechanism $\mathscr{R}$ with an example to highlight how the fairness and strategic properties hold, to shed some light on the proof of these properties.

Example 2. Consider the following simple problem Π:

$$N = \{1,2,3\}, \ T = 7, \ [b_t, b_{t+1}] = [t, t+1], \ \beta_t^B = 1, \text{ and } \Theta_t^B = \{\emptyset, [b_t, b_{t+1}]\}, \tag{6}$$

$$\Rightarrow E_t = \left\{0, \frac{1}{3}, \frac{1}{2}, 1\right\} \Rightarrow \delta_t = \frac{1}{6}, \ \forall 0 \leq t \leq 6 \text{ and } \chi_t = t, \ \forall 1 \leq t \leq 6,$$

which yield a following candidate pair of sequences

$$\rho = \left\{\frac{1}{13}, \frac{1}{40}, \frac{1}{161}, \frac{1}{806}, \frac{1}{4837}, \frac{1}{33860}\right\} \text{ and } \Delta = \left\{\frac{1}{6}, \frac{1}{13}, \frac{1}{40}, \frac{1}{161}, \frac{1}{806}, \frac{1}{4837}, \frac{1}{33860}\right\}.$$

Table 2: Period and cumulative payoffs given by Mechanism $\mathscr{R}$ for preference profile $\hat{\theta}$

$i\backslash t$	0	1	2	3	4	5	6	u
$1y$	1	0	0	0.3333333	0.5000000	0.4999483	0	NA
$1x$	0	1	1	1	1.3333333	1.8333333	2.3332816	2.3332816
$2y$	0	1	0	0.3333333	0.5000000	0	0.5000000	NA
$2x$	0	0	1	1	1.3333333	1.8333333	1.8333333	2.3333333
$3y$	0	0	1	0.3333333	0	0.5000517	0.5000000	NA
$3x$	0	0	0	1	1.3333333	1.3333333	1.8333850	2.3333850

Now, consider the following two preference profiles:

$$\begin{cases} \theta_1 = [0,1] \cup [4,6],\ \theta_2 = [1,2] \cup [3,5] \cup [6,7],\ \theta_3 = [2,4] \cup [5,7] \quad [\theta]\,; \\ \hat{\theta}_1 = [0,1] \cup [3,6],\ \hat{\theta}_2 = [1,2] \cup [3,5] \cup [6,7],\ \hat{\theta}_3 = [2,4] \cup [5,7] \quad \left[\hat{\theta}\right]. \end{cases}$$

The payoffs given by Mechanism $\mathscr{R}$ for the two profiles are summarized in Table 1 for θ and Table 2 for $\hat{\theta}$. Unshaded cells contain the period payoffs while shaded cells contain cumulatives.

To demonstrate the fairness property of Mechanism $\mathscr{R}$, the interpersonal comparisons of allocations and payoffs are given in Table 3. The left panel corresponds to the preference profile θ and the right to $\hat{\theta}$. Cell (i, j) represents player i's payoff if she were allocated with player j's allocation. Clearly, entries on the diagonals are greater than those off the diagonals, representing envy-free-ness. Finally, strategy-proof-ness is presented in Table 4 with the cross preference and cumulative-payoff comparisons for player 1 between types θ_1 and $\hat{\theta}_1$. Each cell in rows 1, 2, and 4 contains the cumulative payoff corresponding to the preference types and allocations in the first column. Rows 3 and 5 are the differences between rows 1 and 2 and 1 and 4, respectively, representing the cumulative allocation and payoff processes for player 1 with type θ_1 to misrepresent

Table 3: Fairness property: interpersonal preference-payoff comparisons

	1	2	3
$u_1(\cdot\|\theta_1)$	2.0003101	0.4996898	0.5000000
$u_2(\cdot\|\theta_2)$	0.5003102	2.4996898	1.0000000
$u_3(\cdot\|\theta_3)$	0.5000000	1.0000000	2.5000000

	1	2	3
$u_1\left(\cdot\|\hat{\theta}_1\right)$	2.3332816	0.8333333	0.8333850
$u_2\left(\cdot\|\hat{\theta}_2\right)$	0.8333333	2.3333333	0.8333333
$u_3\left(\cdot\|\hat{\theta}_3\right)$	0.8332816	0.8333333	2.3333850

himself with type $\hat{\theta}_1$. The first two periods are ignored: they are equal for both $\mathscr{R}(\theta)$ and $\mathscr{R}\left(\hat{\theta}\right)$.

From rows 3 and 5, by misrepresenting her type, player 1 will initially be allocated more goods in the period of deviation, which, nevertheless, does not generate any payoff for her since the interval $[3,4]$ is not contained in her type θ_1. This initial bump in the cumulative allocation decays away as time goes by, demonstrated by the decreasing trend in the absolute value of the entries in row 3, indicating that in each period after deviation, player 1 is retaining less payoff with misrepresentation. Row 5 yields the same conclusion with a different narrative. It measures the cumulative *payoff* differences between truth-telling and misrepresentation, with all entries being positive and showing an increasing trend.

Example 2 turns out to be extendable to the general context of any unilateral misrepresentation of types by any player: the observations of the initial bump in the amount of goods/resources allocated to the deviating player, which later tapers off in terms of payoffs (a hill-shaped trend or sorts) as demonstrated in Table 4, are maintained through any arbitrary deviations made by one single player. The pursuit of such observations motivates limiting the sequence of ρ_t's to be strictly less than 1 in each iteration and serves as the basis of the proof of strategy-proof-ness. On the other hand, the choice of player 1 only deviating for one single period is certainly no accident. In any problem Π within the scope of this paper, the "one-shot deviation principle" introduced in Blackwell (1965) directly applies.

Not equally evident in Example 2 is the reason for the choice of the deviation $\hat{\theta}_1$ being a superset of the true type θ_1. This proves to be a sufficient choice, along with considering deviations as subsets of the true types, for strategy-proof-ness, a result previously documented as Lemma 2 in Tian (2015). The current context requires a very mild extension of this previous result in the sequential and dynamic settings. However, the applicability of Lemma 2 in Tian (2015) is evident. Hence, I shall only briefly present the argument for the sufficiency of only considering deviations as super and subsets instead of documenting a formal result—readers should feel free to refer to Tian (2015) for the details of the proof.

Without loss of generality, let player 1 be deviating from her true type θ_1 to some $\hat{\theta}_1$ such that $\theta_1 \not\subseteq \hat{\theta}_1$ and $\hat{\theta}_1 \not\subseteq \theta_1$. Moreover, by the one-shot deviation principle, let θ_1 and $\hat{\theta}_1$ differ in only one period t. Define $\tilde{\theta}_1 \equiv \left(\theta_1 \cap \hat{\theta}_1\right)$. Also, since other players' preference profile is constant, let $\hat{\theta} \equiv \left(\hat{\theta}_1, \theta_{-1}\right)$ represent the preference profile obtained by combining $\hat{\theta}_1$ with other players' constant preference profile and similarly for $\tilde{\theta}$. Clearly, player 1's payoff before period t will be the same across θ_1, $\hat{\theta}_1$, and $\tilde{\theta}_1$ since Mechanism $\mathscr{R}$ is only history dependent in terms of payoffs. On the other hand, let CV, $\hat{CV}$, and $\tilde{CV}$ represent the continuation values from period $(t+1)$ when player 1 is reporting θ_1, $\hat{\theta}_1$, and $\tilde{\theta}_1$, respectively. Different feasible divisions from period $(t+1)$ can be compared just with the continuation value for player 1 because she reports the same type beyond period t. Now, recall the premise for the sufficiency result is that neither deviations to supersets nor deviations to subsets of the true type can benefit the deviating agent. The goal is to show that, in this case, $\hat{\theta}_1$ can never be a profitable deviation for player 1. Since any goods assigned to player 1 outside her type does not contribute to her utility,

$$u_{1t}\left(\mathscr{R}\left(\hat{\theta}\right)|\theta_1\right) = u_{1t}\left(\mathscr{R}\left(\hat{\theta}\right)|\tilde{\theta}_1\right) \text{ and } u_{1t}\left(\mathscr{R}\left(\tilde{\theta}\right)|\tilde{\theta}_1\right) = u_{1t}\left(\mathscr{R}\left(\tilde{\theta}\right)|\theta_1\right). \tag{7}$$

Now, consider when player 1's type is $\tilde{\theta}_1$. No profitable deviation to supersets implies that

$$u_{1t}\left(\mathscr{R}\left(\hat{\theta}\right)|\tilde{\theta}_1\right)+\hat{CV} \leq u_{1t}\left(\mathscr{R}\left(\tilde{\theta}\right)|\tilde{\theta}_1\right)+\tilde{CV}. \tag{8}$$

Moreover, between θ_1 and $\tilde{\theta}_1$, no profitable deviation to subsets implies that

$$u_{1t}\left(\mathscr{R}\left(\tilde{\theta}\right)|\theta_1\right)+\tilde{CV} \leq u_{1t}\left(\mathscr{R}\left(\theta\right)|\theta_1\right)+CV. \tag{9}$$

Combining (7), (8), and (9) yields

$$\begin{aligned} u_{1t}\left(\mathscr{R}\left(\hat{\theta}\right)|\theta_1\right)+\hat{CV} &\underbrace{=}_{\text{by (7)}} u_{1t}\left(\mathscr{R}\left(\hat{\theta}\right)|\tilde{\theta}_1\right)+\hat{CV} \underbrace{\leq}_{\text{by (8)}} u_{1t}\left(\mathscr{R}\left(\tilde{\theta}\right)|\tilde{\theta}_1\right)+\tilde{CV} \\ &\underbrace{=}_{\text{by (7)}} u_{1t}\left(\mathscr{R}\left(\tilde{\theta}\right)|\theta_1\right)+\tilde{CV} \underbrace{\leq}_{\text{by (9)}} u_{1t}\left(\mathscr{R}\left(\theta\right)|\theta_1\right)+CV, \end{aligned} \tag{10}$$

showing that $\hat{\theta}_1$ can never be a profitable deviation from θ_1. This result, along with the one-shot deviation principle, reduces the potentially very intractable problem of checking incentive compatibility in a sequential or dynamic setting into a comparative statics problem where all potentially profitable deviations are completely and linearly ordered so that Theorem 1 can be applied. Observe that the one-shot deviation principle is invoked here only for convenience, since (7) applies to any comparison between types and even for the cumulative payoffs. I document this result with the following lemma for the ease of future reference and defer the brief proof to Appendix B.

Lemma 10 (Super-Sub)**.** *For any* $(i,(\theta_i,\theta_{-i})) \in \left(N\times\Theta^B\times\left(\Theta^B\right)^{n-1}\right)$,

$$u_i\left(\mathscr{R}\left(\theta_i,\theta_{-i}\right)|\theta_i\right) \geq u_i\left(\mathscr{R}\left(\underline{\theta}_i,\theta_{-i}\right)|\theta_i\right),\ \forall\underline{\theta}_i \subset \theta_i \in \Theta^B$$

Table 4: Strategic property: individual cross preferences-payoff comparisons

	3	4	5	6	7
$u_1(\mathscr{R}(\theta)\|\theta_1)$	1.0000000	1.0000000	1.5003102	2.0003101	2.0003101
$u_1\left(\mathscr{R}\left(\hat{\theta}\right)\|\hat{\theta}_1\right)$	1.0000000	1.3333333	1.8333333	2.3332816	2.3332816
Row 1 − Row 2	0.0000000	−0.3333333	−0.3330232	−0.3329715	−0.3329715
$u_1\left(\mathscr{R}\left(\hat{\theta}\right)\|\theta_1\right)$	1.0000000	1.0000000	1.5000000	1.9999483	1.9999483
Row 1 − Row 4	0.0000000	0.0000000	+0.0003101	+0.0003618	+0.0003618

and

$$u_i\left(\mathscr{R}\left(\theta_i,\theta_{-i}\right)|\theta_i\right) \geq u_i\left(\mathscr{R}\left(\overline{\theta}_i,\theta_{-i}\right)|\theta_i\right), \ \forall\overline{\theta}_i \supset \theta_i \in \Theta^B$$

jointly implies that $u_i\left(\mathscr{R}\left(\theta_i,\theta_{-i}\right)|\theta_i\right) \geq u_i\left(\mathscr{R}\left(\theta_i',\theta_{-i}\right)|\theta_i\right), \ \forall\theta_i' \neq \theta_i \in \Theta^B$.

Proof. See Appendix B for the proof of Lemma 10. ■

To apply Theorem 1, I provide the following lemma verifying that the conditions in Theorem 1 are met with deviations to supersets of true types.

Lemma 11 (Applicability of Theorem 1)**.** *For all preference profiles $\theta \neq \hat{\theta} \in \left(\Theta^B\right)^n$ such that there exists a unique $i \in N$ with $\theta_i \subseteq \hat{\theta}_i$ and $\theta_j = \hat{\theta}_j$ for all $j \neq i$,*

$$\textit{1. } i \notin S \Rightarrow v(S|\theta) = v\left(S|\hat{\theta}\right);\ \textit{2. } i \in S \Rightarrow v(S|\theta) \leq v\left(S|\hat{\theta}\right);$$
$$\textit{3. } i \in S \subset T \Rightarrow v\left(T|\hat{\theta}\right) - v(T|\theta) \leq v\left(S|\hat{\theta}\right) - v(S|\theta).$$

for all $S,T \subseteq N$ and $0 \leq t \leq (T-1)$.

Proof. See Appendix B for the proof of Lemma 11. ■

With Lemma 11, I manage to extend Theorem 1 to the dynamic setting and, with the carefully chosen sequences of $\{\rho_t\}_{t=1}^{T-1}$, extrapolate previously known static strategy-proof-ness results to encapsulate cumulative payoffs for any periods after deviation. Specifically, consider two preference profiles where all players' preferences are the same except for i's. Within only one period t, i's type in one profile is a superset of her type in the second profile. Theorem 1, along with the sequence $\{\rho_t\}_{t=1}^{T-1}$ defined in Definition 3, ensures that player i is given more goods/resources in period t and cumulatively for every period after t. This first rules out any profitable deviations to subsets of any player's true type. On the other hand, the difference in the cumulative amount of goods or resources assigned to i between the two profiles will decrease over time (nevertheless, will always stay positive), which implies that the cumulative assignment for i is weakly less in the second profile than in the first in periods following the deviation. Moreover, i's assignment in period t given by the first profile within her type in the second profile will be less than that given by the second profile, a direct result of Proposition 3 in Tian (2015)[17]. Therefore, if player i's type is as hers in the second profile (the subset), then a misrepresentation as in the first profile (the superset) can never benefit her in payoff terms, recovering strategy-proof-ness dynamically.

