

On Bayesian Persuasion with Multiple Senders*

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Abstract

In a multi-sender Bayesian persuasion game, Gentzkow and Kamenica (2012) show that increasing the number of senders cannot decrease the amount of information revealed. They assume: (i) senders reveal information simultaneously, (ii) senders' information can be arbitrarily correlated, and (iii) senders play pure strategies. This paper shows that these three conditions are also necessary to the result. In sequential persuasion games, the order of moves matters, and we show that adding a sender as a first mover and keeping the order of moves fixed for the other senders cannot result in a loss of information.

Keywords: Communication, Bayesian Persuasion, Multiple Senders, Sequential Persuasion.

JEL Classification Codes: D82, D83

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1 Introduction

This paper considers persuasion games with multiple experts. We ask whether consulting additional experts is beneficial for a decision maker. Our results show that the answer is sensitive to the timing of the game, the way the decision maker draws inferences from multiple signals, and the permissibility of mixed strategies. Explicit examples are constructed where adding an expert leads to equilibria that garble the information from the single-expert, implying that the decision maker can be strictly worse off by relying on multiple experts. In two out of three examples there are also more informative equilibria in the case with multiple experts, but if play is sequential it is possible that the unique equilibrium with two experts is less informative than the unique equilibrium with a single expert. This result relies on the second expert's ability to react in response to the information structure chosen by the first expert. We show that, in general, adding an expert who moves first cannot result in a garbling of the information provided in any equilibrium of the game before the expert was added.

A persuasion game is one way to formalize the strategic interaction between decision makers and experts. The key friction in such expert-decision maker relationship is that experts may care directly about how the decision problem is resolved. As pointed out by Crawford and Sobel (1982), this can create incentives for experts to influence decision makers by partially withholding or filtering information. As decision makers are made worse off by such manipulation, a question obviously arises about what can be done to mitigate this problem.

A natural idea is to counteract the incentives to withhold information by soliciting advice from more than one expert.¹ In the cheap talk literature, it has been shown that there are circumstances when the true state is fully revealed and there are other circumstances when asking for a second opinion is useless. Krishna and Morgan (2001) use a multi-sender extension of the model in Crawford and Sobel (1982) to demonstrate that information is fully revealed when experts have opposing biases, whereas it is useless to ask two experts with similar biases.² Based on this and other results along the same lines, the conventional wisdom is that competition among experts with opposing preferences is the best way to induce information revelation.

More recently, Gentzkow and Kamenica (2012) have analyzed the effects of adding experts in the context of a multi-sender version of a persuasion game.³ Remarkably, they find that adding more

¹Some early contributions exploring this line of inquiry include Milgrom and Roberts (1986) and Austen-Smith (1993).

²Battaglini (2002) shows that if the state space is a multidimensional Euclidean space, then, generically, a fully-revealing perfect Bayesian equilibrium can be constructed. Ambrus and Takahashi (2008) demonstrate that if the state space is a closed subset of a Euclidean space, then, again, full revelation may not be an equilibrium.

³Persuasion games differ from cheap talk in that a sender constructs a *signal structure* that the decision maker uses to update information as opposed to disclosing information after observing the state. See Milgrom and Roberts (1986), Glazer and Rubinstein (2001), Glazer and Rubinstein (2004), Glazer and Rubinstein (2006), and Kamenica and Gentzkow (2011).

senders can only increase the information available for the receiver. No conditions on the preferences of the senders are needed for the result, and, while full revelation is always an equilibrium of their game with two or more senders, the result does not rely on picking the fully revealing equilibrium.

An important part of the intuition is that senders in Gentzkow and Kamenica (2012) model can always refine the information provided by the others into any “aggregate signal” that is more informative in the sense of Blackwell (1953) than the information provided by the other senders. For this to work it is important that senders use pure strategies and that senders move simultaneously. In addition, a more subtle assumption about the signal structure plays a critical role: signals are *coordinated* as opposed to *independent*. More precisely, it is assumed that each sender partitions the product space of a true state space and a unit interval that may be interpreted as the support of a sunspot variable. In the single-sender case any such partition generates information in exactly the same way as a standard noisy signal. However, with two or more senders it is assumed that the decision maker updates beliefs on the basis of the intersections of individual signals. All senders use the same sunspot variable, implying that individual senders can only add to the information that already exists by further partitioning the state space and the sunspot variable. Together, these three assumptions guarantee that every sender can unilaterally refine an equilibrium information structure in any conceivable way. In equilibrium no sender can have an incentive to provide any extra information, so if an equilibrium with an extra sender garbles an equilibrium without the additional sender, the garbling must also have been an equilibrium before the addition of the new sender.

This paper analyzes persuasion games with sequential moves, independent signals, and mixed strategies. Hence, one may view our paper as asking whether the assumptions in Gentzkow and Kamenica (2012) are also necessary for the result. The answer is mostly affirmative. We find in each case that we can construct an example with an equilibrium in which the decision maker becomes less informed when a second sender is added. We also believe that sequential moves and independent signals are interesting in their own right. Sequentiality is ubiquitous in news reporting, jury trials, scientific research, political campaigning, and many other strategic settings involving multiple experts. Additionally, the order of moves may be part of a design problem and, in the explicit example we consider, the sequential move equilibrium is strictly better than the simultaneous move equilibrium for the senders.⁴

Coordinated and independent signals are two extreme cases, so we also briefly consider an intermediate case. In a jury trial, it seems reasonable to assume that evidence from the prosecution and the defense can sometimes be coordinated. Character evidence, however, seems more or less independent of physical evidence collected at the crime scene. See Lester, Persico, and Visschers (2012) for a fuller discussion of correlations across different pieces of evidence. We will not try to defend mixed strategies in terms of

⁴Glazer and Rubinstein (2001) compare simultaneous and sequential debates and show that sequential persuasion may be better for the decision maker.

real world examples, but, in our example, the mixed strategy equilibrium we construct is strictly better than the unique pure strategy equilibrium for both senders.

Each modification of the benchmark persuasion game results in a setup where senders can no longer refine information unrestrictedly. In the sequential case, only the last mover can have the final word. When signals are independent it is no longer feasible to create arbitrary refinements of the existing information. Randomizations make it impossible to refine the pure signals in the support of the mixed strategy independently. In each case we show that it is possible that the decision maker is strictly less informed in an equilibrium with two senders than in the single-sender equilibrium. We don't explore it in this paper, but we conjecture that incomplete information about preferences would have similar effects.

Our most straightforward example is with sequential moves. Sender 1 prefers the actions the decision maker takes when information is rather precise, but he does not like the actions taken under full information. The example is constructed so that sender 1's most preferred outcome is implemented by the decision maker in an essentially unique equilibrium with a precise, but not fully informative, signal. We then add a sender who moves after sender 1 and strictly prefers the actions taken under full revelation to the single-sender equilibrium, but who is even better off if both senders babble. Given that sender 1 prefers babbling to full information, which we assume, sender 1 will babble (or provide an outcome-equivalent signal structure) in order to avoid having sender 2 respond to signals of intermediate informativeness by fully revealing the state.

Note that the example is driven by the fact that the new sender has the final word. Should the order of moves be reversed, sender 1 could always refine babbling by the other sender. In general, we demonstrate that if a new sender is added and this extra sender moves first, then there can be no equilibrium with the additional sender that is less informative than some equilibrium in the model without the extra sender unless the garbling is also an equilibrium without the new sender.⁵ The proof of this result uses the fact that all "old senders" can refine information in the same way they did before the new sender was added. This setup therefore retains some of the flavor of the simultaneous move model.

The sequential model differs from the simultaneous model also in that full revelation is not always an equilibrium outcome. The intuition for this is much like the difference between simultaneous and sequential voting. Full revelation is an equilibrium outcome in the simultaneous model even if all senders agree that this is the worst possible outcome. This follows as there is no way to scramble a fully informative signal, so while full revelation is a weakly dominated strategy it is nevertheless supportable as a Nash equilibrium. In contrast, if every sender agrees that full information is the worst, providing full information is not subgame perfect (or even Nash) in the sequential case as the first sender who reveals the truth is pivotal given (sequentially) rational continuation strategies.

Providing too much information can also make a sender vulnerable if signals are independent, but

⁵It is also assumed that the order of the existing senders is kept the same when a new sender is added.

for different reasons than in the sequential move model. Unlike the coordinated signal case, senders can no longer refine the information provided by other senders on a realization-by-realization basis. That is, with coordinated signals each sender can construct a best response by conditioning one by one on the sets in the partitioning created by the other senders. This is not feasible when signals are independent. Best responses, when signals are independent, must take into consideration the uncertainty over the sunspot realization of the competitor's signal, creating uncertainty that is not present in the case when signals are coordinated. Senders therefore face uncertainty about the information provided by other senders, which may create disincentives to provide information. This result is robust to introducing some coordination of the signals.

The mixed strategy analysis is more complicated, but again the idea is that providing intermediate levels of information may be dangerous. Our example exploits the fact that, when playing against a mixed strategy, opponents are no longer certain about the consequences of playing any particular individual signal. This allows us to construct an example where the addition of a new sender makes the original sender play a babbling signal to protect himself against bad outcomes.

2 The Model

Consider an environment with $n + 1$ agents, where $n \geq 1$ agents are called *senders* and one is referred to as the *decision maker*. Each sender $i \in \{1, \dots, n\}$ has a payoff function $u_i : A \times \Omega \rightarrow R$, where $a \in A$ is the action taken by the decision maker and $\omega \in \Omega$ is the state of the world. Both A and Ω are finite sets. We denote the decision maker's payoff function by $u_D : A \times \Omega \rightarrow R$ and the common prior belief of the state by $\mu_0 \in \Delta(\Omega)$. Payoff functions are common knowledge and all senders evaluate lotteries using expected utilities.

Both the decision maker and the senders are uninformed about the state of the world, but the senders may provide information to the decision maker by sending signals. A *signal* π_i is a finite partition of $\Omega \times [0, 1]$ with the property that each element $s_i \in \pi_i$ is a non-empty Lebesgue measurable subset of $\Omega \times [0, 1]$. We refer to an element $s_i \in \pi_i$ as a signal realization and let Π be the set of all permissible signals. Exactly how the decision maker makes inferences from a signal in the case with multiple senders depends on certain details that are explained below.

2.1 The Single-Sender Model

It is useful to first consider the single-sender model first analyzed in Kamenica and Gentzkow (2011). Dropping the subscripts we write π for a generic signal and $s \in \pi$ for a signal realization. Defining $p(s|\omega) = \lambda(\{z \in [0, 1] | (\omega, z) \in s\})$ as the conditional probability of realization $s \in \pi$ given that the

state is ω , where $\lambda(Z)$ denotes the Lebesgue measure for each measurable $Z \subset [0, 1]$. One may interpret $z \in [0, 1]$ as a sunspot variable that is independent of the true state and, without loss, uniformly distributed. The unconditional probability of signal realization $s \in \pi$ is thus given by $p(s) = \sum_{\omega \in \Omega} \mu_0(\omega) p(s|\omega)$ and

$$p(\omega'|s) = \frac{\mu_0(\omega') p(s|\omega')}{\sum_{\omega \in \Omega} \mu_0(\omega) p(s|\omega)}, \quad (1)$$

is the perceived probability of state $\omega' \in \Omega$ given signal realization s . For notational ease we suppress π and let $\mu(s) = (p(\omega'|s))_{\omega \in \Omega} \in \Delta(\Omega)$ denote the posterior belief distribution generated by signal π and realization s . The payoff function of the single sender (expert) is denoted by $u_E : A \times \Omega \rightarrow R$.

The timing is as follows. First, the sender posts a signal π . Then, the decision maker observes π and a signal realization $s \in \pi$ before taking an action $a \in A$. Our concept of equilibrium may be interpreted as either perfect Bayesian or subgame perfection depending on whether nature moves before or after the decision maker observes the signal (which is irrelevant for incentives).

Definition 1. A (pure strategy) equilibrium is a signal $\pi^* \in \Pi$ and decision rule $\sigma^* : \Delta(\Omega) \rightarrow A$ such that

$$\sum_{\omega \in \Omega} \sum_{s \in \pi^*} u_E(\sigma^*(\mu(s)), \omega) p(s|\omega) \mu_0(\omega) \geq \sum_{\omega \in \Omega} \sum_{s' \in \pi'} u_E(\sigma^*(\mu(s')), \omega) p(s'|\omega) \mu_0(\omega)$$

for each $\pi' \in \Pi$, where

$$\sigma^*(\mu(s')) \in \arg \max_{a \in A} \sum_{\omega \in \Omega} u_D(a, \omega) p(\omega'|s')$$

for every $\pi' \in \Pi$ and $s' \in \pi'$, where the posterior belief $\mu(s') \in \Delta(\Omega)$ follows from (1) for every $\pi' \in \Pi$.

In principle, a strategy for the decision maker could be an arbitrary mapping from signals and signal realizations to action, but no sender has any direct preferences over the signals, which is why we have opted for this reduced form where we express actions as a function of beliefs. Also note that the decision maker, unlike decision makers in cheap talk games, does not need to make any inferences about the type of the sender as the sender is uninformed about the state and the decision maker observes the partition of $\Omega \times [0, 1]$ provided by the sender. One may interpret this as a sender who commits to perform a number of costless experiments with no possibility to withhold experimental results.

Notice that Bayes rule provides a direct mapping from realized signals to beliefs *on and off the equilibrium path* because every conceivable signal starts a subgame. Hence, we express a sequentially rational response by the decision maker as a mapping from beliefs to actions rather than as a map from signals to actions.

As the decision maker is non-strategic we take σ^* as a primitive in most of the analysis.

2.2 The Multiple-Sender Model

2.2.1 Coordinated Signals

Gentzkow and Kamenica (2012) propose a generalization of the single-sender model with an analytically convenient lattice structure. As in the single-sender case, each sender sends a signal π_i which is a finite partition of $\Omega \times [0, 1]$ consisting of measurable sets. Given that the senders post $(\pi_1, \dots, \pi_n)$ the decision maker observes a realized “joint signal” $\cap_{i=1}^n s_i$ and forms beliefs in accordance. That is, the decision maker understands that $p(\cap_{i=1}^n s_i | \omega) = \lambda(\{z \in [0, 1] | (\omega, z) \in s_i \text{ for each } i\})$ is the conditional probability of the joint signal and can use Bayes’ rule on each $\cap_{i=1}^n s_i$ exactly as in the single sender case. In what follows we refer to this signal structure interchangeably as *arbitrarily correlated* or *coordinated*.

The extensive form is as follows. In the first stage, senders simultaneously post signals. The decision maker observes the signals $(\pi_1, \dots, \pi_n)$ and the joint realization $\cap_{i=1}^n s_i$ before taking an action $a \in A$.

