Common Belief in Maximin-Rationality[1]

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Keywords: Maximin expected utility, solution concepts, bounded rationality, epistemic game theory, decision theory, reasoning.

Journal of Economic Literature Classification: C72

1 Introduction

Classically game theoretic solution concepts modeled players as subjective expected utility (SEU) players in the sense of Savage (1954) or Anscombe and Aumann (1963) (AA). Take for example the well-known concepts Nash equilibrium (Nash, 1951) and rationalizability (Bernheim, 1984; Pearce, 1984). The framework of SEU assume that the agents' preferences satisfy certain standard axioms, which makes it possible to represent them by a utility function and a single subjective probability distribution over the states of the world. Ellsberg's (1961) famous paradox, however, suggests that especially the independence axiom used in AA might be too strong since it rules out hedging, a quite important phenomenon in economics.

Hence, Schmeidler (1989) and Gilboa and Schmeidler (1989) suggested frameworks with weaker versions of the independence axiom. These lead in the case of Schmeidler (1989) to Choquet expected utility and in Gilboa and Schmeidler (1989) to maximin expected utility (MMEU). MMEU is a very interesting framework, since it allows for multiple subjective beliefs instead of just a single one. To obtain MMEU, Gilboa and Schmeidler (1989) add another axiom called uncertainty aversion. Together with their weakened independence axiom and the classical axioms of AA, agents act as if they minimize their SEU over convex and compact sets of beliefs. Thus, MMEU allows us to model agents that have multiple beliefs instead of only one.

There are several reasons besides hedging why multiple beliefs might be of interest. For example, where should one single belief come from? Generally, people have great difficulties to give an exact percentage of how likely a certain event is. Naming an interval on the other hand seems much more natural. Consider forecasting for example. Since point estimates are neither very informative nor very likely to occur, forecasts are usually stated in confidence intervals, i.e. there is a 95 percent chance that a certain outcome will be in a given interval.

Given the fact that MMEU can account for the phenomenon of hedging and that considering multiple beliefs of agents seems to be natural, we suggest a game theoretic solution concept that describes how MMEU players would act in a strategic setting. Some works such as for example Eichberger and Kelsey (2000) have already analyzed equilibrium based solution concepts with MMEU players. Here, however, we are modeling games as Bayesian decision problems in the tradition of Tan and Werlang (1988) using an epistemic model.

We introduce an epistemic model that models hierarchies of ambiguous belief based on Ahn (2007). Using the epistemic model, we define maximin-rationality and common belief in maximin-rationality (CBMMR) to formulate a model that generalizes common belief in rationality (CBR) (Tan and Werlang, 1988). Players who believe in CBR can simply be accommodated by only allowing them to have belief hierarchies with single beliefs. In this case the requirements of CBMMR reduce exactly to those of CBR. It is also possible that a CBMMR player believes that his opponent only has a belief hierarchy consisting of single beliefs. Hence the player has a belief about his opponent that is as if he believes that his opponent only believes in CBR. Furthermore, there are strong similarities between the to concept of CBR and CBMMR.

First of all, it is important to note that randomized choices play a special role in the maximin setting because contrary to CBR, randomized choices can be strictly better than pure choices under CBMMR.[3] Our first main result shows that maximin-rational randomized choices are exactly those randomized choices that are not strictly dominated by another randomized choice. Interestingly, this characterization is very close

[1] I am grateful to Andrés Perea and Elias Tsakas for their valuable feedback.

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[3] See the discussion Example 1 in the next section.

to CBR, where rational choices are not strictly dominated by a randomized choice. But not only is their characterization very similar, we also find that all choices in the support of a maximin-rational randomized choice are not strictly dominated by a randomized choice. Therefore, they are optimal under CBR, which implies that CBR and common belief in maximin-rationality are behaviorally equivalent. In any one-shot setting one cannot tell an CBR and a CBMMR player apart due to their behavior. Note also that this is true for any non-strategic decision problem. So whenever we only observe a single choice of a player, we cannot tell if he is an CBR or an CBMMR type. This is not only of interest for theorists but also for behavioral economists who aim at identifying the two types in the lab.

This also relates to the fact that for any choice that is maximin-rational for a set of beliefs we can find a single belief in this set for which the choice is rational. The single belief for which a player's maximin-rational choice is optimal can be found by assuming that the player believes that his opponents jointly try to minimize his maximum SEU.

Finally, we show that the algorithm of iterated elimination of strictly dominated randomized choices yields exactly the choices that a player can make under CBMMR.[4]

In the following, we will first introduce the epistemic model and define common belief in maximin-rationality. With these two ingredients we can characterize the randomized choices that players can make under common belief in maximin-rationality, which then gives us the means to introduce an algorithm to find these randomized choices. The proofs supporting these results can be found in the appendix of this paper. Finally we discuss an example.

2 Epistemic Model

Before introducing the formal model we want to discuss some of the main differences between MMEU players and SEU players. In Example 1 the two choices b and c are rational for a SEU maximizing *Row Player*. Choice b yields the highest SEU for any belief $p(d) \in [\frac{1}{2}, 1]$, and c for any belief $p(d) \in [0, \frac{1}{2}]$ about the *Column Player*'s choice d, where by $p(d)$ we mean the *Row Player*'s belief about the *Column Player*'s choice d. Hence, there is no belief for which choice a yields the highest utility, and therefore choice a can never be an optimal choice for a SEU player.

