

Allocating divisible and indivisible resources according to conflicting claims: collectively rational solutions*

Albin Erlanson[†] and Karol Szwagrzak[‡]

November 2, 2014

Abstract

We consider the problem of allocating multiple divisible and indivisible resources according to conflicting claims on these resources. We prove that choosing allocations maximizing a separable social welfare function is a consequence of three basic principles: consistency, resource monotonicity, and the independence of irrelevant alternatives.

Keywords: Indivisible objects; Consistency; Resource monotonicity; Independence of irrelevant alternatives

JEL classification: D70, D63, D61

1 Introduction

How should an organization - a government, a university, a school, or a research department - distribute scarce resources among its members? The problem studied here is that of distributing an endowment of divisible and indivisible resources among a

*We thank Özgür Kıbrıs, William Thomson, and John Stovall for very useful discussions. We are also grateful audiences at Sabancı University and in the 12th Meeting of the Society for Social Choice and Welfare for a number of insightful comments. This paper was first presented under the title of “Utilitarian resource allocation”. Albin Erlanson gratefully acknowledges financial support from the European Research Council and the Jan Wallander and Tom Hedelius Foundation.

[†]University of Bonn. E-mail: aerlanso@uni-bonn.de

[‡]University of Southern Denmark. E-mail: karolszw@sam.sdu.dk

group of agents holding claims or expressing needs for these resources. The potential solutions or distribution methods are evaluated on the basis of properties reflecting the rationality of an arbitrator weighing the claims on the available resources.

The first of these properties, *resource monotonicity*, specifies that an increase in the availability of resources should make no individual worse off. This can be interpreted as having the arbitrator view assigning each agent a further unit of one of the resources a normal good: an increase in the availability of resources does not decrease the amount awarded to any agent.

The second property, *consistency*, specifies that the amounts received by a group of agents should depend solely on the claims of those agents and on the available resources. This can be expressed in terms of sequential choice: suppose that the arbitrator first chooses some distribution of resources among a group of agents. Thereafter, a subgroup of agents pools their received shares and brings them back to the arbitrator to reassess the distribution within the subgroup. The requirement on the solution is that the arbitrator again chooses the same distribution for the agents in the subgroup. If this did not hold, the initially chosen allocation would be dominated by the revealed preferences of the arbitrator.

The third and final property, *independence of irrelevant alternatives* (IIA) was first introduced by Nash (1950) in bargaining problems. It specifies that, if a solution selects an alternative and this alternative is still feasible after a contraction in the feasible set, then the alternative is still chosen after the contraction. In rational choice terms, this property is equivalent to a basic axiom of individual choice (Chernoff, 1954) often referred to as Sen's α (Sen, 1970).

Resource monotonicity, consistency, and IIA imply that the solution to our resource allocation problem obtains by maximizing a separable social welfare function over the possible alternatives, i.e., the possible distributions of the resource. Thus, the three properties characterize a class of solutions embodying a central and long-standing idea in social choice, collective rationality. Its basic premise, starting with Condorcet in the 18th century, is that social decisions ought to reflect the principles of individual rational choice by maximizing an ordering over the set of alternatives. This is a defining property of Bergson-Samuelson social welfare functions and the subject of Arrow's impossibility theorem.

The solutions introduced in this paper are thus called collectively rational. This follows Lensberg (1987) where collectively rational solutions, defined by maximizing a separable social welfare function over the set of feasible alternatives, were first described in bargaining problems. Most prominently among these, the Nash and non-symmetric Nash solutions (Kalai, 1977) can also be expressed in terms of the maximization of a separable social welfare function.

However, unlike in bargaining theory, the resource allocation problem studied here is ordinal. Its data only includes physical feasibility constraints on the possible allocations. Thus, the characterization of the collectively rational solutions for this domain of problems is also a contribution to the extension of bargaining theory to natural economic domains. (See de Clippel (2014) and the references therein.)

Related literature The resource allocation model introduced in this paper, featuring divisible and indivisible resources, has so far not been studied. However, in the special case of our analysis where a single divisible resource is to be distributed, our model encompasses the classical claims problem introduced by O’Neill (1982). The claims problem has been the subject of extensive axiomatic studies.¹ Most of this literature (see Young, 1987, 1988) is concerned with symmetric solutions, solutions awarding claimants with equal claims equal shares of the resource. In our setting with indivisibilities this is incompatible with feasibility. For this reason, and since the appropriateness of any given equity notion depends on the application at hand, we avoid imposing symmetry requirements.

More recently, a growing body of work on the claims problem has started to axiomatically evaluate potentially asymmetric solutions. Most of this literature still concerns the division of a single divisible resource (see Kıbrıs, 2012; Stovall, 2014a,b, and the references therein). A notable exception is Moulin (2000),² studying the allocation of a single resource that may either be divisible or be available only in indivisible units. Moulin’s axioms however narrows the menu of allocation mechanisms to those based on sequential priorities. Here, the agent with the highest priority is assigned all available resources up to her claim; thereafter the second highest priority agent receives all remaining resources up to her claim, and so forth.

Our class of rules is broader, even in the single resource context. It allows us to implement a wide gamut of distributional objectives ranging from the highest degree of egalitarianism possible subject to the indivisibility constraints while not excluding the solutions based on sequential priorities.

Moreover, the presence of multiple resource may enable the arbitrator in charge of the distribution to balance the treatment of agents across the multiple resources. For instance, due to indivisibility an agent may receive non or relatively few units of the first resource relative to another agent. The planner could thus favor this disadvantaged agent by increasing his priority to receive another resource. Our

¹See Thomson (2003) and Thomson (2014) for a survey.

²See Moulin and Stong (2002), Herrero and Martínez (2008), and Chen (2011) for further axiomatic studies of the claims problem featuring the assignment of a single resource available in indivisible units.

proposed class of solutions can readily account for this objective.

Our collectively rational solutions are closest to those proposed in Stovall (2014b) and Erlanson and Szwagrzak (2014b). Working within the standard claims problem, Stovall (2014b) proves that the class of resource-monotonic and consistent solutions satisfying IIA³ coincides with those obtained by maximizing a separable social welfare function over the feasible divisions of the resource at which no agent receives more than her claim.

Erlanson and Szwagrzak (2014b) introduce a resource allocation problem incorporating preference information and indivisibilities. The main requirement studied there is strategy-proofness, giving agents dominant strategy incentives to report their preferences truthfully. The class of strategy-proof, consistent, resource monotonic solutions is described in terms of the maximization of a separable social welfare function.

Outline Section 2 introduces the resource allocation problem studied here, the multidimensional claims problem. It then formally introduces the main properties of this study. Section 3 introduces the collectively rational solutions and establishes that these solutions are characterized by resource monotonicity, consistency, and IIA. Section 4 contains the proofs of the main results. An Appendix gathers the proofs not included in the body of the paper.

2 Framework

2.1 Multidimensional claims problems

A number of divisible and indivisible resources are to be allocated among a group of claimants drawn from the finite set A .⁴ Let $\mathcal{N}$ denote the subsets of A . The resource kinds that are available in indivisible units are indexed by I while those available in divisible units are indexed by D . Let K denote the union of I and D . Let $\mathcal{C} \equiv \mathbb{R}_+^D \times \mathbb{Z}_+^I$ denote the space of possible resource profiles.

For every group of agents $N \in \mathcal{N}$, a (multidimensional) **claims problem** is the pair (C, E) , where $C \in \mathcal{C}^N$ and $E \in \mathcal{C}$ are such that $\sum_N C_i \geq E$. For each $N \in \mathcal{N}$, let $\mathcal{P}^N$ denote the claims problems involving the agents in N . An **allocation** for

³IIA is introduced in the analysis of standard claims problems by Kibris (2012).

⁴The basic mathematical notation is as follows: Let $\{Y_i\}_{i \in I}$ be a family of sets Y_i indexed by I . Let $Y^I \equiv \times_{i \in I} Y_i$. For each $y \in Y^I$ and each $J \subseteq I$, we denote by y_J the projection of y onto Y^J . If $x, y \in \mathbb{R}^I$, then $x \geq y$ means that, for each $i \in I$, $x_i \geq y_i$. For each $i \in I$, $e_i \in \mathbb{R}^I$ denotes the i th standard basis vector, the vector with a one in the i th coordinate and zeros elsewhere.

the claims problem $(C, E) \in \mathcal{P}^N$ is a profile $z \in \mathcal{C}^N$ such that $\sum_N z_i = E$ and, for each $i \in N$, $z_i \leq C_i$. Let $Z(C, E)$ denote the collection of all allocations for claims problem (C, E) . A **solution** is a function φ recommending allocations for all possible claims problems: for each $N \in \mathcal{N}$ and each $(C, E) \in \mathcal{P}^N$, $\varphi(C, E) \in Z(C, E)$.