Definition 2. A (pure strategy) equilibrium is a signal vector $(\pi_1^*, \dots, \pi_N^*) \in \Pi^N$ and a decision rule $\sigma^* : \Delta(\Omega) \rightarrow A$ such that

$$\begin{aligned} & \sum_{\omega \in \Omega} \sum_{(s_1^*, \dots, s_n^*) \in \times_{j=1}^n \pi_j^*} u_i(\sigma^*(\mu(\cap_{j=1}^n s_j^*)), \omega) p(\cap_{i=1}^n s_j^* | \omega) \mu_0(\omega) \\ & \geq \sum_{\omega \in \Omega} \sum_{(s_{-i}^*, s_i') \in \times_{j \neq i} \pi_j^* \times \pi_i'} u_i(\sigma^*(\mu(\cap_{j \neq i} s_j^* \cap s_i')), \omega) p(\cap_{j \neq i} s_j^* \cap s_i' | \omega) \mu_0(\omega) \end{aligned} \quad (2)$$

for each $i \in \{1, \dots, N\}$ and $\pi_i \in \Pi$ and

$$\sigma^*(\mu(\cap_{j=1}^n s_j)) \in \arg \max_{a \in A} \sum_{\omega \in \Omega} u_D(a, \omega) p(\omega' | \cap_{j=1}^n s_j) \quad (3)$$

for every $(\pi_1, \dots, \pi_N) \in \Pi^N$ and $s \in \times_{j=1}^n s_j$ where the posterior beliefs $\mu(\cap_{j=1}^n s_j) \in \Delta(\Omega)$ are given by

$$p(\omega' | \cap_{j=1}^n s_j) = \frac{\mu_0(\omega') p(\cap_{j=1}^n s_j | \omega')}{\sum_{\omega \in \Omega} \mu_0(\omega) p(\cap_{j=1}^n s_j | \omega)}. \quad (4)$$

for every $(\pi_1, \dots, \pi_N) \in \Pi^N$ and $s \in \times_{j=1}^n s_j$

A signal is a *babbling signal* if $\mu(s_i) = \mu_0$ for any $s_i \in \pi_i$. A special case of a babbling signal is the *trivial signal* $\pi_i = \{\Omega \times [0, 1]\}$. The reader should notice that a babbling signal is qualitatively different from the trivial signal because a babbling signal may provide additional information when it is combined with another signal.

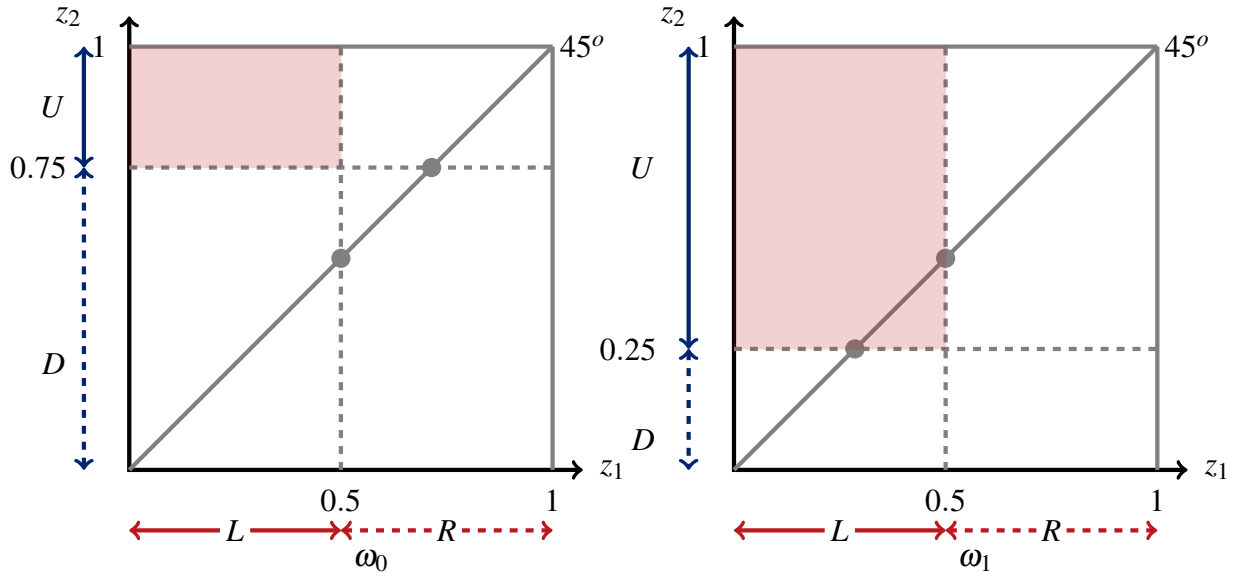


Figure 1: Coordinated versus Independent Signals

2.2.2 Independent Signals

An alternative multiple sender generalization is as follows. Again, let a signal π_i be defined as a finite partition of $\Omega \times [0, 1]$, but let $\{z_i\}_{i=1}^N$ be independent uniform random variables on $[0, 1]$. Conditional on state ω the probability of realization s_i is $p(s_i|\omega) = \lambda(\{z_i \in [0, 1] | (\omega, z_i) \in s_i\})$, and by independence, $s = (s_1, \dots, s_n)$ has probability $\times_{i=1}^n p(s_i|\omega)$. The posterior belief that the state is ω' is denoted

$$p(\omega'|s) = \frac{\mu_0(\omega') \times_{i=1}^n p(s_i|\omega')}{\sum_{\omega \in \Omega} \mu_0(\omega) \times_{i=1}^n p(s_i|\omega)}. \quad (5)$$

Everything else is as it is in the case with coordinated signals, so an equilibrium may be defined as in Definition 2, with the only change being that beliefs are generated following (5) instead of (4).

Figure 1 provides a geometric interpretation of the difference between coordinated and independent signals.⁶ It shows an example with two states, $\Omega = \{\omega_0, \omega_1\}$ and signal $\pi_1 = \{L, R\}$ for sender 1 and $\pi_2 = \{U, D\}$ for sender 2. The areas in the graph indicate the probability of the signal conditional on the state. In Gentzkow and Kamenica (2012) senders are forced to use the same sunspot variable, which corresponds to signals that are restricted to the 45 degree line, whereas independence corresponds to pairs of signals that are in the full unit square.

⁶We thank Santiago Oliveros for contributing the idea for this graph.

2.2.3 Sequential Persuasion and Mixed Strategies

Besides the case with independent signals we analyze two additional variations that retain the coordinated signal structure. First, we will consider the case where the senders move sequentially. Second, we will introduce mixed strategies to the simultaneous move persuasion game. In each case it should be obvious how to adjust the equilibrium concept in Definition 2 appropriately.

2.3 Informativeness

We order equilibria by the garbling criterion from Blackwell (1953). Let $\pi = \{\pi_i\}_{i=1}^n$ and $\pi' = \{\pi'_i\}_{i=1}^n$ denote two signal profiles. Signal π is said to be weakly more informative than signal π' , which we denote by $\pi \succeq \pi'$, if there exists a garbling $g : S \times S \rightarrow [0, 1]$ such that $\sum_{s' \in \pi'} g(s', s) = 1$ for all $s \in \pi$ and $p(s'|\omega) = \sum_{s \in \pi} g(s', s)p(s|\omega)$ for all ω and all $s' \in \pi'$. We write $\pi \succ \pi'$ if signal π is strictly more informative than π' (if $\pi \succeq \pi'$ and not $\pi' \succeq \pi$), and $\pi \sim \pi'$ if the signals are equivalent (when $\pi \succeq \pi'$ and $\pi' \succeq \pi$).

In general, we need to compare sets of equilibria. Suppose Σ and Σ' are two sets of signals. We say Σ is *weakly more informative* than Σ' if for any $(\pi, \pi') \in \Sigma \times \Sigma'$, (i) $\exists \hat{\pi} \in \Sigma : \hat{\pi} \succeq \pi'$, and (ii) $\exists \hat{\pi}' \in \Sigma' : \pi \succeq \hat{\pi}'$. If (i) and (ii) are both strict, then Σ is *strictly more informative* than Σ' .⁷

3 Results

3.1 A Single-Sender Example

In most of the analysis that follows, we assume that there are two states: $\{\omega_0, \omega_1\}$ with prior $\Pr(\omega = \omega_0) = \mu_0 = \frac{1}{2}$, and $A = \{a_i\}_{i=0}^3$. We first consider a single-sender model, and assume that the decision maker follows the rule,

$$\sigma^*(\mu) = \begin{cases} a_0 & \text{if } \mu \in [\frac{99}{100} - k, \frac{99}{100}] \\ a_1 & \text{if } \mu \in [\frac{1}{100}, \frac{1}{100} + k] \\ a_2 & \text{if } \mu \in [\frac{1}{3} - k, \frac{1}{3} + k] \text{ or } \mu \in [\frac{1}{2} - k, \frac{1}{2} + k] \text{ or } \mu \in [\frac{3}{5} - k, \frac{3}{5} + k] \\ a_3 & \text{otherwise} \end{cases}, \quad (6)$$

where k is a small enough number to make (6) well defined.

This decision rule should be interpreted as a reduced form of one where the decision maker takes more than four actions. The reason is that it is inconsistent with expected utility theory for the optimal de-

⁷We work directly with the signal space. This is because the belief space characterization in Gentzkow and Kamenica (2012) does not apply when signals are not perfectly coordinated.

cision as a function of the beliefs to flip back and forth between a_2 and a_3 in a non-monotonic way. However, one can justify this as a reduced form of a decision rule with one action per interval and sender preferences where the senders are indifferent between the (distinct) actions on $[\frac{1}{3} - k, \frac{1}{3} + k]$, $[\frac{1}{2} - k, \frac{1}{2} + k]$ and $[\frac{3}{5} - k, \frac{3}{5} + k]$ as well as over the (distinct) actions on $[0, \frac{1}{100})$, $(\frac{1}{100} + k, \frac{1}{3} - k)$, $(\frac{1}{3} + k, \frac{1}{2} - k)$, $(\frac{1}{2} + k, \frac{3}{5} - k)$, $(\frac{3}{5} + k, \frac{99}{100} - k)$ and $(\frac{99}{100}, 1]$. Any decision rule with all distinct actions is consistent with some utility function for the decision maker. Lumping the first three of these actions together as a_2 and the final six together as a_3 we have a model with equivalent sender incentives, but considerably cleaner notation.

Sender 1's payoff is

$$u_1(a, \omega) = \begin{cases} 1 & \text{if } a = a_0 \text{ and } \omega = \omega_0 \\ 1 & \text{if } a = a_1 \text{ and } \omega = \omega_1 \\ \frac{99}{100} - \varepsilon & \text{if } a = a_2 \\ 0 & \text{otherwise} \end{cases}, \quad (7)$$

where ε is a small number (for now $0 < \varepsilon < \frac{99}{100}$). In other words, sender 1's favorite action is to match the state $a_i = \omega_i$ for $i = 1, 2$, and his second best action is a_2 regardless of the state. Denote by Σ_1 the set of equilibria in the single-sender persuasion game. We can then show that for any $\pi \in \Sigma_1$, the posterior takes on value $\frac{1}{100}$ with probability $\frac{1}{2}$ and $\frac{99}{100}$ with probability $\frac{1}{2}$, implying that sender 1's expected payoff is $\frac{99}{100}$.

Consider signal $\pi^{**} = \{s_0, s_1\}$, where

$$\begin{aligned} s_0 &= \left(\omega_0, \left[0, \frac{99}{100}\right] \right) \cup \left(\omega_1, \left(\frac{99}{100}, 1\right] \right) \\ s_1 &= \left(\omega_1, \left[0, \frac{99}{100}\right] \right) \cup \left(\omega_0, \left(\frac{99}{100}, 1\right] \right), \end{aligned} \quad (8)$$

and a routine calculation shows that the probability of state ω_0 is $\mu(s_0) = \frac{99}{100}$ and $\mu(s_1) = \frac{1}{100}$ respectively. While there are infinitely many alternative ways to partition the sunspot variable, all other equilibria give the same equilibrium outcome and are informationally equivalent in the sense of Blackwell (1953).

Proposition 1. *The signal π^{**} is an equilibrium. All equilibria are outcome equivalent and if $\pi' = \{s_k\}_{k=1}^K$ is an equilibrium, then there is a partitioning of $\{S_L, S_H\}$ of $\{s_k\}_{k=1}^K$ such that $\mu(s_k) = \frac{1}{100}$ for every $s_k \in S_L$ and $\mu(s_k) = \frac{99}{100}$ for every $s_k \in S_H$. Moreover, the ex ante probability of $\cup_{s_k \in S_L} s_k$ is $\frac{1}{2}$.*

The proof can be found in Section A.1 of the Appendix.

3.2 Multiple Senders in Gentzkow and Kamenica's (2012) Model

Suppose that there is another sender whose (state-independent) payoff function is

$$u_2(a, \omega_i) = \begin{cases} -1 & \text{if } a \in \{a_0, a_1\} \\ 0 & \text{if } a = a_3 \\ 1 & \text{if } a = a_2 \end{cases} \quad (9)$$

Proposition 2. *In the two-sender persuasion game with payoffs given by (7) and (9) and decision rule given by (6), an equilibrium exists whenever both senders reveal the state. Moreover, information is fully revealed in any pure strategy equilibrium.*

The proof is in Section A.2 of the appendix.

Denote by Σ_2 the set of equilibria in the two-sender persuasion game and let Σ_1 be the set of single-sender equilibria from Section 3.1. It should be rather clear that Σ_2 is strictly more informative than Σ_1 . In each case, the equilibrium signals must be equivalent, so to compare Σ_2 and Σ_1 it is sufficient to compare any pair (π^*, π^{**}) with $\pi^* \in \Sigma_2$ and $\pi^{**} \in \Sigma_1$.

One equilibrium is the signal profile $\pi^* = \{\pi_1^*, \pi_2^*\}$ with $\pi_1^* = \pi_2^* = \{s_{20}, s_{21}\}$ and

$$\begin{aligned} s_{20} &= (\omega_1, [0, 1]) \cup (\omega_0, \emptyset) \\ s_{21} &= (\omega_1, \emptyset) \cup (\omega_0, [0, 1]), \end{aligned}$$

Clearly, $p(s_{20}|\omega_0) = p(s_{21}|\omega_1) = 1$, and $\mu(s_{20}) = 1, \mu(s_{21}) = 0$. Recall that π^{**} is a garbling of π^* if

$$\begin{aligned} p(s_{10}|\omega_0) &= g(s_{10}, s_{20})p(s_{20}|\omega_0) + g(s_{10}, s_{21})p(s_{21}|\omega_0) \\ p(s_{11}|\omega_1) &= (1 - g(s_{10}, s_{20}))p(s_{20}|\omega_1) + (1 - g(s_{10}, s_{21}))p(s_{21}|\omega_1) \end{aligned}$$

for $g(s_{10}, s_{20}), g(s_{10}, s_{21}) \in [0, 1]$ with either $0 < g(s_{10}, s_{20}) < 1$ or $0 < g(s_{10}, s_{21}) < 1$, or both. Simple algebra yields that $g(s_{10}, s_{20}) = g(s_{10}, s_{21}) = \frac{99}{100}$. Hence, π^{**} is a garbling of π_1^* . Clearly, π_1^* is not a garbling of π^{**} , so the two-sender equilibrium is strictly more informative in the sense of Blackwell. As all equilibria are informationally equivalent in each case, Σ_2 is strictly more informative than Σ_1 .

