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Probabilistic Voting in Models of Electoral Competition

by

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Abstract

The pioneering model of electoral competition was developed by Harold Hotelling and Anthony Downs. The model developed by Hotelling and Downs and many subsequent models in the literature about electoral competition have assumed that candidates embody policies and, if a voter is not indifferent between the policies embodied by two candidates, then the voter's choices are *fully determined* by his preferences on possible polices. More specifically, those models have assumed that if a voter prefers the policies embodied by one candidate then the voter will definitely vote for that candidate. Various authors have argued that i) factors other than policy can affect a voter's decision and ii) those other factors cause candidates to be uncertain about who a voter will vote for. These authors have modeled the candidates' uncertainty by using a probabilistic description of the voters' choice behavior. This paper provides a framework that is useful for discussing the model developed by Hotelling and Downs and for discussing other models of electoral competition. Using that framework, the paper discusses work that has been done on the implications of candidates being uncertain about whom the individual voters in the electorate will vote for.

1. An overview

The initial step toward the development of the first model of electoral competition was taken by Hotelling (1929), who developed a model of duopolists in which each firm chooses a location for its store. Near the end of his paper, he briefly described how his duopoly model could be reinterpreted as a model of competition between two political parties. Downs (1957) later sought to "borrow and elaborate upon an apparatus invented by Hotelling" (p. 115) and made explicit the assumptions of a model of electoral competition at which Hotelling had hinted. The resulting Hotelling-Downs (HD) model has subsequently become the "central model" for research on electoral competition, in the sense that alternative models commonly include many of the assumptions used by Hotelling and Downs and are often explicitly presented as variations on the HD model.

The HD model (and many of its variants) assume that candidates embody policies and, if a voter is not indifferent between the policies embodied by two candidates, a voter's choice is *fully determined* by his preferences on these polices; more specifically, the voter is certain to cast his vote for the candidate with the preferred policies. But in some models candidates are uncertain about who the individual voters will vote for, and this uncertainty has been formulated by assuming that, from a candidate's perspective, voters' choices are *probabilistic* in nature. Accordingly, these models of electoral competition are commonly called "probabilistic voting models".

The rest of the paper is organized as follows. Section 2 discusses the rationale for probabilistic voting models. Section 3 provides a framework for presenting results from the literature on electoral competition. The subsequent sections then present some probabilistic

voting models and their implications for candidate strategies.

2. Reasons for probabilistic voting models

Researchers have become interested in the implications of candidate uncertainty about voters' choices primarily because there are good empirical reasons for believing that actual candidates often are uncertain about the choices that voters are going to make on election day. Candidates tend to rely on polls for information about how voters will vote, but as Ordeshook (1986, p. 179) states, "information from public opinion surveys is not error-free and is best represented as statistical". More generally, according to Fiorina (1981, p. 155): "In the real world choices are seldom so clean as those suggested by formal decision theory. Thus, real decision makers are best analyzed in probabilistic rather than deterministic terms". Therefore scholars have developed models in which candidates are assumed to have probabilistic (rather than deterministic) expectations about voters' choices. As Ordeshook puts it, "if we want to design models that take cognizance of the kind of data that the candidates are likely to possess, probabilistic models seem more reasonable" (1986, p. 179). Similarly, Calvert (1986, pp. 28-29) argues that, for any given voter, one candidate may have an advantage "due to extraneous, non-policy considerations that are unmeasurable to the candidates … Each voter may know exactly how he should vote and why, but the candidate, not having access to [those extraneous considerations] can only estimate". Furthermore, Hinich and Munger (1997, pp. 172-173) point out that: "Research on vote choice, including much of the work specifically based on the spatial model … suggests that more than spatial position matters. Other important factors include the character of the candidate, perceptions of competence and probity, and loyalty to party or

influence by campaign advertising. Probabilistic voting takes account of the multivariate aspects of political choice, but allows the observable factors in the spatial model to have predictable impacts".

Probabilistic voting models are thus especially appropriate for elections in which candidates have incomplete information about voters' preferences and/or there are random factors that can potentially affect voters' decisions. Because most elections have these features, the assumption "that candidates cannot perfectly predict the response of the electorate to their platforms is appealing for its realism" (Calvert 1986, p. 14).