Column Player

Row Player		$\boldsymbol{d}$	$\boldsymbol{e}$
	$\boldsymbol{a}$	1	1
	$\boldsymbol{b}$	3	0
	$\boldsymbol{c}$	0	3
	$(\frac{1}{2}\,\boldsymbol{b} + \frac{1}{2}\,\boldsymbol{c})$	1.5	1.5

Example 1: *Row Player's utilities*

On the other hand, if a MMEU player deems the whole set of beliefs $\Delta(\{d, e\})$ possible, then a is rational[5] because the minimum utility from choosing a is 1 whereas the minimum from choosing either b or c is 0. Hence, the uncertainty that a MMEU player faces gives rise to different rational choices than for a SEU player.

Remember that for a SEU player, we do not need to talk about randomized choices explicitly because each of the choices over which a player randomizes is optimal for some belief. Consider the randomized choice $(\frac{1}{2}b + \frac{1}{2}c)$. This choice yields a higher utility than choice a for any belief but also for any belief either choice b or choice c will yield a higher utility than choice a. However, this is not true for a MMEU player. The randomized choice $(\frac{1}{2}b + \frac{1}{2}c)$ yields a higher minimum utility than choices a, b and c if the *Row Player*

[4]We find the choices that players can rationally make under CBR by iterative elimination of strictly dominated choices.

[5]For the moment, we only consider pure choices.

deems the whole set of belief $\Delta(\{d,e\})$ possible. Consequently, randomized choices need special attention when modeling MMEU players.

To simplify notation throughout the paper, let X_i be any set for player i then $X := \times_{i\in I} X_i$ and $X_{-i} := \times_{j\neq i} X_i$. Our example showed that randomized choices play a special role in MMEU. Since pure choices are a special case of randomized choices, we will restrict our attention solely to randomized choice. Hence, we define a normal form game where we specifically define the set of randomized choices that are allowed. Let $\Gamma = (I,(C_i)_{i\in I},(R_i)_{i\in I},(U_i)_{i\in I})$, where I is the finite set of players, C_i is the finite choice set of player i, where $R_i \subseteq \Delta(C_i)$ is the set of randomized choices, and $U_i : \times_{j\in I} C_j \to \mathbb{R}$ denotes player i's payoff function. If we do not mention any restriction on R_i, we take $R_i = \Delta(C_i)$. Moreover, we assume that players' can use randomization devices that choose for them according to the probability distribution assigned by $r_i \in R_i$, and receive the following utility

$$U_i(r_i,r_{-i}) := \sum_{c_i\in C_i} r_i(c_i) \sum_{c_{-i}\in C_{-i}} r_{-i}(c_i)U_i(c_i,c_{-i}),$$

where $r_{-i}(c_{-i}) = \prod_{j\neq i} r_j(c_j)$.

On top of the normal form game we define a epistemic model that describes various epistemic mental states of players. Ahn (2007) shows that hierarchies of sets of beliefs can be encoded by a type in such a model.

Let $\mathcal{K}$ indicate families of sets, then by $\mathcal{K}(\Delta(R_{-i}\times T_{-i}))$ we denote the family of compact and convex sets of probability measures on $R_{-i}\times T_{-i}$ with finite support[6], which are within the convex hull of finitely many points.

Definition 1 (Epistemic Model)**.** An epistemic model of a game Γ is a tuple $\mathcal{M}^\Gamma = ((T_i)_{i\in I},(B_i)_{i\in I})$, where

- T_i is a set of types for player $i\in I$, and
- $B_i : T_i \to \mathcal{K}(\Delta(\times_{j\neq i} R_j \times T_j))$ assigns to every type $t_i \in T_i$ a convex and compact set of probability measures with finite support on the set of opponents' randomized-choice-type combinations.

The set $B_i(t_i)$ represents t_i's set of beliefs about the opponents' randomized-choice-type pairs. Type t_i's maximin expected utility from a randomized choice $r_i \in R_i$ is given by

$$u_i(r_i,t_i) := \min_{b_i\in B_i(t_i)} \left(\sum_{(r_{-i},t_{-i})\in supp(b_i)} b_i(r_{-i},t_{-i})U_i(r_i,r_{-i}) \right).$$

At some points we will discuss convex and compact sets of beliefs $B_i \in \mathcal{K}(\Delta(R_{-i}))$ independently of an epistemic model. In this case, the players' utility will be given by

$$u_i(r_i,B_i) := \min_{b_i\in B_i} \left(\sum_{(r_{-i},t_{-i})\in supp(b_i)} b_i(r_{-i},t_{-i})U_i(r_i,r_{-i}) \right).$$

If $|B_i| = 1$ with $B_i = \{b_i\}$ we write $u_i(r_i,b_i)$ to simplify notation. At all points in the paper the domain of the utility function will be clear from the context.

3 Common Belief in Maximin-Rationality

The epistemic model gives us the means to fully describe the players' mental states. To make this model meaningful, we need to define what it means for a player to be maximin-rational.

Definition 2 (Maximin-Rationality)**.** A randomized choice $r_i \in R_i$ is optimal for player i's type t_i if we have that

$$u_i(r_i,t_i) \geq u_i(r_i^{'},t_i)$$

holds for all $r_i^{'} \in R_i$. Furthermore, we call a randomized choice r_i maximin-optimal if there is some epistemic model $\mathcal{M}^\Gamma = ((T_i)_{i\in I},(b_i)_{i\in I})$ and $t_i \in T_i$ such that r_i is optimal for t_i.