Notation For each $N \in \mathcal{N}$, each $(C, E) \in \mathcal{P}^N$ and each $k \in K$, let C^k and E^k denote the projections of C and E onto the k th coordinates of $\mathcal{C}^N$ and $\mathcal{C}$, respectively. Thus, C^k is in $\mathbb{R}_+^N$ (in $\mathbb{Z}_+^N$ if $k \in I$) and E^k is in $\mathbb{R}_+$ (in $\mathbb{Z}_+$ if $k \in I$). Similarly, for each $x \in Z(C, E)$, let x^k denote the projection of x onto the k th coordinates of $\mathcal{C}^N$.

2.2 Properties of solutions

The objective of this paper is to investigate the consequences of the following properties.

Resource monotonicity For each pair $(C, E), (\bar{C}, \bar{E}) \in \mathcal{P}^N$ such that $\bar{C} = C$ and $\bar{E} \geq E$, $\varphi(C, E) \geq \varphi(\bar{C}, \bar{E})$.

Resource monotonicity is a sufficient condition for an increase in the availability of resources to make each agent at least as well off as before. In a sense it is also a necessary condition for this to be the case: multidimensional claims problems lack information on how agents assess the tradeoffs between the different resources, i.e. they lack agents' preference information on the assignment space $\mathcal{C}$. All we know about this information is that more is better. Thus, if we were to specify that, under any monotone preferences over $\mathcal{C}$, an increase in the available resources makes all agents at least as well off as before the increase, we would be left with the above *resource monotonicity* requirement.

Consistency For each pair $N, N' \in \mathcal{N}$ such that $N' \subseteq N$, each $(C, E) \in \mathcal{P}^N$, and each $i \in N'$, $\varphi_i(C_{N'}, \sum_{i \in N'} \varphi_i(C, E)) = \varphi_i(C, E)$.

Consistency has been one of the most thoroughly studied properties in the axiomatic resource allocation literature since it was introduced in the analysis of bargaining problems by Harsanyi (1959).⁵ Balinski (2005) argues that *consistency* is a

⁵Harsanyi refers to the property as “bilateral stability”. See Thomson (2011) for a survey on the applications of *consistency*.

fundamental component of equitable resource allocation.⁶ Thomson (2012) provides further normative arguments for consistency.

Independence of irrelevant alternatives (IIA) For each pair $(C, E), (\bar{C}, \bar{E}) \in \mathcal{P}^N$ such that $Z(C, E) \supseteq Z(\bar{C}, \bar{E})$ and $\varphi(C, E) \in Z(\bar{C}, \bar{E})$, $\varphi(C, E) = \varphi(\bar{C}, \bar{E})$.

IIA has been, since it was investigated by Nash and Arrow, one of the most controversial and influential properties in social choice. Though there is little to add to the controversy, the class of problems studied here lends itself to a new interpretation of IIA: if claims are viewed as demands for the various resources or as expressions of agents' needs, then IIA is equivalent to the condition that agents do not gain by exaggerating their demands.

3 Collectively rational solutions

Let $\mathcal{U}$ consist of all profiles of functions $U \equiv \{U_i^k : i \in A, k \in K\}$ such that each $U_i^k : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly concave and continuous. A solution φ is **collectively rational** if there is a $U \in \mathcal{U}$ such that, for each $N \in \mathcal{N}$ and each $(C, E) \in \mathcal{P}^N$,

$$\varphi(C, E) = \arg \max \{ \sum_{i \in N} \sum_{k \in K} U_i^k(z_i^k) : z \in Z(C, E) \}.$$

Let φ^U denote the collectively rational solution specified by $U \in \mathcal{U}$.

The collectively rational solutions are well defined when all resources are divisible: the maximization problem defining them has a strictly concave objective and a convex and compact constraint set, thus a unique solution.

Theorem 1. *If all resources are divisible, a solution is consistent, resource-monotonic, and satisfies IIA if and only if it is collectively rational.*

Theorem 1 is a corollary of Theorem 2 below, dealing with both indivisible and divisible resources. In contrast to the case where all resources are divisible, it is not immediately clear that any profile $U \in \mathcal{U}$ yields a well defined solution. Though the optima of the maximization problem defining a collectively rational solution's recommendation are guaranteed to be feasible allocations, the optima may contain more than one allocation. Thus, we refine the definition of the collectively rational solutions to clarify exactly what profiles of concave functions specify collectively rational solutions.

⁶Balinski refers to *consistency* as “coherence”.

Let $\mathcal{I}$ denote the profiles of functions $(U_i)_{i \in A}$ such that each $U_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly concave and continuous and for each $(c, e) \in \mathbb{Z}_+^A \times \mathbb{Z}_+$ such that $\sum_N c_i \geq e$,

$$\arg \max \{ \sum_A U_i(z_i) : \sum_A z_i = e, 0 \leq z_i \leq c_i \} \text{ is in } \mathbb{Z}_+^A. \quad (1)$$

Let $\mathcal{U}^*$ denote the profile of functions $U \equiv \{U_i^k : i \in A, k \in K\}$ in $\mathcal{U}$ such that, for each $k \in I$, $(U_i^k)_{i \in A} \in \mathcal{I}$. Note that, $\mathcal{U}^* \subseteq \mathcal{U}$ and that, if all resources are divisible, $\mathcal{U}^*$ and $\mathcal{U}$ coincide.

We can now *refine* our definition of collectively solutions to account for indivisibilities: a solution φ is **collectively rational** if and only if there is $U \in \mathcal{U}^*$ such that $\varphi = \varphi^U$. The following lemma establishes that an allocation chosen by the maximization problem defining a collectively rational solution is, in fact, the unique solution to a maximization problem over the convex hull of the feasible set. Since this convex hull contains the feasible set, there is in fact a single solution to the optimization problem over the feasible set.

Lemma 1. *For each $U \in \mathcal{U}^*$, each $N \in \mathcal{N}$, and each $(C, E) \in \mathcal{P}^N$,*

$$\varphi^U(C, E) = \arg \max \{ \sum_{i \in N} \sum_{k \in K} U_i^k(z_i^k) : z \in \text{co}Z(C, E) \}$$

where $\text{co}Z(C, E)$ is the convex hull of $Z(C, E)$.

Lemma 1 tells us that, once we have a profile $U \in \mathcal{U}^*$, we can treat the optimization problem defining the collectively rational solution φ^U as if all resources were divisible, yet we are guaranteed integral feasible assignments of resources available in indivisible units.

Now we can state the result characterizing the class of collectively rational solutions for situations involving indivisibilities as well as divisibilities.

Theorem 2. *A solution is consistent, resource-monotonic, and satisfies IIA if and only if it is collectively rational.*

To deepen our understanding of the collectively rational solutions, we characterize the structure of the profiles of concave functions in $\mathcal{I}$. Lemma 2, below, shows that we can restate the optimization problem in (1) as an integer linear program in a higher dimensional space. Lemma 2 is also used to prove Lemma 1 (see the Appendix). In effect, Lemma 2 shows that a profile in $\mathcal{I}$ can be approximated by a profile of piece-wise linear functions with decreasing slopes.

Lemma 2. *Let $\mathcal{M}$ denote the class of matrices*

$$M \equiv \{m_{in} \in \mathbb{R}_{++} : i \in A, n \in \mathbb{N}\} \text{ such that}$$

(i) for each $i \in A$, $m_{i1} > m_{i2} > \cdots > m_{in} > \cdots$, and

(ii) all entries in M are distinct.

If $(U_i)_{i \in A} \in \mathcal{I}$ ($M \in \mathcal{M}$), then there is $M \in \mathcal{M}$ ($(U_i)_{i \in A} \in \mathcal{I}$) such that, for each $n \in \mathbb{Z}_+$, if

$$\begin{aligned} x &= \arg \max \left\{ \sum_{i \in A} U_i(z_i) : \sum_{i \in A} z_i = n, 0 \leq z_i \leq n \right\} \quad \text{and} \\ y &= \arg \max \left\{ \sum_{i \in A} \sum_{h=1}^n m_{ih} z_{ih} : \sum_{i \in A} \sum_{h=1}^n z_{ih} = n, 0 \leq z_{ih} \leq 1 \right\} \end{aligned}$$

then, for each $i \in A$, $x_i = \sum_{h=1}^n y_{ih}$ and $y \in \{0, 1\}^{|A| \times n}$.