3. A framework for models of electoral competition

Because the Hotelling-Downs model has been the central model in the literature on electoral competition, most of its assumptions will be adopted here. One noteworthy difference between this framework and the HD model stems from the fact that neither Hotelling nor Downs used the language of game theory -- even though, as Arrow (1987, p. 670) points out, Hotelling's "paper was in fact a study in game theory". In what follows, (as in much of the literature on electoral competition) I explicitly treat electoral competition as a non-cooperative game.

A non-cooperative game in strategic form is specified by (1) the set of *players*, (2) the possible *strategies* for each player, and (3) the *payoff function* for each player.

3.1 The players

The Hotelling-Downs model represents an election of a public official. Just as Hotelling (1929) modelled competition between two firms, the HD model analogously assumes that there are two

candidates (from two different political parties) competing.[1] As Mueller (2003, p. 180) notes, in the HD model "the words 'candidate' or 'party' can be used interchangeably ... for the implicit assumption when discussing parties is that they take a single position in the voter's eyes." Accordingly, I retain the assumption that there are two competitors for a particular public office and refer to these players as candidates (or parties) c1 and c2.

3.2 Strategies

In Hotelling's spatial model of firm competition, each firm must decide where to locate its store along the main street of a town. Thus the possible strategies for each firm are points along a line.

In his political interpretation of the model, Hotelling (1929) described the possible locations for political parties as positions on issues. Downs (1957), in contrast, interpreted the possible locations for political parties as "party ideologies". The following aspects of the framework being used here include these two possibilities (and some other possibilities as well). There is a set of (potential) 'policy alternatives' or 'political outcomes', which is a geometrical space (of one dimension or possibly more). This set will be represented by S. The elements of S will be the possible strategies for a candidate.Strategies for c1 and c2 will be represented by s_1 and s_2 respectively. In this framework, as in the HD model, candidates choose their strategies simultaneously.[2]

Ordeshook (1986, p. 98) observed that "an equilibrium is a statement … about the actions that people choose" and added that "an equilibrium corresponds to the empirical regularities that our models predict". Ordeshook (1986, p. 118) also argued that "the concept of a Nash equilibrium … is perhaps *the* most important idea in non-cooperative game theory". In general,

a *Nash equilibrium* is a set of strategies, one for each player, such that no player can gain a higher payoff by selecting another strategy, given the strategy choice of the other player(s). Ordeshook (1986, p. 118) pointed out that when "we are analyzing candidates' election strategies ... predictions about events reduce to a search for and description of equilibria." More specifically, analyses of the HD and related models seek to identify pairs of strategies, one for each candidate, that meet the definition of a Nash equilibrium. This requires that payoff functions for the candidates be specified. But first some assumptions about the social choice rule and the voters and their possible choices are needed.

3.3 The social choice rule

In describing the way in which the votes would determine the outcome of the election, Downs (1957, pp. 23-24) stated that: a "single party ... is chosen by popular election to run the government apparatus… [and a] party ... receiving the support of a majority of those voting is entitled to take over the powers of government." The same rule is applied here, so that (i) if one of the parties gets more votes than the other party, then the party with more votes wins and (ii) if each party gets the same number of votes, then the two parties tie.

For a set of two candidates, this is the social choice rule that results from combining 1) Arrow's (1963, p. 15) assumption about how social preferences should be used to determine the alternative or alternatives chosen from a set and 2) the preference aggregation rule that Arrow (1963, pp. 46-48) referred to as the "method of majority decision". His Possibility Theorem for Two Alternatives established that, when there are two candidates (as in the two-party elections considered here), the method of majority decision satisfies all of the normative conditions he

postulated. As Arrow pointed out, "(this theorem) is, in a sense, the logical foundation of the Anglo-American two-party system." [Arrow (1963, p. 48)].[3]

In this framework, it is assumed that ties are broken by using a chance device (such as a coin toss).