[6]We assume finite support to avoid additional topological assumptions. Our results hold also for probability measures with infinite support.

Every type $t_i \in T_i$ for player i represents an ambiguous[7] belief hierarchy. For player i to be maximin-rational then means that a rational randomized choice needs to maximize MMEU given his type t_i, i.e. his ambiguous belief hierarchy. Sometimes we want to express what it means for a randomized choice $r_i \in R_i$ to be maximin-rational. In this case we require that there exists an epistemic model $\mathcal{M}^\Gamma$ including a $t_i \in T_i$ for which r_i is optimal.

To allow the players in our model to make inferences about each other, we need to require mutual belief in maximin-rationality. Hence, it is important to define what it means to believe in the opponents' maximin-rationality.

Definition 3 (Believe in the Opponents' Maximin-Rationality)**.** A type t_i of player i believes in the opponents' maximin-rationality if for all $b_i \in B_i(t_i)$ and all $j \neq i$ we have for every $(r_j, t_j) \in \text{supp}(\text{marg}_{R_j \times T_j} b_i)$ that the randomized choice r_j is optimal for t_j.

If player i's type t_i believes in the rationality of his opponent j, then type t_i will only consider beliefs where the opponent's randomized choices are optimal for the opponent's ambiguous belief hierarchy. Hence, in his set of beliefs about opponent j, expressing his uncertainty, player i only considers beliefs about his opponent where his opponent acts rationally. Thus, player i rules out that player j will act irrationally, i.e. that j will make a randomized choice that is not optimal. This already gives player i some information about how to respond to player j's behavior.

Further inferences about an opponent require, however, slightly more information. Therefore, we will require a player to not only believe in the opponents' rationality but also believe that the opponents' believe that he will act rationally. By using this additional information, we not only can rule out that the opponent will not make irrational choices but also that the opponent believes that his opponents' will make irrational choice. So we know that our opponent already restricts his attention to certain choices of his opponents and again we can use this fact to make more concise inferences about his behavior. The following definition formalizes this idea.

Definition 4 (K-Fold Belief in Maximin-Rationality)**.** Consider a player $i \in I$ and one of his types $t_i \in T_i$, then

- Type t_i expresses 1-fold belief in maximin-rationality, if t_i believes in the opponents' maximin-rationality.
- Type t_i expresses k-fold belief in maximin-rationality, if for all $b_i \in B_i(t_i)$, and for all $j \neq i$ every $t_j \in \text{supp}(\text{marg}_{T_j} b_i)$ expresses (k-1)-fold belief in maximin-rationality.
- Type t_i expresses common belief in maximin-rationality if t_i expresses k-fold belief in maximin-rationality for all $k \geq 1$.

The definition captures the idea that a type t_i who expresses k-fold belief in maximin-rationality, only considers beliefs about his opponents within which the opponents satisfy (k-1)-fold belief in maximin-rationality. However, types that express common belief in maximin-rationality only consider beliefs in which their opponents also express common belief in maximin-rationality. This will allow the players to make inferences about their opponents' behavior since they believe that they act rationally, that their opponents believe that they act rationally, that their opponents believe that they believe that their opponents act rationally and so on.

4 Characterization

We will characterize maximin-rational randomized choices in terms of strictly dominated randomized choices. Thus, we will use strictly dominated randomized choices in the conventional way.

Definition 5 (Strictly Dominated by Randomized Choice)**.** A randomized choice $r_i^* \in R_i$ is strictly dominated by a randomized choice $r_i \in R_i \setminus \{r_i^*\}$ if

$$U_i(r_i^*, r_{-i}) < U_i(r_i, r_{-i}) \text{ for all } r_{-i} \in R_{-i}.$$

[7] Where by ambiguous we mean that a first-order belief is not a singleton but a set of beliefs.

Interestingly, it turns out that we can characterize maximin-rational choices in a very similar way as rational choices under CBR. The only difference is that in the maximin setting we need to pay special attention to randomized choices because in the maximin setting it can be strictly better to make a randomized choice.

Theorem 6 (Characterization of Maximin-Rational Choices)**.** *Maximin-rational randomized choices are exactly those randomized choices that are not strictly dominated by a randomized choice.*

This means we can find all randomized choices that a maximin player can rationally make by deleting all randomized choices from the game which are strictly dominated by another randomized choice. Again, remember that under CBR it doesn't matter whether or not we look at randomized choices as all choices in the support of the randomized choice yield the same SEU. For MMEU players this is clearly not true but we can make another interesting statement.

Corollary 7 (Behavioral Equivalence)**.** *All choices in the support of a maximin-rational randomized choice are not strictly dominated by a randomized choice.*

This means that all choices in the support of a maximin-rational randomized choice are also maximin-rational. Moreover, these are exactly those choices which are not strictly dominated by a randomized choice. Remember, this is the exactly the characterization of rational choices under CBR. Hence, by observing only a single choice, the observer cannot tell a SEU player and a MMEU player apart. Note that this does not only hold for a strategic decision problem but for any one-shot decision problem in general. This is rather surprising since one's intuition seems to suggest that looser requirements for rationality should allow for a broader range of choices. In the MMEU case this is, however, not the case.

