THE BORDA COUNT AND DOMINANCE SOLVABLE VOTING GAMES

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ABSTRACT. We analyse dominance solvability (by iterated elimination of weakly dominated strategies) of voting games with three candidates and provide sufficient and necessary conditions for the Borda Count to yield a unique winner. We find that Borda is the unique scoring rule that is dominance solvable both (i) under unanimous agreement on a best candidate and (ii) under unanimous agreement on a worst candidate and in the absence of a tie. Turning to generalized scoring rules, we find that Approval Voting violates a desirable monotonicity property: a candidate that is the unique dominance solvable winner for some preference profile, may lose the election once she gains further popularity. In contrast, a candidate that is the unique dominance solvable winner under Borda, will always remain so as her popularity increases.
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JEL codes:

1. Introduction

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2. The Model

2.1. **Candidates and voters.** Throughout this paper, we consider a set of three candidates (or alternatives) $A = \{a, b, c\}$ and a finite set of voters I with generic element i. Each voter's preferences are assumed to be given by a strict linear order $\succ_i$ on A. In consequence, there are six distinct sets of voters, characterized by their preferences, that we denote $I_{xyz} = \{i \in I | x, y, z \in A, x \succ_i y \succ_i z\}$ and whose generic element we refer to as i_{xyz}. A preference profile is denoted as $\succ_I = (\succ_i)_{i \in I}$.

2.2. **Voting schemes.** We consider voting schemes that allow each voter i to cast a ballot $v_i = (v_i^a, v_i^b, v_i^c)$ from a set of available ballots V_i. For scoring rules, V_i is taken to be the set of permutation of $(1, s, 0)$, where $s \in [0, 1]$ is a fixed parameter that characterizes the scoring rule.

The most notable scoring rules in our context are *Plurality*, corresponding to $s = 0$, *Antiplurality* ($s = 1$) and the *Borda Count* ($s = \frac{1}{2}$). Under *Approval Voting* there is additional flexibility, in that V_i consists of all permutations of $(1, 1, 0)$ and $(1, 0, 0)$.[1]

Using cartesian products, we define $V = \prod_{i \in I} V_i$ and $V_{-i} = \prod_{j \neq i} V_j$ and denote generic elements as v and v_{-i}. We refer to a $v \in V$ as a ballot profile and denote the associated score of some candidate x as $|v^x| = \sum v_i^x$. To deal with ties, we introduce an additional dummy player t, who is indifferent between the three alternatives, yet has to chose a strict linear order $\triangleright_t$ on A, where

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[1] Some authors (e.g. Buenrostro et al. [2013]) refer to Approval Voting and similarly flexible voting schemes as 'scoring rules'. However, as this breaks the common terminology with social choice theory where for the purpose of preference aggregation a one-to-one mapping between individual preferences and the score vector is required, we follow another group of authors (Cox [1987],Baharad and Nitzan [2003]) and refer only to rules with fixed s as scoring rules.

$\triangleright$ denotes the set of such orders.[2] Then, for given v and $\triangleright_t$, candidate x wins the election whenever she has a weakly higher score than all other candidates and, in case of a tie, is ranked first according to $\triangleright_t$. Formally, x wins if and only if

$$\forall y \neq x : |v^x| \geq |v^y| \text{ and } |v^x| = |v^y| \Rightarrow x \triangleright_t y.$$

Note that for any given ballot profile v and tiebreaker $\triangleright_t$, there exists a unique winner. If we would refrain from breaking ties in a deterministic manner, the outcome of a voting game would take the form of a lottery over alternatives and we would have to amend voters preferences $\succ_I$, for example by specifying von Neumann - Morgenstern utilities. Instead we opt for deterministic tiebraking which allows us to base our analysis exclusively on ordinal preferences over candidates.

2.3. **Dominance Solvability.** Together, the set of candidates, voters' preferences and a voting scheme give rise to a voting game $\Gamma = \Gamma(\succ_I, V)$ with a set of players $I \cup \{t\}$. In each game Γ, players' strategies $(v, \triangleright_t) \in V \times \triangleright$ determine a unique outcome $g(v, \triangleright_t) \in A$.