5 Discussions

5.1 Why are the ρ_t's so small?

In this subsection, I will discuss the significance in the care taken in choosing the sequence $\{\rho_t\}_{t=t}^{T}$ in Mechanism $\mathscr{R}$. I first demonstrate, through a counterexample, how an arbitrary choice of the sequence can result in the breakdown of strategy-proof-ness. On the flip side, I show how to refine any given *problem* into a new one so that the only undominated strategy for any player is truth-telling, which has a remarkable impact on the efficiency comparison between Mechanism $\mathscr{R}$ and a

[17]See Theorem 3 in Appendix B for the full statement of Proposition 3 in Tian (2015), presented here as a theorem for concave cooperative games. Please refer to the original paper for the proof.

simple repeated deployment of Algorithm $\mathscr{E}$. I start with the following example.

Example 3. Consider the following problem Π:

$$N=\{1,2,3\}\,,\; T=25,\; [b_t,b_{t+1}]=[t,t+1],\; \beta_t^B=1,\; \text{and } \Theta_t^B=\{\emptyset,[b_t,b_{t+1}]\}\,. \tag{11}$$

Instead of choosing the sequence of ρ_t according to Definition 3, simply choose $\rho_t=1$ for all $1\le t\le 24$. That is, this alternated version of Mechanism $\mathscr{R}$ will simply run the procedure of Mechanism $\mathscr{R}$ with the cumulative payoffs without being factored by the sequence defined in Definition 3. Fix the types of players 1 and 2 and consider two different preferences for player 3:

$$\begin{cases}\theta_1=[0,11]\cup[20,25],\; \theta_2=[7,17],\; \theta_3=[5,25] \quad [\theta]\,;\\ \hat\theta_1=[0,11]\cup[20,25],\; \hat\theta_2=[7,17],\; \hat\theta_3=[9,25] \quad \left[\hat\theta\right].\end{cases}$$

The outcome of this alternative mechanism is presented in Table 5. Notice that a player with type θ_3 will receive a higher payoff by misrepresenting as $\hat\theta_3$. Observe that player 3 actually deviates in more than one period in this example.

The spirit of this alternative mechanism is actually identical to that of Mechanism $\mathscr{R}$—both build in a reward and punishment mechanism when dividing in period t that is based on players' cumulative payoffs up until b_t. The difference lies in the seemingly insignificantly small reward or punishment given by Mechanism $\mathscr{R}$ that has an ample effect on incentives. Take a deviation to a subset of the true type as an example such as Table 5 but suppose that the deviating player only does so in one period t. Her cumulative payoff in the first t periods decreases as a result, which puts her at an advantage according to both the alternative mechanism and Mechanism $\mathscr{R}$. However, this lower previous cumulative payoff is discounted in Mechanism $\mathscr{R}$ by factors $\rho_{\bar t}$ with $\bar t>t$ just enough so that the resulting reward in periods after t is never enough to compensate for

Table 5: Period and cumulative payoffs by $\rho_t = 1$ for θ and $\hat{\theta}$

$i \backslash t$	5	6	7	8	9	10	11	12	13	14	15	16...	...20	21	22	23	24	u
$1y$	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	.5	NA
$1x$	5	5	5	5	5	5	5	5	5	5	5	5	5	6	7	8	9	9.5
$2y$	0	0	1	1	.5	.5	.5	.5	.5	.5	.5	.5	0	0	0	0	0	NA
$2x$	0	0	0	1	2	2.5	3	3.5	4	4.5	5	5.5	6	6	6	6	6	6
$3y$	1	1	0	0	.5	.5	.5	.5	.5	.5	.5	.5	0	0	0	0	.5	NA
$3x$	0	1	2	2	2	2.5	3	3.5	4	4.5	5	5.5	9	9	9	9	9	9.5
$\hat{1}y$	1	1	0	0	0	0	0	0	0	0	0	0	1	.5	.5	.5	.5	NA
$\hat{1}x$	5	6	7	7	7	7	7	7	7	7	7	7	7	8	8.5	9	9.5	10
$\hat{2}y$	0	0	1	1	0	0	.5	.5	.5	.5	.5	.5	0	0	0	0	0	NA
$\hat{2}x$	0	0	0	1	2	2	2	2.5	3	3.5	4	4.5	5	5	5	5	5	5
$\hat{3}y$	0	0	0	0	1	1	.5	.5	.5	.5	.5	.5	0	.5	.5	.5	.5	NA
$\hat{3}x$	0	0	0	0	0	1	2	2.5	3	3.5	4	4.5	8	8	8.5	9	9.5	10

Periods 1 through 4 and 17 through 19 are ignored since there is only one player (player 1 and 3, respectively)whose type includes these periods. Periods 1 through 4 and 17 through 19 are ignored since there is only one player (player 1 and 3, respectively) whose type includes these periods.

the loss incurred from the deviation in period t.

As simple as it seems, such reward and punishment interpretations actually have more complex implications. Even a one-shot deviation can affect the deviating player's final payoff in at least two different aspects. First of all, the dynamic nature provides the opportunity of recovering previous losses in future periods. For example, in Table 5, player 3's payoff before period 9 stands at a loss of 2 when she deviates to the subset of her true type. However, this loss has already been refilled by half before period 20, when she starts to compete with player 1 again. Clearly, part of the reason of such quick recovery lies in the fact that one unit decrease in previous cumulative payoffs directly translates into one unit of reward when the current period is up for division.

More importantly, when there are more than three players, a unilateral deviation causes changes in not only the deviating player's cumulative payoff but also the overall distribution of cumulative payoffs of potentially *many* other players. That is, the effect of a unilateral distribution on the whole system of cumulative payoffs is *not* a simple practice of increasing or lowering the deviating player's payoff and lowering or increasing one single other player's. Moreover, by the discrete nature of Algorithm $\mathscr{E}$, it is even quite intractable to parse out which other players than the deviating are the "first-order" beneficiary or victim and who else are affected by the more subtle "ripple effects". This is can be seen by focusing on period 20 in Table 5, where players 2 and 3 each contribute one unit of payoff to the two-unit increase for player 1.

Hence, the change in the comparative advantage or handicap between any pair of players as the result of a single player's misrepresentation must be carefully taken into account when devising the reward and punishment schemes to incentivize truth-telling. As the columns beyond period 20 in Table 5 demonstrate, a simplistic one-to-one correspondence between lower previous cumulative payoff and current advantage provides and exacerbates player 3's incentive to sacrifice past payoffs for the future competition against only player 1. Such concerns are exactly what the sequence of ρ_t's in Definition 3 are designed to address: it serves as a uniform upper-bound—maybe as a tad bit too crude as it may seem—on the magnitude of inter-temporal comparative advantage any player can reap in the future by incurring a previous cost.

5.2 Why are the ρ_t's not zero?

Given the discussions in the previous subsection, a natural question arises as to why not simply implementing Algorithm $\mathscr{E}$ for $\Gamma\left(\vec{\theta}_t\right)$ in period t—a special case of Mechanism $\mathscr{R}$ with $\rho_t = 0$ for all t and the exact result from Chen et al. (2013): after all, it is known that such repeated applications of Algorithm $\mathscr{E}$ is Pareto efficient, envy-free, and strategy-proof in the context of fair divisions with dichotomous preferences. In this section, I provide an efficiency motivated answer to the significance of the nonzero ρ_t's. The essence of such choices of ρ_t's are to rule out the voluminous non-truth-telling pure strategy Nash equilibria that induce hefty efficiency loss.

Per usual definitions, in a problem Π, a **pure strategy** of player i is a function $\sigma_i : \Theta^B \to \Theta^B$. Given a type profile θ, let $\sigma_{-i}(\theta_{-i})$ represent the profile of the values of the strategies of all players except for i's, i.e. $\left(\sigma_j\left(\theta_j\right)\right)_{j\neq i}$. A profile of strategies $(\sigma_i(\cdot))_{i\in N}$ is a **pure strategy Nash equilibrium** for a direct mechanism M if and only if, for all player i, there does not exist another pure strategy $\sigma_i' \neq \sigma_i$ such that

$$u_i\left(M\left(\sigma_i'(\theta_i), \sigma_{-i}(\theta_{-i})\right)|\theta_i\right) \geq u_i\left(M\left(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})\right)|\theta_i\right), \forall \theta \in \left(\Theta^B\right)^n,$$

with the inequality being strict for at least one type profile. Now, since M is a direct mechanism and no player's preferences is affected by other players' types, the definition of pure strategy Nash equilibrium is equivalent to switching out the strategy profile of other players' strategies above with the profile of all the other players' types. That is, a profile of pure strategies $(\sigma_i(\cdot))_{i\in N}$ constitute a pure strategy Nash equilibrium if and only if, for all player i, there does not exist another pure strategy $\sigma_i' \neq \sigma_i$ such that

$$u_i\left(M\left(\sigma_i'(\theta_i), \theta_{-i}\right)|\theta_i\right) \geq u_i\left(M\left(\sigma_i(\theta_i), \theta_{-i}\right)|\theta_i\right), \forall \theta \in \left(\Theta^B\right)^n,$$

with the inequality being strict for at least one type profile. By extension, I shall use the symbol $\sigma(\theta)$ to stand for the profile of values for the strategy profile $(\sigma_i(\cdot))_{i\in N}$ when the players' types are given by the profile θ. The goal of the current subsection is to compare the ratios of the maximal to the minimal achievable efficiency between Mechanism $\mathscr{R}$ and repeated applications of Algorithm

$\mathscr{E}$. Hence, let the **refinement** of a problem Π be the problem given by enlarging the segmenting B so that any possible type in Θ^B is a union of the intervals included in the new segmenting. I present this definition along with its existence result with the following lemma.

Lemma 12 (Refinement of a problem)**.** *For any problem $\Pi = \left(b, B, T, \Theta^B, \beta^B\right)$, there exists a problem $\bar{\Pi} = \left(b, \bar{B}, \bar{T}, \Theta^B, \beta^{\bar{B}}\right)$ where $\bar{B}$ is a strictly increasing sequence with $\bar{b}_0 = 0$, B is a subsequence of $\bar{B}$, $\beta_{t'}^{\bar{B}} = \beta_t^B$ if $\left[\bar{b}_{t'}, \bar{b}_{t'+1}\right] \subseteq [b_t, b_{t+1}]$, and for each $\theta_i \in \Theta^B$, there exists a subsequence $\bar{B}' \equiv \left\{\bar{b}_t'\right\}_{t=0}^{\bar{T}'}$ of $\bar{B}$ such that*

$$\theta_i = \bigcup_{t=0}^{\bar{T}'} \left[\bar{b}_t', \bar{b}_{t+1}'\right]. \tag{12}$$

Proof. See Appendix C for the proof of Lemma 12. ■

Remark 3. Lemma 12 is *not* an extra assumption about the type space Θ^B. It is simply stating that, given any problem Π, there exists a finer segmenting of the batches or periods in Π so that the batches or periods in the refined problem become the building blocks of players' preferences so that any player's type compatible with the original problem can be represented by the union of the batches or periods in the refined problem. Hence, notice that type space in the refined problem is equal to that in the original problem. Also when a problem is defined, the finiteness of Θ_t^B in the original problem is indispensable and will prove crucial in the proof of Lemma 12.

On the other hand, it should be quite obvious that any given problem has infinitely many refinements since a refinement of the original problem is a new problem while a refinement of the new problem is definitely yet another refinement of the original problem. However, the existence of *a* refinement will be completely constructive—hence, the proof in Appendix C offers the "coarsest" refinement of the original problem. Furthermore, if a problem Π already has the property that (12) is satisfied for all types in Θ^B, I will refer to Π as a **refined problem**. As will be seen shortly, refined problems are the simplest form of division problems and possess the strongest incentive and efficiency implications when combined with their corresponding nonzero sequences of ρ_t's. Since any problem can be refined into a refined problem (see the proof of Lemma 12 in Appendix C), I shall focus on refined problems in this subsection and use the symbol $\bar{\Pi}$ for such refined problems.

From now on, I shall also focus on problems where players' types are unbounded. When players' type are unbounded, there *never* exists a final period that is included in the players' types, where the incentive property and efficiency performance of Mechanism $\mathscr{R}$ simply replicates that of the repeated deployment of Algorithm $\mathscr{E}$. Within this scope, I manage to strengthen the strategy-proof-ness of Mechanism $\mathscr{R}$ so that truth-telling is the only pure strategy Nash equilibrium. I document this result along with its efficiency implications with Proposition 2, before which I first discuss the measure of efficiency adopted in this model.

Given any problem Π and direct mechanism M, let $\Sigma(\Pi, M)$ be the set of all pure strategy Nash equilibria. Let the **price of anarchy of M for Π** be defined as the ratio between maximum collective payoff when the players' types are public information to that of the minimum collective payoff among all pure strategy Nash equilibria. That is,

$$\mathrm{POA}(\Pi, M) \equiv \sup_{\theta \in \Theta^B} \left\{ \frac{\displaystyle\sum_{t=0}^{T-1} \beta_t^B \int_{\left(\bigcup_{i \in N} \theta_i\right) \cap [b_t, b_{t+1}]} 1 \, dr}{\displaystyle\inf_{\sigma \in \Sigma(\Pi, M)} \sum_{i \in N} u_i\left(M\left(\sigma(\theta)\right) | \theta_i\right)} \right\}.$$

The order of computation of the price of anarchy is to fix a type profile first, then select the worst performing pure strategy Nash equilibrium and compute the ratio between the maximal achievable collective payoff under perfect information for that profile, then select the type profile that results in the highest said *ratio*. Supremum and infimum are adopted since both type space and the set of pure strategy Nash equilibria can potentially be infinite. However, the maximum achievable collective payoff is well-defined and only depends on the type profile.

Clearly, the current definition allows the worst performing equilibrium to vary for different type profiles, which offers a lower bound on the collective welfare in equilibrium for any equilibrium-type-profile combination. However, this does not imply that the current definition provides the upper bound on the ratio between maximal achievable collective payoffs versus that in any equilibrium-type-profile combination. In the current definition, the numerator is tied to the type profile first, then is the worst performing equilibrium selected to calibrate the efficiency loss due to private information withheld by the players. This order of computation is a sensible choice in this mechanism design problem since it is unnecessary for the mechanism designer to be concerned about the worst outcome for the worst type profile while unreasonably expecting the best type profile when computing the achievable collective payoff. Finally, the definition implicitly stays agnostic on the whether should mechanisms be measured in terms of efficiency when simply no player is demanding any resources—an extreme scenario safely assumed away within reasons.