Moreover, we have that a single belief b_i^*, for which a maximin-rational choice r_i^* is optimal, is necessarily in the convex and compact set of beliefs B_i^* that makes r_i^* maximin-rational. To understand why this is the case, suppose there would be no belief $b_i^* \in B_i^*$ for which r_i^* is rational. Then by Pearce's Lemma 3 (1984) r_i^* would be strictly dominated for this restricted set of beliefs B_i^* and thus by Theorem 6 the randomized choice r_i would be maximin-irrational for B_i^*. The question at hand is: *How we can find this belief?*

Theorem 8 (Single Optimal Belief)**.** *A choice $r_i^* \in R_i$ that is maximin-rational for a set of beliefs $B_i^* \in \mathcal{K}(\Delta(R_{-i}))$ is also optimal for a single belief*

$$b_i^* \in \underset{b_i \in B_i}{\operatorname{argmin}} \max_{r_i \in R_i} U_i(r_i, b_i).$$

By Theorem 8 we see that the single optimal belief b_i^* for r_i^* is as if player i's opponents would jointly minimize i's maximum SEU within the scenarios that player i considers possible, i.e. his set of beliefs B_i^*.

5 Algorithm

Now we turn to the question of finding all the randomized choices that players can make under common belief in maximin-rationality. By Theorem 6 a maximin-rational choice is not strictly dominated by a randomized choice. As already discussed, this is very similar to the characterization of rational choice under CBR. CBR then uses iterative elimination of strictly dominated choices to identify the choices that players can make under CBR. For CBMMR it turns out that also the algorithm is quite similar, only we need to account for the special role of randomized choices. Therefore, we first define k-fold elimination of strictly dominated randomized choices. To fully describe k-fold elimination we need to define the set of degenerate randomized choices to be

$$C_i^k := \left\{ r_i \in R_i^k : r_i(c_i) = 1 \text{ for some } c_i \right\}.$$

Definition 9 (k-fold Elimination of Strictly Dominated Randomized Choices)**.** *1-fold elimination.* Consider the full game $\Gamma^0 = \Gamma$ with the set of randomized choices $R^0 := R$. Now for every player $i \in I$, consider the set of strictly dominated randomized choices

$$D_i^1 := \left\{ r_i \in R_i^0 : U_i(r_i, r_{-i}) < U_i(r_i', r_{-i}) \text{ for some } r_i' \in R_i^0 \text{ and every } r_{-i} \in R_{-i}^0 \right\}.$$

Then define the set of randomized choices that survive 1-fold elimination of strictly dominated randomized choices by $R_i^1 := R_i^0 \setminus D_i^1$. Let Γ^1 be the reduced game with the pure choices C^1 and the randomized choice set R^1.

k-fold elimination. Consider the game Γ^{k-1} that only includes randomized choices that survived (k-1)-fold elimination of strictly dominated randomized choices, denoted by R^{k-1}. Then for every player $i \in I$, consider the set of dominated randomized choices in Γ^{k-1},

$$D_i^k := \left\{ r_i \in R_i^{k-1} : U_i(r_i, r_{-i}) < U_i(r_i', r_{-i}) \text{ for some } r_i' \in R_i^{k-1} \text{ and every } r_{-i} \in R_{-i}^{k-1} \right\}.$$

Then Γ^k is the reduced game with the pure choices C^k and the set of randomized choices equal to R^k, where $R_i^k := R_i^{k-1} \setminus D_i^k$.

As defined in k-fold elimination of strictly dominated randomized choices, the algorithm keeps on eliminating randomized choices until there are no randomized choices left to delete.

Definition 10 (Algorithm Iterative Elimination of Strictly Dominated Randomized Choices)**.** For each $k \in \mathbb{N}_+$ perform k-fold elimination of strictly dominated randomized choices until $D^k = \emptyset$.

The algorithm will terminate after finitely many rounds because after eliminating all strictly dominated choices,[8] it only needs to perform one additional round of elimination of strictly dominated randomized choices.

Finally, Theorem 11 shows that the algorithm does what we claim, i.e. it finds those randomized choices that players can make under common belief in maximin-rationality.

Theorem 11 (The Algorithm Works)**.** *Consider a static game with finitely many choices for each player. Then the randomized choices that can rationally be made under common belief in maximin-rationality are exactly those randomized choices that survive iterated elimination of strictly dominated randomized choices.*

6 Example

Column Player (columns), *Row Player* (rows)

	e	***f***	***g***
a	2 , 4	2 , 5	1 , 3
b	3 , 3	0 , 2	2 , 1
c	0 , 6	3 , 5	7 , 2
d	0 , 5	1 , 4	5 , 6

Example 2: Two-player 4×3 game

Column Player (columns), *Row Player* (rows)

	e	***f***
a	2 , 4	2 , 5
b	3 , 3	0 , 2
c	0 , 6	3 , 5

Example 2: Reduced game Γ'

To illustrate the features of CBMMR we will now discuss the two-player 4×3 game in Example 2. We initially allow for all possible randomized choices for both players. Then we start off by applying the algorithm of iterative elimination of strictly dominated choices. This is possible because by Corollary 7 there can be no maximin-rational randomized choice that have strictly dominated choices in their support. First of all, notice that the *Row Player*'s choice d is strictly dominated by $(\frac{1}{4}b + \frac{3}{4}c)$. Then in the reduced game Γ^1 with $C_R^1 = \{a, b, c\}$, the *Column Player*'s choice g is strictly dominated by his choices e and f respectively. Hence, the reduced game Γ^2 after the second round contains the choices $C_R^2 = \{a, b, c\}$ for the *Row Player* and $C_C^2 = \{e, f\}$ for the *Column Player*. Here the algorithm of iterative elimination of strictly dominated choices stops.