Gentzkow and Kamenica (2012) show that in general:

1. Full revelation is always an equilibrium outcome if there are two or more senders, and that;
2. Increasing the number of senders must weakly increase the amount of information that is revealed in the least informative pure strategy equilibrium.

Full revelation is an equilibrium outcome, even when this leads to the lowest possible payoff for all senders. The reason is that no sender is pivotal if at least two senders provide signals that are fully revealing by themselves.

More interesting is the finding that adding senders cannot result in a less informative equilibrium set. It is important to note that the logic that drives this result is that every sender can always make information more precise, but no sender can reduce the informativeness relative to what the decision maker can infer from the signals posted by the other senders. Hence, if a sender is added and information is lost, one of the original senders can deviate and replicate the more informative outcome before the new sender was added. In addition, at least one sender must have an incentive to deviate to a more informative signal as otherwise the signals with less information would also be an equilibrium before the new sender was added.

3.3 Sequential Persuasion

This section considers a sequential persuasion model. The game is as follows. First, sender 1 plays a signal π_1 . Then, sender 2 observes π_1 and plays a signal π_2 . Finally, the decision maker observes π_1 and π_2 and a pair of signals $s_1 \in \pi_1$ and $s_2 \in \pi_2$ and infers that the posterior probability of state ω_0 is $\mu(s_1 \cap s_2)$ before taking an action a in A . We find that, in the context of our example, the amount of information that is revealed is unambiguously less than in the single sender case.

Proposition 3. *There exists a subgame perfect equilibrium π^* in which both senders play the trivial signal and no information is revealed. Moreover, in any equilibrium, the decision maker takes action a_2 for sure.*

The proof is in Section A.3 in the appendix. The idea is that sender 2 can further partition any possible realization of the signal played by sender 1. Hence, if some $s_1 \in \pi_1$ induces a posterior that does not make the decision maker play a_2 , sender 2 can always partition this signal further in a way that makes the decision maker plays a_2 with positive probability and $a \in \{a_0, a_1\}$ with probability zero. It follows that there is no way that sender 1 can induce the decision maker to take action in $\{a_0, a_1\}$, so sender 1 may as well babble. For an illustration, in the graph to the left in Figure 2 we consider the case where the posterior beliefs given a signal realization s_1 are in between the lowest and the highest range that gives sender 2 his maximal payoff. The only constraint on the beliefs from a further partition of s_1 is that they must be consistent with Bayes' rule, so a partition of s_1 that decreases the posterior to any value on $[0, \mu(s_1))$ with some probability and increases the posterior to any value on $(\mu(s_1), 1]$ is feasible. Hence, sender 2 can partition any s_1 with $\mu(s_1)$ in the intermediate range in a way that ensures the maximal payoff.

To the right in Figure 2 we illustrate the case when $\mu(s_1)$ is below $\frac{1}{3}$ (the case with $\mu(s_1)$ above $\frac{3}{5}$ is symmetric). In this case sender 2 cannot get his maximal probability for sure, but it is always possible to partition s_1 so that the posterior is sent to 0 with some probability and to $\frac{1}{3}$ with some probability. This is the optimal response to any s_1 in the range under consideration.

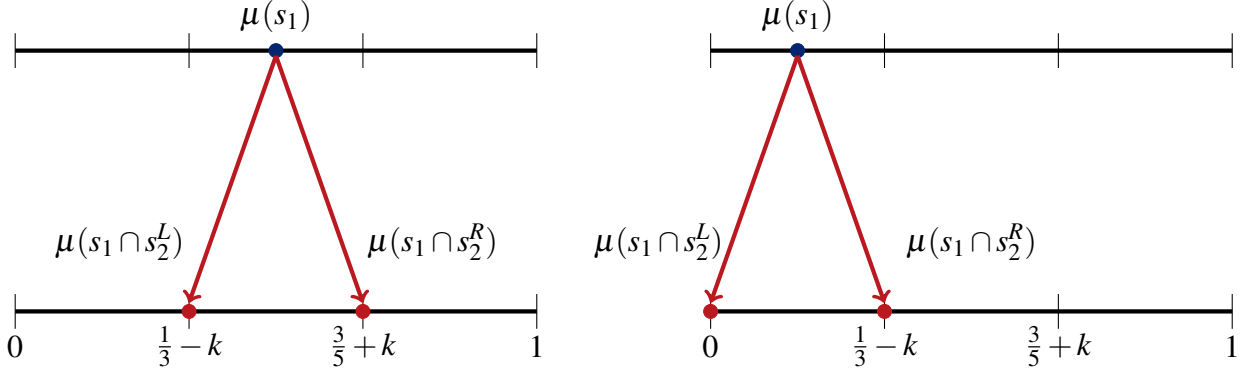


Figure 2: Dynamic Best Responses

Notice that there exists no fully revealing equilibrium. To understand this it is useful to note that in the simultaneous move persuasion game, each belief in support of the equilibrium posterior distribution must be unimprovable for each sender in the sense that no sender can gain by adding information. This rests on the fact that under simultaneous persuasion, all senders are pivotal in the determination of the final distribution of the posterior. In contrast, only the last mover can add information in an unconstrained fashion in a sequential persuasion game. Senders moving earlier in the game are constrained to deviations over possible continuation equilibria. This difference makes it possible to generate examples such as the one analyzed above, in which babbling cannot be supported as an equilibrium in the single sender case, but where it is an equilibrium in the sequential game with multiple senders.

3.3.1 Multiple Rounds of Persuasion

In the analysis above, senders are only allowed to move once. This is only for notational convenience as one gets the same result even when allowing for multiple rounds of persuasion.

Assume that there are $2M$ stages with sender 1(2) moving at every odd (even) stage. Babbling is still an equilibrium. This is because that, sender 1 anticipates that, as long as he induces a posterior $\hat{\mu} \notin [\frac{1}{3} - k, \frac{3}{5} + k] \cup \{0\} \cup \{1\}$, the continuation play by sender 2 will generate beliefs such that a_3 is chosen by the decision maker with positive probability and actions $a \in \{a_0, a_1\}$ are never played. Consequently, sender 1 has no incentive to deviate from babbling. In addition, babbling makes the decision maker take the best action from the point of view of sender 2, so this is clearly a best response. Just like when each sender moves once there are other equilibria, but they all induce the decision maker to take the same action.⁸

⁸This is very different from sequential cheap talk where multiple rounds may support fully revealing equilibria in cases

An alternative setup has no deadline and allows each sender to add information as long as he or she wants, but we maintain the assumption that there are no simultaneous moves.⁹ Again the result is unchanged as sender 1 understands that sender 2 will respond in a way that makes him worse off than if he had babbled should he try to induce his favorite outcome.

This reasoning generalizes. Hence, there can be no equilibrium with sequential moves that is Pareto dominated by babbling from the point of view of the senders. Moreover, if full revelation is Pareto dominated by an equilibrium, then full revelation is not an equilibrium. The proofs are immediate and are left to the reader.

3.3.2 Adding Senders as First Movers

Adding a sender who moves last resulted in a loss of information in the example above. Now, consider a sequence of sequential persuasion games:

$$\{\Gamma^n\}_{n \in \mathbb{N}} = \{\Omega, (\Pi_1, \dots, \Pi_n), A, u_D, (u_1, \dots, u_n)\}_{n \in \mathbb{N}},$$

where the state space Ω is kept fixed, where $\Pi_i = \Pi$ which we defined as the set of Lebesgue measurable partitions of $\Omega \times [0, 1]$, and where A represents the set of actions available to the decision maker, which is also the same for each game in the sequence. The utility function for the decision maker $u_D : \Omega \times A \rightarrow R$ is also held fixed, as is $u_i : \Omega \times A \rightarrow R$ for every $i \in N$. Hence, the sequence is one in which a new sender is added to an existing set of senders in each step. Finally, we assume that the sequence is such that the new sender (sender n in game Γ^n) moves first and the remaining senders move in the same order as in Γ^{n-1} . For our next result we use the following fact:

Lemma 1. *For any $\pi \succeq \pi'$, there exists a π'' which is a finer partition of π' such that π'' and π are outcome equivalent.*

Lemma 1 restates Lemma 4 from Gentzkow and Kamenica (2012) directly in terms of signals and outcomes (as opposed to decision maker's belief).

Proposition 4. *Suppose that $\{\Gamma^n\}_{n \in \mathbb{N}}$ is a sequence of persuasion games where in each game n , sender n moves first and the remaining agents move in the same order as in Γ^{n-1} . Let π^n be an equilibrium in Γ^n , π^{n+1} an equilibrium in Γ^{n+1} , and assume that π^{n+1} is less informative than π^n in the sense of Blackwell (1953). Then, there exists an equilibrium in game Γ^{n+1} that is outcome equivalent with π^n .*

Proof. Let π_i^n and π_i^{n+1} denote equilibrium path signals played by i in games Γ^n and Γ^{n+1} . Also, let S_i^n be the implied partitioning of $\Omega \times [0, 1]$ after senders $i, \dots, n$ have played $(\pi_i^n, \dots, \pi_n^n)$ but before $i-1$ plays.

where a single round does not. See Krishna and Morgan (2001).

⁹See Matthews and Postlewaite (1994) for a discussion about the end-round effect in a cheap talk model.

Consider sender 1. If S_1^{n+1} is a garbling of S_1^n it is feasible for sender 1 to obtain the same payoff as in equilibrium $\pi^n = (\pi_n^n, \dots, \pi_1^n)$ when it is sender 1's turn to play. Hence, either it is a best response for sender 1 to play a signal that is outcome equivalent with S_1^n or sender 1 is strictly better off given the less informative partition S_1^{n+1} because S_1^{n+1} is always feasible by Lemma 1. For $\pi^{n+1} = (\pi_{n+1}^{n+1}, \dots, \pi_1^{n+1})$ to be an equilibrium it has to be that S_1^{n+1} is weakly preferred to S_1^n by sender 1 and for $\pi^n = (\pi_n^n, \dots, \pi_1^n)$ to be an equilibrium S_1^n must be weakly preferred to any S' for any S' that is more informative than S_1^n . For induction, assume that:

1. S_1^{n+1} is weakly preferred to S_1^n by senders $1, \dots, i$
2. Each sender $j \in \{1, \dots, i\}$ weakly prefers S_1^n to any S' that is more informative than S_1^n and a continuation equilibrium outcome after history $(\pi_n^n, \dots, \pi_{j+1}^n)$.

Then, we note that there exists a continuation equilibrium in which no sender $j \in \{1, \dots, i\}$ would provide any additional information if sender $i+1$ plays a signal that is equivalent to S_1^n . Moreover, such a signal is feasible as S_{i+1}^{n+1} is a garbling of S_1^{n+1} and S_1^{n+1} is a garbling of S_1^n . Hence, it follows $i+1$ weakly prefers S_1^{n+1} to S_1^n . By induction, it follows that:

1. S_1^{n+1} is weakly preferred to S_1^n by senders $1, \dots, n$
2. Each sender $j \in \{1, \dots, n\}$ weakly prefers S_1^n to any S' that is more informative than S_1^n and a continuation equilibrium outcome after history $(\pi_n^n, \dots, \pi_{j+1}^n)$.

It follows that there is an equilibrium in Γ^n that is outcome equivalent to $\pi^{n+1} = (\pi_{n+1}^{n+1}, \dots, \pi_1^{n+1})$. Repeated use of this fact establishes the result for any n and m with $m < n$. $\square$

Proposition 4 says that if we add a sender and if the new sender moves first and an equilibrium in the model with the new sender is a garbling of an equilibrium that exists before the sender was added, then the garbling must also have been an equilibrium before the new sender was added. Hence, adding a sender as a first mover cannot reduce the informativeness of the least informative equilibrium. However, even if the new sender moves first it is easy to construct examples where equilibria cannot be compared.

3.4 Independent Signals

We now consider the multi-sender generalization with independent signals introduced in Section 2.2.2. Again, a signal π_i is defined as a finite partition of $\Omega \times [0, 1]$, but now we assume that sender 1 and

sender 2 have two independent sunspot variables available. The posterior that the state is ω_0 conditional on $s_1 \in \pi_1$ and $s_2 \in \pi_2$ is

$$p(\omega_0|s_1, s_2) = \frac{p(s_1|\omega_0)p(s_2|\omega_0)}{p(s_1|\omega_0)p(s_2|\omega_0) + p(s_1|\omega_1)p(s_2|\omega_1)}, \quad (10)$$

where we have built in the assumption that $\mu_0 = 1 - \mu_0 = \frac{1}{2}$ in (10). Everything else is as in the benchmark model. Senders move simultaneously and we consider equilibria in pure strategies.

Again, the single sender outcome is not an equilibrium because sender 2 can always add information to avoid posteriors in $\mu \in [\frac{1}{100}, \frac{1}{100} + k] \cup [\frac{99}{100} - k, \frac{99}{100}]$, thereby avoiding a negative payoff. One such deviation is simply to reveal the state. Full revelation may also be supported as an equilibrium as any signal, including full revelation, is a best response to full revelation regardless of whether signals are independent or arbitrarily correlated.

Unlike the coordinated case, it is now possible to construct an equilibrium with two senders that is less informative than the single-sender equilibrium.

Proposition 5. *There exists a pair $(k^*, \varepsilon^*) \in \mathbb{R}_{++}^2$ such that for any $k \in [0, k^*)$ and $\varepsilon \in (0, \varepsilon^*)$, there exists an equilibrium $\pi^* = (\pi_1^*, \pi_2^*)$ in which sender 1 sends the trivial signal $\pi_1^* = \{\{\omega_0, \omega_1\} \times [0, 1]\}$ and sender 2 sends any signal $\pi_2^* = \{s_{21}, s_{22}\}$ such that*

$$p(s_{21}|\omega_0) = \frac{1}{4} \text{ and } p(s_{21}|\omega_1) = \frac{1}{2}.$$

The posterior beliefs for the decision maker are

$$p(\omega_0|s_1, s_2) = \begin{cases} \frac{1}{3} & \text{if } s_2 = s_{21} \\ \frac{3}{5} & \text{if } s_2 = s_{22}. \end{cases},$$

implying that the decision maker responds with $\sigma^(\frac{1}{3}) = \sigma^*(\frac{3}{5}) = a_2$. Any equilibrium signal in the single-sender game is strictly more informative than π^* .*

See the proof in Section A.4 of the appendix for details. The idea of the construction is simple. sender 1 would ideally like the decision maker to have beliefs near $\frac{1}{100}$ or $\frac{99}{100}$. However, if sender 2 plays in accordance to the proposition and sender 1 deviates in such a way that the decision maker takes the action that sender 1 seeks when s_{21} (s_{22}) is realized, then the outcome must be bad for sender 1 when s_{22} (s_{21}) is realized. sender 1 is therefore better off babbling.

The crucial difference between independent and arbitrarily correlated signals is that senders can no longer add information on a set by set basis. That is, with arbitrarily correlated signals each sender can construct a best response by conditioning one by one on the partitions created by the signals played by the other senders. This falls apart when signals are independent. Then, best responses must take into consideration the uncertainty over the sunspot realization of the competitor's signal, creating risk that is not present in the case when signals are coordinated.