3.4 The voters

In Hotelling's model of competition between two firms, each firm tries to appeal to consumers. In an electoral competition, each candidate tries to appeal to voters. Each voter has a utility function, $U_i(s)$, on S (where *i* represents an individual voter). When a voter has a unique point of maximum preference, it is called his "ideal point". In models where each voter has an ideal point, there is a corresponding distribution of ideal points. Downs (1957, pp. 115-116) assumed the voters' preferences are single-peaked, i.e., given any two alternatives on the same side of his ideal point, a voter prefers the closer one. That is one of the possible assumptions about voter preferences that will be considered here.

In his model of firm competition, Hotelling (1929, p. 45) assumed that "the buyers of a commodity will be supposed uniformly distributed along a line." The basic HD model assumes that there is a continuous uniform distribution of ideal points[4] and thus the number of voters is infinite. Subsequent scholarship has considered other continuous univariate or multivariate distributions of ideal points (e.g., Davis and Hinich 1966; Riker and Ordeshook 1973) and has also considered finite set of voters (e.g., Ordeshook 1986, pp. 160-163; 1992, pp. 103-105; Osborne 1995, Section 8).

3.5 The possible choices for a voter

In his model of competition between two firms, Hotelling assumed that each consumer buys a "unit quantity" of some commodity from one of the two firms. In the context of an electoral competition, this assumption is equivalent to each voter casting a vote for one of the two candidates. In other words, there is no abstention.[5] Hotelling also assumed that the consumers learn the locations of the firms' stores before making their choices. The HD model analogously assumes that voters learn the strategies (positions) chosen by the candidates before they vote. I adopt the same assumptions here.

3.6 Candidate expectations about voter choices

The Hotelling-Downs model and many related models assume that candidates embody policies and, if a voter is not indifferent between the policies, then a voter's choice is *fully determined* by his preferences between them. More specifically, these models assume that a voter who prefers the policies embodied by one of the candidates will definitely vote for that candidate. In addition, the choice for a voter who is indifferent between the policies is treated as being equivalent to the toss of a fair coin.

In order to consider candidate uncertainty, I will assume that, for each candidate strategy pair (s_1, s_2), each voter has some probability P^1_i of voting for candidate 1 and a corresponding probability $P^2_i = 1 - P^1_i$ of voting for candidate 2.[6] Under deterministic voting, these probabilities are defined as follows:

$$P^1_i(s_1, s_2) = \begin{cases} 1 & \text{if } U_i(s_1) > U_i(s_2) \\ \tfrac{1}{2} & \text{if } U_i(s_1) = U_i(s_2) \\ 0 & \text{if } U_i(s_1) < U_i(s_2) \end{cases} \qquad (1)$$

and similarly for $P^2_i(s_1, s_2)$ (with the inequalities in (1) reversed).

It is useful to compare (1) with the conclusions about voter choices when electoral competition is modeled as a two-stage game where both candidates and voters are players .[7] In such games, candidates select their strategies simultaneously in the first stage and voters cast their votes simultaneously based on the known candidate positions in the second stage. While the second stage can have multiple Nash equilibria, some of them are more plausible than others. For instance, whenever a set of voter choices gives one of the candidates a margin of at least three votes, those choices will be a Nash equilibrium (because, in any such case, no individual voter can change the outcome by changing his vote) -- even if every voter prefers the candidate who would then lose. The concept of a Nash equilibrium can be refined to include only equilibria in which no voter uses a weakly dominated strategy.[8] If a voter prefers one candidate's position to the other's position, voting for the less preferred candidate is a weakly dominated strategy, so (1) is consistent with assuming that (i) each voter's payoff is based entirely on the policies embodied by the winning candidate and (ii) in the second stage of the game, no voter uses a weakly dominated strategy.

3.7 Some possible objectives

The elements set out thus far provide us with a set of players and the possible strategies for each player. In addition, the assumptions concerning the social choice rule, the voters, and candidate expectations about voter choices provide important steps toward the formation of payoff functions for the candidates. Adding an objective for each candidate will complete the specification of a non-cooperative game in strategic form.