Using Corollary 7 we can already conclude that all $r_R \in R_R$ with $r_R(d) > 1$ are not maximin-rational under CBMMR. Also all $r_C \in R_C$ with $r_C(g) > 1$ are not maximin-rational under CBMMR. Hence, we focus on the reduced game Γ' with $C_R' = \{a, b, c\}$, $C_C' = \{e, f\}$, $R_R' = \Delta(C_R')$ and $R_C' = \Delta(C_C')$. To fully

[8] Remember, the number of choices is assumed to be finite. Moreover, we assume that in every round we eliminate all strictly dominated randomized choices.

characterize all maximin-rational randomized choices, we need to find all randomized choice $r_R^{'} \in R_R^{'}$ and all randomized choices in $r_C^{'} \in R_C^{'}$ that are not strictly dominated by another randomized choice. For the *Column Player* no $r_C^{'} \in R_C^{'}$ is strictly dominated by another randomized choice because there are only two choices left and any mixture of two choices cannot dominate any other mixture or the pure choices.

For the *Row Player*, it is important to remember that not any mixture of non-dominated choices is non-dominated[9]. In fact, any randomized choice $r_R \in \Delta(\{b, c\})$ is strictly dominated. Take any $\beta \in [0, 1]$ and a randomized choice $(\beta b + (1-\beta)c)$ then we can find some $\alpha \in (0, 1)$ such that

$$U_R\left((\beta b + (1-\beta)c), c_C\right) = U_R\left((\alpha a + (1-\alpha)\left[\frac{1}{2}b + \frac{1}{2}c\right]), c_C\right)$$

for all $c_C \in C_C^{'}$, but we have

$$U_R\left((\alpha b + (1-\alpha)\left[\frac{1}{2}b + \frac{1}{2}c\right]), c_C\right) < U_R\left((\alpha b + (1-\alpha)a), c_C\right)$$

for any $\alpha \in (0, 1)$ and all $c_C \in C_C^{'}$. Thus, any strictly randomized choice over b and c is strictly dominated by some randomized choice over a and b or over a and c. By the same argument we can exclude any mixture of a, b and c. Also no mixture of a and b can dominate a mixture of a and c and vice versa. Hence, we can conclude that randomized choices $r_R \in \Delta(\{a, b\}) \cup \Delta(\{a, c\})$ cannot be strictly dominated. Hence, for the *Row Player* all randomized choices $r_R \in \Delta(\{a, b\}) \cup \Delta(\{a, c\})$ are maximin-rational under CBMMR and for the *Column Player* all randomized choices $r_C \in \Delta(\{e, f\})$.

The following table illustrates how part of an epistemic model for the game could look like. All types express CBMMR since they choose rationally given their type and only consider types that also choose rationally given their type. Choice a is maximin-rational under CBMMR for t_R^a, who considers a set of beliefs. By Theorem 8 we know that choice a must also be optimal for the single belief $(\frac{1}{2}e + \frac{1}{2}f)$.

Types	$T_R = \{t_R^a, t_R^b\}$ $T_c = \{t_C^e, t_C^f\}$
Row Player's sets of beliefs	$B_R(t_R^a) = \Delta\left(\left\{(r_C^e, t_C^e), (r_C^f, t_C^f)\right\}\right)$ $B_R(t_R^b) = \{(r_C^e, t_C^e)\}$
Column Player's sets of beliefs	$B_C(t_C^e) = \{(r_R^b, t_R^b)\}$ $B_C(t_C^f) = \{(r_R^a, t_R^a)\}$

Epistemic Model: Part of an epistemic model for Γ

Finally note, that a *Row Player* expressing CBMMR can never choose the van Neumann maximin strategy $(\frac{6}{8}a + \frac{1}{8}b + \frac{1}{8}c)$ since we showed that this mixture cannot be maximin-rational under CBMMR.

Appendix

Proof of Theorem 6. We have to show that (a) every strictly dominated randomized choice is maximin-irrational and that (b) every maximin-irrational randomized choice is strictly dominated.

Let us start with (a). Assume that $r_i^* \in \Delta(C_i)$ is strictly dominated by a randomized choice $r_i \in \Delta(C_i)$. Then for every epistemic model $\mathcal{M}^\Gamma = ((T_i)_{i \in I}, (B_i)_{i \in I})$ and every $t_i \in T_i$ there is a corresponding belief $b_i^{r_i} \in B_i(t_i)$ and a belief $b_i^{r_i^*} \in B_i(t_i)$ such that $u_i(r_i^*, t_i) = U_i(r_i^*, b_i^{r_i^*})$ and $u_i(r_i, t_i) = U_i(r_i, b_i^{r_i})$. It follows

[9] See discussion of Example 1.

that

$$\begin{aligned} u_i(r_i^*, t_i) &= u_i(r_i^*, b_i^{r_i^*}) \\ &\leq u_i(r_i^*, b_i^{r_i}) \\ &< u_i(r_i, b_i^{r_i}) \\ &= u_i(r_i, t_i). \end{aligned}$$

The first inequality holds because $b_i^{r_i^*}$ minimizes i's utility from choosing r_i^* and the second inequality holds because r_i^* is strictly dominated by r_i. Consequently, r_i^* cannot be maximin-rational, which concludes (a).