I now set out to compare the prices of anarchy of Mechanism $\mathscr{R}$ and repeated application of Algorithm $\mathscr{E}$ for each period, effectively setting $\rho_t = 0$ for all periods. I shall refer to the latter mechanism as $\mathscr{R}_0$. The main result in this subsection below shows that the price of anarchy of $\mathscr{R}_0$ for any *refined* infinite problem with a sufficiently rich type spaces is *at least n*, the size of the set of players, while Mechanism $\mathscr{R}$ achieves full efficiency with a price of anarchy of 1.

Proposition 2 (Price of anarchy)**.** *Let* $\bar{\Pi}\left(+\infty, \bar{B}, +\infty, \Theta^{\bar{B}}, \beta^{\bar{B}}\right)$ *be an infinite refined problem with a finite set of players N, where* $\Theta^{\bar{B}}$ *is the set of all infinite unions of intervals in* $\{[b_t, b_{t+1}]\}_{t=0}^{+\infty}$*. The price of anarchy of* $\mathscr{R}_0$ *for* $\bar{\Pi}$ *is at least* $n = |N|$*. The price of anarchy of Mechanism* $\mathscr{R}$ *for* $\bar{\Pi}$ *is* 1.

Proof. See Appendix C for the proof of Proposition 2. ■

The outcome of $\mathscr{R}_0$ is very straightforward to calculate for refined problems. Any period or batch is simply divided among players who demonstrate demand with equal shares. The proof in Appendix C even contains direct computation on a candidate lower bound of the price of anarchy of $\mathscr{R}_0$ for each type profile. On the other hand, the outcome of Mechanism $\mathscr{R}$ is much more convoluted for any given period, compounded with the limitless type space, making direct computation impossible. Nevertheless, refined problems offer relatively easy characterization of candidate sequences of ρ_t's based on very straightforward computations of the sequence of δ_t's as in Definition 3. The proof of Proposition 2 bypasses all these issues by ensuring that in Mechanism $\mathscr{R}$, truth-telling is a *strictly* dominant strategy for all players, making all player telling the truth the only pure strategy Nash equilibrium. Since Mechanism $\mathscr{R}$ is Pareto efficient when players are honest, its price of anarchy is naturally 1.

Proposition 2 highlights the value of nonzero ρ_t's: most importantly, it eliminates numerous undesirable equilibria the static mechanism, even as delicate as Algorithm $\mathscr{E}$ is, cannot rule out. Even more interesting is the magnitude of the working ρ_t's implied by Proposition 2. Larger population of players and richer type space result in lower δ_t's simply by definition in Definition 3. However, such complications introduce *larger* advantage in efficiency performance of Mechanism $\mathscr{R}$ as compared to $\mathscr{R}_0$. In some sense, Proposition 2 bears the "leverage" interpretation on the role of ρ_t's in the working mechanics of Mechanism $\mathscr{R}$: with larger population and richer type space, the mechanism designer is only allowed to move within an even tighter wiggle room, which they, if maneuvering carefully, can eventually leverage into a more sizable gain in performance. It is worth pointing out that the recursive and dynamic but not simply repeated nature of Mechanism $\mathscr{R}$ is the key to such possibilities of equilibrium selection.

The focus on refined problems, on the other hand, is barely motivated by sequential or dynamic concerns[18]. Recall that deviations that are neither supersets nor subsets of the true types

[18]Nevertheless, it is worth reiterating that refinement of problems is not an additional assumption of any sort on the type space—it is merely a rephrase or restructure of the same environment to provide tractability in computing prices of anarchy of the mechanisms in comparison.

are dismissed only with weak inequalities in payoffs when establishing strategy-proof-ness. The unimportant difference between weak and strict inequalities in incentive properties become significant in efficiency measures. Players may be indifferent between deviations and truth-telling but the efficiency consequence of such indifference can be considerable, precisely the reason for the substantial price of anarchy for $\mathscr{R}_0$. Notice that such concerns do not exist in a refined problem since whenever the first period of deviation is, it has to be to either a superset or subset of the true type. Deviations in future periods are suppressed by the same argument as in Lemma 10 once at least one strict inequality is established for the first period of deviation in payoff terms.

Furthermore, deviations that are neither supersets nor subsets of the true types induce ambiguous changes in the feasible set of divisions faced by the mechanism, the comparative statics implications on the amount of goods assigned is yet untraceable, not to mention those on the eventual payoffs to the deviating player. Consequently, the premises of Lemma 8 collapse hence the result no longer applies even in the first period of deviation. Compounded with deviations in potentially multiple periods, the efficiency implications in unrefined problems quickly becomes intractable. At the core of this impasse is Mechanism $\mathscr{R}$ staying agnostic on which compatible *feasible division* to choose once a payoff vector is determined—Chen et al. (2013) also acknowledges this obstacle, to which this current work owns a great deal. It goes without saying that this unanswered question offers a crucial challenge that any future research is more than encouraged to tackle.

6 Concluding Remarks

I studied the problem of dynamic and sequential fair divisions for players with dichotomous preferences. The major contributions of this project include a systematic design of Pareto efficient, envy-free, and strategy-proof dynamic division mechanisms for any generic problem. The mechanism devised in this paper possesses strong incentive properties which enforce truth-telling by all players to be the only pure strategy Nash equilibrium in infinite problems with unbounded types. Such strong incentive properties translate into a factor of performance improvement over simple repeated applications of static strategy-proof mechanisms equal to the size of the set of players.

On a more technical and theoretical level, this paper contributes a novel comparative statics result on the egalitarian solutions to monotone and concave cooperative games with transferable utilities in characteristic function form. I also discovered the duality-like property of the algorithms locating the egalitarian solutions to reconcile the conflict between the process and result in the pursuit of the egalitarian solutions.

As mentioned before, one immediate future research direction is to develop a tangible way of translating payoff vectors into appropriate selection of feasible divisions to ridden the shackle placed on the proof of strategy-proof-ness by Lemma 8. This will most likely be a purely technical endeavor. For purposes closer to applications, development of measures or indices of envy-free-ness and strategy-proof-ness, similar to the notion of price of anarchy for efficiency, should be noted as a valuable potential for later research.

Were the foregoing development of performance measures successful, tradeoffs among these measures (and, by extension, potentially even a "efficiency-incentive-fairness" frontier of sorts) are to be better explored. Cho (2014) already represents some of the earliest adventures in such territories although, unsurprisingly, all of the results so far are negative. However, the ultimate goal of this direction should not be enumerating theoretical impossibilities with a top-down approach. On the contrary, a characterization of the exchange competitive economy market mechanism as the ultimate institute for assignment and division problems on a maximal domain where efficiency, incentive, and fairness are all compatible should be set as a finish line in sight.

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A Omitted proofs from section 3

Lemma 1 (Preliminary observations of $e(\cdot)$)**.** *Given any monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function given by $v(\cdot)$,*

1. $\forall R \subset S \subset T \subseteq N$, $e(T-R) = e(S-R) \Rightarrow e(T-R) = e(S-R) = e(T-S)$;

2. $\forall R \subset T \subseteq N$ *and* $R \subset S \subseteq N$, *where* $R \subset (T \cap S)$ *and* $T \not\subseteq S$ *and* $S \not\subseteq T$, *such that* $e(S-R) = e(T-R)$, *then* $\min\{e[(T \cup S) - R], e[(T \cap S) - R]\} \leq e(S-R) = e(T-R)$;

3. $\forall R \subset T \subseteq N$ *and* $R \subset S \subseteq N$, *where* $R = (T \cap S)$ *and* $T \not\subseteq S$ *and* $S \not\subseteq T$, *such that* $e(S-R) = e(T-R)$, *then* $e[(T \cup S) - R] \leq e(S-R) = e(T-R)$.

Proof. 1. $R \subset S \subset T \Rightarrow |R| < |S| < |T|$. Therefore,

$$\begin{aligned}
e(T-R) =& \frac{v(T)-v(R)}{|T|-|R|} = \frac{v(T)-v(S)+v(S)-v(R)}{|T|-|R|} \\
=& \frac{v(T)-v(S)}{|T|-|S|} \cdot \frac{|T|-|S|}{|T|-|R|} + \frac{v(S)-v(R)}{|S|-|R|} \cdot \frac{|S|-|R|}{|T|-|R|} \\
=& e(T-S) \cdot \frac{|T|-|S|}{|T|-|R|} + e(S-R) \cdot \frac{|S|-|R|}{|T|-|R|}.
\end{aligned}$$

Recall that $e(T-R) = e(S-R)$, which implies

$$e(T-S) \cdot \frac{|T|-|S|}{|T|-|R|} = e(T-R) - e(S-R) \cdot \frac{|S|-|R|}{|T|-|R|} = e(T-R) \cdot \frac{|T|-|R|-(|S|-|R|)}{|T|-|R|}$$

$$\Rightarrow e(T-S) = e(T-R).$$

2. By concavity of $v(\cdot)$, $v(T \cap S) + v(T \cup S) \leq v(T) + v(S)$, which implies

$$v(T \cap S) - v(R) + v(T \cup S) - v(R) \leq v(T) - v(R) + v(S) - v(R).$$

Dividing both sides by $(|T \cap S| + |T \cup S| - 2|R|) = (|T| + |S| - 2|R|)$,

$$\begin{aligned}\Rightarrow & e[(T \cap S) - R] \cdot \frac{|T \cap S| - |R|}{|T \cap S| + |T \cup S| - 2|R|} \\ & + e[(T \cup S) - R] \cdot \frac{|T \cap S| - |R|}{|T \cup S| + |T \cup S| - 2|R|} \leq e(S - R).\end{aligned} \tag{13}$$

Notice that the left-hand side of (13) is a convex combination of $e[(T \cap S) - R]$ and $e[(T \cup S) - R]$, thus completing the proof.

3. By concavity of $v(\cdot)$, $\underbrace{v(T \cap S) - v(R)}_{=0} + v(T \cup S) - v(R) \leq v(T) - v(R) + v(S) - v(R)$. Thus,

$$\begin{aligned} e[(T \cup S) - R] \cdot (|T \cup S| - |R|) = & v(T \cup S) - v(R) \leq v(T) - v(R) + v(S) - v(R) \\ = & e(T - R) \cdot (|T| - |R|) + e(S - R) \cdot (|S| - |R|) \\ = & \underbrace{e(T - R)}_{=e(S-R)} \cdot (|T| + |S| - 2|R|). \end{aligned}$$

Recall that $(T \cap S) = R \Rightarrow \underbrace{|T \cap S| - |R|}_{=0} + |T \cup S| - |R| = |T| + |S| - 2|R|$. Thus,

$$e[(T \cup S) - R] = e(T - R). \qquad \square$$

Lemma 2 (First observation from Algorithm $\mathscr{E}$)**.** *Let $S_1 \subset S_2 \subset \cdots \subset S_K = N$ be the sequence of subsets of N generated by Algorithm $\mathscr{E}$ with $K \leq n = |N|$ for the monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic* *function $v(\cdot)$. Then $\forall 2 \leq k \leq (K-1),\ e(S_k - S_{k-1}) \leq e(S_{k+1} - S_k)$.*

Proof. By definition, $S_{k-1} \subset S_k \subset S_{k+1}$. Suppose there is a k such that $e(S_{k+1} - S_k) < e(S_k - S_{k-1})$.

$$
\begin{aligned}
e(S_{k+1} - S_{k-1}) =& \frac{v(S_{k+1}) - v(S_k) + v(S_k) - v(S_{k-1})}{|S_{k+1}| - |S_{k-1}|} \\
=& \frac{v(S_{k+1}) - v(S_k)}{|S_{k+1}| - |S_k|} \cdot \frac{|S_{k+1}| - |S_k|}{|S_{k+1} - S_{k-1}|} + \frac{v(S_k) - v(S_{k-1})}{|S_k| - |S_{k-1}|} \cdot \frac{|S_k| - |S_{k-1}|}{|S_{k+1} - S_{k-1}|} \\
=& \underbrace{e(S_{k+1} - S_k)}_{< e(S_k - S_{k-1})} \cdot \frac{|S_{k+1}| - |S_k|}{|S_{k+1} - S_{k-1}|} \\
& + e(S_k - S_{k-1}) \cdot \frac{|S_k| - |S_{k-1}|}{|S_{k+1}| - |S_{k-1}|} < e(S_k - S_{k-1}).
\end{aligned}
$$

This is contradiction to the definition of S_k having the least $e(S - S_{k-1})$ among all $S \supset S_{k-1}$. $\square$

Lemma 3 (Finest partition of N)**.** *Let y^* be the egalitarian solution given by Algorithm $\mathscr{E}$ that terminates in K steps for the monotone and concave game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$. Let $\mathscr{P}^*$ be the partition of N implied by Algorithm $\mathscr{E}$, namely $\mathscr{P}^* \equiv \{(S_k \backslash S_{k-1})\}_{k=1}^{K}$.*

*A partition $\mathscr{P}$ of N is y^*-**compatible** if and only if*

1. *For all $i \in N$ and $j \in N$, $i \in S \in \mathscr{P}$ and $j \in S \in \mathscr{P}$ implies that $y_i^* = y_j^*$;*
2. *And for any $S \in \mathscr{P}$, $\sum_{j \in (\lhd_i(y^*) \cup S)} y_j^* = v\left(\lhd_i(y^*) \cup S\right)$, $\forall i \in S$.*

$\mathscr{P}^*$ is the finest among all y^*-compatible partitions of N. That is, let $\mathscr{P}$ be a y^*-compatible partition of N. Then any element of $\mathscr{P}$ is a union of some elements of $\mathscr{P}^*$.