See Theorem 1 in Erlanson and Szwagrzak (2014a) for a proof of Lemma 2.

4 Proof of Theorems

Lemma 3. *A collectively rational solution is consistent, resource-monotonic, and satisfies IIA.*

Note that a collectively rational solution “separates” the allocation of each of the resources. The next lemma, establishes that any *consistent* and *resource-monotonic* solution has this property. Before proceeding with the lemma, we formalize precisely how this separation takes place. For each $N \in \mathcal{N}$, each claims problem $(C, E) \in \mathcal{P}^N$, and each $k \in K$, let ψ^k denote a function mapping (C^k, E^k) into $Z(C, E)^k$, i.e., the set of feasible allocations of the endowment E^k . Let Ψ denote the collection of profiles $\{\psi^k : k \in K\}$ of such functions.

Lemma 4. *Let φ denote a consistent and resource monotonic solution. Then, there is a profile $\{\psi^k : k \in K\} \in \Psi$ such that, for each $N \in \mathcal{N}$, and each $(C, E) \in \mathcal{P}^N$,*

$$\varphi(C, E) = \{\psi^k(C^k, E^k) : k \in K\}.$$

Proof. Let φ denote a *consistent* and *resource monotonic* solution. We will first establish Lemma 4 when $N = A$, and then show that it holds for each $N \in \mathcal{N}$. For each $k \in K$ and each $(c, e) \in [\mathcal{C}^N \times \mathcal{C}]^k$ such that $\sum_A c_i \geq e$, let $\psi^k(c, e) = \varphi(C, E)^k$ where $(C, E) \in \mathcal{P}^A$ is such that

- i. $C^k = c$ and $E^k = e$;
- ii. for each $l \in K \setminus \{k\}$ and each $i \in A$, $E^l = 0$ and $C_i^l = 0$.

By construction, $\{\psi^k : k \in K\}$ is in Ψ . We now prove that $\{\psi^k : k \in K\}$ is as claimed in the statement of Lemma 4. Let $k \in K$, $(C, E) \in \mathcal{P}^A$, and $x \equiv \varphi(C, E)$. Let $(\bar{C}, \bar{E}) \in \mathcal{P}^A$ be such that

$$[\text{for each } l \in K \setminus \{k\} \text{ and each } i \in A, \bar{E}^l = 0 \text{ and } \bar{C}_i^l = 0].$$

Let $y \equiv \varphi(\bar{C}, \bar{E})$. By the definitions of ψ^k and $(\bar{C}, \bar{E})$, $y^k = \psi^k(C^k, E^k)$. By resource monotonicity, for each $i \in A$, $y_i^k \leq x_i^k$. Since $\sum_{i \in A} x_i^k = E^k = \bar{E}^k = \sum_{i \in A} y_i^k$, for each $i \in A$, $\varphi_i(C, E)|^k = x_i^k = y_i^k = \psi^k(C^k, E^k)$. We can repeat this argument for each $k \in K$ to conclude that,

$$\text{for each } k \in K \text{ and each } (C, E) \in \mathcal{P}^A, \varphi(C, E)|^k = \psi^k(C^k, E^k).$$

To finish the proof we have to establish the analogous result for each $N \in \mathcal{N}$. Let $N \in \mathcal{N}$ and $(C, E) \in \mathcal{P}^N$. Let $(\bar{C}, \bar{E}) \in \mathcal{P}^A$ be such that

$$[\text{for each } k \in K \text{ and each } i \in A \setminus N, \bar{E}^k = E^k, \bar{C}_i^k = 0, \text{ and for each } i \in N, \bar{C}_i = C_i].$$

Let $y \equiv \varphi(\bar{C}, \bar{E})$ and $x \equiv \varphi(C, E)$. By definition, for each $k \in K$ and each $i \in A \setminus N$, $y_i^k = 0$. Thus, $\sum_N y_i = \sum_N x_i$ and recall that, for each $i \in N$, $\bar{C}_i = C_i$. Thus, by consistency, for each $i \in N$, $x_i = y_i$. Thus, for each $k \in K$, $\varphi(C, E)|^k = \psi^k(C^k, E^k)$. $\blacksquare$

For clarity, we first prove Theorem 1 in Section 4.1 dealing only with divisible resources. Section 4.2 then extends the arguments to account for indivisibilities and prove Theorem 2.

4.1 Divisible resources

Suppose, for the remainder of this section, that all resources are divisible, i.e. $K = D$.

Lemma 5. *Let α denote a positive number. Let φ denote a consistent and resource monotonic solution. Let $\{\psi^k : k \in K\} \in \Psi$ denote the corresponding profile of functions described in Lemma 4. Then, for each $k \in K$ and each $i \in A$, there is a strictly concave and continuous function $U_i^k : [0, \alpha] \rightarrow \mathbb{R}$ such that, for each $\beta \in [0, |A| \cdot \alpha]$, $\psi^k(\alpha \mathbf{1}, \beta) = \arg \max \{\sum_{i \in A} U_i^k(z_i) : \sum_A z_i = \beta, z \in [0, \alpha]^A\}$, where $\mathbf{1}$ is a vector of ones in $\mathbb{R}^A$.*

Proof. Let all of the notation be as in the statement of the Lemma. Let $\gamma \equiv |A| \cdot \alpha$ and $k \in K$.

Step 1. *Constructing a continuous monotone path g in $\mathbb{R}_+^A$.*

For each $\beta \in [0, \gamma]$, let $g : [0, \gamma] \rightarrow \mathbb{R}_+^A$ be defined by $g(\beta) \equiv \psi^k(\alpha \mathbf{1}, \beta)$. By the *resource monotonicity* of φ , for each pair $\beta, \beta' \in [0, \gamma]$, $\beta' \geq \beta$ implies,

$$g(\beta') = \psi^k(\alpha \mathbf{1}, \beta') \geq \psi^k(\alpha \mathbf{1}, \beta) = g(\beta).$$

Thus, for each $i \in A$, each g_i is increasing in β , $g(0) = (0)_{i \in A}$ and $g(\gamma) = (\alpha)_{i \in A}$. It is straightforward to show that g is continuous as well.

Step 2. *Constructing $(U_i^k)_{i \in A}$ from g .*

For each $\beta \in [0, \gamma]$ and each $i \in A$, let $h_i : [0, \alpha] \rightarrow \mathbb{R}$ denote a strictly increasing function such that $h_i(0) = 0$, $h_i(\alpha) = \gamma$, and

$$\begin{aligned} \text{for each } x_i \in (0, \alpha), \quad x_i = g_i(\beta) & \text{ if and only if } \lim_{z \uparrow x_i} h_i(z) \leq \beta \leq \lim_{z \downarrow x_i} h_i(z), \\ 0 = g_i(\beta) & \text{ if and only if } 0 \leq \beta \leq \lim_{z \downarrow 0} h_i(z), \text{ and} \\ \alpha = g_i(\beta) & \text{ if and only if } \lim_{z \uparrow \alpha} h_i(z) \leq \beta \leq \gamma. \end{aligned} \quad (2)$$

For each $x_i \in [0, \alpha]$, let $f_i(x_i) \equiv \int_0^{x_i} h_i(t) dt$. Then, $f_i : [0, \alpha] \rightarrow \mathbb{R}$ is a well-defined, closed, and proper convex function.⁷ Additionally, because h_i is strictly increasing, f_i is strictly convex. For each $i \in A$, let $U_i^k \equiv -f_i$. Hence, each $U_i^k : [0, \alpha] \rightarrow \mathbb{R}$ is strictly concave.

Step 3. *Verifying that $(U_i^k)_{i \in A}$ is as claimed in the Lemma.*

For each $i \in A$, let $f_i \equiv -U_i^k$. It suffices to establish that, for each $\beta \in [0, \gamma]$,

$$g(\beta) = \arg \min \left\{ \sum_A f_k(z_k) : \sum_A z_k = \beta, z \in [0, \alpha]^A \right\}. \quad (3)$$

Case 1: $\beta = 0$ or $\beta = \gamma$. If $\beta = 0$, $\{z \in [0, \alpha]^A : \sum_A z_i = \beta\} = (0)_{i \in A}$. Thus, $\arg \min \{\sum_A f_k(z_k) : \sum_A z_k = \beta, z \in [0, \alpha]^A\} = (0)_{i \in A} = g(0)$, as desired. A symmetric argument establishes (3) when $\beta = \gamma$.