3.4.1 Partially Coordinated Signals

Arguably, independent and perfectly coordinated signals are two extreme cases, and a natural question is whether Proposition 5 is robust to introducing some correlation. There are several ways one may create such a hybrid model, but a simple version is as follows. Just like in the previous two cases a signal π_i is a finite partition of $\{\omega_0, \omega_1\} \times [0, 1]$, but we assume that there are two independently drawn sunspot variables $z_1 \in [0, 1]$ and $z_2 \in [0, 1]$ with probability $\alpha \in (0, 1)$, whereas there is a common sunspot variable $z \in [0, 1]$ with the complementary probability. As it is the perception of the decision maker that matters we assume that the decision maker don't know if signals are independent or coordinated.¹⁰ The posterior belief given a realized pair (s_1, s_2) is then

$$p(\omega_0 | s_1, s_2) = \alpha \frac{p(s_1 | \omega_0)p(s_2 | \omega_0)}{p(s_1 | \omega_0)p(s_2 | \omega_0) + p(s_1 | \omega_0)p(s_2 | \omega_1)} + (1 - \alpha) \frac{p(s_1 \cap s_2 | \omega_0)}{p(s_1 \cap s_2 | \omega_0) + p(s_1 \cap s_2 | \omega_1)}. \quad (11)$$

The only restriction on the second term in (11) is Bayesian plausibility, which, since the question is whether sender 1 can break the equilibrium in Proposition 5, can be ignored. For simplicity, suppose $k = 0$ (by continuity the logic extends to $k > 0$ provided that it is sufficiently small) and assume that the posterior beliefs when s_{21} is realized is

$$\frac{\alpha}{1 + 2 \frac{p(s_1 | \omega_1)}{p(s_1 | \omega_0)}} + (1 - \alpha) \frac{p(s_1 \cap s_{21} | \omega_0)}{p(s_1 \cap s_{21} | \omega_0) + p(s_1 \cap s_{21} | \omega_1)} = \frac{1}{100}. \quad (12)$$

When $s_2 = s_{22}$, the posterior belief

$$\frac{\alpha}{1 + 2 \frac{p(s_1 | \omega_1)}{p(s_1 | \omega_0)}} + (1 - \alpha) \frac{p(s_1 \cap s_{22} | \omega_0)}{p(s_1 \cap s_{22} | \omega_0) + p(s_1 \cap s_{21} | \omega_2)} \rightarrow \frac{1}{34}$$

as $\alpha \rightarrow 0$. Hence, we can extend Proposition 5 as long as signals are not too correlated.

3.5 Mixed Strategy Equilibria in the Simultaneous Model with Coordinated Signals

In our final example we consider the exact same model as in Section 3.2, but allow senders to play mixed strategies. A mixed strategy equilibrium is constructed that sometimes results in babbling and sometimes results in some information being revealed, but regardless of which pure signal in the support of the randomized equilibrium is played, the resulting signal structure is always less informative than the equilibria in the single-sender game.

¹⁰The decision maker must observe some reduced form signal realizations and not the actual partitions for this to work.

The construction is more complicated, but the idea is similar to the independent case in section 3.4. The key idea is that sender 2 plays a randomization, so that anything that sender 1 plays that makes the beliefs of the decision maker close to $\frac{1}{100}$ or $\frac{99}{100}$ when played against some signal in the support of sender 2's mixed strategy, there is always another signal in the support that makes the decision maker take the action that sender 1 likes the least with positive probability. The difficult part of the proof is the construction of such a randomized strategy.

3.5.1 Some Heuristics

As the construction is notationally heavy and the formal proof involves a large number of different cases, we here provide an informal explanation of the construction below. The idea is to create a randomization by sender 2 such that sender 1's best reply is the trivial signal. In addition, we seek to construct an example where information is lost when adding a sender, so the pure signals in the support of the randomization are all less informative than the single-sender equilibrium. Finally, sender 2 achieves his maximal payoff for sure (if sender 1 babbles), while sender 1 achieves the second best.

The complicated part of the proof is to make sure that sender 1 cannot profitably deviate from playing the trivial signal. Our mixed strategy has support on 10 different pure signal structures. We tried simpler examples before converging on this one, but in order to induce babbling by sender 1 we had to use a somewhat complicated randomization.

To understand this and how our equilibrium works we note that any profitable deviation for sender 1 must have at least one signal realization $s_1 \in \pi_1$ such that the beliefs of the decision maker are in $[\frac{1}{100}, \frac{1}{100} + k]$ or $[\frac{99}{100} - k, \frac{99}{100}]$ when combined with some realization of one of the pure signals in the support of the mixed strategy played by sender 2. In addition, for small ε there can exist no realization of any signal played with positive probability by sender 2 that induces the decision maker to play a_3 when combined with s_1 . The construction below provides a rich enough support for the mixed strategy to rule out the existence of such a signal s_1 .

3.5.2 A Mixed Strategy Equilibrium

Let

$$I^A = \left[0, \frac{1}{4}\right], I^B = \left[\frac{1}{4}, \frac{1}{2}\right], I^C = \left[\frac{1}{2}, \frac{3}{4}\right], \text{ and } I^D = \left[\frac{3}{4}, 1\right].$$

For each $J \in \{A, B, C, D\}$ let $\pi_2^J = \{s_{21}^J, s_{22}^J\}$ be the partition given by

$$\begin{aligned}
s_{21}^J &= (\omega_0 \times [0, 1] \setminus I^J) \cup \left(\omega_1 \times \left[0, \frac{1}{2}\right] \right) \\
s_{22}^J &= (\omega_0 \times I^J) \cup \left(\omega_1 \times \left(\frac{1}{2}, 1\right] \right),
\end{aligned} \tag{13}$$

and for $JK \in \{AB, AC, AD, BC, BD, CD\}$ let $\pi_2^{JK} = \{s_{21}^{JK}, s_{22}^{JK}\}$ be the partition given by

$$\begin{aligned}
s_{21}^{JK} &= (\omega_0 \times [0, 1] \setminus (I^J \cup I^K)) \cup \left(\omega_1 \times \left[0, \frac{1}{2}\right] \right) \\
s_{22}^{JK} &= (\omega_0 \times (I^J \cup I^K)) \cup \left(\omega_1 \times \left(\frac{1}{2}, 1\right] \right).
\end{aligned} \tag{14}$$

For ease of notation, let

$$\begin{aligned}
\Pi_2 &= \left\{ \{\pi_2^J\}_{J \in \{A, B, C, D\}}, \{\pi_2^{JK}\}_{JK \in \{AB, AC, AD, BC, BD, CD\}} \right\} \\
S_2 &= \left\{ \{s_{21}^J, s_{22}^J\}_{J \in \{A, B, C, D\}}, \{s_{21}^{JK}, s_{22}^{JK}\}_{JK \in \{AB, AC, AD, BC, BD, CD\}} \right\}
\end{aligned}$$

denote the set of partitions and the set of possible signal realizations, respectively. For a generic partition or signal realization in one of these partitions we use notation π_2 for elements in Π_2 and s_2 for elements in S_2 when this can be done without risk of confusion.

For any set $Y \subseteq [0, 1]$ let $\lambda(Y)$ denote the Lebesgue measure of Y . We note that when a partition π^J with $J \in \{A, B, C, D\}$ is selected the posterior probabilities that the state is ω_0 are

$$\begin{aligned}
\mu(s_{21}^J) &= \frac{\lambda([0, 1] \setminus I^J)}{\lambda([0, 1] \setminus I^J) + \lambda\left[\left[0, \frac{1}{2}\right]\right]} = \frac{\frac{3}{4}}{\frac{3}{4} + \frac{1}{2}} = \frac{3}{5} \\
\mu(s_{22}^J) &= \frac{\lambda(I^J)}{\lambda(I^J) + \lambda\left[\left(\frac{1}{2}, 1\right]\right]} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{2}} = \frac{1}{3},
\end{aligned}$$

while for $JK \in \{AB, AC, AD, BC, BD, CD\}$ the posteriors are

$$\begin{aligned}
\mu(s_{21}^{JK}) &= \frac{\lambda([0, 1] \setminus (I^J \cup I^K))}{\lambda([0, 1] \setminus (I^J \cup I^K)) + \lambda\left[\left[0, \frac{1}{2}\right]\right]} = \frac{1}{2} \\
\mu(s_{22}^{JK}) &= \frac{\lambda(I^J \cup I^K)}{\lambda(I^J \cup I^K) + \lambda\left[\left(\frac{1}{2}, 1\right]\right]} = \frac{1}{2}.
\end{aligned}$$

We will now show that it is an equilibrium for sender 1 to play the trivial partition and for sender 2 to randomize over the 10 partitions defined above with equal probability. To prove this, we first establish an intermediate result.

Lemma 2. Suppose that s_1 is a signal realization in π_1 such that $\sigma^*(\mu(s_1 \cap s_2)) = a_0$ or $\sigma^*(\mu(s_1 \cap s_2)) = a_1$ for some $s_2 \in S_2$. Then, provided that k is sufficiently small, there exists $s'_2 \in S_2$ such that $\sigma^*(\mu(s_1 \cap s'_2)) = a_3$.

A detailed proof is in Section A.5 of the Appendix. There is nothing difficult about the proof but unfortunately, there are numerous cases to take care of. To illustrate why Lemma 2 is true, consider the possibility that sender 1 can create a signal which generates beliefs on $[\frac{1}{100}, \frac{2}{100}]$ against both s_{22}^J and s_{22}^K . Rearranging the Bayes' rule formula, it then follows that we can bound both $\lambda(I^J \cap Z_0)$ and $\lambda(I^K \cap Z_0)$ in terms of $\lambda((\frac{1}{2}, 1] \cap Z_1)$. That is,

$$\begin{aligned} \min \{ \lambda(I^J \cap Z_0), \lambda(I^K \cap Z_0) \} &\geq \lambda \left(\left(\frac{1}{2}, 1 \right] \cap Z_1 \right) \frac{1}{99} \\ \max \{ \lambda(I^J \cap Z_0), \lambda(I^K \cap Z_0) \} &\leq \lambda \left(\left(\frac{1}{2}, 1 \right] \cap Z_1 \right) \frac{2}{98}. \end{aligned}$$

Now, recall that we constructed signal realization s_{22}^{JK} by adding partitions I^J and I^K under state ω_0 and keeping the partition under state ω_1 the same as in s_{22}^J and s_{22}^K . Hence, the posterior belief if s_1 is played against s_{22}^{JK} is

$$\begin{aligned} \mu(s_1 \cap s_{22}^{JK}) &= \frac{\lambda(I^J \cup I^K \cap Z_0)}{\lambda(I^J \cup I^K \cap Z_0) + \lambda((\frac{1}{2}, 1] \cap Z_1)} = \frac{\lambda(I^J \cap Z_0) + \lambda(I^K \cap Z_0)}{\lambda(I^J \cap Z_0) + \lambda(I^K \cap Z_0) + \lambda((\frac{1}{2}, 1] \cap Z_1)} \\ &\in \left[\frac{2\lambda((\frac{1}{2}, 1] \cap Z_1) \frac{1}{99}}{2\lambda((\frac{1}{2}, 1] \cap Z_1) \frac{1}{99} + \lambda((\frac{1}{2}, 1] \cap Z_1)}, \frac{2\lambda((\frac{1}{2}, 1] \cap Z_1) \frac{2}{98}}{2\lambda((\frac{1}{2}, 1] \cap Z_1) \frac{2}{98} + \lambda((\frac{1}{2}, 1] \cap Z_1)} \right]. \end{aligned}$$

For sufficiently small k , the posterior is too large to make the decision maker play a_1 and too small for the decision maker to play a_2 . Hence, if sender 2 plays signal π_2^{JK} and s_{22}^{JK} is realized, then the decision maker takes the least preferred action for sender. As this happens with probability $\frac{1}{20}$ conditional on s_1 being played it is not worth it for sender 1 if ε is small enough.

We also have to rule out cases where s_1 and s_{22}^J induce the decision maker to play the best outcome for sender 1 and some other s_{22}^K leads to a second best outcome. There are a multitude of such cases to consider, and we will also have to consider a number of cases involving all other potential signal realizations in the support of the mixed strategy played by sender 2. All these arguments are along similar lines and the conclusion is always that if the decision maker matches the state against some sender 2 signal realization, we can always find some other realization that is bad for sender 1.

Using Lemma 2 we can prove this section's main result.

Proposition 6. Suppose that $\varepsilon \in (0, \varepsilon^*)$, where $\varepsilon^* = \frac{99}{100} - \frac{77}{80}$. Then, there exists $k^* > 0$ such that when $k \in [0, k^*)$, it is an equilibrium for sender 1 to submit the trivial partition $\pi_1 = \{(\omega_0 \times [0, 1]) \cup (\omega_1 \times [0, 1])\}$ and for sender 2 to randomize over the partitions in (13) and (14) with equal probabilities.

Proof. Under the candidate equilibrium strategy, the decision maker takes action a_2 with probability 1. As this is the action that sender 2 prefers, sender 2 has no profitable deviation. Sender 1, on the other hand, receives a payoff of $\frac{99}{100} - \varepsilon$ in the candidate equilibrium, which is less than the highest possible payoff of $\frac{99}{100}$. Hence, we must make sure that no profitable deviation exists. By Lemma 2, the probability that sender 1 gets a payoff of zero is bounded away from zero. Each partition is played with probability $\frac{1}{10}$ and each signal in every partition is realized with probability no smaller than

$$\Pr(s_{22}^J) = \lambda(I^J) \Pr(\omega = \omega_0) + \lambda\left(\left(\frac{1}{2}, 1\right]\right) \Pr(\omega = \omega_1) = \frac{3}{8},$$

which is the least likely signal realization for partitions A, B, C and D . The other partitions have ex ante probabilities $(\frac{1}{2}, \frac{1}{2})$. Hence, the probability that sender 1 earns payoff 0 is bounded below by $\frac{3}{80}$, so the expected payoff conditional on the signal s_1 in signal π_1 being realized is no more than $\frac{77}{80}$. (it must be strictly less, but we don't need that for the argument) This is true for every signal that makes the decision maker take action a_0 or a_1 with positive probability. We define

$$S_1^* = \left\{ s_1 \in \pi_1 \mid \text{there is } s_2 \in S_2 \text{ such that } \mu(s_1 \cap s_2) \in \left[\frac{1}{100}, \frac{1}{100} + k \right] \cup \left[\frac{99}{100} - k, \frac{99}{100} \right] \right\},$$

and, for each signal realization $s_1 = (\omega_0 \times Z_0) \cup (\omega_1 \times Z_1)$ where $Z_0, Z_1 \subseteq [0, 1]$ let

$$\Pr[s_1] = \lambda(\omega_0 \times Z_0) \frac{1}{2} + \lambda(\omega_1 \times Z_1) \frac{1}{2}$$

be the ex ante probability for the state/sunspot draw to be in s_1 . Summing over all signals we thus have an upper bound of the expected payoff for sender 1 given by

$$\sum_{s_1 \in S_1^*} \Pr(s_1) \frac{77}{80} + \left[1 - \sum_{s_1 \in S_1^*} \Pr[s_1] \right] \left[\frac{99}{100} - \varepsilon \right] < \frac{99}{100} - \varepsilon$$

provided that $\varepsilon < \frac{99}{100} - \frac{77}{80}$. We conclude that no profitable deviation from the trivial partition exists for sender 1. $\square$

4 Concluding Remarks

Whether considering a medical procedure or a potential auto repair, it is typically assumed that asking for a second opinion will always improve the decision maker's information. This paper demonstrates that this is not always correct in a world where the experts have a stake in the decision. If the experts move simultaneously, signals can be arbitrarily coordinated, and mixed strategies are ruled out, a second

opinion cannot lead to a loss of information. If any of these assumptions is modified the decision maker may be worse off by seeking out a second opinion.