The original HD model assumed that each candidate tries to maximize his vote share. Because this framework allows for probabilistic voting, a candidate may be uncertain about his potential vote share for any given candidate strategy pair. Thus this objective will be generalized here to maximizing *expected* vote share. Other possible objectives include maximizing the expected number of votes received or the expected plurality of (i.e., difference in) votes between the candidates. Because there are no abstentions in this framework, these three objectives will be equivalent in what follows. A fourth objective that has been assumed in the literature on electoral competition is maximizing the probability of winning.[9] When the objective for each of the candidates is one of the objectives discussed above, candidates will be said to be "office-seeking". This framework does not require the candidates to be office-seeking and other possible objectives will also be considered.

In this framework (as in the HD model and the subsequent literature), I will assume that the candidates have the same type of objective (e.g., it could be that each candidate wants to maximize his expected vote share). In what follows, I will identify the specific assumptions about the candidates' objectives that have been used in various studies. However, the results from different studies can be compared, even when their specific assumptions about candidates' objectives vary.

4. One-dimensional models with probabilistic voting

The most famous result for one-dimensional models satisfying the basic assumptions set out in the previous section and with office-seeking candidates is the *Median Voter Theorem for Electoral Competition*. This theorem states that if each voter has an ideal point and single-peaked preferences, then a pair of candidate strategies is a Nash equilibrium if and only if each

candidate's strategy is a median for the distribution of voter ideal points.[10] The theorem establishes that, when both candidates locate at a median for the distribution of ideal points, neither candidate can increase his payoff by moving to a different location while his opponent's location stays fixed. The theorem also reveals that, if the candidates do not both choose median locations, then at least one of the candidates can increase his payoff unilaterally.

4.1 Candidate uncertainty can change the equilibrium strategies

Comaner (1976) and Hinich (1977) independently analyzed one-dimensional models with single-peaked preferences and showed that, when there is candidate uncertainty about voter choices, choosing a median of the distribution of voter ideal points might not be an equilibrium strategy after all. Comaner provided examples of skewed distributions of ideal points in which an equilibrium exists at an alternative that is not the median, and showed that the distance from the median depends on the degree of skewness of the distribution. Hinich showed that candidate uncertainty about voter choices can produce an equilibrium at either the mean or the mode for the distribution of ideal points, rather than at the median.

4.2 The impact of a very small amount of candidate uncertainty

Hinich (1977) also considered whether even a *very small amount* of candidate uncertainty about voter choices could cause median positions to no longer be equilibrium strategies. He observed that expression (1) for deterministic voters can rewritten as a function of the *utility difference for candidate 1* (i.e., as a function of $U_i(s_1) - U_i(s_2)$) giving

$$P^1_i\,(U_i(s_1) - U_i(s_2)) = \begin{cases} 1 & \text{if } U_i(s_1) - U_i(s_2) > 0 \\ ½ & \text{if } U_i(s_1) - U_i(s_2) = 0 \\ 0 & \text{if } U_i(s_1) - U_i(s_2) < 0 \end{cases} \quad (2)$$

with a similar formulation for P^2_i found by reversing s_1 and s_2 in (2). Hinich assumed that if *i* is a probabilistic voter, then P^1_i has the following properties (the first two of which also hold for deterministic voter probabilities): (1) $P^1_i = ½$ when the utility difference is zero, (2) P^1_i never decreases as the utility difference for candidate 1 increases, (3) P^1_i is a differentiable (and, hence, continuous) function of the utility difference, and (4) P^1_i is strictly increasing in some range of the utility difference.

He then observed that (2) can be approximated as closely as desired when this formulation of probabilistic voting is used. In particular, for any positive number δ (no matter how small), there exist P^1_i functions for probabilistic voters which have (1) $P^1_i = 1$ whenever the utility difference for candidate 1 exceeds δ, (2) has $P^1_i = ½$ when the utility difference is zero and (3) $P^1_i = 0$ whenever the utility difference for candidate 1 is below -δ (or, equivalently, has $P^1_i = 0$ whenever the utility difference for candidate 2 exceeds δ). For any such function, the choice behavior of a probabilistic voter will differ from (2) only on the interval (-δ, δ).