Case 2: $0 < \beta < \gamma$. Let $a \equiv \arg \min \{\sum_A f_i(z_i) : \sum_A z_i = \beta, z \in [0, \alpha]^A\}$. For each $i \in A$, let $F_i : \mathbb{R} \rightarrow \mathbb{R}$ be the function such that, for each $x_i \in [0, \alpha]$, $F_i(x_i) = f_i(x_i)$ and, for each $x_i \notin [0, \alpha]$, $F_i(x_i) = \infty$. Note that, under the standard convention that the convex combination of a finite number and ∞ is itself ∞ , $F_i : \mathbb{R} \rightarrow \mathbb{R}$ is convex, closed, and proper. Moreover,

⁷See Theorem 24.2 in (Rockafellar, 1970, page 230). In this context, the closedness of f_i is equivalent to its lower semi-continuity.

$$a = \arg \min \{ \sum_A F_i(z_i) : \sum_A z_i = \beta \}.$$

Clearly, there is z in the relative interior of $[0, \alpha]^A$ such that $\sum_A z_i = \beta$ and $\sum_A F_i(z_i) = \sum_A f_i(z_i) \neq -\infty$. Thus, by Corollary 28.2.2 in Rockafellar (1970), there is a Kuhn-Tucker coefficient $\lambda^* \in \mathbb{R}$ for problem $\min \{ \sum_A F_i(z_i) : \sum_A z_i = \beta \}$. For each $(z, \lambda) \in \mathbb{R}^A \times \mathbb{R}$, let $L(z, \lambda) \equiv \sum_A F_i(z_i) + \lambda[\beta - \sum_A z_i]$. By Theorem 28.3 in Rockafellar (1970),

$$\begin{aligned} \min_{x \in \mathbb{R}^A} L(x, \lambda^*) &= \lambda^* \beta + \sum_{i \in A} \min \{ F_i(x_i) - \lambda^* x_i : x_i \in \mathbb{R} \} \\ &= \lambda^* \beta + \sum_{i \in A} \{ F_i(a_i) - \lambda^* a_i \}. \end{aligned} \tag{4}$$

Thus, for each $i \in A$ and each $x_i \in \mathbb{R}$,

$$F_i(x_i) \geq F_i(a_i) + \lambda^*(x_i - a_i).$$

Thus, λ^* is in the sub-differential of F_i at a_i . That is, for each $i \in A$, $\lambda^* \in \partial F_i(a_i)$. By the definitions of f_i and F_i in Step 2 and Theorem 24.2 in Rockafellar (1970), (i) if a_i is in $(0, \alpha)$, $\partial F_k(a_i) = [\lim_{z \uparrow a_i} h_i(z), \lim_{z \downarrow a_i} h_i(z)]$, (ii) if $a_i = 0$, $\partial F_k(a_i) = (-\infty, \lim_{z \downarrow a_i} h_i(z)]$, and (iii) if $a_i = \alpha$, $\partial F_k(a_i) = [\lim_{z \uparrow a_i} h_i(z), \infty)$. Moreover, since $\gamma > \beta > 0$, there are $i, j \in A$ such that $a_i < \alpha$ and $0 < a_j$. Thus, since h_i and h_j are strictly increasing,

$$h_i(a_i) \leq \lim_{z \downarrow a_i} h_i(z) < h_i(\alpha) = \gamma \text{ and } 0 = h_j(0) < \lim_{z \uparrow a_j} h_j(z) \leq h_j(a_j).$$

Thus, $0 < \lambda^* < \gamma$. Thus, by (2), for each $h \in A$, $g_h(\lambda^*) = a_h$. Thus, $\beta = \sum_A a_h = \sum_A g_h(\lambda^*) = \lambda^*$. Thus, $\beta = \lambda^*$ and $g(\beta) = a$, confirming (3). $\blacksquare$

For each $\alpha \in \mathbb{R}_+$, let $\mathcal{U}_\alpha$ consist of all profiles $\{U_i^k : i \in A, k \in K\}$ such that $U_i^k : [0, \alpha] \rightarrow \mathbb{R}$ is strictly concave and continuous. Let $\mathcal{C}_\alpha \equiv \mathcal{C} \cap [0, \alpha]$ and, for each $N \in \mathcal{N}$, let $\mathcal{P}_\alpha^N$ denote the class of claims problems (C, E) such that $(C, E) \in \mathcal{C}_\alpha^N \times \mathcal{C}_\alpha$.

Lemma 6. *Let φ denote a consistent and resource monotonic satisfying IIA and let $\alpha \in \mathbb{R}_+$. Then, for each $N \in \mathcal{N}$ with $|N| = 2$ and each $(C, E) \in \mathcal{P}_\alpha^N$ there is a profile $U \in \mathcal{U}_\alpha$ such that $\varphi(C, E) = \arg \max \{ \sum_{i \in N} \sum_{k \in K} U_i^k(z_i^k) : z \in Z(C, E) \}$.*

Proof. Let all of the notation be as in the statement of the Lemma and denote by $\mathbf{1}$ a vector of ones in $\mathbb{R}^A$. By Lemma 4, φ can be decomposed by each resource type $k \in K$, $\varphi(C, E) = \{ \psi^k(C^k, E^k) : k \in K \}$. Thus, $\varphi(C, E)|^k = \psi^k(C^k, E^k)$. Note that the feasible set has a product structure $Z(C, E) = \times_{k \in K} Z^k(C^k, E^k)$, where $Z^k(C^k, E^k) = \{ z_i \in \mathbb{R}_+ : \text{for each } i \in N, 0 \leq z_i \leq C_i^k, \sum_N z_i = E^k \}$. By Lemma 4 and the fact that the feasible set has the above described product structure it is enough to show that, for each $k \in K$,

$$\varphi(C, E)|^k = \arg \max \{ \sum_{i \in N} U_i^k(z_i) : z \in Z^k(C^k, E^k) \}.$$

Let $k \in K$, and $\beta \leq \alpha|A|$, and define $\bar{C} \in \mathcal{C}^A$ such that, for each $j \in K$ and each $i \in A$, $\bar{C}_i^j = \alpha$. By Lemma 5, there is a $U \in \mathcal{U}_\alpha$ such that,

$$\varphi(\bar{C}, \beta \mathbf{1}_K)|^k = \psi(\alpha \mathbf{1}, \beta) = \arg \max \{ \sum_{i \in A} U_i^k(z_i) : z \in Z^k(\alpha \mathbf{1}, \beta) \}.$$

By Lemma 4 and the construction of $(U_i^k)_{i \in A}$ in the proof of Lemma 5, there is a β' such that $\sum_{i \in N} \varphi_i(\alpha \mathbf{1}, \beta' \mathbf{1}_K)|^k = E^k$. By consistency, for each $i \in N$,

$$\varphi_i(\alpha \mathbf{1}_N, E)|^k = \varphi_i(\alpha \mathbf{1}, \beta' \mathbf{1}_K)|^k.$$

Thus, by Lemma 4,

$$x \equiv \varphi(\alpha \mathbf{1}_N, E)|^k = \arg \max \{ \sum_{i \in N} U_i^k(z_i) : z \in Z^k(\alpha \mathbf{1}_N, E^k) \}.$$

Let $y \equiv \varphi(C, E)|^k$. We will now show that $x = y$.

First, consider the case where $x \in Z^k(C^k, E^k)$. Since $(C, E) \in \mathcal{P}_\alpha^N$, $Z(C, E) \subseteq Z(\bar{C}, E)$. By Lemma 4 and IIA, $x = \varphi(C, E)|^k = y^k$.