For (b) we have to show that every maximin-irrational randomized choice is strictly dominated. A randomized choice r_i^* is maximin-irrational if there is no epistemic model $\mathcal{M}^\Gamma = ((T_i)_{i\in I}, (B_i)_{i\in I})$ and $t_i \in T_i$ such that r_i^* is optimal for t_i. Now consider only types $t_i^* \in T_i^*$ with $T_i^* := \{t_i \in T_i : |b_i(t_i)| = 1\}$ that hold a single belief. From Pearce's Lemma 3 (1984) we know that a randomized choice r_i^* that is not optimal for any type t_i^* with a single belief is strictly dominated, which completes (b). □

Proof of Corollary 7. Consider a maximin-rational randomized choice $r_i \in R_i$. Then by Theorem 6 r_i is not strictly dominated. Hence, it follows from Pearce's Lemma 3 (1984) that there exist a belief p^* for which

$$U_i(r_i, p^*) \geq U_i(r_i^{'}, p^*)$$

for all $r_i^{'} \in R_i$. Therefore, we must also have for all $c_i \in \text{supp}(r_i)$ that

$$U_i(c_i, p^*) = U_i(r_i, p^*).$$

Suppose this was not true, then we had at least one $c_i^* \in \text{supp}(r_i)$ such that $U_i(c_i^*, p^*) > U_i(c_i, p^*)$. But then we could find a $r_i^{'}$ with $c_i \notin \text{supp}(r_i^{'})$ with $U_i(r_i^{'}, p^*) > U_i(r_i, p^*)$, which cannot be the case. Hence, for all $c_i \in \text{supp}(r_i)$ we have $U_i(c_i, p^*) \geq U_i(r_i^{'}, p^*)$ for all $r_i^{'} \in \Delta(C_i)$. Therefore, by Pearce's Lemma 3 (1984), every $c_i \in \text{supp}(r_i)$ is not strictly dominated. □

Proof of Theorem 8. Take a choice $r_i^* \in R_i$ and a set of beliefs $B_i \in \mathcal{K}(\Delta(R_{-i}))$ for which it is maximin-rational. Hence, we have that

$$\min_{b_i \in B_i} U_i(r_i^*, b_i) \geq \min_{b_i \in B_i} U_i(r_i, b_i)$$

for all $r_i \in R_i$. Remember that any set B_i is generated by N corner points, such that $B_i = \text{conv}\left(\left\{b_i^1, \ldots, b_i^N\right\}\right)$. Now we construct a two player zero-sum game Γ^z such that player 1 can choose from $C_1 := C_i$ and player 2 from $C_2 := \left\{b_i^1, \ldots, b_i^N\right\}$. We define player 1's utility by $U_1(c_1, c_2) := U_i(c_1, b_i^{c_2})$, where $b_i^{c_2}$ is the belief that corresponds to $c_2 \in C_2$. Moreover, we define $U_2(c_1, c_2) := -U_1(c_1, c_2)$. Finally, we allow all randomized choices and therefore take $R_1 := \Delta(C_1)$ and $R_2 := \Delta(C_2)$. Note that R_1 is exactly equivalent to R_i and that for every $r_2 \in R_2$ we can find a $b_i^{r_2} = \sum_{c_2 \in C_2} r_2(c_2) b_i^{c_2}$ by convexity for B_i. Take r_1^* such that $r_1^* = r_i^*$ then we must have that

$$\min_{r_2 \in R_2} U_1(r_1^*, r_2) \geq \min_{r_2 \in R_2} U_1(r_1, r_2)$$

for all $r_1 \in R_1$. Hence, r_1^* is a maximin strategy for player 1 in Γ^z. Choose $r_2^* \in R_2$ such that

$$\max_{r_1 \in R_1} U_1(r_1, r_2^*) \leq \max_{r_1 \in R_1} U_1(r_1, r_2)$$

for all $r_2 \in R_2$, then r_2^* is a maximin strategy for player 2. This follows from the fact that $U_2(c_1, c_2) = -U_1(c_1, c_2)$. By van Neumann's Minimax Theorem (1928) we know that every finite two-player zero-sum game has a value in mixed strategies. Moreover, Theorem 4.44 in Maschler et al. (2013) tells us that if a two-player zero-sum game has a value, then an optimal randomized choice for player 1 and an optimal randomized choice for player 2 constitute a Nash equilibrium. Thus, r_1^* is optimal for r_2^* but we know that r_2^* has a corresponding belief in $b_i^{r_2^*} \in B_i$ and therefore we found a belief $b_i^{r_2^*} \in B_i$ for which r_i^* is optimal. Note that by the same argument any $r_2^* \in \text{argmin}_{r_2 \in R_2} \max_{r_1 \in R_1} U_1(r_1, r_2)$ is optimal for r_1^* and thus for r_i^*. By our construction and the fact that we can think of any $b_i \in B_i$ as a probability distribution over $\{b_i^1, \ldots, b_i^N\}$ we then have that r_i^* must be optimal for any $b_i^* \in \text{argmin}_{b_i \in B_i} \max_{r_i \in R_i} U_i(r_i, b_i)$. □

Lemma 12 (Optimality Principle). *Let $k \geq 1$. Consider a randomized choice $r_i \in R_i^k$ that survives k-fold elimination of strictly dominated randomized choices. Then there is some compact and convex set of beliefs $B_i^{r_i} \in \mathcal{K}(\Delta(R_{-i}))$ such that $supp(b_i) \subseteq R_i^{k-1}$ for all $b_i \in B_i^{r_i}$ and that r_i is maximin-optimal for $B_i^{r_i}$ among all randomized choices in the original game.*