Where we want to highlight the choice of voting scheme, we write Γ_{AV} for approval voting games and Γ_s when a scoring rule with parameter s is used. Moreover, we want to consider restricted games, that is games where each voter's strategies are restricted to some set $V'_i \subseteq V_i$, which we denote as $\Gamma(\succ_I, V')$. In particular, we will focus on restricted games where weakly dominated strategies have been removed.

Definition 1. A strategy $v_i \in V'_i$ is *weakly dominated* in $\Gamma(\succ_I, V')$ iff there exists $w_i \in V'_i$ such that for all $v_{-i} \in V'_{-i}$, $\triangleright_t \in \triangleright$

$$g(w_i, v_{-i}, \triangleright_t) \succ_i g(v_i, v_{-i}, \triangleright_t) \text{ or } g(w_i, v_{-i}, \triangleright_t) = g(v_i, v_{-i}, \triangleright_t)$$

with $g(w_i, v_{-i}, \triangleright_t) \succ_i g(v_i, v_{-i}, \triangleright_t)$ for at least one $v_{-i} \in V'_{-i}$ and $\triangleright_t \in \triangleright$.

Strategies $\triangleright_t \in \triangleright$ are never dominated, as the dummy player t is assumed to be indifferent between all outcomes $g(v, \triangleright_t) \in A$. Hence, in order to iteratively remove dominated strategies, we can focus on voters $i \in I$. First, we define the set of undominated strategies as $V_i^1 = V_i \backslash \{v_i \in V_i | v_i \text{ is weakly dominated in } \Gamma(\succ_I, V)\}$. We will make use of the following useful

Fact 1. In approval voting games, the set of undominated strategies V_i^1 for a voter of type i_{xyz} consists of all ballots $v_i \in V_i$ such that $v_i^x = 1$ and $v_i^z = 0$ [Brams and Fishburn, 1978]. For scoring rule voting games, i_{xyz}'s undominated strategies are all ballots $v_i \in V_i$, such that $v_i^x \geq s$ and $v_i^z \leq s$ (Proposition 1 in [Buenrostro et al., 2013]).

Next, we move to the iterative elimination of dominated strategies and define

$$V_i^{m+1} = V_i^m \backslash \{v_i \in V^m | v_i \text{ is weakly dominated in } \Gamma(\succ_I, V^m)\}, \text{ for } m \in \mathbb{N}.$$

Clearly, $V_i^{m+1} \neq \varnothing$, as it is impossible for all strategies in V_i^m to be dominated by one another.[3] Also, as V is finite, there exists some $\overline{m}$, such that no further restrictions are possible; $V^m = V^{\overline{m}}$, for all $m \geq \overline{m}$.

Definition 2. A game $\Gamma(\succ_I, V)$ is *dominance solvable* iff there exists a sequence $V \supset V^1 ... \supset V^{\overline{m}}$, s.t.

$$\forall i \in I, v_i, v'_i \in V_i^{\overline{m}}, v_{-i} \in V_{-i}^{\overline{m}}, \triangleright_t \in \triangleright : g(v_i, v_{-i}, \triangleright_t) = g(v'_i, v_{-i}, \triangleright_t).$$

In words, a game $\Gamma(\succ_I, V)$ is dominance solvable, iff in the restricted game $\Gamma(\succ_I, V^{\overline{m}})$, all players $I \cup \{t\}$ are indifferent between the outcomes associated with their individual strategies, holding opponents strategies fix.

[2] If one objects to the introduction of an additional player, another option would be to break ties by a multiplayer version of "matching pennies": ask each voter to report a number $t_i \in \{0, 1, .., 5\}$, set $t = \sum t_i \bmod 6$ and let each of the 6 possible outcomes $t = \{0, 1, ..., 5\}$ correspond to one of the 6 possible linear orders $\triangleright_t$. For our purposes, the two approaches are essentially equivalent, as neither the dummy player's set of possible reports, nor the voters' set of possible reports t_i can be reduced using elimination of weakly dominated strategies.

[3] Recall that $\succ_i$ is a strict linear order.