Hinich then used an example along the following lines to show that each candidate choosing a median ideal point can fail to be an equilibrium even when the amount of uncertainty about voter choices is arbitrarily small. Consider three voters (1, 2, and 3) who have distinct ideal points, with voter 2's in the median position. If all three voters are deterministic, the equilibrium is for both candidates to position themselves at voter 2's ideal point; in this event, each voter has probability of ½ of voting for either candidate, so the expected vote for each candidate is 3/2 votes. Now suppose that voter 3 continues to vote deterministically but voter 1 and voter 2 vote probabilistically in the manner described above. If c1 moves ever so slightly towards voter 3,

then voter 3 now votes for c1 with certainty, while voter 1 and voter 2 vote for c1 with a probability only slightly less than ½. Candidate 2's expected vote therefore increases to almost two, so the previous equilibrium at the median no longer holds. Moreover, this remains true no matter how closely the probabilistic voting of voter 1 and voter 2 approaches (2), provided c1's movement toward voter 3 is small enough.

4.3 Locations of candidate choices as the range of uncertainty shrinks

Another important question is whether a small amount of uncertainty about voter choices can cause large changes in the policies candidates choose to embody. Kramer (1978) addressed this question by proving "a general result which characterizes the limiting behavior of candidate equilibria in a wide range of situations of the type Hinich considers." Kramer worked within the framework that is being used here, although (unlike Hinich) he assumed that each candidate maximizes his probability of winning. Kramer considered two-candidate games which have the following features: (1) for each voter *i*, there is a non-negative number δ_i such that, for any particular game, candidates are uncertain about *i*'s vote choice only when *i*'s absolute utility difference is less than δ_i times a parameter λ that can range from 0 to 1 (which allows the range of utility differences producing candidate uncertainty to vary from voter to voter); (2) the only difference between any pair of games is the value of λ. He proved that, for any candidate strategy distinct from the median position, there is some value of λ greater than 0 such that the strategy is weakly dominated for all smaller values of λ. In other words, for any policy distinct from the median, there exists a degree of proximity to the deterministic case for which this policy is weakly dominated. So, for a sequence of electoral games in which λ decreases and converges to 0, there will be a shrinking neighborhood of the median for which it is the case that candidate

strategies outside of this neighborhood are weakly dominated by strategies inside it. Kramer (1978) described this result as follows: "We can thus expect the candidates to choose policies close to the median when voter behavior is nearly deterministic".

4.4 Existence of Equilibria

Coughlin (1990, pp. 149-150) used an example to show that, when voters have single-peaked preferences and Hinich's formulation of candidate uncertainty is used for some of the voters, electoral competition can fail to have a Nash equilibrium in pure strategies[10].

Laussell and Le Breton (2002) addressed the question of whether a pure-strategy equilibrium exists when there is only a very small amount of uncertainty about voter choices. In their model, each voter has preferences on a one-dimensional policy space S but may alsoexhibit bias in favor of one of the candidates (reflecting, for example, personal characteristics of the candidates or voter partisanship). What's more, they leave open the possibility that the candidates are uncertain about these biases, and thus, are also uncertain about how the individuals will vote.

Laussell and Le Breton assume that each voter has an ideal point in S, which can be represented by θ. Let a voter's bias for c2 (which can be positive, negative or zero) be represented by b. Laussell and Le Breton assume that there is a function $u(\theta, x)$ which, for any given value of θ, is a utility function on S for every voter whose ideal point is the given value of θ. In addition, they assume $u(\theta, x)$ is differentiable and single peaked with respect to the policy space and continuous with respect to the ideal point. For example, the function could be $u(\theta, x) = 100 - (\theta - x)^2$.

They assume that, if the specific values of θ and b for a particular voter *i* are used, then

$$P^1_i(s_1, s_2) = \begin{cases} 1 & \text{if } u(\theta, s_1) > u(\theta, s_2) + b \\ \frac{1}{2} & \text{if } u(\theta, s_1) = u(\theta, s_2) + b \\ 0 & \text{if } u(\theta, s_1) < u(\theta, s_2) + b \end{cases} . \quad (3)$$

Equation (3) implies that a similar representation holds for $P^2_i(s_1, s_2)$ (with the inequalities in (3) reversed).