Proof. Consider some randomized choice $r_i \in R_i^k$. By construction R_i^k contains exactly those randomized choices in R_i^{k-1} which are not strictly dominated in the reduced game Γ^{k-1}. By applying Theorem 6 to the reduced game Γ^{k-1} we know that every $r_i \in R_i^k$ is optimal for some set of beliefs $B_i^{r_i}$ such that $\text{supp}(b_i) \subseteq R_{-i}^{k-1}$ for all $b_i \in B_i^{r_i}$. Then we have

$$u_i(r_i, B_i^{r_i}) \geq u_i(r_i{}', B_i^{r_i}) \text{ for all } r_i' \in R_i^{k-1}.$$

We want to show that

$$u_i(r_i, B_i^{r_i}) \geq u_i(r_i{}', B_i^{r_i}) \text{ for all } r_i' \in R_i^0.$$

Now suppose there is some $r_i' \in R_i^0$ with $u_i(r_i, B_i^{r_i}) < u_i(r_i{}', B_i^{r_i})$. Let r_i^* be an optimal choice for $B_i^{r_i}$ from the randomized choices in R_i^0. Then,

$$u_i(r_i^*, B_i^{r_i}) \geq u_i(r_i', B_i^{r_i}) > u_i(r_i, B_i^{r_i}).$$

Since r_i^* is optimal for $B_i^{r_i}$, we can apply Theorem 6 to the reduced game Γ^{k-1} and conclude that r_i^* is not strictly dominated by a randomized choice in Γ^{k-1}, and hence $r_i^* \in R_i^k$. In particular, $r_i^* \in R_i^{k-1}$. Therefore, r_i^* is a randomized choice such that $r_i^* \in R_i^{k-1}$ and $u_i(r_i^*, B_i^{r_i}) > u_i(r_i, B_i^{r_i})$, which contradicts our assumption. □

Lemma 13 (Degenerate Beliefs). *For any $r_i \in R_i^k$ we must have that $supp(r_i) \subseteq C_i^k$.*

Proof. Take any $r_i \in R_i^k$ and take any $c_i \in \text{supp}(r_i)$. We need to show that we have $c_i \in C_i^k$. By Corollary 7 we know that c_i cannot be strictly dominated in the reduced game Γ^k. Hence, there is an $r_i^{c_i} \in R_i^k$ such that $r_i^{c_i}(c_i) = 1$. By construction, $r_i^{c_i}$ is not strictly dominated in Γ^k. Therefore, we need to have $c_i \in C_i^k$ for all $c_i \in \text{supp}(r_i)$. □

Proof of Theorem 11. Let BR_i^k denote the set of randomized choices that player i can rationally make under expressing up to k-fold belief in maximin-rationality and R_i^k the set of choices, which survive k-fold elimination of strictly dominated randomized choices. Hence, we have to show (1) that $BR^\infty \subseteq R^\infty$ and (2) that $R^\infty \subseteq BR^\infty$. Let us consider (1) first. We will use induction to show that $BR^k \subseteq R^{k+1}$.

Induction start. Set $k = 1$ and consider some player i and a randomized choice $r_i \in BR_i^1$. Then r_i is optimal for some type t_i that expresses 1-fold belief in maximin-rationality. Then for t_i and any opponent $j \neq i$ we have that for all $b_i \in B_i(t_i)$ and every $(r_j, t_j) \in \text{supp}(\text{marg}_{R_j \times T_j} b_i)$ that r_j is optimal for t_j. From Theorem 6 we know that these choices r_j cannot be strictly dominated in the original game Γ^0 and hence that $r_j \in R_j^1$. But then r_i is optimal for t_i precisely when r_i is optimal for some set of beliefs $b_i \in B_i(t_i)$ that player i can hold about the opponents' randomized choices in the reduced game Γ^1. That is when r_i is rational within the reduced game Γ^1 but by Theorem 6 these are exactly those randomized choices $r_i \in R_i^2$ that are not strictly dominated within the reduced game Γ^1.

Induction step. Take some $k \geq 2$, and assume that $BR^{k-1} \subseteq R^k$. We need to show that $BR^k \subseteq R^{k+1}$. Consider a player i and any randomized choice $r_i \in BR_i^k$. Then r_i is optimal for some type t_i that expresses up to k-fold belief in maximin-rationality. Then for t_i and any opponent $j \neq i$ we have that for all $b_i \in B_i(t_i)$ and every $t_j \in \text{supp}(\text{marg}_{T_j} b_i)$ that t_j expresses up to (k-1)-fold belief in maximin-rationality. Hence, t_i only considers beliefs $b_i \in B_i(t_i)$ such that for all $r_j \in \text{supp}(\text{marg}_{R_j} b_i)$ we have that $r_j \in BR_j^{k-1}$. By the induction assumption we have $BR_j^{k-1} \subseteq R_j^k$ for all $j \neq i$. It follows that t_i only considers beliefs that assign positive probability to the opponents' randomized-choice-combinations in R_{-i}^k. Since r_i is optimal for t_i, it follows that r_i is optimal for some set of beliefs $b_i \in B_i(t_i)$ such that for all $r_j \in \text{supp}(\text{marg}_{R_j} b_i)$, we have that $r_j \in R_j^k$ for all $j \neq i$. But by Theorem 6, r_i cannot be strictly dominated in the reduced game Γ^k since it is maximin-rational. Hence, by construction r_i must be in R_i^{k+1}. So we have that every $r_i \in BR_i^k$ must be in R_i^{k+1} for all $i \in I$. Since $BR^k \subseteq R^{k+1}$ for all k, we also have $BR^\infty \subseteq R^\infty$, which concludes (1).