Where $\Gamma(\succ_I, V)$ is dominance solvable, we refer to the set of possible outcomes of $\Gamma(\succ_I, V^{\overline{m}})$ as *d-solution*, denoted as

$$ds(\succ_I, V) = \{x \in A | \exists v \in V^{\overline{m}}, \rhd_t \in \rhd : x = g(v, \rhd_t)\}.$$

Note that, $ds(\succ_I, V)$ is not necessarily a singleton:

Example 1. Consider a plurality voting game with two voters, $I = \{1, 2\}$ and preferences $a \succ_1 b \succ_1 c$, $b \succ_2 a \succ_2 c$. By Fact 1, voting for c is dominated for both voters, so $V_i^1 = \{(1,0,0),(0,1,0)\}$. But then, the game is reduced to a vote over just two alternatives, so that $V_1^2 = \{(1,0,0)\}$, $V_2^2 = \{(0,1,0)\}$. As all voters have just one remaining strategy, the game is dominance solvable. However, depending on t's choice of $\rhd_t$, the outcome of the game may be either a or b.

While in the above example, each voter is eventually left with a unique strategy, a game may also be deemed dominance solvable if a voter has multiple strategies remaining:

Example 2. Consider an approval voting game with two voters, and preferences $a \succ_1 b \succ_1 c$, $a \succ_2 c \succ_2 b$. By Fact 1, after eliminating dominated strategies, each voter is sure to vote for her first choice and sure not to vote for her least preferred candidate. But then, for all $v \in V^1$, the score of candidate a is $|v^a| = 2$ while $|v^b|, |v^c| \le 1$. Hence, candidate a is sure to win the election, no matter whether individual voters vote for their second most preferred candidate or not. Thus, the game is dominance solvable even though no further strategy can be eliminated based on weak dominance; effectively, each voter is left with two duplicate strategies in the game $\Gamma(\succ_I, V^1)$.

2.4. **Order Independence.** In defining dominance solvability, we followed Moulin [1979] in that we eliminated *all* weakly dominated strategies when moving from V^m to V^{m+1}.[4] This raises the question, whether a different order of elimination, where only some individuals' dominated strategies are eliminated at each step, might yield a different solution.

Fortunately, Marx and Swinkels [1997] assure us that this is not the case. More precisely, their Theorem 1 ensures that once we reach a restricted game $\Gamma(\succ_I, V')$ such that no further strategy can be eliminated based on weak dominance, $\Gamma(\succ_I, V')$ will be equivalent to $\Gamma(\succ_I, V^{\overline{m}})$ up to the elimination of duplicate strategies and the renaming of strategies. In particular, the set of possible outcomes of both games will be the same.

This is because, in our voting games, the elimination of dominated strategies satisfies what Marx and Swinkels [1997] call 'transference of decisionmaker indifference': whenever a voter i, for a given opposing strategy profile, is indifferent between outcomes $g(v_i, v_{-i}, \rhd_t)$ and $g(v_i', v_{-i}, \rhd_t)$, then so is every other player. This is of course satisfied, as i will only be indifferent if both outcomes coincide.[5]

3. Results

Our first result concerns a sufficient and necessary condition on $\succ_I$ for the Borda Count voting game $\Gamma_{1/2}(\succ_I, V)$ to be dominance solvable.

Proposition 1. *A candidate, say a, is the unique dominance solvable winner of a Borda Count voting game, if and only if there exists another candidate, say c, such that*

$$\mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| > |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|. \tag{1}$$

[4] Other authors in the context of voting theory, most notably [Farquharson, 1969], have used the same solution concept under the name of 'sophisticated voting'.

[5] Indifference of the dummy player does not transfer to indifference of other voters. However, this is unproblematic, as the principle of 'transference of decisionmaker indifference' is only required to hold for players whose strategies are eliminated (see Definition 2 in [Marx and Swinkels, 1997]).

Proof. We first prove sufficiency. For that, the selection of a as the unique winner is obtained in two steps. First, given (1), c is shown to never win the election once weakly dominated strategies have been eliminated. Second, in the restricted voting game with essentially two remaining candidates, condition (1) ensures that a wins against b.