Proof. Let $\mathscr{P}$ be any y^*-compatible partition of N. It suffices to show that for all $i \in N$, $i \in S \in \mathscr{P}^*$ and $i \in S' \in \mathscr{P}$ implies that $S \subseteq S'$. Note that $i \in (S \cap S')$ hence $(S \cap S') \neq \emptyset$. Suppose $S \not\subseteq S'$. Then

$(S \cap S') \subset S$. By the second requirement of Lemma 3 and Lemma 1,

$$e\left[((S \cap S') \cup \lhd_i(y^*)) - \lhd_i(y^*)\right] = e\left[(S \cup \lhd_i(y^*)) - \lhd_i(y^*)\right],$$

which means that S is *not* a smallest candidate for the step in Algorithm $\mathscr{E}$ for $\lhd_i(y^*)$, contradictory to the definition of $\mathscr{P}^*$, the elements of which are all smallest candidates. □

Lemma 4 (Feasibility and dominance of y^*)**.** *Given any monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$ with V being the feasible set given by $v(\cdot)$, the egalitarian solution y^* defined by Algorithm $\mathscr{E}$ generalized Lorenz dominates any $y \in V$ and $y^* \in V$.*

Proof. I first show the feasibility of y^*. That is,

$$\sum_{i \in R} y_i^* \leq v(R), \forall R \subseteq N. \tag{14}$$

This is done by running an induction on any $R \subseteq N$ as compared to the S_k in the steps of Algorithm $\mathscr{E}$. Specifically, let $K \leq n$ be the number steps the Algorithm terminates in. Take any $R \subseteq N$. First let $R \subseteq S_1$. Clearly, if $R = S_1$, $\sum_{i \in R} y_i^* = v(S_1)$ by the definition of y^*. Hence, suppose, instead, $R \subset S_1$. Keep in mind that, in this case, $y_i^* = e(S_1 - \emptyset)$, $\forall i \in R$. I want to show that $\sum_{i \in R \subset S_1} y_i^* \leq v(R)$. By the definition of S_1,

$$e(S_1 - \emptyset) \leq e(R - \emptyset) = \frac{v(R)}{|R|} \Rightarrow |R| \cdot e(S_1 - \emptyset) \leq |R| \cdot e(R - \emptyset) = v(R).$$

Therefore, $\sum_{i \in R} y_i^* = |R| \cdot e(S_1 - \emptyset) \leq |R| \cdot e(R - \emptyset) = v(R)$. Thus, the initiation step in the induction is completed. Now, suppose for all $R \subseteq S_k$ for some $k \leq (K-1)$, (14) holds. In the induction step, I want to show that (14) holds for all $R \subseteq S_{k+1}$. If $R \subseteq S_k$, then $R \subseteq S_{k+1}$ and (14) holds trivially by the assumption for step k. Suppose otherwise: $R \not\subseteq S_k$. Then it must be the case that $S_k \subset (R \cup S_k)$. In this case, there are two possibilities to consider: $R \cap S_k = \emptyset$ and $R \cap S_k \neq \emptyset$.

- $R \cap S_k = \emptyset$ and $R \subseteq S_{k+1}$ imply that $R \subseteq (S_{k+1} \backslash S_k)$. Moreover, $|S_k \cup R| = |S_k| + |R|$. Then, $y_i^* = e(S_{k+1} - S_k)$, $\forall i \in R$. Then, by the definition of S_{k+1},

$$\begin{aligned}
\sum_{i \in R} y_i^* =& |R| \cdot e(S_{k+1} - S_k) \leq |R| \cdot e[(S_k \cup R) - S_k] \\
=& |R| \cdot \frac{v(S_k \cup R) - v(S_k)}{|S_k \cup R| - |S_k|} = |R| \cdot \frac{v(S_k \cup R) - v(S_k)}{|R|} = v(S_k \cup R) - v(S_k) \\
& \underbrace{\leq}_{\text{by concavity of } \Gamma} \quad v(R) - \underbrace{v(S_k \cap R)}_{=0 \text{ by } S_k \cap R = \emptyset} = v(R).
\end{aligned}$$

- Now, for $R \cap S_k \neq \emptyset$, remember that for all $i \in (R \backslash S_k)$, $y_i^* = e(S_{k+1} - S_k) \leq e[(S_k \cup R) - S_k]$ by the definition of S_{k+1}. Mover, $|R \backslash S_k| = (|R \cup S_k| - |S_k|)$. Then,

$$\begin{aligned}
\sum_{i \in R} y_i^* =& \sum_{i \in (R \backslash S_k)} y_i^* + \sum_{i \in (R \cap S_k)} y_i^* = |R \backslash S_k| \cdot e(S_{k+1} - S_k) + \underbrace{\overbrace{\sum_{i \in (R \cap S_k)} y_i^*}^{\leq v(S_k \cap R)}}_{\text{by assumption for step } k} \\
\leq& |R \backslash S_k| \cdot e[(S_k \cup R) - R] + v(S_k \cap R) \\
=& v(S_k \cup R) - v(S_k) + v(S_k \cap R) \underbrace{\leq}_{\text{by concavity of } \Gamma} v(R).
\end{aligned}$$

Hence, the induction for step $k+1$ is complete and so is the proof for $y^* \in V$. ■

I will now turn to showing that y^* generalized Lorenz dominates every $y \in V$. Take any $y \in V$. Recall that $\mathbf{y}$ and $\mathbf{y}^*$ are generated from y and y^* by rearranging the arguments in a weakly increasing order, respectively. Among alternative rearrangement, select the ones that respects the indices of the players. Formally, let ν and ν^* be the permutations of N for the pair of vectors $(y, \mathbf{y})$

and $(y^*, \mathbf{y}^*)$ such that for all $i, j \in N$

$$\begin{cases} y_i < y_j \Rightarrow \nu(i) < \nu(j); & \text{When } y_i = y_j,\, i < j \iff \nu(i) < \nu(j); \\ y_i^* < y_j^* \Rightarrow \nu^*(i) < \nu^*(j); & \text{When } y_i^* = y_j^*,\, i < j \iff \nu^*(i) < \nu^*(j). \end{cases}$$

Let μ and μ^* be the inverse of ν and ν^*, respectively. Thus, $\mu(i)$ and $\mu^*(i)$ are the identities of the players in N holding the i-th entries in y and y^*, respectively. Without loss of generality, let $\mathbf{y}_0 = \mathbf{y}_0^* = 0$. Now, supposed y^* does *not* generalized Lorenz dominates y. The goal of this proof by contradiction is to show that $y \notin V$.

The assumption of y^* not generalized Lorenz dominating y means that there exists an $1 \le L \le N$ such that $\sum_{l=1}^{L} \mathbf{y}_l > \sum_{l=1}^{L} \mathbf{y}_l^*$. Let $(\tilde{L}+1)$ be the smallest of such L if there are more than one candidate—$0 \le \tilde{L} \le (N-1)$. This implies that $\sum_{l=1}^{\tilde{L}} \mathbf{y}_l \le \sum_{l=1}^{\tilde{L}} \mathbf{y}_l^*$, which still holds even when $\tilde{L} = 0$. Therefore,

$$\mathbf{y}_{\tilde{L}+1} - \mathbf{y}_{\tilde{L}+1}^* > \sum_{l=1}^{\tilde{L}} \mathbf{y}_l^* - \sum_{l=1}^{\tilde{L}} \mathbf{y}_l \ge 0. \tag{15}$$

Now, let S_k be the largest subset of N such that $y_i^* \le \mathbf{y}_{\tilde{L}+1}^*$, $\forall i \in S_k$. Such S_k is well-defined and clearly $|S_k| \ge (\tilde{L}+1) \Rightarrow \left(|S_k| - \tilde{L}\right) \ge 1$ by statement 1 in Lemma 6, the proof of which will be presented shortly afterwards. Since π is a permutation, $|S| = \tilde{L}$. Let $S \equiv \left\{\pi(l) : 1 \le l \le \tilde{L}\right\}$ (by definition, $S = \emptyset$ when $\tilde{L} = 0$). Let $\kappa \equiv |S \cap S_k|$. For $i \in S \backslash (S \cap S_k)$, $y_i \le \mathbf{y}_{\tilde{L}+1}$ by the definition of $\mathbf{y}$, which implies that $\sum_{l=1}^{\tilde{L}} \mathbf{y}_l = \sum_{i \in (S \cap S_k)} y_i + \sum_{j \in (S \backslash (S \cap S_k))} y_i$

$$\Rightarrow \sum_{i \in (S \cap S_k)} y_i \ge \sum_{l=1}^{\tilde{L}} \mathbf{y}_l - (|S| - \kappa) \cdot \mathbf{y}_{\tilde{L}+1} = \sum_{l=1}^{\tilde{L}} \mathbf{y}_l - \left(\tilde{L} - \kappa\right) \cdot \mathbf{y}_{\tilde{L}+1}.$$

Now, for each $i \in (S_k \backslash S)$, $y_i \geq \mathbf{y}_{\tilde{L}+1}$. Therefore,

$$\sum_{i \in S_k} y_i \geq (|S_k| - \kappa)\, \mathbf{y}_{\tilde{L}+1} + \sum_{l=1}^{\tilde{L}} \mathbf{y}_l - \left(\tilde{L} - \kappa\right) \cdot \mathbf{y}_{\tilde{L}+1} = \sum_{l=1}^{\tilde{L}} \mathbf{y}_l + \left(|S_k| - \tilde{L}\right) \cdot \mathbf{y}_{\tilde{L}+1}.$$

Similar to S, let $S^* \equiv \left\{\pi^*(l) : 1 \leq l \leq \tilde{L}\right\}$. By the definition of S_k, for any $i \in (S_k \backslash S^*)$, $y_i = \mathbf{y}^*_{\tilde{L}+1}$.

$$\begin{aligned}
\sum_{i \in S_k} y_i - \sum_{i \in S_k} y_i^* &\geq \sum_{l=1}^{\tilde{L}} \mathbf{y}_l + \left(|S_k| - \tilde{L}\right) \cdot \mathbf{y}_{\tilde{L}+1} - \sum_{l=1}^{\tilde{L}} \mathbf{y}_l^* - \left(|S_k| - \tilde{L}\right) \cdot \mathbf{y}^*_{\tilde{L}+1} \\
&= \sum_{l=1}^{\tilde{L}} \mathbf{y}_l - \sum_{l=1}^{\tilde{L}} \mathbf{y}_l^* + \underbrace{\left(|S_k| - \tilde{L}\right)}_{\geq 1} \cdot \underbrace{\left(\mathbf{y}_{\tilde{L}+1} - \mathbf{y}^*_{\tilde{L}+1}\right)}_{\geq 0 \text{ by } (15)} \\
&\geq \sum_{l=1}^{\tilde{L}} \mathbf{y}_l - \sum_{l=1}^{\tilde{L}} \mathbf{y}_l^* + \left(\mathbf{y}_{\tilde{L}+1} - \mathbf{y}^*_{\tilde{L}+1}\right) > 0.
\end{aligned}$$

Now, recall, by (3), $\sum_{i \in S_k} y_i^* = v(S_k)$. Therefore, $\sum_{i \in S_k} y_i > v(S_k)$ hence $y \notin V$. □

Lemma 6 (Second observation of Algorithm $\mathscr{E}$)**.** *Let $S_1 \subset S_2 \subset \cdots \subset S_K = N$ be the sequence of subsets of N generated by Algorithm $\mathscr{E}$ with $K \leq n = |N|$ for the monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$. Then $\forall 0 \leq k \leq (K-1)$, $[S \supset S_k$ and $|S| > |S_{k+1}|] \Rightarrow e(S - \emptyset) \geq e(S_{k+1} - \emptyset)$.*

Proof. The proof will be carried out on the induction on k. Clearly, for $k = 0$, the statement is trivially true by the definition of S_1. Suppose the statement is true for some $k \leq (K-2)$. For $k+1$, I want to show that for any $S \subseteq N$ such that $S \supset S_{k+1}$ and $|S| > |S_{k+2}|$, $e(S - \emptyset) \geq e(S_{k+2} - \emptyset)$. First of all, recall that both S and S_{k+2} are proper supersets of S_k, which means that $e(S - \emptyset) \geq e(S_{k+1} - \emptyset)$

and $e(S_{k+2}-\emptyset) \geq e(S_{k+1}-\emptyset)$.

$$\begin{cases} e(S-\emptyset) = \dfrac{v(S)}{|S|} = \dfrac{v(S)-v(S_{k+1})+v(S_{k+1})}{|S|} \\ \qquad\qquad = \dfrac{|S|-|S_{k+1}|}{|S|} \cdot e(S-S_{k+1}) + \dfrac{|S_{k+1}|}{|S|} \cdot \underbrace{e(S_{k+1}-\emptyset)}_{\leq e(S-\emptyset)}; \\ e(S_{k+2}-\emptyset) = \dfrac{v(S_{k+2})}{|S_{k+2}|} = \dfrac{v(S_{k+2})-v(S_{k+1})+v(S_{k+1})}{|S_{k+2}|} \\ \qquad\qquad = \dfrac{|S_{k+2}|-|S_{k+1}|}{|S_{k+2}|} \cdot \underbrace{e(S_{k+2}-S_{k+1})}_{\leq e(S-S_{k+1})} + \dfrac{|S_{k+1}|}{|S_{k+2}|} \cdot \underbrace{e(S_{k+1}-\emptyset)}_{\leq e(S_{k+2}-\emptyset)}. \end{cases}$$

Notice that in both equations above, the left-hand side is a convex combination of the two components on the right-hand side. Hence, $e(S-S_{k+1}) \geq e(S_{k+1}-\emptyset)$ and $e(S_{k+2}-S_{k+1}) \geq e(S_{k+1}-\emptyset)$. Moreover, $|S| > |S_{k+2}|$ implies that $\dfrac{|S|-|S_{k+1}|}{|S|} > \dfrac{|S_{k+2}|-|S_{k+1}|}{|S_{k+2}|}$. Therefore, we have

$$\begin{aligned} e(S-\emptyset) &\geq \frac{|S|-|S_{k+1}|}{|S|} \cdot e\left(S_{k+2}-S_{k+1}\right) + \frac{|S_{k+1}|}{|S|} \cdot e(S_{k+1}-\emptyset) \\ &\geq \frac{|S_{k+2}|-|S_{k+1}|}{|S_{k+2}|} \cdot e\left(S_{k+2}-S_{k+1}\right) + \frac{|S_{k+1}|}{|S_{k+2}|} \cdot e(S_{k+1}-\emptyset) = e\left(S_{k+2}-S_{k+1}\right). \end{aligned}$$ □