Now consider (2). Suppose the procedure of iterated elimination of strictly dominated randomized choices terminates after K rounds, hence $R^{K+1} = R^K$. To show $R^\infty \subseteq BR^\infty$ we first (a) construct for every randomized choice $r_i \in R^\infty$ some belief $p_{-i}^{r_i}$ for which it is optimal. Then we (b) use this belief $p_{-i}^{r_i}$ to construct the epistemic model $\mathcal{M}^\Gamma$, where we define for every randomized choice $r_i \in R^\infty$ some type $t_i^{r_i}$ for which r_i is optimal. Finally we (c) show that for every randomized choice $r_i \in R_i^\infty$, the associated type $t_i^{r_i}$ expresses common belief in maximin-rationality.

(a) Construction of beliefs. Consider some randomized choice $r_i \in R_i^K$ that survives the full procedure. Then $r_i \in R_i^{K+1}$ and therefore by Lemma 12 we can find some convex and compact set of beliefs $B_i^{r_i} \in \mathcal{K}(\Delta(R_{-i}))$ for which r_i is optimal among all choices in the original game and such that $\text{supp}(b_i) \subseteq R_{-i}^K$ for all $b_i \in B_i^{r_i}$. Furthermore, by Theorem 8 there is a single belief $p_{-i}^{r_i} \in \text{argmin}_{b_i \in B_i^{r_i}} \max_{r_i \in R_i} U_i(r_i, b_i)$ for which r_i is optimal among all choices in the original game and such that $\text{supp}(b_i^{r_i}) \subseteq R_{-i}^K$.

(b) Construction of types. We now use the belief from step (a) to construct for every randomized choice $r_i \in R_i^K$, some type $t_i^{r_i}$ for which r_i is optimal. For every player i, let the set of types be given by

$$T_i^* = \left\{ t_i^{r_i} : r_i \in R_i^K \right\}.$$

We construct the set of beliefs $B_i(t_i^{r_i})$ for $t_i^{r_i}$ such that $B_i(t_i^{r_i})$ contains only one belief. To fulfill the finite support condition of the epistemic model, we have to construct a belief that only assigns positive probability to finitely many randomized-choice-type pairs. To construct this belief take $p_{-i}^{r_i}$ from the previous step (a). By Lemma 13 we know that there exists a set C_{-i}^K that contains all degenerate beliefs needed to construct all $r_{-i} \in R_{-i}^k$. Then consider $p_j \in \text{marg}_{R_j} p_{-i}^{r_i}$ and let $\left(p_j^*(c_j)\right)_{c_j \in C_j^K}$ such that $p_j^*(c_j) := \sum_{r_j \in R_j^K} p_j(r_j) r_j(c_j)$ for all $c_j \in C_j^K$. Hence, for each player $j \neq i$ we can construct a belief of i about j's choices with finite support p_j^* that mimics p_j. Now let us denote degenerate randomized choices by $r_j^{c_j}$, with $r_j^{c_j}(c_j) = 1$ and the set of all non-degenerate choices by $\bar{R}_j := \left\{ r_j \in R_j^K : r_j(c_j) \neq 1 \text{ for any } c_j \in C_j^K \right\}$. Then let $b_i \in \Delta(\times_{j \neq i} R_j \times T_j)$ be such that b_i assigns probability p_j^* to the degenerate-randomized-choice-type combinations $\times_{c_j \in C_j}(r_j^{c_j}, t_j^{c_j})$ and probability 0 to all other randomized-choice-type combinations $\times_{r_j \in \bar{R}_j}(r_j, t_j)$ for all $j \neq i$, and $b_i(t_i^{r_i}) = \{b_i\}$. By construction r_i is maximin-optimal for $b_i(t_i^{r_i})$. Since by step (a) $p_{-i} \in \Delta(R_{-i}^K)$, and type $t_i^{r_i}$ only considers one belief $b_i \in b_i(t_i^{r_i})$ that assigns positive probability to randomized-choice-type pairs with $t_j = t_j^{c_j}$, it follows that $t_i^{r_i}$ only assigns positive probability to the opponents' types $t_j^{c_j}$ with $c_j \in C_j^K$. That is, $t_i^{r_i}$ only assigns positive probability to the opponents' types in T_j^*.

(c) Common belief in maximin-rationality. Finally we have to show that every $t_i \in T_i^*$ expresses common belief in maximin-rationality. First of all, note that every $t_i \in T_i^*$ expresses 1-fold belief in maximin-rationality. By our construction in (b) we know that for all $t_i \in T_i^*$ we have $t_i = t_i^{r_i}$ for some $r_i \in R_i^K$. Since the types for all players are construct in the same way, for every $b \in b_i(t_i^{r_i})$ we must have for all $j \neq i$ that $(r_j, t_j^{r_j}) \in \text{supp}(\text{marg}_{\Delta(C_j) \times T_j} b)$ that r_j is maximin-optimal for $t_j^{r_j}$. Hence, t_i expresses 1-fold belief in maximin-rationality. Thus all $t_i \in T_i^*$ express 1-fold belief in maximin-rationality for all $i \in I$. Moreover, every $t_i \in T_i^*$ is constructed such that he assigns positive probability only to types in T_{-i}^*, but since these types all express 1-fold belief in maximin-rationality, all types in T^* express common believe in maximin-rationality. □

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