After removing all strategies that are weakly dominated in the game $\Gamma_{1/2}(\succ_I, V)$, we are left with strategies $v_i \in V_i^1$ where each voter awards at least $\frac{1}{2}$ points to her most preferred candidate and at most $\frac{1}{2}$ to her least preferred candidate (see Fact 1). For these strategies we have

$$\min_{v \in V^1} |v^a| - |v^c| = 1/2|I_{abc}| - 1/2|I_{bac}| - 1/2|I_{acb}| - |I_{bca}| - |I_{cab}| - |I_{cba}|.$$

If $\min_{v\in V^1} |v^a| - |v^c| > 0$, then c is not among the possible outcomes of $\Gamma_{1/2}(\succ_I, V^1)$. Moreover, in the absence of voters I_{acb}, $\min_{v\in V^1} |v^a| - |v^c| > 0$ is equivalent to condition (1).

Next, assume that condition (1) holds and that there is a voter i_{acb} so that (1) is equivalent to

$$1/2 + 1/2|I_{abc}| - 1/2|I_{bac}| - 1/2|I_{acb}| - |I_{bca}| - |I_{cab}| - |I_{cba}| > 0$$

or simply $\min_{v\in V^1} \ |v^a| - |v^c| \geq 0$. Note that for any $v \in \operatorname{argmin}_{V^1} |v^a| - |v^c|$, $v_{i_{acb}} = (\frac{1}{2}, 0, 1)$. But this strategy is dominated in $\Gamma_{1/2}(\succ_I, V^1)$ by $\tilde{v}_i = (1, 0, \frac{1}{2})$:

(i) for a ballot profile such that $|v^a| \geq |v^c|$ and $|v^a| \geq |v^b| - \frac{1}{2}$, a switch of an individual voter i_{abc} from v_i to $\tilde{v}_i$ changes candidates' scores such that $|\tilde{v}^a| > |\tilde{v}^c|, |\tilde{v}^b|$ handing victory to a regardless of the choice of $\rhd_t$;

(ii) for a ballot profile such that $|v^a| \geq |v^c|$ and $|v^a| < |v^b| - \frac{1}{2}$, candidate b is the winner under the original ballot v_i, so that a switch to $\tilde{v}_i$ cannot be detrimental to i_{acb}.

Hence we can eliminate i_{acb}'s strategy $v_i = (\frac{1}{2}, 0, 1) \in V_i^1$ and move to a restricted game $\Gamma_{\frac{1}{2}}(\succ_I, V')$ where $V'_{i_{acb}} = V^1_{i_{acb}} \setminus \{(\frac{1}{2}, 0, 1)\} = \{(1, 0, \frac{1}{2}), (1, \frac{1}{2}, 0)\}$ and $V'_{i \neq i_{acb}} = V^1_{i \neq i_{acb}}$, so that in $\Gamma_{\frac{1}{2}}(\succ_I, V')$ outcome c is ruled out.[6]

Once we have ruled out the election of candidate c (after one or two rounds of eliminating some dominated strategies), the election comes down to a choice over candidates a and b. Hence, in the game $\Gamma_{\frac{1}{2}}(\succ_I, V')$, for any voter who prefers a over b, $v_i = (1, 0, 1/2) \in V'_i$ is a best reply for any opposing strategy profile $(v_{-i}, \rhd_t) \in V'_{-i} \times \rhd$ as it maximizes the impact that i has on $|v^a| - |v^b|$. If another ballot $\tilde{v}_i \neq v_i$ is also a best reply against any $(v_{-i}, \rhd_t)$, then $\tilde{v}_i$ and v_i are duplicate strategies. If on the other hand $\tilde{v}_i$ is a worse reply than v_i against some $(v_{-i}, \rhd_t)$, it is dominated and will be eliminated when moving to V'', defined by

$$V''_i = V'_i \setminus \{v_i \in V' | v_i \text{ is weakly dominated in } \Gamma(\succ_I, V')\}, \text{ for all } i \in I.$$