This representation reveals that when a voter's policy preferences and bias are *both* known, the voter's behavior is deterministic in nature. However, since Laussel and Le Breton's model allows for the possibility of the candidates being uncertain about the value of b, (3) implies that the candidates can be uncertain about the voters' choices.

The only difference between (3) and the assumption of deterministic voting discussed in Section 3.6 is the role that voter bias can potentially play. If voter bias doesn't play any role in the voters' decisions (i.e., each candidate is certain that b = 0 for every voter), then the assumption of deterministic voting that was discussed in Section 3.6 is satisfied and an equilibrium exists where both candidates choose the median of the distribution of ideal points. However, Laussel and Le Breton's model also allows for the possibility of the candidates not being certain that b = 0 for every voter. More specifically, they allow for settings where, from the candidates' perspective, b is a random variable which has a cumulative distribution function which is symmetric around 0 and has a strictly positive derivative at 0. In these settings, voting no longer appears deterministic to the candidates. At the same time, their formulation includes cases that closely approximate the deterministic case, in the sense that the proportion of voters for which the bias term is not arbitrarily small is negligible. Laussel and Le Breton established

that that there is a neighborhood of the "degenerate distribution" (i.e., the distribution where b is always 0) in which electoral competition does not have an equilibrium in pure strategies.

Banks and Duggan (2005, see p. 48 & p. 54) obtained similar results for one-dimensional probabilistic voting models where (1) each P^1_i is a function of the utility difference for candidate 1, (2) the voters have quadratic utility functions, (3) the number of voters is odd, and (4) the median for the distribution of voter ideal points is not equal to the mean. In particular, they showed that when this type of probabilistic voting model is close enough to the deterministic model (in a specific sense of "closeness" that they define precisely) the probabilistic voting model does not have an equilibrium in pure strategies.

Laussell and Le Breton (2002) and Banks and Duggan (2005) also both observed that (under the assumptions that they used) when a probabilistic voting model is close to the deterministic case (1) there is an equilibrium in mixed strategies[11] and (2) the outcome from the mixed strategy equilibrium can be expected to be close to the median for the distribution of voter ideal points.

4.5 The convergence of candidate strategies and alternative objectives

An important implication of the Median Voter Theorem for Electoral Competition is that, if the distribution of ideal points has a unique median (which occurs when there is an odd number of voters), the candidate strategies converge; specifically, both candidates choose the median strategy. A second implication is that, if the distribution of ideal points does not have a unique median (which may occur if there is an even number of voters), multiple equilibria exist where the candidate strategies converge at a median location, and also non-convergent equilibria exist in which the candidates choose different median locations.

The Median Voter Theorem for Electoral Competition assumes that both candidates have one

of the objectives discussed in Section 3.7, but other assumptions about the objectives of a political candidate have also been considered. Wittman (1977), Calvert (1985), Roemer (2001) and others have analyzed models where candidates are to some degree "policy-seeking" indicating they are willing to make a tradeoff between policy outcomes and the margin (or probability) of victory. Significantly, Calvert (1985, p. 73) has established that, when voting is deterministic, such policy motivations for candidates do not affect the conclusions. Similar results are in Wittman (1977, Proposition 5) and Roemer (2001, Theorem 2.1).

Hansson and Stuart (1984) proved that, if candidates are willing to make a tradeoff between policy outcome and the margin of victory and are uncertain about voter choices, it is possible to have an equilibrium only when the candidate strategies do *not* converge. Similar result are demonstrated by Calvert (1985, p. 85) and Roemer (2001, Theorem 3.4). Calvert (1985) also showed that, if candidate uncertainty about voter choices is small, departure from convergence is likewise small.[12]

5. A finite-dimensional model with probabilistic voting

Scholars have also considered models of electoral competition where the candidates' strategy set is not required to be one-dimensional. This section reviews finite-dimensional models where the candidates have expectations that are based on an influential model of probabilistic choice originally developed by Luce (1959) (and which provides the foundation for the logit model in econometrics (McFadden (1974)).