We believe that the sequential persuasion model and the case with independent signals deserve further study. This paper takes a small step towards a characterization of the sequential model by establishing that a decision maker can avoid a loss of information from adding an expert by asking the new expert first. However, a more serious study of the design aspects with multiple experts is beyond the scope of this paper. In the case with independent signals it could be interesting to add more structure to the payoffs and investigate whether insights diverge from the cheap talk literature.

A Appendix: Proofs

A.1 Proof of Proposition 1

Proof. Suppose that the sender plays the signal π^{**} . The posterior probabilities relevant for the decision makers actions are

$$\begin{aligned}\mu(s_0) &= \frac{p(s_0|\omega_0)^{\frac{1}{2}}}{p(s_0|\omega_0)^{\frac{1}{2}} + p(s_0|\omega_1)^{\frac{1}{2}}} = \frac{99}{100} \\ \mu(s_1) &= \frac{p(s_1|\omega_1)^{\frac{1}{2}}}{p(s_1|\omega_0)^{\frac{1}{2}} + p(s_1|\omega_1)^{\frac{1}{2}}} = \frac{1}{100}.\end{aligned}$$

It follows that the optimal response is a_0 if s_0 is realized and a_1 if s_1 is realized. Thus, the sender gets expected utility $\frac{99}{100}$ in each state. The only way the sender could be better off would be if the probability that the decision maker would match the state would increase but this is impossible as it would require $\mu(s) > \frac{99}{100}$ or $\mu(s) < \frac{1}{100}$ for some signal, which would induce the decision maker to pick a_3 , the least preferred action. It follows that $\pi^{**} = \{s_0, s_1\}$ together with σ^* is an equilibrium. As there are many other signals generating the same posterior distribution, there are multiple equilibria. However, they are outcome-equivalent because, to the sender, any signal generating a different distribution of posteriors is inferior. $\square$

A.2 Proof of Proposition 2

Proof. First, full revelation is an equilibrium as no unilateral deviation affects the decision maker's belief when each sender posts a signal that is fully revealing by itself. In the rest of the proof we will establish that if $\pi^* = (\pi_1^*, \pi_2^*)$ is an equilibrium, then there can be no pair of signal realizations $s_1 \in \pi_1^*, s_2 \in \pi_2^*$ such that $\mu(s_1 \cap s_2) \in (0, 1)$. There are three cases to consider: $\mu(s_1 \cap s_2) \in (0, \frac{1}{100} + k]$, $\mu(s_1 \cap s_2) \in (\frac{1}{100} + k, \frac{99}{100} - k)$ and $\mu(s_1 \cap s_2) \in [\frac{99}{100} - k, 1)$. We will begin with the middle range as this is the most instructive case:

CASE 1: Suppose there is an equilibrium $\pi^* = (\pi_1^*, \pi_2^*)$ such that $\mu(s_1 \cap s_2) \in (\frac{1}{100} + k, \frac{99}{100} - k)$ for some $s_1 \in \pi_1^*, s_2 \in \pi_2^*$. We will show that sender 1 has a profitable deviation π_1' by demonstrating that s_1 can be partitioned into two sets such that the posterior beliefs for the two new signal realizations are either $\frac{1}{100}$ or $\frac{99}{100}$ with probability t and $1 - t$ respectively. Hence, these two new signals give sender 1 the best possible outcome when they are played against sender 2 and nothing else changes. Hence, this is a profitable deviation. For the details, let

$$\begin{aligned}Z_0 &= \{x \in [0, 1] | (\omega_0, x) \in s_1 \cap s_2\} \\ Z_1 &= \{x \in [0, 1] | (\omega_1, x) \in s_1 \cap s_2\}.\end{aligned}$$

For $i = 0, 1$ we let $\{Z_i^L, Z_i^R\}$ be any partition of Z_i satisfying

$$\begin{aligned}\lambda(Z_0^L) &= y^0 \lambda(Z_0) \\ \lambda(Z_0^R) &= (1 - y^0) \lambda(Z_0) \\ \lambda(Z_1^L) &= y^1 \lambda(Z_1) \\ \lambda(Z_1^R) &= (1 - y^1) \lambda(Z_1),\end{aligned}$$

where $y^0, y^1 \in (0, 1)$ solve¹¹

$$\begin{aligned}\mu(s_1^L \cap s_2) &= \frac{\mu_0 y^0 \lambda(Z_0)}{\mu_0 y^0 \lambda(Z_0) + (1 - \mu_0) y^1 \lambda(Z_1)} = \frac{y^0 \lambda(Z_0)}{y^0 \lambda(Z_0) + y^1 \lambda(Z_1)} = \frac{1}{100} \\ \mu(s_1^R \cap s_2) &= \frac{(1 - y^0) \lambda(Z_0)}{(1 - y^0) \lambda(Z_0) + (1 - y^1) \lambda(Z_1)} = \frac{99}{100}.\end{aligned}\tag{A1}$$

Consider a deviation for sender 1, π'_1 , constructed by removing signal realization s_1 and replacing it with s_1^L, s_1^R and s_1^N given by

$$\begin{aligned}s_1^L &= (\omega_0, Z_0^L) \cup (\omega_1, Z_1^L) \\ s_1^R &= (\omega_0, Z_0^R) \cup (\omega_1, Z_1^R) \\ s_1^N &= s_1 \setminus (s_1 \cap s_2).\end{aligned}$$

Hence, the posterior beliefs are unchanged relative (π_1^*, π_2^*) for all realizations except $s_1 \cap s_2$. Instead of $s_1 \cap s_2$ the deviation contains two new realizations $s_1^L \cap s_2$ and $s_1^R \cap s_2$ and the expected payoff for sender 1 from playing π'_1 against π_2^* is

$$\begin{aligned}& \sum_{\omega \in \{\omega_0, \omega_1\}} \left[\sum_{(s'_1, s'_2) \in \pi'_1 \times \pi_2^*} u_1(\sigma^*(\mu(s'_1 \cap s'_2)), \omega) p(s'_1 \cap s'_2 | \omega) \right] \mu_0(\omega) \\ &= \sum_{\omega \in \{\omega_0, \omega_1\}} \left[\sum_{(s'_1, s'_2) \in \pi_1^* \times \pi_2^*} u_1(\sigma^*(\mu(s'_1 \cap s'_2)), \omega) p(s'_1 \cap s'_2 | \omega) \right] \mu_0(\omega) \\ &+ \sum_{\omega \in \{\omega_0, \omega_1\}} \left[\sum_{K=L,R} u_1(\sigma^*(\mu(s_1^K \cap s_2)), \omega) p(s_1^K \cap s_2 | \omega) \right] \mu_0(\omega) \\ &- \sum_{\omega \in \{\omega_0, \omega_1\}} u_1(\sigma^*(\mu(s_1 \cap s_2)), \omega) p(s_1 \cap s_2 | \omega) \mu_0(\omega),\end{aligned}\tag{A2}$$

¹¹The solution is $y^0 = \frac{99 - 100\mu(s_1 \cap s_2)}{9800\mu(s_1 \cap s_2)}$ and $y^1 = \frac{9801 - 9900\mu(s_1 \cap s_2)}{9800[1 - \mu(s_1 \cap s_2)]}$. Both expressions equal 0 when evaluated at $\mu(s_1 \cap s_2) = \frac{99}{100}$, sender 1 when evaluated at $\mu(s_1 \cap s_2) = \frac{1}{100}$, and are strictly in between 0 and 1 for $\mu(s_1 \cap s_2) \in (\frac{1}{100}, \frac{99}{100})$.

where

$$\begin{aligned}
& \sum_{\omega \in \{\omega_0, \omega_1\}} \left[\sum_{K=L,R} u_1(\sigma^*(\mu(s_1^K \cap s_2)), \omega) p(s_1^K \cap s_2 | \omega) \right] \mu_0(\omega) \tag{A3} \\
&= (1 - y^0) \lambda(Z_0) \frac{1}{2} + y^1 \lambda(Z_1) \frac{1}{2} \\
&= \frac{1}{2} \frac{99}{100} [\lambda(Z_0) + \lambda(Z_1)] \\
&> \left(\frac{99}{100} - \varepsilon \right) \left[\frac{1}{2} \lambda(Z_0) + \frac{1}{2} \lambda(Z_1) \right] \\
&\geq \sum_{\omega \in \{\omega_0, \omega_1\}} u_1(\sigma^*(\mu(s_1 \cap s_2)), \omega) p(s_1 \cap s_2 | \omega) \mu_0(\omega).
\end{aligned}$$

The first equality in (A3) uses the fact that the decision maker's response to (s_1^R, s_2) is a_0 which gives payoff 1 when the state is ω_0 which occurs with probability $(1 - y^0) \lambda(Z_0) \frac{1}{2}$. Symmetrically, the decision maker's response to (s_1^L, s_2) is a_1 which gives payoff 1 when the state is ω_1 which occurs with probability $y^1 \lambda(Z_1) \frac{1}{2}$. The second equality comes from (A1) and the final inequality holds because $\mu(s_1 \cap s_2) \in (\frac{1}{100} + k, \frac{99}{100} - k)$. Combining (A2) with (A3), we see that the expected payoff from playing π'_1 is strictly greater than playing π_1^* .

CASE 2: If $\mu(s_1 \cap s_2) \in (0, \frac{1}{100} + k]$, sender 2 has a profitable deviation π'_2 with s_2 partitioned into (s_2^L, s_2^R, s_2^N) given by

$$\begin{aligned}
s_1^L &= (\omega_1, Z_1^L) \\
s_1^R &= (\omega_0, Z_0) \cup (\omega_1, Z_1^R)
\end{aligned}$$

and $s_1^N = s_1 \setminus (s_1 \cap s_2)$ where

$$\mu(s_1^R \cap s_2) = \frac{\lambda(Z_0)}{\lambda(Z_0) + \lambda(Z_1^R)} = \frac{1}{3} - k,$$

and all other signals are unchanged. The calculations follow case 1 step by step and are left to readers.

CASE 3: Similarly, when $\mu(s_1 \cap s_2) \in [\frac{99}{100} - k, 1)$, sender 2 has a profitable deviation π'_2 with s_2 partitioned into (s_2^L, s_2^R, s_2^N) given by

$$\begin{aligned}
s_1^L &= (\omega_0, Z_0^L) \cup (\omega_1, Z_1) \\
s_1^R &= (\omega_0, Z_0^R)
\end{aligned}$$

and $s_1^N = s_1 \setminus (s_1 \cap s_2)$ where

$$\mu(s_1^L \cap s_2) = \frac{\lambda(Z_0^L)}{\lambda(Z_0^L) + \lambda(Z_1)} = \frac{3}{5} + k,$$

and all other signals are unchanged. The calculations follow case 1 step by step and are left to readers.

Combining the three cases we have that $\mu(s_1 \cap s_2) \in \{0, 1\}$ for any pair of potential signal realizations. $\square$

A.3 Proof of Proposition 3

Proof. For any π_1 let $r(\pi_1)$ be an optimal response to π_1 . Partition the signal realizations in π_1 into $S_1^A = \{s_1 \in \pi_1 \mid \sigma^*(\mu(s_1)) = a_2\}$ and $S_1^B = \{s_1 \in \pi_1 \mid \sigma^*(\mu(s_1)) \neq a_2\}$. Let

$$r(\pi_1) = \left(\{s_2(s_1)\}_{s_1 \in S_1^A}, \{s_2^L(s_1)\}_{s_1 \in S_1^B}, \{s_2^R(s_1)\}_{s_1 \in S_1^B} \right),$$

where $s_2(s_1) = s_1$ for each $s_1 \in S_1^A$. For $s_1 \in S_1^B$ there are three distinct cases. First, if $s_1 = \omega_0 \times Z_0 \cup \omega_1 \times Z_1 \in \pi_1$ is such that $\mu(s_1) \in (0, \frac{1}{3} - k)$ let $s_2^L(s_1) = (\omega_1, Z_1^L)$ and $s_2^R(s_1) = (\omega_0 \times Z_0) \cup (\omega_1 \times Z_1^R)$, where $\{Z_1^L, Z_1^R\}$ is a partitioning of Z_1 satisfying

$$\mu(s_1 \cap s_2^R(s_1)) = \frac{\lambda(Z_0)}{\lambda(Z_0) + \lambda(Z_1^R)} = \frac{1}{3} - k. \quad (\text{A4})$$

Second, for realizations such that $\mu(s_1) \in (\frac{1}{3} + k, \frac{3}{5} - k) \setminus [\frac{1}{2} - k, \frac{1}{2} + k]$ let $s_2^L(s_1) = (\omega_0 \times Z_0^R) \cup (\omega_1 \times Z_1^L)$ and $s_2^R(s_1) = (\omega_0 \times Z_0^R) \cup (\omega_1 \times Z_1^R)$, where $\{Z_0^L, Z_0^R\}, \{Z_1^L, Z_1^R\}$ are partitionings of Z_0 and Z_1 satisfying.

$$\begin{aligned} \mu(s_1 \cap s_2^L(s_1)) &= \frac{\lambda(Z_0^L)}{\lambda(Z_0^L) + \lambda(Z_1^L)} = \frac{1}{3} \\ \mu(s_1 \cap s_2^R(s_1)) &= \frac{\lambda(Z_0^R)}{\lambda(Z_0^R) + \lambda(Z_1^R)} = \frac{3}{5}. \end{aligned} \quad (\text{A5})$$

Third, for realizations $\mu(s_1) \in (\frac{3}{5} + k, 1)$ let $s_2^L(s_1) = (\omega_0 \times Z_0^R) \cup (\omega_1 \times Z_1)$ and $s_2^R(s_1) = (\omega_0, Z_0^R)$ where

$$\mu(s_1 \cap s_2^L(s_1)) = \frac{\lambda(Z_0^L)}{\lambda(Z_0^L) + \lambda(Z_1)} = \frac{3}{5} + k. \quad (\text{A6})$$

We will now argue that $r(\pi_1)$ is an optimal response to π_1 . To this we will first argue that $r(\pi_1)$ is a valid partition. Since $r(\pi_1)$ partitions signal realizations one by one, we do so by checking that each s_1 is partitioned in a valid way:

CASE 0: For each $s_1 \in S_1^A$ we have $s_2(s_1) = s_1$, which is obviously fine.