To determine the possible outcomes in $\Gamma_{\frac{1}{2}}(\succ_I, V'')$, we can assume that all $i \in I_{abc} \cup I_{acb} \cup I_{cab}$ cast ballot $v_i = (1, 0, 1/2)$ - any other remaining strategy would be a duplicate strategy and produce the same outcome. But then, a wins against b if

$$|I_{abc}| + |I_{acb}| + |I_{cab}| > |I_{bac}| + |I_{bca}| + |I_{cba}|. \tag{2}$$

Finally, (2) follows from condition (1) as

$$\begin{aligned}
|I_{abc}| + |I_{acb}| + |I_{cab}| &= |I_{abc}| + \mathbb{1}_{\{|I_{acb}|>0\}} - \mathbb{1}_{\{|I_{acb}|>0\}} + |I_{acb}| + |I_{cab}| \\
&> |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - \mathbb{1}_{\{|I_{acb}|>0\}} + |I_{acb}| + |I_{cab}| \\
&= |I_{bac}| + 2|I_{bca}| + 2|I_{cba}| \underbrace{- \mathbb{1}_{\{|I_{acb}|>0\}} + 2|I_{acb}|}_{\geq 0} + 3|I_{cab}| \\
&\geq |I_{bac}| + |I_{bca}| + |I_{cba}|
\end{aligned}$$

where condition (1) is used in the first inequality.

Next, we show that the condition is also necessary...[left out for now, as it is not needed for Proposition 2.] □

[6]Choosing to not eliminate all dominated strategies does not influence the set of outcomes in our final, maximally reduced game, see subsection 2.4.

In light of Proposition 1, it is natural to ask how the Borda Count compares to other scoring rules; how often are the corresponding voting games dominance solvable and when does the associated winner of the election conform with our intuition on what is the 'best candidate' given voters' preferences? For that, we define two desirable properties a voting scheme might satisfy.

Definition 3. A voting scheme is said to satisfy *Unanimity (**U**)*, iff for any set of voters $I = I_{abc} \cup I_{acb}$, candidate a is the unique dominance solvable winner of the associated voting game $\Gamma(\succ_I, V)$.

A voting scheme is said to satisfy *Independence of Least preferred Alternatives (**ILA**)*, iff for any set of voters such that $I = I_{abc} \cup I_{bac}$ and $|I_{abc}| > |I_{bac}|$, candidate a is the unique dominance solvable winner of the associated voting game $\Gamma(\succ_I, V)$.

Unanimity significantly weakens pareto-efficiency. Independence of least preferred alternatives demands, that whenever an alternative is unanimously considered worst, the voting game reduces to a pairwise choice between the remaining candidates that is decided by simple majority.

Corollary 1. *The Borda Count satisfies both **U** and **ILA**.*

Proof. Assume that $I = I_{abc} \cup I_{acb}$. Without loss of generality, we can assume $|I_{abc}| \geq |I_{acb}|$. By Proposition 1, a is the unique dominance solvable winner as $\mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| > |I_{acb}| = |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|$.

Assume on the other hand that $I = I_{abc} \cup I_{bac}$ and $|I_{abc}| > |I_{bac}|$. Then by Proposition 1, a is the unique dominance solvable winner, as $\mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| = |I_{abc}| > |I_{bac}| = |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|$. □

The above corollary also follows from results in Buenrostro et al. [2013]. The only extension with respect to their Theorem 1 and Theorem 2, is the inclusion of the case $I = I_{abc} \cup I_{acb}$, $|I_{abc}| = |I_{acb}|$. What might be remarkable though, is the exceptional position among scoring rules that Corollary 1 grants to the Borda Count:

Proposition 2. *The Borda Count is the unique scoring rule that satisfies **U** and **ILA**. In particular, scoring rules with $s < \frac{1}{2}$ violate unanimity, while scoring rules with $s > \frac{1}{2}$ violate independence of least least preferred alternatives.*

Proof. We first consider scoring rules with $s < \frac{1}{2}$ and show that for any given such s, there exist preference profiles with $I = I_{abc} \cup I_{acb}$, where the induced voting game fails to elect a after iterative elimination of dominated strategies.