Lemma 7 (Third observation of Algorithm $\mathscr{E}$)**.** *Given a monotone and concave cooperative game* $\Gamma = (N, v(\cdot))$ *with transferable utilities in characteristic function form and let* $\{S_k\}_{k=1}^{K}$ *be any sequence of subsets generated by Algorithm* $\mathscr{E}$*. Then*

$$\forall 1 \leq k \leq K,\ e(S_k - S_{k-1}) = \max_{S \subset S_k} e(S_k - S).$$

Proof. The proof is by induction on k as the index of S_k. For $k=1$, by definition $e(S_1-\emptyset) \leq$

$e(S-\emptyset)$ for any $S \subseteq N$. Let $S \subset S_1$. Then,

$$e(S_1-\emptyset) = \frac{v(S_1)}{|S_1|} = \frac{v(S_1)-v(S)+v(S)}{|S_1|} = \frac{|S_1|-|S|}{|S_1|}\cdot e(S_1-S) + \frac{|S|}{|S_1|}\cdot e(S-\emptyset),$$

$$\Rightarrow \frac{|S_1|-|S|}{|S_1|}\cdot e(S_1-S) = e(S_1-\emptyset) - \frac{|S|}{|S_1|}\cdot \underbrace{e(S-\emptyset)}_{\geq e(S_1-\emptyset)} \leq \frac{|S_1|-|S|}{|S_1|}\cdot e(S_1-\emptyset).$$

So the initiation step is complete. Now suppose the statement is true for $1 \leq k \leq (K-1)$. I want to show in the induction step that

$$e\left(S_{k+1}-S_k\right) \geq e(S_{k+1}-S),\ \forall S \subset S_{k+1}.$$

First of all, the statement is trivially true if $S = S_k$. Hence, let $S \neq S_k$. If $S \subset S_k$, then

$$\begin{aligned} e(S_{k+1}-S) =& \frac{v(S_{k+1})-v(S_k)+v(S_k)-v(S)}{|S_1|-|S|} \\ =& \frac{|S_{k+1}|-|S_k|}{|S_{k+1}|-|S|}\cdot e(S_{k+1}-S_k) + \frac{|S_k|-|S|}{|S_{k+1}|-|S|}\cdot e(S_k-S). \end{aligned}$$

Again, the left-hand side is a convex combination of the two terms on the right-hand side so it must be less than or equal to the larger one of the two on the right-hand side. By the assumption in step k, $e(S_k-S) \leq e(S_k-S_{k-1})$ and by Lemma 2, $e(S_k-S_{k-1}) \leq e(S_{k+1}-S_k)$, completing this step.

Now, suppose $S_{k+1} \supset S \supset S_k$. Recall that by the definition of S_{k+1}, $e(S_{k+1}-S_k) \leq e(S-S_k)$.

$$e(S_{k+1}-S_k) = \frac{|S_{k+1}|-|S|}{|S_{k+1}|-|S_k|}\cdot e\left(S_{k+1}-S\right) + \frac{|S|-|S_k|}{|S_{k+1}|-|S_k|}\cdot \underbrace{e\left(S-S_k\right)}_{\geq e(S_{k+1}-S_k)};$$

$$\Rightarrow \frac{|S_{k+1}|-|S|}{|S_{k+1}|-|S_k|}\cdot e\left(S_{k+1}-S\right) = e(S_{k+1}-S_k)-\frac{|S|-|S_k|}{|S_{k+1}|-|S_k|}\cdot e\left(S-S_k\right)$$
$$\leq \frac{|S_{k+1}|-|S|}{|S_{k+1}|-|S_k|}\cdot e\left(S_{k+1}-S_k\right).$$

Finally, let $S \not\supseteq S_k$ and $S \not\subseteq S_k$ and $S \subset S_{k+1}$. Then $(S \cup S_k) \supset S_k$ and $(S \cap S_k) \subset S_k$. Thus,

$$e(S_{k+1}-(S\cup S_k)) = \frac{v(S_{k+1})-v(S\cup S_k)}{|S_{k+1}|-|S\cup S_k|} \underbrace{\geq}_{\text{by concavity of } \Gamma} \frac{v(S_{k+1})-v(S)-v(S_k)+v(S\cap S_k)}{|S_{k+1}|-|S\cup S_k|};$$

$$\Rightarrow \underbrace{e(S_{k+1}-(S\cup S_k))}_{\leq e(S_{k+1}-S_k) \text{ since } (S\cup S_k)\supset S_k} + \frac{|S_k|-|S\cap S_k|}{|S_{k+1}|-|S\cup S_k|}\cdot \underbrace{e\left(S_k-(S\cap S_k)\right)}_{\leq e(S_{k+1}-S_k) \text{ since } (S\cap S_k)\subset S_k}$$
$$\geq \frac{|S_{k+1}|-|S|}{|S_{k+1}|-|S\cup S_k|}\cdot e\left(S_{k+1}-S\right).$$

Therefore, for this last case, $e(S_{k+1}-S_k) \geq e(S_{k+1}-S)$ for S that is neither a superset nor a subset of S_k. Hence, the induction for step $(k+1)$ is complete and so is the whole proof. □

Theorem 1 (Maximal increase in payoffs)**.** *Let $\Gamma = (N, v(\cdot))$ and $\hat{\Gamma} = (N, \hat{v}(\cdot))$ be two monotone and concave cooperative games with transferable utilities in characteristic function form such that there exists a unique $i \in N$ such that, for all $S \subseteq N$,*

1. $i \notin S \Rightarrow v(S) = \hat{v}(S)$; 2. $i \in S \Rightarrow v(S) \leq \hat{v}(S)$; 3. $i \in S \subset T \Rightarrow \hat{v}(T) - v(T) \leq \hat{v}(S) - v(S)$.

Let y^* and $\hat{y}^*$ be the egalitarian solutions of Γ and $\hat{\Gamma}$, respectively. Then, for all $j \in N$,

$$\hat{y}^*_i - y^*_i \geq \hat{y}^*_j - y^*_j. \tag{4}$$

Proof. I first establish, very quickly, an quite obvious fact:

$$i \in S \subset T \Rightarrow \frac{\hat{v}(T)-\hat{v}(S)}{|T|-|S|} \leq \frac{v(T)-v(S)}{|T|-|S|}. \tag{16}$$

Simply rearrange the third condition in the statement of the Theorem 1 and (16) is proved.

As mentioned in the main text, any player who has lower payoff than i at y^* will receive exactly the same payoff at $\hat{y}^*$. Also, i's payoff is weakly increasing from y^* to $\hat{y}^*$ by Proposition 3 in Tian (2015). Therefore, it suffices to focus on the players who receive strictly more payoffs than i at $\hat{y}^*$. Take any such player j. It suffices to show that $\hat{y}^*_j \leq y^*_j$. Suppose $j \in (S_{k+1} \backslash S_k)$ at y^* generated by Algorithm $\mathscr{E}$ for Γ. By Definition 1,

$$y^*_j = \frac{v(S_{k+1}) - v(S_k)}{|S_{k+1}| - |S_k|}.$$

Now, focus on the set of players $(S_{k+1} \backslash S_k)$. Let $\hat{S}_{l+1}$ be the smallest set in the sequence $\{\hat{S}_l\}_{l=1}^{L}$ generated by Algorithm $\mathscr{E}$ for $\hat{\Gamma}$ that contains the subset $(S_{k+1} \backslash S_k)$. That is $(S_{k+1} \backslash S_k) \subseteq \hat{S}_{l+1}$ but $(S_{k+1} \backslash S_k) \not\subseteq \hat{S}_l$. Thus,

$$\left(\hat{S}_l \cup S_{k+1}\right) \supset S_l \text{ and } \left(\hat{S}_l \cap S_{k+1}\right) \subset S_{k+1}. \tag{17}$$

Recall that $\hat{y}^*_j > \hat{y}^*_i$ by assumption. Also, by Lemma 2, $\dfrac{\hat{v}\left(\hat{S}_{l+1}\right) - \hat{v}\left(\hat{S}_l\right)}{\left|\hat{S}_{l+1}\right| - \left|\hat{S}_l\right|} \geq \hat{y}^*_j > \hat{y}^*_i$, implying that $i \in \hat{S}_l$. By the definition of $\hat{S}_{l+1}$ according to Algorithm $\mathscr{E}$,

$$\hat{y}^*_j \leq \frac{\hat{v}\left(\hat{S}_{l+1}\right) - \hat{v}\left(\hat{S}_l\right)}{\left|\hat{S}_{l+1}\right| - \left|\hat{S}_l\right|} \underbrace{\leq}_{\text{by Definition 1}} \frac{\hat{v}\left(\hat{S}_l \cup S_{k+1}\right) - \hat{v}\left(\hat{S}_l\right)}{\left|\hat{S}_l \cup S_{k+1}\right| - \left|\hat{S}_l\right|} \underbrace{\leq}_{\text{by (16)}} \frac{v\left(\hat{S}_l \cup S_{k+1}\right) - v\left(\hat{S}_l\right)}{\left|\hat{S}_l \cup S_{k+1}\right| - \left|\hat{S}_l\right|}.$$

By concavity of Γ, Lemma 7, and (17),

$$\begin{aligned}\frac{v\left(\hat{S}_l \cup S_{k+1}\right) - v\left(\hat{S}_l\right)}{\left|\hat{S}_l \cup S_{k+1}\right| - \left|\hat{S}_l\right|} &\underbrace{\leq}_{\text{by concavity of } \Gamma} \frac{v(S_{k+1}) - v\left(S_{k+1} \cap \hat{S}_l\right)}{|S_{k+1}| - \left|S_{k+1} \cap \hat{S}_l\right|} \\ &\underbrace{\leq}_{\text{by Lemma 7}} \frac{v(S_{k+1}) - v(S_k)}{|S_{k+1}| - |S_k|} = y^*_j.\end{aligned}$$

□

Lemma 8 (Unilateral disturbance)**.** *Let $\Gamma = (N, v(\cdot))$ and $\hat{\Gamma} = (N, \hat{v}(\cdot))$ be two monotone and concave cooperative games with transferable utilities in characteristic function form such that there exists a unique $i \in N$ such that, for all $S \subseteq N$,*

1. $i \notin S \Rightarrow v(S) = \hat{v}(S)$; 2. $i \in S \Rightarrow v(S) \leq \hat{v}(S)$; 3. $i \in S \subset T \Rightarrow \hat{v}(T) - v(T) \leq \hat{v}(S) - v(S)$.

Let y^* and $\hat{y}^*$ be the egalitarian solutions and $\mathscr{P}^*$ and $\hat{\mathscr{P}}^*$ generated by Algorithm $\mathscr{E}$ for Γ and $\hat{\Gamma}$, respectively. Then $y_i^* = \hat{y}_i^* \Rightarrow \left[y^* = \hat{y}^* \text{ and } \mathscr{P}^* = \hat{\mathscr{P}}^*\right]$. Otherwise,

$$\left[\hat{y}_j^* \geq \hat{y}_i^* \text{ and } j \notin Q_i(\hat{y}^*)\right] \Rightarrow \hat{y}_j^* \leq y_j^*.$$

Proof. First of all, for any j such that $y_j^* < y_i^*$ or $y_j^* = y_i^*$ and $y \notin Q_i(y^*)$, $y_j^* = \hat{y}_j^*$, since there is a sequence of subsets of players that can include all such players before selecting i according to Algorithm $\mathscr{E}$ and the order of the subsets selected do not change the egalitarian solution. Let $\bar{S}$ be the last subset of players selected before i is in Γ and notice that there exists a sequence of subsets of selected players for $\hat{\Gamma}$ that replicates the sequences for Γ up to $\bar{S}$. Hence, for Γ, $\bar{S}$ is the largest union of cliques excluding i that contains players with weakly lower payoffs than i at y^*. Notice that since $i \notin \bar{S}$, $v(S) = \hat{v}(S)$ for any $S \subseteq \bar{S}$.

Now, consider any $\bar{T} \subseteq N$ that is a proper superset of $\bar{S}$. I want to show that

$$\bar{T} \neq \left(Q_i(y^*) \cup \bar{S}\right) \Rightarrow \frac{\hat{v}(\bar{T}) - \hat{v}\left(\bar{S}\right)}{|\bar{T}| - \left|\bar{S}\right|} > y_i^* = \hat{y}_i^*.$$

That is, the clique of i at $\hat{y}^*$ will be exactly the same as that of i at y^*. First of all, for any $\bar{T} \supset \bar{S}$,

$$\frac{\hat{v}(\bar{T}) - \hat{v}\left(\bar{S}\right)}{|\bar{T}| - \left|\bar{S}\right|} \geq \max_{j \in \bar{S}} \left\{\hat{y}_j^*\right\} = \max_{j \in \bar{S}} \left\{y_j^*\right\}$$

by Lemma 2. Observe that, by the definition of Algorithm $\mathscr{E}$ in Definition 1,

$$\bar{T} \neq \left(Q_i\left(y^*\right) \cup \bar{S}\right) \Rightarrow e\left(\bar{T} - \bar{S}\right) > y_i^* = \hat{y}_i^*.$$

Now, if $i \notin \bar{T}$, then $\hat{v}(\bar{T}) = v(\bar{T})$. Hence, $\dfrac{\hat{v}(\bar{T}) - \hat{v}\left(\bar{S}\right)}{|\bar{T}| - |\bar{S}|} = e\left(\bar{T} - \bar{S}\right) > y_i^* = \hat{y}_i^*$. Thus, let $\bar{T} \ni i$. Then $\hat{v}(\bar{T}) \geq v(\bar{T})$, which implies that

$$\frac{\hat{v}(\bar{T}) - \hat{v}\left(\bar{S}\right)}{|\bar{T}| - \left|\bar{S}\right|} = \frac{\hat{v}(\bar{T}) - v\left(\bar{S}\right)}{|\bar{T}| - \left|\bar{S}\right|} \geq \frac{v(\bar{T}) - v\left(\bar{S}\right)}{|\bar{T}| - \left|\bar{S}\right|} = e\left(\bar{T} - \bar{S}\right) > y_i^* = \hat{y}_i^*.$$

Therefore, the clique of i at $\hat{y}^*$, $Q_i\left(\hat{y}^*\right)$, must be the same as $Q_i\left(y^*\right)$ when $y_i^* = \hat{y}_i^*$, which implies

$$v\left(\bar{S} \cup Q_i\left(y^*\right)\right) = \hat{v}\left(\bar{S} \cup Q_i\left(y^*\right)\right) \Rightarrow v(T) = \hat{v}(T)$$

for any $T \supset \left(\bar{S} \cup Q_i\left(y^*\right)\right)$. Hence, the subsets of players selected in Algorithm $\mathscr{E}$ will be the same for Γ and $\hat{\Gamma}$, resulting in $y^* = \hat{y}^*$. However, this does not imply that the two games are identical: beyond the sequences of players selected by Algorithm $\mathscr{E}$, no statement is ever made regarding the values of the characteristic functions $v(\cdot)$ and $\hat{v}(\cdot)$.