CASE 1: For s_1 such that $\mu(s_1) \in (0, \frac{1}{3} - k)$ we need to check that there exists $Z_1^R \subseteq Z_1$ such that (A4) is satisfied. But this is obvious as

$$\mu(s_1) = \frac{\lambda(Z_0)}{\lambda(Z_0) + \lambda(Z_1)} < \frac{1}{3} - k$$

and the posterior on the left-hand side in (A4) is strictly decreasing in $\lambda(Z_1^L)$. Hence, any posterior on $[\mu(s_1), 1]$ can be generated by subsets $Z_1^R \subseteq Z_1$.

CASE 2: $\{Z_0^L, Z_0^R\}, \{Z_1^L, Z_1^R\}$ are valid partitionings of Z_0 and Z_1 . Solving the two equations in (A5) with the additional equation

$$\frac{\lambda(Z_0)}{\lambda(Z_0) + \lambda(Z_1)} = \mu(s_1),$$

we get a unique solution

$$\begin{aligned}\lambda(Z_0^L) &= \frac{3-5\mu(s_1)}{4\mu(s_1)}\lambda(Z_0) \\ \lambda(Z_1^L) &= \frac{3-5\mu(s_1)}{2(1-\mu(s_1))}\lambda(Z_1) \\ \lambda(Z_0^R) &= \left[1 - \frac{3-5\mu(s_1)}{4\mu(s_1)}\right]\lambda(Z_0) \\ \lambda(Z_1^R) &= \left[1 - \frac{3-5\mu(s_1)}{2(1-\mu(s_1))}\right]\lambda(Z_1),\end{aligned}$$

where $\frac{3-5\mu(s_1)}{4\mu(s_1)} \in [0, 1]$ and $\frac{3-5\mu(s_1)}{2(1-\mu(s_1))} \in [0, 1]$ for every $\mu(s_1) \in [\frac{1}{3}, \frac{3}{5}]$. Hence, there are valid partitions of s_1 satisfying (A5) for every s_1 with $\mu(s_1) = (\frac{1}{3} + k, \frac{3}{5} - k) \setminus [\frac{1}{2} - k, \frac{1}{2} + k]$.

CASE 3: Finally, for the case with $\mu(s_1) \in (\frac{3}{5} + k, 1)$ it is obvious that we can find $Z_0^L \subseteq Z_0$ satisfying (A6) because the posterior is decreasing in $\lambda(Z_0^L)$.

Next, we need to check optimality. In CASE 1 and CASE 3, this is obvious as sender 2 gets a_2 , the most preferred action, for sure. For CASE 2, suppose that there is a better response against such a signal. It is without loss to consider an alternative partitioning of the given signal s_1 and leave everything else the same. Clearly $s_2(s_1) = s_1$ is worse than the candidate equilibrium partition as this gives sender 2 the most preferred outcome with probability 0. In addition, in order for sender 2 to improve on the candidate equilibrium outcome, she must be able to guarantee the most preferred outcome with a strictly higher probability. That is, there is a partition $\{s_2^k\}_{k=1}^K$ of s_1 such that

$$\sum_{k=1}^K I(s_2^k) \Pr[s_2^k] > \Pr(s_2^R(s_1)) = \frac{\lambda(Z_0) + \lambda(Z_1^R)}{2} = \frac{\lambda(Z_0)}{\frac{2}{3} - 2k} \quad (\text{A7})$$

where $I(s_2^k) = 1$ if $\sigma^*(\mu(s_1 \cap s_2^k)) = a_2$ and 0 otherwise. Write $s_2^k = (\omega_0 \times Z_0^k) \cup (\omega_1 \times Z_1^k)$ and order the signals in such a way that $\sigma^*(\mu(s_1 \cap s_2^k)) = a_2$ for $k \in \{1, \dots, K^*\}$. It immediately follows that

$$\mu(s_1 \cap s_2^k) = \frac{\lambda(Z_0^k)}{\lambda(Z_0^k) + \lambda(Z_1^k)} \geq \frac{1}{3} - k \quad (\text{A8})$$

for each $k \in \{1, \dots, K^*\}$. Hence, the left inequality holds as $\{Z_0^k\}_{k=1}^K$ is a partition of Z_0 and the right inequality comes from (A8). Hence

$$\sum_{k=1}^K I(s_2^k) \Pr[s_2^k] = \sum_{k=1}^{K^*} \Pr[s_2^k] = \sum_{k=1}^{K^*} \frac{\lambda(Z_0^k) + \lambda(Z_1^k)}{2} \leq \sum_{k=1}^{K^*} \frac{\lambda(Z_0^k)}{\frac{2}{3} - 2k} \leq \frac{\lambda(Z_0)}{\frac{2}{3} - 2k},$$

where the first inequality comes from (A8) and the second inequality holds because $\{Z_0^k\}_{k=1}^K$ is a partition of Z_0 . Hence, we have a contradiction. The optimality of the response for CASE 4 is proved by a symmetric argument. We conclude that we have shown that $r(\pi_1)$ is an optimal response given any finite signal π_1 .

We note that $r(\pi_1)$ is such that a_0 and a_1 are never played. Hence, π_1^* is a best response to $r(\pi_1)$ if and only if $\sigma^*(s_1 \cap s_2(s_1))$ for every signal $s_2(s_1) \in r(\pi_1^*)$, which completes the proof of the first part of the proposition. $\square$

A.4 Proof of Proposition 5

Proof. Sender 2 has no incentive to deviate as the equilibrium outcome is the most preferred action for sender 2. We thus only need to show that sender 1 has no incentive to deviate. As a_2 is the second best to sender 1, we only need to rule out any profitable deviations that generate decision maker beliefs on $[\frac{1}{100}, \frac{1}{100} + k] \cup [\frac{99}{100} - k, \frac{99}{100}]$ with positive probability. Hence, if there is a profitable deviation π_1 , there must be a signal $s_1 \in \pi_1$ and $s_2 \in \pi_2^* = \{s_{21}, s_{22}\}$ such that

$$\begin{aligned} p(\omega_0 | s_1, s_2) &= \frac{p(s_1 | \omega_0) p(s_2 | \omega_0)}{p(s_1 | \omega_0) p(s_2 | \omega_0) + p(s_1 | \omega_1) p(s_2 | \omega_1)} \\ &= \frac{1}{1 + \frac{p(s_1 | \omega_1) p(s_2 | \omega_1)}{p(s_1 | \omega_0) p(s_2 | \omega_0)}} \in [\frac{1}{100}, \frac{1}{100} + k], \end{aligned} \tag{A9}$$

for some $s_2 \in \{s_{21}, s_{22}\}$. Notice that

$$\frac{p(s_2 | \omega_1)}{p(s_2 | \omega_0)} = \begin{cases} 2 & \text{if } s_2 = s_{21} \\ \frac{2}{3} & \text{if } s_2 = s_{22}. \end{cases}$$

We need to consider two cases depending on whether s_2 is s_{21} or s_{22} :

CASE I. $p(\omega_0 | s_1, s_{21}) \in [\frac{1}{100}, \frac{1}{100} + k]$. Then Equation (A9) implies that

$$\frac{p(s_1 | \omega_1) p(s_{21} | \omega_1)}{p(s_1 | \omega_0) p(s_{21} | \omega_0)} = 2 \frac{p(s_1 | \omega_1)}{p(s_1 | \omega_0)} \in [\frac{99 - 100k}{1 + 100k}, 99],$$

implying that $\frac{p(s_1|\omega_1)}{p(s_1|\omega_0)} \in [\frac{99-100k}{2+200k}, \frac{99}{2}]$. Since $p(\omega_0|s_1, s_{21}) \in [\frac{1}{100}, \frac{1}{100} + k]$ the decision maker responds with $\sigma^*(p(\omega_0|s_1, s_{21})) = a_1$ and sender 1's payoff is $\frac{99}{100}$. However, the unconditional probability that sender 2 plays s_{22} is $\frac{1}{2}$ and

$$\frac{p(s_1|\omega_1)p(s_2|\omega_1)}{p(s_1|\omega_0)p(s_2|\omega_0)} = \frac{2}{3} \frac{p(s_1|\omega_1)}{p(s_1|\omega_0)}.$$

Simple algebra implies that the posterior

$$p(\omega_0|s_1, s_{22}) = \frac{1}{1 + \frac{2p(s_1|\omega_1)}{3p(s_1|\omega_0)}} \in [\frac{1}{1 + \frac{99}{3}}, \frac{1}{1 + \frac{99-100k}{3+300k}}].$$

Because $\lim_{k \rightarrow 0} p(\omega_0|s_1, s_{22}) = \frac{1}{34}$, when $k > 0$ is small enough, $\sigma^*[p(\omega_0|s_1, s_{22})] = a_3$, and sender 1's payoff is zero. Hence, sender 1's expected payoff when s_1 is realized is $\frac{99}{200}$, which is smaller than his equilibrium payoff $\frac{99}{100} - \varepsilon$ for small ε .

CASE II. $s_2 = s_{22}$. Similarly, Equation (A9) implies that $\frac{2}{3} \frac{p(s_1|\omega_1)}{p(s_1|\omega_0)} \in [\frac{99-100k}{1+100k}, 99]$, and the decision maker chooses a_1 and sender 1's payoff is $\frac{99}{100}$. However, with equal probability, $s_2 = s_{21}$, then the likelihood ratio is $2 \frac{p(s_1|\omega_1)}{p(s_1|\omega_0)} \in [\frac{297-300k}{1+100k}, 297]$, and the posterior is

$$p(\omega_0|s_1, s_{21}) = \frac{1}{1 + \frac{2p(s_1|\omega_1)}{p(s_1|\omega_0)}} \in [\frac{1}{298}, \frac{1}{1 + \frac{297-300k}{100k+1}}].$$

Because $\lim_{k \rightarrow 0} p(\omega_0|s_1, s_{21}) = \frac{1}{298} < \frac{1}{100}$, when $k > 0$ is small enough, $\sigma^*[p(\omega_0|s_1, s_{21})] = a_3$, and sender 1's payoff is zero. Hence, sender 1's expected payoff when s_1 is realized is $\frac{99}{200}$, which is smaller than his equilibrium payoff $\frac{99}{100} - \varepsilon$ for sufficiently small ε .

In sum, conditional on any s_1 such that $p(\omega_0|s_1, s_2) \in [\frac{1}{100}, \frac{1}{100} + k]$ for some $s_2 \in \{s_{21}, s_{22}\}$, the posterior must be such that a_3 is chosen for the other signal played by sender 2. The expected payoff to sender 1 conditional on s_1 is thus at most $\frac{99}{200} < \frac{99}{100} - \varepsilon$ for ε small enough. Similarly, one can show that there is no profitable deviation for sender 1 such that $p(\omega_0|s_1, s_2) \in [\frac{99}{100} - k, \frac{99}{100}]$ with positive probability. Hence, the strategy profile specified in the Proposition is an equilibrium. Recall that $\pi_2^* = (s_{21}, s_{22})$ is a garbling of the single-sender equilibrium signal $\pi_1 = (s_{10}, s_{11})$ if

$$\begin{aligned} p(s_{21}|\omega_0) &= g(s_{21}, s_{10})p(s_{10}|\omega_0) + g(s_{21}, s_{11})p(s_{11}|\omega_0) \\ p(s_{22}|\omega_0) &= (1 - g(s_{22}, s_{10}))p(s_{10}|\omega_0) + (1 - g(s_{22}, s_{11}))p(s_{11}|\omega_0) \end{aligned}$$

for $g(s_{21}, s_{10}) \in [0, 1]$, $g(s_{21}, s_{11}) \in [0, 1]$ with either $0 < g(s_{21}, s_{10}) < 1$ or $0 < g(s_{21}, s_{11}) < 1$. As $p(s_{21}|\omega_0) = \frac{1}{4}$, $p(s_{21}|\omega_1) = \frac{1}{2}$, $p(s_{10}|\omega_0) = \frac{99}{100}$, and $p(s_{11}|\omega_0) = \frac{1}{100}$ we have that $(g(s_{21}, s_{10}), g(s_{21}, s_{11}))$ must solve

$$\begin{aligned} \frac{1}{4} &= g(s_{21}, s_{10}) \frac{99}{100} + g(s_{21}, s_{11}) \frac{1}{100} \\ \frac{1}{2} &= (1 - g(s_{22}, s_{10})) \frac{99}{100} + (1 - g(s_{22}, s_{11})) \frac{1}{100}, \end{aligned}$$

which has solution $(g(s_{21}, s_{10}), g(s_{21}, s_{11})) = (\frac{97}{392}, \frac{197}{392})$. Hence, $\pi_2^* = (s_{21}, s_{22})$ is a garbling of $\pi_1 = (s_{10}, s_{11})$. $\square$

A.5 Proof of Lemma 2

The proof of Lemma 2 consists of many cases.

Lemma A1. *Suppose that there exists J, K such that $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{2}{100}]$ and $\mu(s_1 \cap s_{22}^K) \in [\frac{1}{100}, \frac{2}{100}]$. Then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$ provided that $k < \frac{2}{101} - \frac{1}{100}$.*

Proof. If $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{2}{100}]$ and $\mu(s_1 \cap s_{22}^K) \in [\frac{1}{100}, \frac{2}{100}]$, then

$$\begin{aligned} \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{1}{99} &\leq \lambda(I^J \cap Z_0) \leq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{2}{98} \\ \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{1}{99} &\leq \lambda(I^K \cap Z_0) \geq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{2}{98}. \end{aligned} \quad (\text{A10})$$

Hence, since I^J and I^K are disjoint and since $f(x, y) = \frac{x}{x+y}$ is increasing in x provided that both x and y are strictly positive

$$\begin{aligned} \mu(s_1 \cap s_{22}^{JK}) &= \frac{\lambda(I^J \cup I^K \cap Z_0)}{\lambda(I^J \cup I^K \cap Z_0) + \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right)} \\ &= \frac{\lambda(I^J \cap Z_0) + \lambda(I^K \cap Z_0)}{\lambda(I^J \cap Z_0) + \lambda(I^K \cap Z_0) + \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right)} \\ &\in \left[\frac{2\lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{1}{99}}{2\lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{1}{99} + \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right)}, \frac{2\lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{2}{98}}{2\lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{2}{98} + \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right)} \right] \\ &= \left[\frac{\frac{2}{99}}{\frac{2}{99} + 1}, \frac{\frac{4}{98}}{\frac{4}{98} + 1} \right] = \left[\frac{2}{101}, \frac{2}{51} \right], \end{aligned} \quad (\text{A11})$$

which proves the claim. $\square$

Lemma A2. *Suppose that there exists J, K such that $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{2}{100}]$ and $\mu(s_1 \cap s_{22}^J) \in [\frac{49}{149}, \frac{50}{149}]$ and that $k < \frac{\frac{1}{99} + \frac{49}{100}}{\frac{1}{99} + \frac{49}{100} + 1} - \frac{1}{3}$. Then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$.*

Proof. If $\mu(s_1 \cap s_{22}^J) \geq \frac{1}{100}$ and $\mu(s_1 \cap s_{22}^K) \geq \frac{49}{149}$ then $\lambda(I^J \cap Z_0) \geq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{1}{99}$ and $\lambda(I^K \cap Z_0) \geq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{49}{100}$, and

$$\mu(s_1 \cap s_{22}^{JK}) \geq \frac{\frac{1}{99} + \frac{49}{100}}{\frac{1}{99} + \frac{49}{100} + 1} > \frac{\frac{1}{100} + \frac{49}{100}}{\frac{1}{100} + \frac{49}{100} + 1} = \frac{1}{3}.$$

If $\mu(s_1 \cap s_{22}^J) \leq \frac{2}{100}$ and $\mu(s_1 \cap s_{22}^K) \leq \frac{50}{149}$ then $\lambda(I^J \cap Z_0) \leq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{2}{98}$ and $\lambda(I^K \cap Z_0) \leq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{50}{99}$, so that

$$\mu(s_1 \cap s_{22}^{JK}) \leq \frac{\frac{2}{98} + \frac{50}{99}}{\frac{2}{98} + \frac{50}{99} + 1} < \frac{52}{150} = \frac{26}{75},$$

so if $k < \frac{1}{3} - \frac{\frac{1}{99} + \frac{49}{100}}{\frac{1}{99} + \frac{49}{100} + 1}$ we have that $\frac{1}{3} + k < \mu(s_1 \cap s_{22}^{JK}) < \frac{1}{2} - k$, implying that $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$. $\square$

Lemma A3. Suppose that there exists J, K such that $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{2}{100}]$ and $\mu(s_1 \cap s_{22}^J) \in [\frac{199}{399}, \frac{201}{399}]$ and that $k < \frac{\frac{1}{99} + \frac{199}{200}}{\frac{1}{99} + \frac{199}{200} + 1} - \frac{1}{2}$. Then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$.