Assume that $|I_{abc}| = |I_{acb}| = n > \frac{2-2s}{1-2s} \geq 2$. We will show that the ballot profile v, given by $v_{i_{abc}} = (s, 1, 0)$ and $v_{i_{acb}} = (s, 0, 1)$ respectively, survives the iterative elimination of dominated strategies.

Consider $\Gamma(\succ_I, V^1)$ and assume that all voters chose $v_{i_{abc}} = (s, 1, 0)$ and $v_{i_{acb}} = (s, 0, 1)$ respectively. Then $|v^b| = |v^c| = n$ while $|v^b| - |v^a| = n - 2ns = n(1 - 2s) > 2 - 2s > 1$. Thus, the winner is either b or c, depending on $\rhd_t$. If i_{abc} would switch to a different strategy, $(1, s, 0), (1, 0, s) \in V^1_{i_{abc}}$, that awards fewer points to candidate b, c would win the election independent of $\rhd_t$. Hence, neither $(1, s, 0)$ nor $(1, 0, s)$ dominate $(s, 1, 0)$ for voter i_{abc}, so that $v_{i_{abc}} = (s, 1, 0) \in V^2_{i_{abc}}$.

A symmetric argument applies to i_{acb} so that $v_{i_{acb}} = (s, 0, 1) \in V^2_{i_{acb}}$. But then, we can again consider the strategy profile v in $\Gamma(\succ_I, V^2)$ and show that neither strategy is dominated and eliminated as we move to V^3. By induction it follows that the two strategies are never eliminated.

Moreover, we have already seen that for strategy profile v, candidate a does not win the election which concludes the proof for the case $s < \frac{1}{2}$.

Next, we consider the case of Antiplurality, i.e. $s = 1$. Assume that all voters agree on the ranking $a \succ_i b \succ_i c$, so that $V^1_i = \{(1, 1, 0), (1, 0, 1)\}$. If in $\Gamma(\succ_I, V^1)$ all voters $j \neq i$ chose

$v_j = (1,1,0)$, then i can ensure the election of a by casting the ballot $v_i = (1,0,1)$, whereas $v_i' = (1,1,0)$ would lead to the election of b whenever $b \rhd_t a$. Hence, $(1,0,1)$ is not dominated. Similarly, if all $j \neq i$ cast ballot $v_j = (1,0,1)$ and the dummy player choses $b \rhd_t a$, i's unique best reply is $v_i = (1,1,0)$. But then, voters' strategy sets cannot be narrowed down any further than $V_i^1 = \{(1,1,0),(1,0,1)\}$, so that a is not the unique dominance solvable winner.

Last, consider the case $s \in (\frac{1}{2},1)$. Assume that $I = I_{abc} \cup I_{bac}$ and $|I_{abc}| = n+1 > n = |I_{bac}|$ with $n > \max\{\frac{1}{2s-1},\frac{1}{1-s}\} \geq 3$. We will show that in process of iterative elimination, strategies $v_{i_{abc}} = (1,s,0),(1,0,s)$ and $v_{i_{bac}} = (s,1,0),(0,1,s)$ are never weakly dominated and hence not eliminated. But then, b remains a possible outcome throughout the sequence of restricted games: if all i_{abc} vote $(1,s,0)$ while all i_{bac} vote $(0,1,s)$, candidates scores are

$$|v^a| = n+1, \qquad |v^b| = n + s(n+1), \qquad |v^c| = sn.$$

As $s > \frac{1}{2}$ and $n > 3$, candidate b then wins the election.

First, let us remind ourselves, that the sets of undominated strategies in $\Gamma_s(\succ_I, V)$ are

$$V_{abc}^1 = \{(1,s,0),(1,0,s),(s,1,0)\} \text{ and } V_{bac}^1 = \{(s,1,0),(0,1,s),(1,s,0)\}.$$

To show that $\{(1,s,0),(1,0,s)\} \subseteq V_{abc}^{m+1} \subseteq V_{abc}^m \subseteq V_{abc}^1$ and $\{(s,1,0),(0,1,s)\} \subseteq V^{m+1} \subseteq V_{bac}^m \subseteq V_{bac}^1$ for all $m > 1$ we consider 6 cases.