5.1 Electoral competition with candidate expectations that are based on Luce's model

When each voter is assumed to vote, he is making a binary choice between two candidates. In

this setting, the appropriate version of Luce's model is (to use the terminology of Becker, DeGroot and Marschak (1963, p. 44)) the "binary Luce model".

Stated in the context of electoral competition models, the binary Luce model for the individuals' choice probabilities assumes that each voter *i* has a positive, real-valued "scaling function", $f_i(x)$, on S such that,

$$P^1{}_i(s_1, s_2) = \frac{f_i(s_1)}{f_i(s_1) + f_i(s_2)} \tag{4}$$

and similarly for $P^2{}_i(s_1, s_2)$ (with $f_i(s_2)$ in the numerator).

Using this assumption for the candidates' expectations, Coughlin (1992) proved that, when there is a finite set of *n* voters where each voter's utility function is concave and continuously differentiable and S is compact and convex, an equilibrium exists if and only if both candidate locations maximize

$$F(x) = \ln(f_1) + \cdots + \ln(f_n) \tag{7}$$

on S (where $\ln(v)$ denotes the natural logarithm of v).

Coughlin (1992) observed that this result implies (1) when each voter's scaling function is his utility function, a strategy pair for the candidates is an equilibrium if and only if both candidate locations maximize the "Nash Social Welfare Function"

$$N(x) = \ln(U_1) + \cdots + \ln(U_n)$$

and (2) when the scaling function satisfies the assumptions in McFadden's choice-theoretic foundation for the logit model (where $f_i(x) = \exp[U_i(s)]$) a strategy pair for the candidates is an equilibrium if and only if both candidate locations maximize the "Benthamite Social Welfare Function"

$$B(x) = U_1 + \cdots + U_n$$

Coughlin (1992) also proved that there is always at least one equilibrium under these conditions. In addition, he proved there is a unique equilibrium if at least one voter has a strictly concave scaling function.

Significantly, there is an important connection between the models discussed in this section and those in which voters have additively separable utility functions (as in Laussell and Le Breton (2002)). More specifically, when a voter's total utility can be represented by the sum of a utility function on policies and a random term that depends on something other than policies (such as a voter's bias for a candidate) the candidates can have expectations consistent with the binary Luce model. In particular, this consistency will occur when the random term has a logistic distribution. Thus the conclusions in this section also apply to certain models where voters have additively separable policy-related and non-policy related utilities.

5.2 Implications for one-dimensional models

In order to easily compare the implications of these results to corresponding models under deterministic voting, I will assume that the set of possible policies is a closed interval on a line and that each voter's utility function on policies is positive, strictly concave and continuously differentiable. These assumptions imply that the voters' preferences are single-peaked and each voter has an ideal point (although, of course, these assumptions do not include all cases where the voters' preferences are single-peaked and each voter has an ideal point). For simplicity, I will also assume that, in the probabilistic voting model, each scaling function is the individual's utility function.

First consider the cases where the number of voters is odd. Under deterministic voting, the

only equilibrium is where each candidate chooses the unique median for the distribution of ideal points. In the probabilistic voting model, the only equilibrium is where each candidate chooses the unique location that maximizes the Nash social welfare function. Unless the location that maximizes the Nash social welfare function happens to coincide with the median for the distribution of ideal points, equilibrium strategies in the two models differ.

Now consider what happens when the number of voters is even. Under deterministic voting, if there is a unique median for the distribution of ideal points, the only equilibrium is where each candidate chooses the unique median for the distribution of most-preferred alternatives; but if there is not a unique median for the distribution of ideal points, then there are multiple equilibria where each candidate chooses any location from within the median interval. However, having an even number of voters does not alter the conclusion that there is a unique location which maximizes the Nash Social Welfare Function. Thus the implications from the probabilistic voting model are unaltered. There remains only one equilibrium where each candidate chooses the unique location that maximizes the Nash social welfare function. Thus, only the probabilistic voting model always has a unique prediction for the candidates' equilibrium strategies. Furthermore, the equilibrium typically will *not* entail a median location.