Proof. If $\mu(s_1 \cap s_{22}^J) \geq \frac{1}{100}$ and $\mu(s_1 \cap s_{22}^K) \geq \frac{199}{399}$ then $\lambda(I^J \cap Z_0) \geq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{1}{99}$ and $\lambda(I^K \cap Z_0) \geq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{199}{200}$. Hence, we have that

$$\mu(s_1 \cap s_{22}^{JK}) = \frac{\frac{1}{99} + \frac{199}{200}}{\frac{1}{99} + \frac{199}{200} + 1} > \frac{\frac{1}{100} + \frac{199}{200}}{\frac{1}{100} + \frac{199}{200} + 1} = \frac{201}{401}.$$

Moreover, if $\mu(s_1 \cap s_{22}^J) \leq \frac{2}{100}$ and $\mu(s_1 \cap s_{22}^K) \leq \frac{201}{399}$ then $\lambda(I^J \cap Z_0) \leq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{2}{98}$ and $\lambda(I^K \cap Z_0) \leq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{201}{198}$, so that

$$\mu(s_1 \cap s_{22}^{JK}) \leq \frac{\frac{2}{98} + \frac{201}{198}}{\frac{2}{98} + \frac{201}{198} + 1} < \frac{55}{100},$$

which implies that $\frac{1}{2} + k < \mu(s_1 \cap s_{22}^{JK}) < \frac{3}{5} - k$ under the condition that $k < \frac{\frac{1}{99} + \frac{199}{200}}{\frac{1}{99} + \frac{199}{200} + 1} - \frac{1}{2}$, implying that $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$. $\square$

Lemma A4. Suppose that there exists J, K such that $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{2}{100}]$ and $\mu(s_1 \cap s_{22}^J) \in [\frac{239}{399}, \frac{240}{399}]$. Then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$

Proof. If $\mu(s_1 \cap s_{22}^J) \geq \frac{1}{100}$ and $\mu(s_1 \cap s_{22}^K) \geq \frac{239}{399}$ then $\lambda(I^J \cap Z_0) \geq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{1}{99}$ and $\lambda(I^K \cap Z_0) \geq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{239}{160}$ then

$$\mu(s_1 \cap s_{22}^{JK}) = \frac{\frac{1}{99} + \frac{239}{160}}{\frac{1}{99} + \frac{239}{160} + 1} > \frac{\frac{1}{160} + \frac{239}{160}}{\frac{1}{160} + \frac{239}{160} + 1} = \frac{3}{5}.$$

If $\mu(s_1 \cap s_{22}^J) \leq \frac{2}{100}$ and $\mu(s_1 \cap s_{22}^K) \leq \frac{240}{399}$ then $\lambda(I^J \cap Z_0) \leq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{2}{98}$ and $\lambda(I^K \cap Z_0) \leq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{240}{159}$ and

$$\mu(s_1 \cap s_{22}^{JK}) \leq \frac{\frac{2}{98} + \frac{239}{159}}{\frac{2}{98} + \frac{239}{159} + 1} < \frac{9}{10},$$

implying that $\frac{3}{5} + k < \mu(s_1 \cap s_{22}^{JK}) < \frac{9}{10} - k$ under the condition that $k < \frac{\frac{1}{99} + \frac{239}{160}}{\frac{1}{99} + \frac{239}{160} + 1} - \frac{3}{5}$, implying that $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$. $\square$

Lemma A5. Suppose that there exists J, K such that $\mu(s_1 \cap s_{22}^J) \geq \frac{50}{51}$ and $\mu(s_1 \cap s_{22}^K) \geq \frac{50}{51}$. Then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$

Proof. If $\mu(s_1 \cap s_{22}^J) \geq \frac{50}{51}$ and $\mu(s_1 \cap s_{22}^K) \geq \frac{50}{51}$ then $\lambda(I^J \cap Z_0) \geq \lambda((\frac{1}{2}, 1] \cap Z_1) 50$ and $\lambda(I^K \cap Z_0) \geq \lambda((\frac{1}{2}, 1] \cap Z_1) 50$, and

$$\mu(s_1 \cap s_{22}^{JK}) \geq \frac{100}{100+1} = \frac{100}{101} > \frac{99}{100},$$

implying that $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$. $\square$

Lemma A6. Suppose that there exists J, K such that $\mu(s_1 \cap s_{22}^J) \geq \frac{395}{400}$ and $\mu(s_1 \cap s_{22}^K) \geq \frac{1}{4}$. Then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$

Proof. If $\mu(s_1 \cap s_{22}^J) \geq \frac{395}{400}$ and $\mu(s_1 \cap s_{22}^K) \geq \frac{1}{4}$ then $\lambda(I^J \cap Z_0) \geq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{395}{4}$ and $\lambda(I^K \cap Z_0) \geq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{1}{3}$, so that

$$\mu(s_1 \cap s_{22}^{JK}) \geq \frac{\frac{395}{4} + \frac{1}{3}}{\frac{395}{4} + \frac{1}{3} + 1} = \frac{1189}{1201} > \frac{99}{100}.$$

Lemma A7. Suppose that there exists J, K such that $\mu(s_1 \cap s_{22}^{JK}) \leq \frac{199}{10099}$ and $\mu(s_1 \cap s_{22}^J) \geq \frac{1}{100}$. Then $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$

Proof. If $\mu(s_1 \cap s_{22}^{JK}) \leq \frac{199}{10099}$ and $\mu(s_1 \cap s_{22}^J) \geq \frac{1}{100}$

$$\lambda(I^J \cup I^K \cap Z_0) = \lambda(I^J \cap Z_0) + \lambda(I^K \cap Z_0) \leq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{199}{9900} \quad (\text{A12})$$

$$\lambda(I^J \cap Z_0) \geq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{1}{99},$$

it follows that

$$\begin{aligned} \lambda(I^K \cap Z_0) &\leq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{199}{9900} - \lambda(I^J \cap Z_0) \leq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \left[\frac{199}{9900} - \frac{1}{99}\right] \\ &= \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{1}{100} \end{aligned} \quad (\text{A13})$$

Hence,

$$\mu(s_1 \cap s_{22}^K) = \frac{\lambda(I^K \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda((\frac{1}{2}, 1] \cap Z_1)} \leq \frac{\frac{1}{100}}{\frac{1}{100} + 1} = \frac{1}{101} < \frac{1}{100}, \quad (\text{A14})$$

implying that $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$. $\square$

Lemma A8. Suppose that there exists J, K such that $\mu(s_1 \cap s_{22}^{JK}) \leq \frac{99}{100}$ and $\mu(s_1 \cap s_{22}^J) \geq \frac{9899}{9999}$. Then $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$

Proof. If $\mu(s_1 \cap s_{22}^{JK}) \leq \frac{99}{100}$ and $\mu(s_1 \cap s_{22}^J) \geq \frac{9899}{9999}$ we have that $\lambda(I^J \cap Z_0) + \lambda(I^K \cap Z_0) \leq \lambda((\frac{1}{2}, 1] \cap Z_1) 99$ and $\mu(I^J \cap Z_0) \geq \lambda((\frac{1}{2}, 1] \cap Z_1) \frac{9899}{100}$ which implies that

$$\begin{aligned} \lambda(I^K \cap Z_0) &\leq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) 99 - \lambda(I^J \cap Z_0) \\ &\leq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \left[99 - \frac{9899}{100}\right] \\ &= \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) \frac{1}{100} \end{aligned}$$

Hence,

$$\mu(s_1 \cap s_{22}^K) = \frac{\lambda(I^K \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda((\frac{1}{2}, 1] \cap Z_1)} \leq \frac{\frac{1}{100}}{\frac{1}{100} + 1} = \frac{1}{101} < \frac{1}{100},$$

so that $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$. □

Lemma A9. Suppose that there exists J, K such that $\mu(s_1 \cap s_{22}^{JK}) \geq \frac{98}{99}$ and $\mu(s_1 \cap s_{22}^J) \leq \frac{3}{4}$. Then $\mu(s_1 \cap s_{22}^K) \geq \frac{95}{96}$

Proof. If $\mu(s_1 \cap s_{22}^{JK}) \geq \frac{98}{99}$ and $\mu(s_1 \cap s_{22}^J) \leq \frac{3}{4}$

$$\begin{aligned} \lambda(I^J \cup I^K \cap Z_0) &= \lambda(I^J \cap Z_0) + \lambda(I^K \cap Z_0) \geq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) 98 \\ \lambda(I^J \cap Z_0) &\leq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) 3, \end{aligned} \tag{A15}$$

it follows that

$$\lambda(I^K \cap Z_0) \geq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) 98 - \lambda(I^J \cap Z_0) \geq \lambda\left(\left(\frac{1}{2}, 1\right] \cap Z_1\right) 95$$

Hence,

$$\mu(s_1 \cap s_{22}^K) = \frac{\lambda(I^K \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda((\frac{1}{2}, 1] \cap Z_1)} \geq \frac{95}{96}. \tag{A16}$$

□

Lemma A10. There exists $k^* > 0$ such that if $k \leq k^*$ and $\mu(s_1 \cap s_{21}^J) \in [\frac{99}{100} - k, \frac{99}{100}]$, then there is at least one L such that $\sigma^*(\mu(s_1 \cap s_{21}^{JL})) = a_3$.

Proof. Without loss let $\lambda(I^K \cap Z_0) \leq \lambda(I^L \cap Z_0) \leq \lambda(I^M \cap Z_0)$

$$\begin{aligned}\mu(s_1 \cap s_{21}^J) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \geq \frac{98}{99} \\ \Rightarrow \lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) &\geq 98\lambda\left[Z_1 \cap \left[0, \frac{1}{2}\right]\right],\end{aligned}$$

which implies that

$$\begin{aligned}\mu(s_1 \cap s_{21}^{JK}) &\geq \mu(s_1 \cap s_{21}^{JL}) \\ &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \\ &\geq \frac{\lambda(I^M \cap Z_0)}{\lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \\ &\geq \frac{\frac{1}{3}[\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0)]}{\frac{1}{3}[\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0)] + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \\ &\geq \frac{98}{101}\end{aligned}$$

Hence, either $\sigma^*(\mu(s_1 \cap s_{21}^{JL})) = a_3$ or

$$\begin{aligned}\mu(s_1 \cap s_{21}^{JK}) &= \frac{\lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \in \left[\frac{99}{100} - k, \frac{99}{100}\right] \\ \mu(s_1 \cap s_{21}^{JL}) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \in \left[\frac{99}{100} - k, \frac{99}{100}\right]\end{aligned}$$

In addition a necessary condition for $\sigma^*(\mu(s_1 \cap s_{21}^{JM})) \neq a_3$ is that

$$\begin{aligned}\mu(s_1 \cap s_{21}^{JM}) &= \frac{\lambda(I^L \cap Z_0) + \lambda(I^K \cap Z_0)}{\lambda(I^L \cap Z_0) + \lambda(I^K \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \geq \frac{1}{100} \\ \Rightarrow \lambda(I^L \cap Z_0) + \lambda(I^K \cap Z_0) &\geq \frac{1}{99}\lambda\left[Z_1 \cap \left[0, \frac{1}{2}\right]\right],\end{aligned}$$

which implies that

$$\lambda(I^L \cap Z_0) \geq \frac{1}{198}\lambda\left[Z_1 \cap \left[0, \frac{1}{2}\right]\right].$$

Together

$$\begin{aligned}
\mu(s_1 \cap s_{21}^{JL}) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \geq \frac{99}{100} - k \\
&= \frac{\lambda(I^K \cap Z_0) + \frac{1}{198}\lambda[Z_1 \cap [0, \frac{1}{2}]] + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \frac{1}{198}\lambda[Z_1 \cap [0, \frac{1}{2}]] + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \\
&\leq \frac{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \\
&= \mu(s_1 \cap s_{21}^J) \leq \frac{99}{100}.
\end{aligned}$$

For k small enough the inequalities

$$\begin{aligned}
\frac{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} &\geq \frac{99}{100} - k \\
\frac{\lambda(I^K \cap Z_0) + \frac{1}{198}\lambda[Z_1 \cap [0, \frac{1}{2}]] + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \frac{1}{198}\lambda[Z_1 \cap [0, \frac{1}{2}]] + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} &\leq \frac{99}{100}
\end{aligned}$$

cannot hold at the same time, which completes the proof. $\square$

Lemma A11. Suppose that there exists J such that $\mu(s_1 \cap s_{21}^J) \leq \frac{3}{201}$. Then there exists $K \neq J$ such that $\mu(s_1 \cap s_{21}^{JK}) < \frac{1}{100}$.