Second, consider the case where $x \notin Z^k(C^k, E^k)$. Then, there is $i \in N$ such that $y_i \leq C_i^k < x_i$, and since $y_i + y_j = E^k = x_i + x_j$, there is $j \in N \setminus \{i\}$ such that $y_j > x_j$. Let $a = \arg \max \{ \sum_{i \in N} U_i^k(z_i) : z \in Z(C_N^k, E^k) \}$. We need to prove that $y = a$. Because $(C, E) \in \mathcal{P}_\alpha^N$, for each $i \in N$, $C_i^k \leq \alpha$. Thus, $Z^k(C_N^k, E^k) \subseteq Z^k(\alpha \mathbf{1}_N, E^k)$. Thus, since $\sum_{i \in N} U_i^k$ is strictly concave, $a_i = C_i^k$ and $\sum_{i \in N} U_i^k(x_i) > \sum_{i \in N} U_i^k(a_i) \geq \sum_{i \in N} U_i^k(y_i)$ where the last inequality holds with equality if and only if $a = y$. Thus, it suffices to prove that $y_i = C_i^k$. Suppose not, i.e. $y_i < C_i^k$. By optimality of y and since $y_i < C_i^k$,⁸

$$\partial_+ U_i^k(y_i) \leq \partial_- U_j^k(y_j) < \partial_+ U_j^k(x_j), \quad (5)$$

where the second equality follows from strict concavity of U_j^k . Similarly by optimality of x and since $x_j < y_j \leq C_j^k \leq \alpha$,

$$\partial_+ U_j^k(x_j) \leq \partial_- U_i^k(x_i) < \partial_+ U_i^k(y_i), \quad (6)$$

where the second equality follows from strict concavity of U_i^k . But (6) contradicts (5), and we conclude that $y_i = C_i^k$. Thus, $y = a$ and we have thereby proven Lemma 6. $\blacksquare$

The last step in the proof of Theorem 2 consists of the following ‘‘Elevator Lemma’’. We require the following definition to proceed:

⁸Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, for each $x \in \mathbb{R}$, let $\partial_+ f(x)$ and $\partial_- f(x)$ denote the right hand and left hand derivatives of f at x , respectively.

Converse consistency For each $N \in \mathcal{N}$ and each $(C, E) \in \mathcal{P}^N$,

$$[x \in Z(C, E) \text{ and, for each } \{i, j\} \subseteq N, x_{\{i, j\}} = \varphi(C_{\{i, j\}}, x_i + x_j)] \Rightarrow x = \varphi(C, E).$$

Such elevator type arguments prove that, if a *consistent* solution coincides with a *conversely consistent* solution for two agent problems then, in fact, the two solutions coincide in general. In Lemma 6, we proved that a solution satisfying the properties in Theorem 2 coincides, on an α -parameterized subdomain of claims problems, with an also α -parameterized solution which is a collectively rational solution for that subdomain of claims problems. The following results “elevates” this coincidence from two agent problems to those with any finite number of agents.

Lemma 7. *The collectively rational solutions are conversely consistent.*

Proof. Let $U \equiv \{U_i^k : k \in K, i \in A\} \in \mathcal{U}$ and denote by φ^U the associated collectively rational solution. Since the collectively rational solutions are *consistent* and *resource monotonic*, by Lemma 4, there is $\{\psi^k : k \in K\} \in \Psi$ such that for each $N \in \mathcal{N}$, and each $(C, E) \in \mathcal{P}^N$,

$$\varphi^U(C, E) = \{\psi^k(C^k, E^k) : k \in K\}.$$

Now, using that φ^U is *resource monotonic* and Lemma 4, implies that, for each $N \in \mathcal{N}$, each $C \in \mathcal{C}^N$, and for each pair $\hat{E}^k, \tilde{E}^k \in \mathcal{C}$,

$$\hat{E}^k > \tilde{E}^k \quad \Rightarrow \quad \psi^k(C^k, \hat{E}^k) \geq \psi^k(C^k, \tilde{E}^k). \quad (7)$$

Let $N \in \mathcal{N}$, and $(C, E) \in \mathcal{P}^N$. Let $x \in Z(C, E)$ be such that, for each $\{i, j\} \subseteq N$, $x_{\{i, j\}} = \varphi^U(C_i, C_j, x_i + x_j)$. Let $y \equiv \varphi^U(C, E)$. Since φ^U is *consistent*, $y_{\{i, j\}} = \varphi^U(C_i, C_j, y_i + y_j)$. Thus, by *consistency*, if there is $\{i, j\} \subseteq N$ such that $y_{\{i, j\}} \neq x_{\{i, j\}}$, $y_i + y_j \neq x_i + x_j$. If so, without loss of generality, there is $k \in K$ such that $y_i^k + y_j^k > x_i^k + x_j^k$. Thus, by (7),

$$y_{\{i, j\}}^k = \psi^k((C_i^k, C_j^k), y_i^k + y_j^k) \geq \psi^k((C_i^k, C_j^k), x_i^k + x_j^k) = x_{\{i, j\}}^k$$

and, without loss of generality, i is such that $y_i^k > x_i^k$. Thus, since $\sum_{h \in N} x_h^k = E^k = \sum_{h \in N} y_h^k$, there is $l \in N \setminus \{i, j\}$ such that $y_l^k < x_l^k$. By *consistency*, $y_{\{i, l\}} = \varphi^U(C_i, C_l, y_i + y_l)$ and, by assumption, $x_{\{i, l\}} = \varphi^U(C_i, C_l, x_i + x_l)$. Thus, if $y_i^k + y_l^k \geq x_i^k + x_l^k$, by (7), $y_l^k \geq x_l^k$, which is not the case. Thus, $y_i^k + y_l^k < x_i^k + x_l^k$. Thus, by (7), $y_{\{i, l\}}^k \leq x_{\{i, l\}}^k$, contradicting $y_i^k > x_i^k$. Thus, φ^U is *conversely consistent*. ■

Lemma 8. *Let α denote a postive number. Let $U \in \mathcal{U}_\alpha$ and let φ be a solution defined on the subdomain of claims problems indexed by α and such that, for each $N \in \mathcal{N}$ and each $(C, E) \in \mathcal{P}_\alpha^N$,*

$$\varphi(C, E) = \arg \max \{ \sum_{i \in N} \sum_{k \in K} U_i^k(z_i^k) : z \in Z(C, E) \}.$$

Then, φ is conversely consistent on the subdomain of claims problems indexed by α .

Proof. The proof is identical to that for Lemma 7. $\blacksquare$

Lemma 9. *Let α denote a positive number. Let φ denote a solution satisfying the properties in Theorem 2. Then, there is $U \equiv \{U_i^k : k \in K, i \in A\} \in \mathcal{U}_\alpha$ such that, for each $(C, E) \in \mathcal{P}_\alpha^N$, $\varphi(C, E) \equiv \arg \max \{ \sum_{k \in K} \sum_{i \in N} U_i^k(z_i^k) : z \in Z(C, E) \}$.*

Proof. Let $N \in \mathcal{N}$, $(C, E) \in \mathcal{P}_\alpha^N$, and $x \equiv \varphi(C, E)$. Clearly, for each $\{i, j\} \subseteq N$, $(C_i, C_j, x_i + x_j) \in \mathcal{P}_\alpha^N$. By consistency, for each $\{i, j\} \subseteq N$, $\varphi(C_i, C_j, x_i + x_j) = x_{\{i, j\}}$. By Lemma 6, there is $U \equiv \{U_i^k : k \in K, i \in A\} \in \mathcal{U}_\alpha$ such that for each $\{i, j\} \subseteq N$, $\varphi(C_i, C_j, x_i + x_j) = \arg \max \{ \sum_{k \in K} U_i^k(z_i^k) + U_j^k(z_j^k) : z \in Z(C_i, C_j, x_i + x_j) \}$. This is the solution defined in Lemma 8, and we know that it is *conversely consistent*. Thus, $x = \varphi(C, E) = \arg \max \{ \sum_{k \in K} \sum_{i \in N} U_i^k(z_i^k) : z \in Z(C, E) \} = x = \varphi(C, E)$ for $U \in \mathcal{U}_\alpha$, as desired. $\blacksquare$

Proof of Theorem 1. Let φ denote a resource monotonic and consistent solution satisfying IIA. Then, by Lemma 9, letting $\alpha \rightarrow \infty$, there is $U \in \mathcal{U}$ such that $\varphi = \varphi^U$. $\blacksquare$

4.2 Extending the argument to indivisible resources

Suppose, for the remainder of this section, that there are resources only available in indivisible units, i.e. $I \neq \emptyset$.

Lemma 10. *Let φ denote a consistent and resource monotonic solution. Let α denote a positive integer. Let $\{\psi^k : k \in K\} \in \Psi$ denote the corresponding profile of functions described in Lemma 4. Then, for each $k \in K$ and each $i \in A$, there is a strictly concave and continuous function $U_i^k : [0, \alpha] \rightarrow \mathbb{R}$ such that, for each $\beta \in [0, |A| \cdot \alpha]$,*

$$\psi^k(\alpha \mathbf{1}, \beta) = \arg \max \{ \sum_{i \in A} U_i^k(z_i) : \sum_A z_i = \beta, z \in [0, \alpha]^A \},$$

where $\mathbf{1}$ is a vector of ones in $\mathbb{R}^A$. Moreover, if $k \in I$ and $\beta \in \{0, 1, \dots, |A| \cdot \alpha\}$,

$$\arg \max \{ \sum_{i \in A} U_i^k(z_i) : \sum_A z_i = \beta, z \in [0, \alpha]^A \} \text{ is in } \mathbb{Z}_+^A. \quad (8)$$

Proof. Let all the notation be as in the statement of the Lemma. If $k \in D$, Lemma 5 immediately establishes the desired conclusions. Suppose instead that $k \in I$ and let $\gamma = |A| \cdot \alpha$.