5.3 Implications for multidimensional models

The implications for multidimensional models may be illustrated with - the following simple example. There are three voters (indexed by $i = 1, 2, 3$). Each candidate proposes an allocation of a particular resource[13]. The total amount of the resource is fixed(e.g., a fixed amount of money to be divided). The proportion of the resource that voter i will receive is denoted by a continuous variable, x_i. The proportion cannot be lower than some small positive amount and the resource is

fully allocated across the three voters. Each voter's utility function is represented by $U_i(x) = x_i$, which implies that each voter cares only about the amount that he receives. The objective for each candidate is to maximize his expected plurality.

Under deterministic voting, there is no equilibrium. The reason is that, no matter what allocation is offered by one candidate, the other candidate can offer greater amounts to two of the voters by reducing the amount to the other voter. However, if instead the candidates have expectations that are based on a binary Luce model, the results stated in section 5.1 both imply that there *is* at least one equilibrium and provide a method for determining the location of any equilibrium. Suppose the scaling function used for a particular voter is the voter's utility function. Then a strategy pair for the candidates will be an equilibrium if and only if each candidate's allocation maximizes the Nash Social Welfare Function $N(x) = \ln(x_1) + \ln(x_2) + \ln(x_3)$.There is a unique solution for this maximization problem, namely $x_1 = x_2 = x_3 = 1/3$. This implies that the strategy pair where $s_1 = s_2 = (1/3, 1/3, 1/3)$ is a unique Nash equilibrium for the game.

Among other things, this example illustrates the important fact that, even when a multidimensional model has no equilibrium under deterministic voting, there can still be an equilibrium in a corresponding probabilistic voting model.

6 Conclusion

The references described in this paper have established several results. First, when there is a Nash equilibrium in a deterministic voting model, there can be a different Nash equilibrium or no equilibrium for a corresponding probabilistic voting model. Second, for some candidate objectives, the assumption of probabilistic voting can affect whether there will be an equilibrium

where the candidates' strategies are the same. Finally, conclusions for models of electoral competition with deterministic voting can sometimes change when candidate uncertainty about voters' choices is introduced into the models even if the amount of uncertainty is very small.

Endnotes

[1] Some scholars have studied the implications of assuming that there are three or more candidates.

[2] However, Downs (1957: pp. 52-62) also discussed the situation in which one candidate is currently in office and the other is the challenger. In this situation the incumbent 'takes a position' by enacting policies, so the challenger can select his strategy knowing the position of his opponent. Such competition has been analyzed in more detail by Wittman (1977), and others.

[3] In related work, May (1952) proved the method of majority decision uniquely meets a set of stronger conditions.

[4] Downs also considered various forms of non-uniformity in his Chapter 8.

[5] Downs (1957, Chapter 14) also examined incentives for voter abstention, which has spawned a huge separate literature,.

[6] These probabilities can be interpreted as objective probabilities, or the candidates' subjective probabilities (provided each candidate has the same expectations).

[7] See, for example, the model in Osborne (1995, Section 8a).

[8] A player's strategy s is weakly dominated by another strategy s′ if in every contingency (set of strategies for the other players) s′ gives at least as high as payoff as s and in at least one contingency s′ gives a strictly higher payoff.

[9] It has been shown that (under fairly general assumptions), if their objective is to maximize the probability of winning, candidates typically choose the same strategies as with the first three objectives that were mentioned (Aranson, Hinich, and Ordeshook 1973; Hinich 1977; Ordeshook 1986).

[10] In the context of an electoral competition, a "pure strategy" is an element of the set S.

[11] Sometimes the term "median" is applied directly to any voter whose ideal point is a median for the distribution of ideal points. Such a voter is called a "median voter".

[12] A mixed strategy is a probability distribution on the set of pure strategies. Some scholars have argued against modeling candidate choices with mixed strategies whereas others have defended this modeling approach (see, for instance, Ordeshook 1986 and Calvert 1986).

[13] Furthermore, Ball (1999) proved that, under the same circumstances, there may be no pure-strategy equilibrium, though there is always a mixed strategy equilibrium.

[14] For more general treatments of distribution problems using probabilistic voting models, see Lindbeck and Weibull (1987) and Coughlin (1992).

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