Now, for the second statement, when $j \notin Q_i\left(y^*\right)$ and $y_j^* \leq y_i^*$, $\hat{y}_j^* = y_j^*$. So the statement is satisfied. Hence, I should only focus on $j \in Q_i\left(y^*\right)$ or those such that $y_j^* > y_i^*$. Note that by the proof of Theorem 1, if $\hat{y}_j^* > \hat{y}_i^*$, then the statement is also satisfied. Hence, the players of interest are only those in i's clique at y^* but not at $\hat{y}^*$ and who receive the same payoff as i does at $\hat{y}^*$.

When j is not in i's clique at $\hat{y}^*$ but receives weakly higher payoff than i, there must exist a sequence of subsets of players chosen by Algorithm $\mathscr{E}$ for $\hat{\Gamma}$ such that $j \in \left(\hat{S}_{l+1} \backslash \hat{S}_l\right)$ and $i \in \hat{S}_l$. This is achieved by running Algorithm $\mathscr{E}$ for $\hat{\Gamma}$ and always choose the clique of i at $\hat{y}^*$ first among all players who are about to receive at least as much payoff as i does. Without loss of generality, suppose $j \in (S_{k+1} \backslash S_k)$ for some k in the sequence of subsets of players generated by Algorithm $\mathscr{E}$ for Γ. Now, as in the proof of Theorem 1, let $\hat{S}_{\bar{l}+1}$ be the first subset in the sequence chosen by Algorithm $\mathscr{E}$ for $\hat{\Gamma}$ that contains all players in S_{k+1}. Since $j \in \left(\hat{S}_{l+1} \backslash \hat{S}_l\right)$, it must be the case that $\bar{l} \geq l$ which implies that $i \in \hat{S}_{\bar{l}}$. A similar sequence of inequalities as that in the proof of Theorem 1 provides the result and completes the proof. □

Lemma 9 (Secondary observations of $e(\cdot)$)**.** *Given any monotone and concave cooperative game $\Gamma = (N, v(\cdot))$ with transferable utilities in characteristic function form with the characteristic function being given by $v(\cdot)$, then*

1. $\forall S \subset T \subset R \subseteq N$, $e(R-T) = e(R-S) \Rightarrow e(R-S) = e(R-T) = e(T-S)$;

2. $\forall T \subset R \subseteq N$ *and* $S \subset R \subseteq N$ *such that* $e(R-S) = e(R-T)$, *then*

$$\max\left\{e\left[R-(T \cup S)\right], e\left[R-(T \cap S)\right]\right\} \geq e(R-S) = e(R-T).$$

Proof. Part of the proof for Lemma 9 will be very similar to that of Lemma 1, especially for the second statement. I start with the first statement and consider the following manipulations that is solely based on the definition of $e(\cdot)$.

1. Simply subtract $\dfrac{|R|-|T|}{|R|-|S|} \cdot e(R-T)$ from the equation below

$$\begin{aligned} e(R-S) =& \frac{v(R)-v(S)}{|R|-|S|} = \frac{v(R)-v(T)+v(T)-v(S)}{|R|-|S|} \\ =& \frac{|R|-|T|}{|R|-|S|} \cdot e(R-T) + \frac{|T|-|S|}{|R|-|S|} \cdot e(T-S). \end{aligned}$$

2. By concavity of Γ, $v(R)-v(T \cup S)+v(R)-v(T \cap S) \geq v(R)-v(T)+v(R)-v(S)$. Follow the same steps in the proof of the second statement in Lemma 1 and the proof is complete. □

Theorem 2 ("Duality"). *Let $\Gamma = (N, v(\cdot))$ be a monotone and concave cooperative game with transferable utilities in characteristic function form with the characteristic function $v(\cdot)$. Then*

1. *Both Algorithm $\mathscr{E}$ and Algorithm $\bar{\mathscr{E}}$ terminate in the same number $K \leq N$ of steps;*

2. *For any sequence of subsets of N, $\{S_k\}_{k=1}^{K}$, generated by Algorithm $\mathscr{E}$, there exists a sequence of subsets of N, $\{T_k\}_{k=1}^{K}$, generated by Algorithm $\bar{\mathscr{E}}$, such that $T_k = S_{K-k}$, $\forall 0 \leq k \leq K$;*

3. *Moreover, the egalitarian solution y^* is equal to the dual egalitarian solution $\bar{y}^*$;*

4. *Furthermore, $\{(T_k \backslash T_{k+1})\}_{k=0}^{K-1}$ generated by Algorithm $\bar{\mathscr{E}}$ coincides with $\mathscr{P}^*$ generated by Algorithm $\mathscr{E}$ as defined in Lemma 3.*

Proof. Note that statement 2 in Theorem 2 implies the rest of the Theorem simply by Definition 1. Let $\{S_k\}_{k=1}^{K}$ be any sequence of subsets generated by Algorithm $\mathscr{E}$. To prove statement 2 of Theorem 2, it suffices to show that for all $1 \leq k \leq K$, S_{k-1} is a *largest* candidate among proper subsets of S_k in Algorithm $\bar{\mathscr{E}}$. Suppose, instead, there exists an $S \subseteq N$ such that $S_{k-1} \subset S \subset S_k$ and $e(S_k - S) = e(S_k - S_{k-1})$. Then, by Lemma 9,

$$e(S - S_{k-1}) = e(S_k - S_{k-1}) = e(S_k - S).$$

Then S_k is *not* a smallest candidate among proper supersets of S_{k-1} in Algorithm $\mathscr{E}$ as defined in Definition 1, thus completing the proof of Theorem 2. $\square$

B Omitted proofs and results from section 4

Lemma 10 (Super-Sub)**.** *For any* $(i,(\theta_i,\theta_{-i})) \in \left(N \times \Theta^B \times \left(\Theta^B\right)^{n-1}\right)$,

$$u_i\left(\mathscr{R}\left(\theta_i,\theta_{-i}\right)|\theta_i\right) \geq u_i\left(\mathscr{R}\left(\underline{\theta}_i,\theta_{-i}\right)|\theta_i\right),\ \forall \underline{\theta}_i \subset \theta_i \in \Theta^B$$

and

$$u_i\left(\mathscr{R}\left(\theta_i,\theta_{-i}\right)|\theta_i\right) \geq u_i\left(\mathscr{R}\left(\overline{\theta}_i,\theta_{-i}\right)|\theta_i\right),\ \forall \overline{\theta}_i \supset \theta_i \in \Theta^B$$

jointly implies that $u_i\left(\mathscr{R}\left(\theta_i,\theta_{-i}\right)|\theta_i\right) \geq u_i\left(\mathscr{R}\left(\theta_i',\theta_{-i}\right)|\theta_i\right),\ \forall \theta_i' \neq \theta_i \in \Theta^B$.

Proof. By the same logic as in (7),

$$\begin{aligned} u_i\left(\mathscr{R}\left(\theta_i',\theta_{-i}\right)|\theta_i\right) =& u_i\left(\mathscr{R}\left(\theta_i',\theta_{-i}\right)|\left(\theta_i\cap\theta_i'\right)\right) \leq u_i\left(\mathscr{R}\left(\left(\theta_i\cap\theta_i'\right),\theta_{-i}\right)|\left(\theta_i\cap\theta_i'\right)\right) \\ =& u_i\left(\mathscr{R}\left(\left(\theta_i\cap\theta_i'\right),\theta_{-i}\right)|\theta_i\right) \leq u_i\left(\mathscr{R}\left(\theta\right)|\theta_i\right). \end{aligned}$$

□

Lemma 11 (Applicability of Theorem 1)**.** *For all preference profiles* $\theta \neq \hat{\theta} \in \left(\Theta^B\right)^n$ *such that there exists a unique* $i \in N$ *with* $\theta_i \subseteq \hat{\theta}_i$ *and* $\theta_j = \hat{\theta}_j$ *for all* $j \neq i$,

$$1.\ i \notin S \Rightarrow v(S|\theta) = v\left(S|\hat{\theta}\right);\ \ 2.\ i \in S \Rightarrow v(S|\theta) \leq v\left(S|\hat{\theta}\right);$$
$$3.\ i \in S \subset T \Rightarrow v\left(T|\hat{\theta}\right) - v(T|\theta) \leq v\left(S|\hat{\theta}\right) - v(S|\theta).$$

for all $S,T \subseteq N$ *and* $0 \leq t \leq (T-1)$.

Proof. Note that the first two conditions are immediately established by the definition of $v(\cdot|\cdot)$. Hence, I will only check the third condition. For any $R \subseteq N$ such that $i \in R$,

$$v\left(R|\hat{\theta}\right) - v(R|\theta) = \int_{\hat{\theta}_i \cup \bigcup_{j\in(R\setminus i)} \theta_j} 1\,dr - \int_{\theta_i \cup \bigcup_{j\in(R\setminus i)} \theta_j} 1\,dr = \int_{\left(\hat{\theta}_i\setminus\theta_i\right)\setminus\left(\bigcup_{j\in(R\setminus\{i\})}\theta_j\right)} 1\,dr.$$

When $i \in S \subset T$,

$$\begin{aligned}(S\backslash\{i\}) \subset (T\backslash\{i\}) &\Rightarrow \bigcup_{j\in(S\backslash\{i\})} \theta_j \subseteq \bigcup_{j\in(T\backslash\{i\})} \theta_j \\ &\Rightarrow \left[(\hat{\theta}_i\backslash\theta_i) \setminus \left(\bigcup_{j\in(S\backslash\{i\})} \theta_j\right)\right] \supseteq \left[(\hat{\theta}_i\backslash\theta_i) \setminus \left(\bigcup_{j\in(T\backslash\{i\})} \theta_j\right)\right].\end{aligned}$$

Therefore, the third condition is established. □

Proposition 1 (C-PE-EF-SP)**.** *For any problem Π for a given set of player N, Mechanism $\mathscr{R}$ as defined in Definition 4 is consistent, Pareto efficient, envy-free, and strategy-proof.*

Proof. I start by showing envy-free-ness. Suppose there exists a preference profile θ such that there exist a pair of players $i \neq j$ such that

$$u_i^B(\mathscr{R}(\theta)|\theta_i) < \sum_{t=0}^{T-1} \left\{\beta_t^B \cdot \left[\int_{\theta_{it}\cap[b_t,b_{t+1})} \mathbb{1}_j^{\mathscr{R}(\theta)}(r)\,dr\right]\right\}.$$

Recall that, by definition,

$$u_i^B(\mathscr{R}(\theta)|\theta_i) = \sum_{t=0}^{T-1} \left\{\beta_t^B \cdot \left[\int_{\theta_{it}\cap[b_t,b_{t+1})} \mathbb{1}_i^{\mathscr{R}(\theta)}(r)\,dr\right]\right\}.$$

Then, there must exist a t such that

$$\beta_t^B \cdot \left[\int_{\theta_{it}\cap[b_t,b_{t+1})} \mathbb{1}_j^{\mathscr{R}(\theta)}(r)\,dr\right] > \beta_t^B \cdot \left[\int_{\theta_{it}\cap[b_t,b_{t+1})} \mathbb{1}_i^{\mathscr{R}(\theta)}(r)\,dr\right]. \tag{18}$$

That is, there must be a period t where j is assigned more within θ_{it} than is i. Of course, there can be more than one such periods hence let the sequence $\mathbf{t} \equiv \{t_1, t_2, \cdots\}$ represent the potentially infinite sequence such that (18) is true. According to Mechanism $\mathscr{R}$, for any t in $\mathbf{t}$, the division of

$[b_t, b_{t+1})$ is determined in step $(t+1)$. Now, in step $(t+1)$, (18) must imply that

$$\begin{aligned}\rho_t x^B_{jt}\left(\mathscr{R}(\theta)|\theta_j\right) + \beta^B_t \cdot \left[\int_{\theta_{it}\cap[b_t,b_{t+1})} \mathbb{1}^{\mathscr{R}(\theta)}_j(r)\,dr\right] \\ \leq \rho_t x^B_{it}\left(\mathscr{R}(\theta)|\theta_i\right) + \beta^B_t \cdot \left[\int_{\theta_{it}\cap[b_t,b_{t+1})} \mathbb{1}^{\mathscr{R}(\theta)}_i(r)\,dr\right]\end{aligned} \tag{19}$$

since if the inequality in (19) is reversed to ">", reassigning some goods from j to i, while still remaining feasible as mentioned before, will increase the value of the objective function $\sum_{i\in N} W_{t+1}(\cdot)$. Since $0<\rho_t<1$ for all t, (19) implies that

$$x^B_{i(t+1)}(\mathscr{R}(\theta)|\theta_i) > x^B_{j(t+1)}(\mathscr{R}(\theta)|\theta_j). \tag{20}$$

Consider the sequences

$$\begin{cases} x_{i\mathbf{t}} \equiv \left\{x^B_{i(t_1+1)}(\mathscr{R}(\theta)|\theta_i), x^B_{i(t_2+1)}(\mathscr{R}(\theta)|\theta_i), \cdots\right\}, \\ x_{j\mathbf{t}} \equiv \left\{x^B_{j(t_1+1)}(\mathscr{R}(\theta)|\theta_j), x^B_{j(t_2+1)}(\mathscr{R}(\theta)|\theta_j), \cdots\right\}, \end{cases}$$

where $x_{i\mathbf{t}}$ is strictly greater than $x_{j\mathbf{t}}$ term-wise. Clearly, both sequences converge with the limit of $x_{i\mathbf{t}}$ strictly less than that of $x_{j\mathbf{t}}$ by the assumption of i envying j. This is a contradiction to (20) when applied to the sequences $x_{i\mathbf{t}}$ and $x_{j\mathbf{t}}$. Hence, Mechanism $\mathscr{R}$ is envy-free. ■

I now turn to strategy-proof-ness. I first define several new notations. I represent the triple $\left(\rho_t, \vec{x}^B_t, \vec{\theta}_t\right)$ with the symbol ϕ_t. Define the game $\Gamma(\phi_t)$ as $\Gamma(\phi_t) \equiv (N, v(\cdot|\phi_t))$ where

$$\forall S\subseteq N,\ v(S|\phi_t) \equiv \rho_t \cdot \sum_{j\in S} x^B_{jt} + \beta^B_t \cdot v\left(S|\vec{\theta}_t\right).$$

Similarly, define $V(\phi_t)$ as $\left\{ x \in \mathbb{R}^n_+ : \sum_{j \in S} x_j \leq v(S|\phi_t),\ \forall S \subseteq N \right\}$. Also, $e(\cdot|\phi_t)$ is similarly defined as $e\left(\cdot|\vec{\theta}_t\right)$ was. Moreover, for any $S \subseteq N$ and $\vec{x}^B_t$, let $\bar{x}^B_{tS}$ represent the average of all x^B_{jt} for all $j \in S$.