Proof. Without loss of generality relabel the intervals so that $\lambda(I^A \cap Z_0) \leq \lambda(I^B \cap Z_0) \leq \lambda(I^C \cap Z_0) \leq \lambda(I^D \cap Z_0)$ and note that

$$\mu(s_1 \cap s_{21}^D) = \frac{\lambda(Z_0 \cap I^A) + \lambda(Z_0 \cap I^B) + \lambda(Z_0 \cap I^C)}{\lambda(Z_0 \cap I^A) + \lambda(Z_0 \cap I^B) + \lambda(Z_0 \cap I^C) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \leq \mu(s_1 \cap s_{21}^D)$$

for $J \in \{A, B, C\}$. Hence, $\mu(s_1 \cap s_{21}^D) \leq \frac{3}{202}$ implying that

$$\lambda(Z_0 \cap I^A) + \lambda(Z_0 \cap I^B) + \lambda(Z_0 \cap I^C) \leq \frac{3}{199}\lambda[Z_1 \cap [0, \frac{1}{2}]]$$

We also have that

$$\lambda(Z_0 \cap I^A) + \lambda(Z_0 \cap I^B) \leq \frac{2}{3}[\lambda(Z_0 \cap I^A) + \lambda(Z_0 \cap I^B) + \lambda(Z_0 \cap I^C)],$$

which implies that

$$\begin{aligned}
\mu(s_1 \cap s_{21}^{CD}) &= \frac{\lambda(Z_0 \cap [0, 1] \setminus (I^C \cup I^D))}{\lambda(Z_0 \cap [0, 1] \setminus (I^C \cup I^D)) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \\
&= \frac{\lambda(Z_0 \cap I^A) + \lambda(Z_0 \cap I^B)}{\lambda(Z_0 \cap I^A) + \lambda(Z_0 \cap I^B) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \leq \frac{\frac{2}{3}\frac{3}{199}}{\frac{2}{3}\frac{3}{199} + 1} = \frac{2}{201} < \frac{1}{100}.
\end{aligned}$$

$\square$

Lemma A12. *There exists $k^* > 0$ such that:*

1. *if $k \leq k^*$ and there exists J, K such that $\mu(s_1 \cap s_{21}^{JK}) \in [\frac{1}{100}, \frac{1}{100} + k]$ and $\mu(s_1 \cap s_{21}^J) \in [\frac{1}{3} - k, \frac{1}{3} + k]$, then there exists some L such that $\sigma^*(\mu(s_1 \cap s_{21}^{JL})) = a_3$;*
2. *if $k \leq k^*$ and there exists J, K such that $\mu(s_1 \cap s_{21}^{JK}) \in [\frac{1}{100}, \frac{1}{100} + k]$ and $\mu(s_1 \cap s_{21}^J) \in [\frac{1}{2} - k, \frac{1}{2} + k]$, then there exists some L such that $\sigma^*(\mu(s_1 \cap s_{21}^{JL})) = a_3$;*
3. *if $k \leq k^*$ and there exists J, K such that $\mu(s_1 \cap s_{21}^{JK}) \in [\frac{1}{100}, \frac{1}{100} + k]$ and $\mu(s_1 \cap s_{21}^J) \in [\frac{3}{5} - k, \frac{3}{5} + k]$, then there exists some L such that $\sigma^*(\mu(s_1 \cap s_{21}^{JL})) = a_3$;*
4. *if $k \leq k^*$ and there exists J, K such that $\mu(s_1 \cap s_{21}^{JK}) \in [\frac{1}{100}, \frac{1}{100} + k]$ and $\mu(s_1 \cap s_{21}^J) \in [\frac{99}{100} - k, \frac{99}{100}]$, then there exists some L such that $\sigma^*(\mu(s_1 \cap s_{21}^{JL})) = a_3$;*

Proof. Assume that there exists J, K such that $\mu(s_1 \cap s_{21}^{JK}) \in [\frac{1}{100}, \frac{1}{100} + k]$ and $\mu(s_1 \cap s_{21}^J) \in [\frac{1}{3} - k, \frac{1}{3} + 3]$ and that $k \leq \frac{1}{99} - \frac{1}{100}$ so that

$$\begin{aligned} \mu(s_1 \cap s_{21}^{JK}) &= \frac{\lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \leq \frac{1}{99} \\ \Rightarrow \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) &\leq \frac{1}{98} \lambda \left[Z_1 \cap \left[0, \frac{1}{2} \right] \right] \end{aligned}$$

and also let $k \leq \frac{1}{3} - \frac{48}{147}$ so that

$$\begin{aligned} \mu(s_1 \cap s_{21}^J) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \geq \frac{48}{147} \\ \Rightarrow \lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) &\geq \frac{48}{99} \lambda \left[Z_1 \cap \left[0, \frac{1}{2} \right] \right] \end{aligned}$$

Combined, these inequalities imply that

$$\begin{aligned} \lambda(I^K \cap Z_0) &\geq \frac{48}{99} \lambda \left[Z_1 \cap \left[0, \frac{1}{2} \right] \right] - \frac{1}{98} \lambda \left[Z_1 \cap \left[0, \frac{1}{2} \right] \right] \\ &> \frac{47}{99} \lambda \left[Z_1 \cap \left[0, \frac{1}{2} \right] \right]. \end{aligned}$$

We conclude that

$$\begin{aligned} \mu(s_1 \cap s_{21}^{JM}) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \in \left[\frac{47}{147}, \frac{\frac{49}{99} + \frac{1}{98}}{\frac{49}{99} + \frac{1}{98} + 1} \right] \\ \mu(s_1 \cap s_{21}^{JL}) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \in \left[\frac{47}{147}, \frac{\frac{49}{99} + \frac{1}{98}}{\frac{49}{99} + \frac{1}{98} + 1} \right] \end{aligned}$$

Hence, in order for $\sigma^*(\mu(s_1 \cap s_{21}^J)) \neq a_3$, $\sigma^*(\mu(s_1 \cap s_{22}^{JM})) \neq a_3$ and $\sigma^*(\mu(s_1 \cap s_{22}^{JL})) \neq a_3$ it must be that

$$\begin{aligned}\mu(s_1 \cap s_{21}^J) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \in \left[\frac{1}{3} - k, \frac{1}{3} + k\right] \\ \mu(s_1 \cap s_{21}^{JM}) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \in \left[\frac{1}{3} - k, \frac{1}{3} + k\right] \\ \mu(s_1 \cap s_{21}^{JL}) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \in \left[\frac{1}{3} - k, \frac{1}{3} + k\right]\end{aligned}$$

In addition

$$\begin{aligned}\mu(s_1 \cap s_{21}^{JK}) &= \frac{\lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \geq \frac{1}{100} \\ \Rightarrow \lambda(I^L \cap Z_0) + \lambda(I^M \cap Z_0) &\geq \frac{1}{99} \lambda[Z_1 \cap [0, \frac{1}{2}]],\end{aligned}$$

Hence, we may label the intervals so that $\lambda(I^L \cap Z_0) \geq \frac{1}{198} \lambda[Z_1 \cap [0, \frac{1}{2}]]$ implying that

$$\begin{aligned}\frac{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \frac{1}{198} \lambda[Z_1 \cap [0, \frac{1}{2}]]}{\lambda(I^K \cap Z_0) + \lambda(I^L \cap Z_0) + \frac{1}{198} \lambda[Z_1 \cap [0, \frac{1}{2}]] + \lambda[Z_1 \cap [0, \frac{1}{2}]]} &\leq \mu(s_1 \cap s_{21}^J) \leq \frac{1}{3} + k \\ \mu(s_1 \cap s_{21}^{JL}) &= \frac{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0)}{\lambda(I^K \cap Z_0) + \lambda(I^M \cap Z_0) + \lambda[Z_1 \cap [0, \frac{1}{2}]]} \geq \frac{1}{3} - k.\end{aligned}$$

For k is small enough these two inequalities cannot be simultaneously satisfied, which proves that there exists some $k_1^* > 0$ such that part 1 holds if $k \leq k_1^*$. Parts 2,3,and 4 are proved by arguments that follow part 1 step by step and these argument generate bounds $k_2^* > 0$, $k_3^* > 0$ and $k_4^* > 0$. The proof is then completed by picking $k^* = \min\{k_1^*, k_2^*, k_3^*, k_4^*\}$. $\square$

We can now prove Lemma 2:

Proof. For contradiction, suppose that there is some s_1 such that $\sigma^*(\mu(s_1 \cap s_2)) = a_0$ or $\sigma^*(\mu(s_1 \cap s_2)) = a_1$ for some $s_2 \in S_2$ but no exists $s'_2 \in S_2$ such that $\sigma^*(\mu(s_1 \cap s'_2)) = a_3$.

A.5.1 Signals $(s_{22}^A, s_{22}^B, s_{22}^C, s_{22}^D)$:

Suppose that $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{1}{100} + k]$ and that k is smaller than the bounds in each Lemma. Lemma A1 establishes that if $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{2}{100}]$ and $\mu(s_1 \cap s_{22}^K) \in [\frac{1}{100}, \frac{2}{100}]$, then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$. Lemma A2 establishes that if $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{2}{100}]$ and $\mu(s_1 \cap s_{22}^L) \in [\frac{49}{149}, \frac{50}{149}]$ then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$. Lemma A3 establishes that if $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{2}{100}]$ and $\mu(s_1 \cap s_{22}^J) \in [\frac{199}{399}, \frac{201}{399}]$ then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) =$

a_3 . Lemma A4 establishes that if $\mu(s_1 \cap s_{22}^J) \in [\frac{1}{100}, \frac{2}{100}]$ and $\mu(s_1 \cap s_{22}^J) \in [\frac{239}{399}, \frac{240}{399}]$ then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$. For k small enough $[\frac{1}{100}, \frac{1}{100} + k] \subseteq [\frac{1}{100}, \frac{2}{100}]$, $[\frac{1}{3} - k, \frac{1}{3} + k] \subseteq [\frac{49}{149}, \frac{50}{149}]$, $[\frac{1}{2} - k, \frac{1}{2} + k] \subseteq [\frac{199}{399}, \frac{201}{399}]$ and $[\frac{3}{5} - k, \frac{3}{5} + k] \subseteq [\frac{239}{399}, \frac{240}{399}]$. We thus conclude:

Case 1. If $\sigma^*(\mu(s_1 \cap s_{22}^J)) = a_1$ and $\sigma^*(\mu(s_1 \cap s_{22}^K)) \in \{a_1, a_2\}$ then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$.

Next, Lemma A5 shows that if $\mu(s_1 \cap s_{22}^J) \geq \frac{50}{51}$ and $\mu(s_1 \cap s_{22}^K) \geq \frac{50}{51}$ then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$ and Lemma A6 that $\mu(s_1 \cap s_{22}^J) \geq \frac{395}{400}$ and $\mu(s_1 \cap s_{22}^K) \geq \frac{1}{4}$ then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$. For $k < \frac{99}{100} - \frac{395}{400}$ this implies that:

Case 2. If $\sigma^*(\mu(s_1 \cap s_{22}^J)) = a_0$ and $\sigma^*(\mu(s_1 \cap s_{22}^K)) \in \{a_0, a_2\}$ then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$.

Now, suppose that $\sigma^*(\mu(s_1 \cap s_{22}^J)) = a_0$ and that there exists no $K \neq J$ such that $\mu(s_1 \cap s_{22}^K) \geq \frac{1}{4}$. Then, either there is some K such that $\mu(s_1 \cap s_{22}^K) \in (\frac{1}{100} + k, \frac{1}{4})$ in which case $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$ or $\mu(s_1 \cap s_{22}^K) \in [\frac{1}{100}, \frac{1}{100} + k]$ for each $K \neq J$. But then, Lemma A1 implies $\sigma^*(\mu(s_1 \cap s_{22}^{KL})) = a_3$ for every $K, L \neq J$. Combining these cases we have now proved that if there exists some J such that $\sigma^*(\mu(s_1 \cap s_{22}^J)) \in \{a_0, a_1\}$, then there exist some K such that $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$ or $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$.

A.5.2 Signals $(s_{22}^{AB}, s_{22}^{AC}, s_{22}^{AD}, s_{22}^{BC}, s_{22}^{BD}, s_{22}^{CD})$

Next, from Lemma A7 we have that if $\mu(s_1 \cap s_{22}^{JK}) \leq \frac{199}{10099}$ and $\mu(s_1 \cap s_{22}^J) \geq \frac{1}{100}$ then $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_3$. Hence (provided that $k < \frac{199}{10099} - \frac{1}{100}$) it follows that if $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_1$ and $\sigma^*(\mu(s_1 \cap s_{22}^J)) \neq a_3$, then $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$. Similarly, from Lemma A8, if $\mu(s_1 \cap s_{22}^{JK}) \leq \frac{99}{100}$ and $\mu(s_1 \cap s_{22}^J) \geq \frac{9899}{9999}$ Then $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$, implying that $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_0$ and $\sigma^*(\mu(s_1 \cap s_{22}^J)) = a_0$, then $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$ under the assumption that $k < \frac{99}{100} - \frac{9899}{9999}$. Finally, Lemma A9 states that if $\mu(s_1 \cap s_{22}^{JK}) \geq \frac{98}{99}$ and $\mu(s_1 \cap s_{22}^J) \leq \frac{3}{4}$ then $\mu(s_1 \cap s_{22}^K) \geq \frac{95}{96}$ which (together with Lemma A8) implies that if $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) = a_0$ and $\sigma^*(\mu(s_1 \cap s_{22}^J)) \neq a_3$, then $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$ (again under the condition that $k < \frac{99}{100} - \frac{9899}{9999}$). Combined we have that if $\sigma^*(\mu(s_1 \cap s_{22}^{JK})) \in \{a_0, a_1\}$ for some JK then $\sigma^*(\mu(s_1 \cap s_{22}^K)) = a_3$ for some $K \in \{A, B, C, D\}$.

A.5.3 Signals $(s_{21}^A, s_{21}^B, s_{21}^C, s_{21}^D)$

Lemma A10 establishes that for k small enough, if $\mu(s_1 \cap s_{21}^J) \in [\frac{99}{100} - k, \frac{99}{100}]$, then there is at least one L such that $\sigma^*(\mu(s_1 \cap s_{21}^{JL})) = a_3$. Hence, if $\sigma^*(\mu(s_1 \cap s_{21}^J)) = a_0$ it follows that $\sigma^*(\mu(s_1 \cap s_{21}^{JL})) = a_3$ for some L . Lemma A11 show that if $\mu(s_1 \cap s_{21}^J) \leq \frac{3}{201}$ then there exists $K \neq J$ such that $\mu(s_1 \cap s_{21}^{JK}) < \frac{1}{100}$, which implies that if $\sigma^*(\mu(s_1 \cap s_{21}^J)) = a_0$ then $\sigma^*(\mu(s_1 \cap s_{21}^{JK})) = a_3$ for some K . Hence, if $\sigma^*(\mu(s_1 \cap s_{21}^J)) \in \{a_0, a_1\}$ there is at least one JK such that $\sigma^*(\mu(s_1 \cap s_{21}^{JK})) = a_3$.

A.5.4 Signals $(s_{21}^{AB}, s_{21}^{AC}, s_{21}^{AD}, s_{21}^{BC}, s_{21}^{BD}, s_{21}^{CD})$

Combining the cases in Lemma A12 we have that if $\mu(s_1 \cap s_{21}^{JK}) \in [\frac{1}{100}, \frac{1}{100} + k]$ and $\sigma^*(\mu s_1 \cap s_{21}^J) \neq a_3$, then $L \neq J, K$ such that $\sigma^*(\mu(s_1 \cap s_{21}^{LL})) = a_3$. If $\mu(s_1 \cap s_{21}^{JK}) \in [\frac{99}{100} - k, \frac{99}{100}]$ then, since $\mu(s_1 \cap s_{21}^J) \geq \mu(s_1 \cap s_{21}^{JK})$ either $\mu(s_1 \cap s_{21}^J) > \frac{99}{100}$ in which case $\sigma^*(\mu(s_1 \cap s_{21}^J)) = a_3$. or $\mu(s_1 \cap s_{21}^J) \in [\frac{99}{100} - k, \frac{99}{100}]$ in which case we know from above that there exists L such that $\sigma^*(\mu(s_1 \cap s_{21}^{LL})) = a_3$.

□

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