(i) *For i_{abc}, $(1,s,0)$ can be a better reply than $(1,0,s)$ in $\Gamma_s(\succ_I, V^m)$:* Consider the situation of $i = i_{abc}$ who faces an opposing strategy profile where

- $n-x$ voters $j \in I_{abc}$ vote $v_j = (1,s,0)$,
- x voters $j \in I_{abc}$ vote $v_j = (1,0,s)$,
- all n voters $j \in I_{bac}$ vote $v_j = (0,1,s)$,
- $x = \lceil \frac{n}{2s} - \frac{1}{2} \rceil$,
- $c \rhd_t b$.

This profile is well defined, as $n > x$: our restriction in n yields

$$(*) \qquad n > \frac{1}{2s-1} \quad \Rightarrow \quad 2sn - n > 1 \quad \Rightarrow \quad s > \frac{n+1}{2n}$$

so that

$$x = \left\lceil \frac{n}{2s} - \frac{1}{2} \right\rceil < \frac{n}{2s} + \frac{1}{2} < \frac{n}{2\frac{n+1}{2n}} + \frac{1}{2} = \frac{n^2}{n+1} + \frac{1}{2} = \frac{2n^2+n+1}{2n+2} < \frac{2n^2+2n}{2n+2} = n$$

If i chooses $v_i = (1,s,0)$, the associated candidates' scores are $|v^a| = n+1$, $|v^b| = s(n-x+1)+n$ and $|v^c| = s(n+x)$. Then, b wins, as its score is larger than a's (as $n > x$) and larger than c's:

$$|v^b| - |v^c| = n + s - 2sx > n + s - 2s\left(\frac{n}{2s} + \frac{1}{2}\right) = 0.$$

If on the other hand i choses $v_i = (1,0,s)$, b's score is at most as high as c's, so that b never wins (ties are broken in favour of c):

$$|v^b| - |v^c| = n - s - 2sx \leq n - s - 2s\left(\frac{n}{2s} - \frac{1}{2}\right) = 0.$$

Moreover, c would also beat i's preferred candiate a, as

$$|v^a| - |v^c| = n + 1 - sn - sx \leq n + 1 - sn - s\left(\frac{n}{2s} - \frac{1}{2}\right) = \underbrace{\frac{n+1}{2}}_{<sn, \text{ see } (*)} - sn < 0.$$

(ii) *For i_{abc}, $(1,s,0)$ can be a better reply than $(s,1,0)$ in $\Gamma_s(\succ_I, V^m)$:* (This case is only relevant if $(s,1,0) \in V_i^m$) Consider the situation of $i = i_{abc}$ who faces an opposing strategy profile where

- n voters $j \in I_{abc}$ vote $v_j = (1,s,0)$,
- n voters $j \in I_{bac}$ vote $v_j = (s,1,0)$.

Ballot $v_i = (1, s, 0)$ would then tip the balance in favour of a, whereas b would win, should i choose $(s, 1, 0)$.

Together, (i) and (ii) imply, that $(1, s, 0) \in V_{abc}^{m+1}$. Next, we show that $(1, 0, s) \in V_{abc}^{m+1}$.

(iii) *For i_{abc}, $(1, 0, s)$ can be the unique best reply in $\Gamma_s(\succ_I, V^m)$:* Consider the situation of $i = i_{abc}$ who faces an opposing strategy profile where

- n voters $j \in I_{abc}$ vote $v_j = (1, s, 0)$,
- 1 voter $j \in I_{bac}$ vote $v_j = (0, 1, s)$,
- $n - 1$ voters $j \in I_{bac}$ vote $v_j = (s, 1, 0)$.

Ballot $v_i = (1, 0, s)$ would then tip the balance in favour of a, as $|v^a| - |v^b| = 1 - s > 0$. Should i choose $(1, s, 0)$, b would win as we would have $|v^a| - |v^b| = 1 - 2s < 0$. Ballot $v_i = (s, 1, 0)$ would only further increase b's lead over a.