Clearly, for each $x \in V(\phi_t)$, there exists a $y \in V\left(\vec{\theta}_t\right)$ such that $x = \left(\rho_t \cdot \vec{x}^B_t + \beta^B_t \cdot y\right)$. In each step $(t+1)$ of Mechanism $\mathscr{R}$, let $y^*\left(\vec{\theta}_t\right)$ represent the egalitarian solution of the game $\Gamma\left(\vec{\theta}_t\right)$ and $x^*(\phi_t)$ represent the egalitarian solution of the game $\Gamma(\phi_t)$. Consider the *cliques* of any player j in these two solutions. I want to show that these cliques are identical for any player j.

Take any $S \subset T \subseteq N$ and $S \subset T' \subseteq N$, by the definition of δ_t,

$$e\left(T-S|\vec{\theta}_t\right) > e\left(T'-S|\vec{\theta}_t\right) \Rightarrow e\left(T-S|\vec{\theta}_t\right) - e\left(T'-S|\vec{\theta}_t\right) \geq \frac{\delta_t}{\beta^B_t}.$$

By the definition of ρ_t with respect to δ_t,

$$\begin{aligned}
e\left(T'-S|\phi_t\right) - e\left(T-S|\phi_t\right) = & \beta^B_t \cdot \left(e\left(T'-S|\vec{\theta}_t\right) - e\left(T-S|\vec{\theta}_t\right)\right) \\
& -\rho_t \cdot \left(\frac{\sum_{j \in T} x^B_{jt} - \sum_{j \in S} x^B_{jt}}{|T|-|S|} - \frac{\sum_{j \in T'} x^B_{jt} - \sum_{j \in S} x^B_{jt}}{|T'|-|S|}\right) \\
\leq & \beta^B_t \cdot \left(e\left(T'-S|\vec{\theta}_t\right) - e\left(T-S|\vec{\theta}_t\right)\right) \\
& +\rho_t \cdot \left(\left|\frac{\sum_{j \in T} x^B_{jt} - \sum_{j \in S} x^B_{jt}}{|T|-|S|}\right| + \left|\frac{\sum_{j \in T'} x^B_{jt} - \sum_{j \in S} x^B_{jt}}{|T'|-|S|}\right|\right) \\
\leq & \beta^B_t \cdot \left(e\left(T'-S|\vec{\theta}_t\right) - e\left(T-S|\vec{\theta}_t\right)\right) + 2\rho_t \chi_t \leq -\delta_t + 2\rho_t \chi_t < 0.
\end{aligned}$$

$$\therefore e\left(T-S|\vec{\theta}_t\right) > e\left(T'-S|\vec{\theta}_t\right) \Rightarrow e\left(T-S|\phi_t\right) > e\left(T'-S|\phi_t\right).$$

As a matter of fact, the definition of the sequence $\{\rho_t\}_{t=1}^{T-1}$ ensures that all *strict* inequalities in period allocations and even cross type profile comparisons will be preserved when combined with cumulative payoffs, a fact that will be often referred to shortly hence useful to keep in mind.

Hence, when applying Algorithm $\mathscr{E}$ to the game $\Gamma(\phi_t)$ and $\Gamma\left(\vec{\theta}_t\right)$, if a subset T' is selected before T when facing S in the game $\Gamma\left(\vec{\theta}_t\right)$ with strictly lower average payoffs, then it must also be so in $\Gamma(\phi_t)$. When two subsets resulting the same payoff in $\Gamma\left(\vec{\theta}_t\right)$, the tie is broken by choosing the one with lower average increase in cumulative payoffs discounted by ρ_t. Hence, the cliques of any player j at $y^*\left(\vec{\theta}_t\right)$ is precisely preserved at $x^*(\phi_t)$. Since Algorithm $\mathscr{E}$ dictates that players in the same clique have the same payoff, it must be

$$x_j^*(\phi_t) = \rho_t \cdot \bar{x}^B_{tQ_j\left(y^*\left(\vec{\theta}_t\right)\right)} + \beta_t^B \cdot y_j^*\left(\vec{\theta}_t\right). \tag{21}$$

Recall that $x_j^*(\phi_t) = \rho_t x_{jt}^B + \beta_t^B y_{j(t+1)}\left(\vec{\theta}_t\right)$, which implies that

$$\beta_t^B y_{j(t+1)}\left(\vec{\theta}_t\right) = \rho_t \cdot \left(\bar{x}^B_{tQ_j\left(y^*\left(\vec{\theta}_t\right)\right)} - x_{jt}^B\right) + \beta_t^B \cdot y_j^*\left(\vec{\theta}_t\right) \tag{22}$$

and converted into the context of θ:

$$x^B_{j(t+1)}(\theta) = x_{jt}^B(\theta) + \beta_t^B y_{j(t+1)}\left(\vec{\theta}_t\right) = (1-\rho_t)\cdot x_{jt}^B(\theta) + \rho_t \cdot \bar{x}^B_{tQ_j\left(y^*\left(\vec{\theta}_t\right)\right)}(\theta) + \beta_t^B \cdot y_j^*\left(\vec{\theta}_t\right). \tag{23}$$

As mentioned before, it suffices to consider two preferences profiles θ and $\hat{\theta}$ such that there exists a unique $i \in N$ and $0 \leq \tau \leq (T-1)$ such that $\theta_j = \hat{\theta}_j$ if $j \neq i$ and $\theta_{i\tau} \subset \hat{\theta}_{i\tau}$ and $\theta_{it} = \hat{\theta}_{it}$ if $t \neq \tau$. Now, since $(\theta \cap [0, b_\tau)) = \left(\hat{\theta} \cap [0, b_\tau)\right)$, by payoff consistency, $x^B_{j\tau}\left(\mathscr{R}(\theta)|\theta_j\right) = x^B_{j\tau}\left(\mathscr{R}\left(\hat{\theta}\right)|\hat{\theta}_j\right)$ for all $j \in N$. Observe that (23) implicitly implies that if two preference profiles are identical up to b_τ and after $b_{\tau+1}$, differ in period t, but share the same $y^*\left(\vec{\theta}_\tau\right) = y^*\left(\vec{\hat{\theta}}_\tau\right)$—which implicitly implies, in turn, the cliques are all the same by Lemma 8—then all the payoff vectors in all periods will be the same. Therefore, assume otherwise, which immediately implies that $y^*\left(\vec{\theta}_\tau\right) < y^*\left(\vec{\hat{\theta}}_\tau\right)$.

Three conditions jointly will establish strategy-proof-ness: the deviating player *cannot* get a lower cumulative amount of assigned goods in the period of deviation or any future period when deviating to a superset; the deviating agent must be getting a weakly lower *payoff* in the period of deviation; and the difference between the cumulative amount of goods assigned must be lower in periods after deviation than in the deviating period for the deviating player. I start with ensuring that player i will *not* get a strictly higher *payoff* in period τ—the second condition. (23) implies

$$\begin{aligned}\beta_\tau^B\left[y_{j(\tau+1)}\left(\hat{\vec{\theta}}_\tau\right)-y_{j(\tau+1)}\left(\vec{\theta}_\tau\right)\right]=&\beta_\tau^B\left[y_j^*\left(\hat{\vec{\theta}}_\tau\right)-y_j^*\left(\vec{\theta}_\tau\right)\right]\\&+\rho_t\left[\bar{x}^B_{\tau Q_j\left(y^*\left(\vec{\theta}_\tau\right)\right)}\left(\hat{\theta}\right)-\bar{x}^B_{\tau Q_j\left(y^*\left(\hat{\vec{\theta}}_\tau\right)\right)}\left(\theta\right)\right].\end{aligned}\tag{24}$$

Again, by the definition of ρ_τ, $y_j^*\left(\hat{\vec{\theta}}_\tau\right)>y_j^*\left(\vec{\theta}_\tau\right)\Rightarrow y_{j(\tau+1)}\left(\hat{\vec{\theta}}_\tau\right)>y_{j(\tau+1)}\left(\vec{\theta}_\tau\right)$. Now, consider the clique of i at $y^*\left(\vec{\theta}_\tau\right)$: if $Q_i\left(y^*\left(\vec{\theta}_\tau\right)\right)=\{i\}$, then i is assigned all she demanded behind the group $\lhd_i\left(y^*\left(\vec{\theta}_\tau\right)\right)$, which will stays the same even when she reports $\hat{\theta}_i$. Hence, her payoff cannot increase. If $Q_i\left(y^*\left(\vec{\theta}_\tau\right)\right)\neq\{i\}$, then everyone in her clique at $y^*\left(\vec{\theta}_\tau\right)$ will receive higher payoff at $y^*\left(\hat{\vec{\theta}}_\tau\right)$ and since the sum of the payoffs to a clique is a constant at any given egalitarian solution, the deviating player i must be strictly worse off in period τ. Furthermore, (24) directly verifies the first condition for strategy-proof-ness for the deviating period since $y_i^*\left(\hat{\vec{\theta}}_\tau\right)>y_i^*\left(\vec{\theta}_\tau\right)$. Now, for the cumulative amount of goods assigned to i, first consider the deviating period.

Recall that according to the definition of ϕ_t, define $\hat{\phi}_\tau=\left(\rho_\tau,\vec{x}_\tau,\hat{\vec{\theta}}_\tau\right)$. Notice now that the games $\Gamma\left(\phi_\tau\right)$ and $\Gamma\left(\hat{\phi}_\tau\right)$ are merely monotone linear transformations of $\Gamma\left(\vec{\theta}_\tau\right)$ and $\Gamma\left(\hat{\vec{\theta}}_\tau\right)$, respectively, and the comparison between the latter two games satisfies the three conditions in the statement of Theorem 1 by Lemma 11. Thus, Theorem 1 directly applies to the former two games, i.e.

$$x^B_{i(\tau+1)}\left(\hat{\theta}\right)-x^B_{i(\tau+1)}\left(\theta\right)\geq x^B_{j(\tau+1)}\left(\hat{\theta}\right)-x^B_{j(\tau+1)}\left(\theta\right),\ \forall j\in N.\tag{25}$$

Moreover, since any j's payoffs are the same before period τ in the two type profiles,

$$\begin{aligned}x_{j(\tau+1)}^{B}\left(\hat{\theta}\right)-x_{j(\tau+1)}^{B}\left(\theta\right)=&\beta_{\tau}^{B}\cdot\left[y_{j(\tau+1)}\left(\vec{\hat{\theta}}_{\tau}\right)-y_{j(\tau+1)}\left(\vec{\theta}_{\tau}\right)\right]\\=&\beta_{\tau}^{B}\left[y_{j}^{*}\left(\vec{\hat{\theta}}_{\tau}\right)-y_{j}^{*}\left(\vec{\theta}_{\tau}\right)\right]\\&+\rho_{\tau}\left[\bar{x}_{\tau Q_{j}\left(y^{*}\left(\vec{\theta}_{\tau}\right)\right)}^{B}\left(\hat{\theta}\right)-\bar{x}_{\tau Q_{j}\left(y^{*}\left(\vec{\hat{\theta}}_{\tau}\right)\right)}^{B}\left(\theta\right)\right]\end{aligned}\tag{26}$$

Since θ and $\hat{\theta}$ will coincide again after $b_{\tau+1}$, take any $\tau' > (\tau+1)$, by (23),

$$\begin{aligned}\underbrace{x_{i\tau'}^{B}\left(\hat{\theta}\right)-x_{i\tau'}^{B}\left(\theta\right)}_{①}=&(1-\rho_{\tau'-1})\cdot\underbrace{\left(x_{i(\tau'-1)}^{B}\left(\hat{\theta}\right)-x_{i(\tau'-1)}^{B}\left(\theta\right)\right)}_{②}\\&+\rho_{\tau'-1}\cdot\underbrace{\left(\bar{x}_{(\tau'-1)Q_{i}\left(y^{*}\left(\vec{\theta}_{\tau'-1}\right)\right)}^{B}\left(\hat{\theta}\right)-\bar{x}_{(\tau'-1)Q_{i}\left(y^{*}\left(\vec{\hat{\theta}}_{\tau'-1}\right)\right)}^{B}\left(\theta\right)\right)}_{③}.\end{aligned}\tag{27}$$

Clearly, both ② and ③ are weakly less than $\max_{j\in N}\left\{\left(x_{j(\tau'-1)}^{B}\left(\hat{\theta}\right)-x_{j(\tau'-1)}^{B}\left(\theta\right)\right)\right\}$, which implies that so must be ① as well. Therefore, the difference in cumulative payoff between the superset and the subset is weakly less in periods after than in the period of the bifurcation, when the maximum difference in payoff between supersets and subsets is captured by the deviating player by (26)! This implies that deviating to supersets will result in weakly lower payoffs in periods after deviation. Combined with the fact already proven that payoff is also weakly lower in the period of deviation, deviation to supersets can never be profitable.