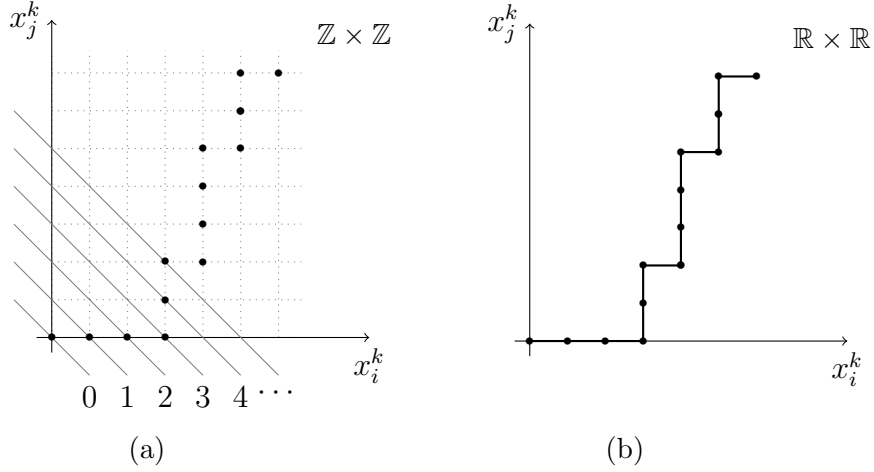


Figure 1: An illustration of the construction of a monotone path in the the integral case: Suppose that $A = \{i, j\}$. (a) As the amount of the resource $k \in K$ increases from 0 to 1 (1 to 2, 2 to 3, ...), by *resource-monotonicity*, the allocation recommended by the mechanism is above or to the right of $(0,0)$ $((1,0), (2,0), \dots)$. (b) We obtain a *continuous* monotone path by connecting the allocations of resource k obtained in part (a). The arguments in Lemma 5, for the divisible resource case, can then be applied to construct a profile of concave functions.

Step 1. *Constructing a continuous monotone path in $\mathbb{R}_+^A$.*

For each $\beta \in \{0, 1, \dots, \gamma\}$, let $G(\beta) \equiv \psi^k(\alpha \mathbf{1}, \beta)$. By the *resource monotonicity* of φ ,

$$(\alpha)_{i \in A} = G(\gamma) \geq G(\gamma - 1) \geq \dots \geq G(1) \geq G(0) = (0)_{i \in A}.$$

where the first and last inequality follow from the fact that ψ^k selects a feasible distribution of each $\beta \in \{0, 1, \dots, \gamma\}$. For the same reason, for each $\beta \in \{0, 1, \dots, \gamma\}$, $G(\beta) \in \mathbb{Z}_+^A$.

Let $g : [0, \gamma] \rightarrow \mathbb{R}_+^A$ be such that

$$\{g(\beta) : \beta \in [0, \gamma]\} = \bigcup_{\beta \in \{0, 1, \dots, \gamma-1\}} \text{co}\{G(\beta), G(\beta + 1)\}.$$

where $\text{co}\{G(\beta), G(\beta + 1)\}$ denote the convex hull of $\{G(\beta), G(\beta + 1)\}$.

Figures 1a and 1b illustrate the construction of g from G when $A = \{i, j\}$. Note that g is a continuous and monotone path in $\mathbb{R}_+^A$. Thus, we can replicate Steps 2 and 3 in the proof of Lemma 5 to construct a profile $(U_i^k)_{i \in A}$ of concave functions such that, for each $\beta \in [0, \gamma]$,

$$g(\beta) = \arg \max \{\sum_A U_i^k(z_k) : \sum_A z_k = \beta, z \in [0, \alpha]^A\}$$

and, since, for each $\beta \in \{0, 1, \dots, \gamma\}$, $g(\beta) = G(\beta) \in \mathbb{Z}_+^A$,

$$g(\beta) = \arg \max \left\{ \sum_{i \in A} U_i^k(z_i) : \sum_A z_i = \beta, z \in [0, \alpha]^A \right\} \text{ is in } \mathbb{Z}_+^A$$

as claimed in the Lemma. ■

Letting $\alpha \rightarrow \infty$ in (8) we obtain condition (1). Lemma 1 tells us that once we have a profile $U \in \mathcal{U}^*$ we can treat the optimization problem defining the collectively rational solution φ^U as if all resources were divisible, yet we are guaranteed integral feasible assignments of resources available in indivisible units. We can thus, use the same arguments in Lemmas 6 and 9 to conclude the the proof of Theorem 2 just as was done for that of Theorem 1.

A Appendix

We will use the following notation. For each $N \in \mathcal{N}$, define,

$$\begin{aligned} \text{for each } (c, e) \in \mathbb{R}_+^N \times \mathbb{R}_+, \quad S(c, e) &\equiv \{z \in \mathbb{R}_+^N : \sum_N z_i = e, 0 \leq z_i \leq c_i\}, \text{ and} \\ \text{for each } (c, e) \in \mathbb{Z}_+^N \times \mathbb{Z}_+, \quad \bar{S}(c, e) &\equiv \{z \in \mathbb{Z}_+^N : \sum_N z_i = e, 0 \leq z_i \leq c_i\}. \end{aligned}$$

Given a set B , let $\text{co}B$ denote its convex hull. We need to define two functions before proving Lemma 1. For each $U \equiv (U_i)_{i \in A} \in \mathcal{I}$, each $M \in \mathcal{M}$, and each $(a, \nu) \in \mathbb{Z}_+^A \times \mathbb{Z}_+$ such that $\sum_A a_i \geq \nu$, let

$$x(a, \nu, U) = \arg \max \left\{ \sum_{i \in A} U_i(z_i) : \sum_{i \in A} z_i = \nu, 0 \leq z_i \leq a_i \right\} \quad \text{and} \quad (9)$$

$$y(\nu, M) = \arg \max \left\{ \sum_{i \in A} \sum_{h=1}^\nu m_{ih} z_{ih} : \sum_{i \in A} \sum_{h=1}^\nu z_{ih} = \nu, 0 \leq z_{ih} \leq 1 \right\}. \quad (10)$$

A.1 Proof of Lemma 1

Before proving Lemma 1 we need to establish the following result.

Lemma 11. *If $(U_i)_{i \in A} \in \mathcal{I}$, then for each $N \in \mathcal{N}$, and each $n \in \mathbb{Z}_+$,*

$$\arg \max \left\{ \sum_N U_i(z_i) : \sum_N z_i = n, 0 \leq z_i \right\} \in \mathbb{Z}_+^N.$$

Proof. Let $U \equiv (U_i)_{i \in A} \in \mathcal{I}$, $N \in \mathcal{N}$, $n \in \mathbb{Z}_+$, $a \in \mathbb{Z}_+^A$ be such that $\sum_N a_i \geq n$, and $b \equiv \sum_A a_i$. Let $x(a, n, U)$ and $y(n, M)$ be defined as in equation (9) and (10). By Lemma 2 there is $M \in \mathcal{M}$ such that, for each $\lambda \in \{0, 1, \dots, b\}$ and for each $i \in A$, $x_i(a, \lambda, U) = \sum_{h=1}^\lambda y_{ih}(\lambda, M)$

First we prove that there is an $\nu \in \mathbb{Z}_+$ such that $\sum_N x_i(a, \nu, U) = n$. Suppose there is $i \in A$ such that $x_i(a, \nu, U) > x_i(a, \nu + 1, U)$. Then there is $j \in A \setminus \{i\}$

such that $x_j(a, \nu + 1, U) \geq x_j(a, \nu, U) + 1$. By optimality at $x(a, \nu, U)$, $m_{i(x_i(a, \nu, U))} > m_{j(x_j(a, \nu, U) + 1)}$. By optimality at $x(a, \nu + 1, U)$, $m_{i(x_i(a, \nu, U))} < m_{j(x_j(a, \nu, U) + 1)}$. But this is a contradiction. Thus, for each $i \in A$, $x_i(a, \nu, U) \leq x_i(a, \nu + 1, U)$. By feasibility $\sum_A x_j(a, \nu + 1, U) = \sum_A x_j(a, \nu, U) + 1$. Hence there is $i \in A$ such that $x_i(a, \nu + 1, U) = x_i(a, \nu, U) + 1$ and, for each $j \in A \setminus \{i\}$, $x_j(a, \nu + 1, U) = x_j(a, \nu, U)$. Restricting attention to N we have that $\sum_N x_i(a, \nu + 1, U) = \sum_N x_i(a, \nu, U) + 1$, if there is $j \in N$ such that $x_j(a, \nu + 1, U) = x_j(a, \nu, U) + 1$, otherwise $\sum_N x_i(a, \nu + 1, U) = \sum_N x_i(a, \nu, U)$. Consider the sequence $(\sum_N x_i(a, h, U))_{h=0}^b$. It is a weakly increasing sequence bounded by $\sum_N a_i$ and, for each $h \in \{0, 1, \dots, b\}$, $\sum_N x_i(a, h + 1, U) - \sum_N x_i(a, h, U) \leq 1$. Thus, there is $0 \leq \nu \leq b$ such that $\sum_N x_i(a, \nu, U) = n$.

Let $\tilde{x} \equiv \arg \max \{\sum_N U_i(z_i) : \sum_N z_i = n, 0 \leq z_i \leq a_i\}$, and suppose, by way of contradiction, that $\tilde{x} \notin \mathbb{Z}_+^N$. From the previous paragraph, there is ν such that $\sum_N x_i(a, \nu, U) = \sum_N \tilde{x}_i = n$. Let $\hat{x} \equiv x(a, \nu, U)$. By Lemma 2, $\hat{x}_N \in \mathbb{Z}_+^N$. Thus, $\tilde{x} \neq \hat{x}_N$. By the optimality of $\tilde{x}$, $\sum_N U_i(\tilde{x}) > \sum_N U_i(\hat{x}_i)$. Thus, $\sum_N U_i(\tilde{x}_i) + \sum_{A \setminus N} U_i(\hat{x}_i) > \sum_N U_i(\hat{x}_i) + \sum_{A \setminus N} U_i(\hat{x}_i)$. Now, since $\sum_N \tilde{x}_i = \sum_N \hat{x}_i = n$, $(\hat{x}_{A \setminus N}, \tilde{x}_N)$ is a feasible solution to the optimization problem defining $\hat{x}$. But this is a contradiction to optimality of $\hat{x}$. Thus, $\tilde{x} \in \mathbb{Z}_+^N$. $\blacksquare$

With Lemma 11 established we have everything needed to prove Lemma 1. In the first part of the proof of Lemma 1 we show that the solutions of the maximization problems in Lemma 1 are feasible, in particular, that when resources are indivisible the corresponding coordinates of the solutions are integral. The final part of the proof shows that these solutions, in fact, coincide with the recommendations made by the collectively rational solutions.

Proof of Lemma 1. Let $U \equiv \{U_i^k : i \in A, k \in K\} \in \mathcal{U}^*$, $N \in \mathcal{N}$, $(C, E) \in \mathcal{P}^N$, $k \in K$, $e \equiv E^k$, $c \equiv C^k$, and $\alpha \in \mathbb{R}_+$ be such that $\alpha > c_i$ for each $i \in A$, and $\alpha \in \mathbb{R}_+^N$ such that $\alpha_i = \alpha$ for each $i \in N$. Let $U_i \equiv U_i^k$ for each $i \in A$ and,

$$x \equiv \arg \max \{\sum_{i \in N} U_i(z_i) : \text{co}Z(C, E)|^k\}$$

and note that $S(c, e)$ is equal to constraint set defining x .

Part 1. Since $Z(C, E)$ has a product structure, it suffices to show that $x \in Z(C, E)|^k$. If k indexes a divisible resource ($k \in D$), then, $Z(C, E)|^k$ is itself convex, implying $\text{co}Z(C, E)|^k = Z(C, E)|^k$. Thus, if all resources are divisible, there is nothing to prove.

It remains to prove that, if $k \in I$, $x \in Z(C, E)|^k$. Let

$$y \equiv \arg \max \{\sum_{i \in N} U_i(z_i) : z \in S(\alpha, e)\}.$$

By Lemma 11, $(U_i)_{i \in A} \in \mathcal{I}$ implies $y \in \mathbb{Z}_+^N$. Since $y \in S(\alpha, e)$, $y \in \bar{S}(\alpha, e)$. Since $S(c, e) \subseteq S(\alpha, e)$, $\sum_{i \in N} U_i(y_i) \geq \sum_{i \in N} U_i(x_i)$. Thus, if $y \in S(c, e)$, since the optimum is unique, $x = y \in Z(C, E)^k$, as desired. It remains to consider the case where $y \notin S(c, e)$. Then, there is $h \in N$ such that $y_h > c_h$.

Step 1. For each $i \in N$, $y_i \geq c_i$ implies $x_i = c_i \in \mathbb{Z}_+$.

Otherwise, because $x \in S(c, e)$, $x_i < c_i$ and, since $\sum_{i \in N} y_i = e = \sum_{i \in N} x_i$ there is $j \in N$ such that $y_j < x_j \leq c_j$. This would lead to a contradiction: it implies there is a real number $\varepsilon > 0$ such that

$$x + \varepsilon(\mathbf{e}_i - \mathbf{e}_j) \in S(c, e) \text{ and } y + \varepsilon(\mathbf{e}_j - \mathbf{e}_i) \in S(\alpha, e).$$

Assume that the above is indeed true. Then, a necessary condition for x and y to maximize $\sum_{i \in N} U_i$ over $S(c, e)$ and $S(\alpha, e)$, respectively, is that, $\partial_+ U_i(x_i) \leq \partial_- U_j(x_j)$ and $\partial_+ U_j(y_j) \leq \partial_- U_i(y_i)$. On the other hand, since U_i and U_j are strictly concave, we obtain the first and last inequalities in

$$\partial_+ U_j(y_j) > \partial_- U_j(x_j) \geq \partial_+ U_i(x_i) > \partial_- U_i(y_i),$$

which contradicts $\partial_+ U_j(y_j) \leq \partial_- U_i(y_i)$. This establishes that, in fact, $x_i = c_i$. This completes Step 1.

Let $N' \equiv \{i \in N : y_i < c_i\}$ and $e' \equiv \sum_{i \in N'} y_i + \sum_{i \in N \setminus N'} (y_i - c_i)$, and

$$y' \equiv \arg \max \{ \sum_{i \in N'} U_i(z_i) : z \in S(\alpha, e') \}.$$

Note that $\sum_{N'} c_i \geq e'$ and $x_{N'} = \arg \max \{ \sum_{i \in N'} U_i(z_i) : z \in S(c_{N'}, e') \}$. By Lemma 11, $(U_i)_{i \in A} \in \mathcal{I}$ implies $y' \in \mathbb{Z}_+^{N'}$. Clearly, since $S(c_{N'}, e') \subseteq S(\alpha_{N'}, e')$, if $y' \in S(c_{N'}, e')$, then, for each $i \in N'$, $x_i = y'_i \in \mathbb{Z}_+$. Combining this with Step 1, would yield $x \in Z(C, E)^k$ as desired. It remains to consider the case where $y' \notin S(c_{N'}, e')$. Then, there is $h \in N'$ such that $y'_h > c_h$.

Step 2. For each $i \in N'$, $y'_i \geq c_i$ implies $x_i \in \mathbb{Z}_+$.

Step 2 is symmetric to Step 1 and is proven symmetrically. We can then move on to Step 3 and so on. At each step, either we establish that $x \in Z(C, E)^k$ or decrease the number of coordinates of x that are possibly non-integer. Since N is finite, the desired conclusion is eventually reached.

Part 2. Let $A \equiv \{z : z \in \text{co}Z(C, E)^{|k}\}$, $B \equiv \{z : z \in Z(C, E)^{|k}\}$, and $w \equiv \varphi^U(C, E)^{|k}$. By Part 1, $x \in B$. Thus, by the definition of φ^U , $\sum_{i \in N} U_i(w_i) \geq \sum_{i \in N} U_i(x_i)$. By the definition of x , since $w \in A$, $\sum_{i \in N} U_i(w_i) \leq \sum_{i \in N} U_i(x_i)$. Thus, w maximizes $\sum_{i \in N} U_i$ over A . Since the maximizer of $\sum_{i \in N} U_i$ over A is unique, in fact, $w = x$. $\blacksquare$

A.2 Proof of Lemma 3

Proof. Let $U \equiv \{U_i^k : i \in A, k \in K\} \in \mathcal{U}^*$ and $N \in \mathcal{N}$.