Now, it only remains to show that deviating to supersets does result in a higher cumulative *amount* of goods assigned, which means that ① must be nonnegative for all $\tau' \geq (\tau+1)$. I will prove this by an induction on τ' about the following statement:

$$x_{i\tau'}^{B}\left(\hat{\theta}\right)-x_{i\tau'}^{B}\left(\theta\right)\geq\Delta_{\tau'}>0.\tag{28}$$

Note that the second inequality is by the definition of the sequence $\{\Delta_t\}_{t=1}^{T}$. When $\tau' = (\tau+1)$,

$$x_{i\tau'}^{B}\left(\hat{\theta}\right) - x_{i\tau'}^{B}\left(\theta\right) \geq \delta_\tau - \rho_\tau\chi_\tau \geq \min\left\{(1-\rho_\tau)\Delta_\tau - \rho_\tau\chi_\tau, \delta_\tau - \rho_\tau\chi_\tau\right\} = \Delta_{\tau+1}$$

by the definition of $\Delta_{\tau+1}$. Now, suppose (28) is true for some $\tau' \geq (\tau+1)$. For $(\tau'+1)$, by (27),

$$\begin{aligned} x_{i(\tau'+1)}^{B}\left(\hat{\theta}\right) - x_{i(\tau'+1)}^{B}\left(\theta\right) &= (1-\rho_{\tau'})\left(x_{i\tau'}^{B}\left(\hat{\theta}\right) - x_{i\tau'}^{B}\left(\theta\right)\right) \\ &+\rho_{\tau'}\left(\vec{x}_{\tau' Q_i\left(y^*\left(\vec{\theta}_{\tau'}\right)\right)}^{B}\left(\hat{\theta}\right) - \vec{x}_{\tau' Q_i\left(y^*\left(\vec{\theta}_{\tau'}\right)\right)}^{B}\left(\theta\right)\right) \\ &\geq (1-\rho_{\tau'})\Delta_{\tau'} - \rho_{\tau'}\chi_{\tau'} \geq \min\left\{(1-\rho_{\tau'})\Delta_{\tau'} - \rho_{\tau'}\chi_{\tau'}, \delta_{\tau'} - \rho_{\tau'}\chi_{\tau'}\right\} = \Delta_{\tau'+1} > 0. \end{aligned} \tag{29}$$

This completes the proof for strategy-proof-ness. $\square$

Theorem 3 (Non-inferiority and complementarity for concave cooperative games)**.** *Let two cooperative games $\Gamma = (N, v(\cdot))$ and $\hat{\Gamma} = (N, \hat{v}(\cdot))$ be concave and such that there exists a unique i such that, for all $S \subseteq N$*

1. $i \notin S \Rightarrow v(S) = \hat{v}(S)$; 2. $i \in S \Rightarrow v(S) \leq \hat{v}(S)$.

For any $j \in N$, let $f_j : \mathbb{R}_+ \to \mathbb{R}$ be strictly increasing, strictly concave, and continuous on $\mathbb{R}_+$. Let

$$y^* \equiv \arg\max_{y \in V} \sum_{j \in N} f_j(y_j) \text{ and } \hat{y}^* \equiv \arg\max_{y \in \hat{V}} \sum_{j \in N} f_j(y_j).$$

Then there exists an $S^* \subseteq N$ such that $i \in S^*$ and $\hat{y}_j^* \geq y_j^*$, $\forall j \in S^*$.

C Omitted proofs from section 5

Lemma 12 (Refinement of a problem)**.** *For any problem $\Pi = \left(b, B, T, \Theta^B, \beta^B\right)$, there exists a problem $\bar{\Pi} = \left(b, \bar{B}, \bar{T}, \Theta^B, \beta^{\bar{B}}\right)$ where $\bar{B}$ is a strictly increasing sequence with $\bar{b}_0 = 0$, B is a sub-*

sequence of $\bar{B}$, $\beta_{t'}^{\bar{B}} = \beta_t^B$ if $\left[\bar{b}_{t'}, \bar{b}_{t'+1}\right] \subseteq [b_t, b_{t+1}]$, and for each $\theta_i \in \Theta^B$, there exists a subsequence $\bar{B}' \equiv \left\{\bar{b}_t'\right\}_{t=0}^{\bar{T}'}$ of $\bar{B}$ such that

$$\theta_i = \bigcup_{t=0}^{\bar{T}'} \left[\bar{b}_t', \bar{b}_{t+1}'\right]. \tag{12}$$

Proof. Notice that since the restrictions of the type space to the different intervals in the original problem Π are mutually exclusive except for the elements in the sequence B, it suffices to show that each interval $[b_t, b_{t+1}]$ can be partitioned into a collection of intervals so that for all θ_i can be represented in the format of (12). Recall that restriction of Θ^B to $[b_t, b_{t+1}]$, Θ_t^B, is finite, consider a sequence of all the restrictions of all the types in Θ^B to $[b_t, b_{t+1}]$, $\{\theta_{it}(p)\}_{p=1}^{\left|\Theta_t^B\right|}$ where no each restriction appears exactly once. For any subset I of $\mathbb{R}_+$ that is the union of a collection of disconnected intervals, let $\mathrm{dp}(I)$ represent the partition of of I into connected intervals that are mutually disconnected. That is, for all $\hat{I}$ and $\tilde{I}$ in $\mathrm{dp}(I)$, there exists an interval $I' \subseteq \mathbb{R}_+$ such that

$$I' \not\subseteq \left[\inf\left\{\hat{I} \cup \tilde{I}\right\}, \sup\left\{\hat{I} \cup \tilde{I}\right\}\right]$$

Note that $\mathrm{dp}(I)$ is a collection of subsets of I. Moreover, for any collection η of subsets of $\mathbb{R}_+$, let $\mathrm{cl}(\eta)$ be the collection of the closures of the elements of η. Now, consider the following sequence of collections of subsets of $[b_t, b_{t+1}]$: let $\eta(1) = \mathrm{dp}(\theta_{it}(1))$. For any $p \geq 1$, let

$$\eta(p+1) = \{I_1 \cap I_2,\ I_1 \backslash I_2, I_2 \backslash I_1 : I_1 \in \eta(p), I_2 \in \mathrm{dp}(\theta_{it}(p+1))\}.$$

Finally, $\eta\left(\left|\Theta_t^B\right| + 1\right) = \mathrm{cl}\left(\eta\left(\left|\Theta_t^B\right|\right)\right)$. Clearly, each $\theta_{it}(p)$ is the union of elements in $\eta(p)$. Note that in $\eta\left(\left|\Theta_t^B\right| + 1\right)$, each element is a closed interval and all restriction of types are closed sets. By construction, the period t restriction of any type θ_{it} is the union of intervals in $\eta\left(\left|\Theta_t^B\right| + 1\right)$. $\square$

Proposition 2 (Price of anarchy)**.** *Let $\bar{\Pi}\left(+\infty, \bar{B}, +\infty, \Theta^{\bar{B}}, \beta^{\bar{B}}\right)$ be an infinite refined problem with* *a finite set of players N, where* $\Theta^{\bar{B}}$ *is the set of all infinite unions of intervals in* $\{[b_t, b_{t+1}]\}_{t=0}^{+\infty}$. *The price of anarchy of* $\mathscr{R}_0$ *for* $\bar{\Pi}$ *is at least* $n = |N|$. *The price of anarchy of Mechanism* $\mathscr{R}$ *for* $\bar{\Pi}$ *is* 1.

Proof. I first show that in an infinite refined problem with the type space being the collection of *all* infinite unions of the batches or periods, truth-telling is a *strictly* dominant strategy for all players in Mechanism $\mathscr{R}$. Hence, all player telling the truth becomes the only pure strategy Nash equilibrium and the price of anarchy of 1 immediately follows. Given any profile θ_{-i} of players' types other than the deviating player i's, consider any $\hat{\theta}_i \neq \theta_i$ where θ_i is i's true type.

Let period t be the first where $\hat{\theta}_i$ and θ_i differ. Note that in a refined problem, either $\hat{\theta}_{it} \subset \theta_{it}$ or $\theta_{it} \subset \hat{\theta}_{it}$. Let $\tilde{\theta}_i = \hat{\theta}_i \cap \theta_i$. If $\hat{\theta}_{it} \subset \theta_i$, by Lemma 10 and strategy-proof-ness of Mechanism $\mathscr{R}$,

$$u_i\left(\mathscr{R}\left(\hat{\theta}_i, \theta_{-i}\right)|\theta_i\right) = u_i\left(\mathscr{R}\left(\hat{\theta}_i, \theta_{-i}\right)|\tilde{\theta}_i\right) \leq u_i\left(\mathscr{R}\left(\tilde{\theta}_i, \theta_{-i}\right)|\tilde{\theta}_i\right) = u_i\left(\mathscr{R}\left(\tilde{\theta}_i, \theta_{-i}\right)|\theta_i\right). \tag{30}$$

Now, let $\bar{\theta}_i$ be given by replacing $\tilde{\theta}_{it} = \hat{\theta}_{it}$ with θ_{it} and keeping all other periods intact in $\tilde{\theta}$. Then

$$u_i\left(\mathscr{R}\left(\theta_i, \theta_{-i}\right)|\theta_i\right) \geq u_i\left(\mathscr{R}\left(\bar{\theta}_i, \theta_{-i}\right)|\theta_i\right) = u_i\left(\mathscr{R}\left(\bar{\theta}_i, \theta_{-i}\right)|\bar{\theta}_i\right). \tag{31}$$

Focus on the comparison between $\bar{\theta}_i$ and $\tilde{\theta}_i$. Note that (28) implies that

$$u_i\left(\mathscr{R}\left(\bar{\theta}_i, \theta_{-i}\right)|\bar{\theta}_i\right) > u_i\left(\mathscr{R}\left(\tilde{\theta}_i, \theta_{-i}\right)|\bar{\theta}_i\right) = u_i\left(\mathscr{R}\left(\tilde{\theta}_i, \theta_{-i}\right)|\theta_i\right). \tag{32}$$

Combining (30) through (32) shows that truth-telling is strictly dominating $\hat{\theta}_i$ when $\hat{\theta}_{it} \subset \theta_{it}$. Now, consider otherwise: $\theta_{it} \subset \hat{\theta}_{it}$. Notice that in this case, $\tilde{\theta}_{it} = \theta_{it}$. Again, (30) still applies. Now, define $\underline{\theta}_i$ by replacing $\tilde{\theta}_{it}$ with $\hat{\theta}_{it}$ and keeping all other periods intact in $\tilde{\theta}_{it}$. Hence, $\theta_{it} \subset \underline{\theta}_{it}$. Define $\vec{\tilde{\theta}}_t$ and $\vec{\underline{\theta}}_t$ by combining $\tilde{\theta}_{it}$ and $\underline{\theta}_{it}$ with the other players' types in period t. Now, it can easily confirmed that $y_i^*\left(\vec{\tilde{\theta}}_t\right) < y_i^*\left(\vec{\underline{\theta}}_t\right)$ but $y_j^*\left(\vec{\tilde{\theta}}_t\right) > y_j^*\left(\vec{\underline{\theta}}_t\right)$ for all $j \neq i$. This implies that the cumulative payoffs before $[b_{t+1}, b_{t+2}]$ is divided is strictly higher only for i at $\underline{\vec{\theta}}_t$ but strictly lower for all other players by (26) and the definition of ρ_t's in Definition 3. This, in turn, implies that, by (27), player i's cumulative payoff after any period including and beyond $(t+1)$ will strictly lower than it was after period t when her report is $\underline{\theta}_i$. Therefore, we have

$$u_i\left(\mathscr{R}\left(\underline{\theta}_i, \theta_{-i}\right)|\theta_i\right) = u_i\left(\mathscr{R}\left(\underline{\theta}_i, \theta_{-i}\right)|\tilde{\theta}_i\right) < u_i\left(\mathscr{R}\left(\tilde{\theta}_i, \theta_{-i}\right)|\tilde{\theta}_i\right) = u_i\left(\mathscr{R}\left(\tilde{\theta}_i, \theta_{-i}\right)|\theta_i\right). \tag{33}$$

Moreover, by strategy-proof-ness of Mechanism $\mathscr{R}$, $u_i\left(\mathscr{R}\left(\tilde{\theta}_i, \theta_{-i}\right)|\theta_i\right) \leq u_i\left(\mathscr{R}\left(\theta_i, \theta_{-i}\right)|\theta_i\right)$. Also recall that Mechanism $\mathscr{R}$ is payoff consistent and $\underline{\theta}_i$ and $\hat{\theta}_i$ coincide up to b_{t+1} while $\underline{\theta}_i$ and $\tilde{\theta}_i$ are the same beyond b_{t+1}. Therefore,

$$u_i\left(\mathscr{R}\left(\hat{\theta}_i, \theta_{-i}\right)|\underline{\theta}_i\right) \leq u_i\left(\mathscr{R}\left(\underline{\theta}_i, \theta_{-i}\right)|\underline{\theta}_i\right) \Rightarrow u_i\left(\mathscr{R}\left(\hat{\theta}_i, \theta_{-i}\right)|\tilde{\theta}_i\right) \leq u_i\left(\mathscr{R}\left(\underline{\theta}_i, \theta_{-i}\right)|\tilde{\theta}_i\right). \tag{34}$$

Combining (33) and (34) and the strategy-proof-ness of Mechanism $\mathscr{R}$ yields truth-telling strictly dominates $\hat{\theta}_i$ when $\theta_{it} \subset \hat{\theta}_{it}$. Thus, truth-telling is a strictly dominant strategy in Mechanism $\mathscr{R}$. ■

I now turn to show the lower bound on the price of anarchy of $\mathscr{R}_0$. Given a preference profile θ, define $N_t(\theta)$ to be the set of players whose types include $[b_t, b_{t+1}]$ as a subset, i.e.

$$N_t(\theta) \equiv \left\{j \in N : [b_t, b_{t+1}] \subset \theta_j\right\}.$$

Now, the division given by $\mathscr{R}_0$ is extremely straightforward: $y_j^*\left(\vec{\theta}_t\right) = \dfrac{b_{t+1} - b_t}{|N_t(\theta)|}$ if $j \in N_t(\theta)$ and zero otherwise. Then, for any player i, any strategy $\sigma_i(\cdot)$ such that

$$\sigma_i(\theta_i) \supseteq \theta_i \tag{35}$$

is a weakly dominant strategy. Consider the strategy profile σ such that $\sigma_j(\theta_j) = I_b$ for all $j \in N$ and $\theta_j \in \Theta^{\bar{B}}$. That is, all players, regardless of their types, always demonstrate demand for the entire set of goods to be divided. This is actually one of the most likely scenarios in division of public resources without monetary transfers, a version of the phenomenon Moulin (2004) referred to as the "tragedy of the commons". Hence, it suffices to propose one type profile that results in ratio in the definition of the price of anarchy being N. This is a very simple task. Simply consider the following profile of types:

$$\theta_j = \bigcup_{z \in (\mathbb{N} \cup \{0\})} [b_{nz+j-1}, b_{nz+j}].$$

In each period, there is only one player who truly demands the goods but all players demonstrate need. $\mathscr{R}_0$ will assign to each player a fair share of $1/n$, which results in a price of anarchy of n. $\square$