Resource monotonicity: We begin by proving that φ^U is *resource-monotonic*. Let $(C, E), (C, \hat{E}) \in \mathcal{P}^N$ and be such that $\hat{E} \geq E$. Let $x \equiv \varphi^U(C, E)$ and $\hat{x} \equiv \varphi^U(C, \hat{E})$. We need to show that $\hat{x} \geq x$. Suppose, instead, that there are $i \in N$ and $k \in K$ such that $x_i^k > \hat{x}_i^k$. By the definition of φ^U , if $E^k = \hat{E}^k$, $x^k = \hat{x}^k$. Thus, $\sum_{j \in N} \hat{x}_j^k = \hat{E}^k > E^k = \sum_{j \in N} x_j^k$. Thus, there is $j \in N$ such that $\hat{x}_j^k > x_j^k$. By the definition of φ^U , for each $h \in N$, $x_h^k \leq C_h^k$ and $\hat{x}_h^k \leq C_h^k$. Further, by the definition of φ^U , x^k and $\hat{x}^k$ maximize $\sum_{h \in N} U_h^k$ over $S(C^k, E^k)$ and $S(C^k, \hat{E}^k)$ respectively. Since $\hat{x}_j^k > x_j^k \geq 0$ and $\hat{x}_i^k < x_i^k < C_i^k$, there is $\varepsilon > 0$ such that $\hat{x}^k + \varepsilon(\mathbf{e}_i - \mathbf{e}_j) \in S(C^k, \hat{E}^k)$. Thus, a necessary condition for $\hat{x}^k$ to maximize $\sum_{h \in N} U_h^k$ over $S(C^k, \hat{E}^k)$ is that $\partial_+ U_i^k(\hat{x}_i^k) \leq \partial_- U_j^k(\hat{x}_j^k)$. Moreover, since U_i^k and U_j^k are strictly concave, we obtain the first and last inequalities in

$$\partial_+ U_j^k(x_j^k) > \partial_- U_j^k(\hat{x}_j^k) \geq \partial_+ U_i^k(\hat{x}_i^k) > \partial_- U_i^k(x_i^k).$$

Thus, $\partial_+ U_j^k(x_j^k) > \partial_- U_i^k(x_i^k)$. Since $x_i^k > \hat{x}_i^k \geq 0$ and $x_j^k < \hat{x}_j^k \leq C_j^k$, there is $\varepsilon > 0$ such that $x^k + \varepsilon(\mathbf{e}_j - \mathbf{e}_i) \in S(C^k, E^k)$. This, in addition to $\hat{x}_j^k > x_j^k \geq 0$, implies that x^k does not maximize $\sum_{h \in N} U_h^k$ over $S(C^k, E^k)$. This contradiction establishes that, in fact, $\hat{x} \geq x$, and we conclude that φ^U is *resource-monotonic*.

Consistency: Now we will show that φ^U is *consistent*. Let $\{N, N'\} \subseteq \mathcal{N}$ be such that $N' \subseteq N$. Let $(C, E) \in \mathcal{P}^N$ and $x \equiv \varphi^U(C, E)$. By way of contradiction, suppose that $y \equiv \varphi^U(C_{N'}, \sum_{N'} x_k) \neq x_{N'}$. Thus, there is $k \in K$ such that $x_{N'}^k \neq y^k$. By the definition of φ^U , for each $i \in N$, $x_i^k \leq C_i^k$ and, for each $i \in N'$, $y_i^k \leq C_i^k$. Then, $x_{N'}^k \neq y^k$, because the maximization problem defining y^k has a unique solution, $\sum_{i \in N'} U_i^k(x_i^k) < \sum_{i \in N'} U_i^k(y_i^k)$. Recall that $\sum_{i \in N'} x_i^k = \sum_{i \in N'} y_i^k$. Thus, $z^k \equiv (y^k, x_{N \setminus N'}^k) \leq C^k$ and $\sum_{i \in N} z_i^k = E^k$. Thus, $\sum_{i \in N} U_i^k(x_i^k) < \sum_{i \in N} U_i^k(z_i^k)$. Thus, $x^k \neq \varphi^U(C, E)^{|k}$, contradicting the definition of φ^U , and we conclude that φ^U is *consistent*.

IIA: It only remains to show that φ^U satisfies *IIA*. Let $N \in \mathcal{N}$, and $(C, E), (\bar{C}, \bar{E}) \in \mathcal{P}^N$ be such that $Z(C, E) \supseteq Z(\bar{C}, \bar{E})$ and $\varphi^U(C, E) \in Z(\bar{C}, \bar{E})$. By way of con-

tradition suppose that $\varphi^U(C, E) \neq \varphi^U(\bar{C}, \bar{E})$. Then, there is $k \in K$ such that $\varphi^U(C, E)|^k \neq \varphi^U(\bar{C}, \bar{E})|^k$. Let $x \equiv \varphi^U(C, E)|^k$ and $\bar{x} \equiv \varphi^U(\bar{C}, \bar{E})|^k$. By definition of φ^U x is the maximizer of $\sum_{i \in N} U_i^k(z_i)$ over $Z(C, E)|^k$. Thus, $\sum_{i \in N} U_i^k(x_i) > \sum_{i \in N} U_i^k(\bar{x}_i)$, since $\bar{x} \in Z(C, E)|^k$. Similarly by definition of φ^U $\bar{x}$ is the maximizer of $\sum_{i \in N} U_i^k(z_i)$ over $Z(\bar{C}, \bar{E})|^k$. Thus, $\sum_{i \in N} U_i^k(x_i) < \sum_{i \in N} U_i^k(\bar{x}_i)$, since by assumption $x \in Z(\bar{C}, \bar{E})|^k$. But this is a contradiction, and we conclude that $\varphi^U(C, E) = \varphi^U(\bar{C}, \bar{E})$. Thus, φ^U satisfies IIA. ■

References

- Balinski, M. (2005). What is just? *American Mathematical Monthly* 112(5), 502–511.
- Chen, S. (2011). Invariance of favorability in claims problems with indivisibilities. working paper.
- Chernoff, H. (1954). Rational selection of decision functions. *Econometrica* 22(4), 422–443.
- de Clippel, G. (2014). On the redundancy of the implicit welfarist axiom in bargaining theory. *working paper*.
- Erlanson, A. and K. Szwagrzak (2014a). Maximizing a separably concave function over a discrete constraint set. working paper.
- Erlanson, A. and K. Szwagrzak (2014b). Strategy-proof assignment of indivisible and divisible resources. working paper.
- Harsanyi, J. (1959). A bargaining model for the cooperative N -person game. In R. Luce and A. Tucker (Eds.), *Contributions to the theory of games*, Volume IV. Princeton University Press.
- Herrero, C. and R. Martínez (2008). Balanced allocation methods for claims problems with indivisibilities. *Social Choice and Welfare* 30(4), 603–617.
- Kalai, E. (1977). Non-symmetric Nash solutions and replications of 2-person bargaining. *International Journal of Game Theory* 6.
- Kibris, O. (2012). A revealed preference analysis of solutions to simple allocation problems. *Theory and Decision* 72, 509–523.
- Lensberg, T. (1987). Stability and collective rationality. *Econometrica* 55, 935–961.
- Moulin, H. (2000). Priority rules and other asymmetric rationing methods. *Econometrica* 68(3), 643–684.
- Moulin, H. and R. Stong (2002). Fair queueing and other probabilistic allocation methods. *Mathematics of Operations Research* 27, 1–30.
- Nash, J. (1950). The bargaining problem. *Econometrica* 18(2), 155–162.

- O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences* 2, 345 – 371.
- Rockafellar, T. (1970). *Convex Analysis*. Princeton University Press.
- Sen, A. (1970). *Collective choice and social welfare*. Holden Day.
- Stovall, J. (2014a). Asymmetric parametric division rules. *Games and Economic Behavior* 84(1), 87 – 110.
- Stovall, J. (2014b). Collective rationality and monotone path division rules. *Forthcoming in the Journal of Economic Theory*.
- Thomson, W. (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences* 45(3), 249 – 297.
- Thomson, W. (2011). Consistency and its converse: an introduction. *Review of Economic Design* 15(4), 257 – 291.
- Thomson, W. (2012). On the axiomatics of resource allocation: Interpreting the consistency principle. *Economics and Philosophy* 28(3), 385–421.
- Thomson, W. (2014). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update.
- Young, P. (1987). On dividing an amount according to claims or liabilities. *Mathematics of Operations Research* 12(3), 397 – 414.
- Young, P. (1988). Distributive justice in taxation. *Journal of Economic Theory* 44, 321 – 335.