(iv) *For i_{bac}, $(0, 1, s)$ can be the unique best reply in $\Gamma_s(\succ_I, V^m)$:* Consider the situation of $i = i_{bac}$ who faces an opposing strategy profile where

- $n + 1$ voters $j \in I_{abc}$ vote $v_j = (1, s, 0)$,
- $n - 1$ voters $j \in I_{bac}$ vote $v_j = (s, 1, 0)$.

Ballot $v_i = (0, 1, s)$ would then tip the balance in favour of b, as $|v^a| - |v^b| = 1 - 2s < 0$. Should i choose $(s, 1, 0)$, a would win, as we would have $|v^a| - |v^b| = 1 - s > 0$. Ballot $v_i = (1, s, 0)$ would only further increase a's lead over b.

From (iv) we learn that $(0, 1, s) \in V_{bac}^{m+1}$. The last two cases establish that that $(s, 1, 0) \in V_{bac}^{m+1}$, which concludes the proof.

(v) *For i_{bac}, $(s, 1, 0)$ can be a better reply than $(1, s, 0)$ in $\Gamma_s(\succ_I, V^m)$:* (only relevant if $(1, s, 0) \in V_{i_{bac}}^m$) Consider the situation of $i = i_{bac}$ who faces an opposing strategy profile where

- $n + 1$ voters $j \in I_{abc}$ vote $v_j = (1, s, 0)$,
- x voters $j \in I_{bac}$ vote $v_j = (1, s, 0)$,
- 1 voters $j \in I_{bac}$ vote $v_j = (0, 1, s)$,
- $n - 2 - x$ voters $j \in I_{bac}$ vote $v_j = (s, 1, 0)$,
- $x = \left\lceil \frac{4s-3}{2-2s} \right\rceil \geq 0$,
- $a \rhd_t b$.

This profile is well defined, as $x < n - 2$:

$$x = \left\lceil \frac{4s - 3}{2 - 2s} \right\rceil < \frac{4s - 3}{2 - 2s} + 1 = \frac{2s - 1}{2 - 2s} < \frac{2s - 1}{1 - s} = \frac{1}{1 - s} - 2 < n - 2$$

If i choses $v_i = (s, 1, 0)$, her favourite candidate b wins the election as its score is larger than $|v^c| = s$ and larger than a's score $|v^a|$:

$$|v^a| - |v^b| = 1 - 2s + (2 - 2s)x < 1 - 2s + (2 - 2s)\left(\frac{4s - 3}{2 - 2s} + 1\right) = 1 - 2s + 4s - 3 + 2 - 2s = 0.$$

If on the other hand, i chooses $v_i = (1, s, 0)$, b's score is weakly less than a's:

$$|v^a| - |v^b| = 3 - 4s + (2 - 2s)x \geq 3 - 4s + (2 - 2s)\left(\frac{4s - 3}{2 - 2s}\right) = 0.$$

As ties are broken in favour of a, b would lose the election.

(vi) *For i_{bac}, $(s, 1, 0)$ can be a better reply than $(0, 1, s)$ in $\Gamma_s(\succ_I, V^m)$:* Consider the situation of $i = i_{bac}$ who faces an opposing strategy profile where

- $n + 1$ voters $j \in I_{abc}$ vote $v_j = (1, 0, s)$,
- $n - x - 1$ voters $j \in I_{bac}$ vote $v_j = (s, 1, 0)$,
- x voters $j \in I_{bac}$ vote $v_j = (0, 1, s)$,
- $x = \left\lceil \frac{n+1}{2s} - \frac{3}{2} \right\rceil$,
- $c \rhd_t a$.

This profile is well defined, since $n > \max\{\frac{1}{2s-1}, \frac{1}{1-s}\} \geq 3$ (and hence $s > \frac{n+1}{2n}$, see (*)) yield

$$x = \left\lceil \frac{n+1}{2s} - \frac{3}{2} \right\rceil < \frac{n+1}{2s} - \frac{1}{2} < \frac{n+1}{2\frac{n+1}{2n}} - \frac{1}{2} = n - \frac{1}{2}$$

and

$$x \geq \frac{\overbrace{n+1}^{>4}}{\underbrace{2s}_{<2}} - \frac{3}{2} > 0.$$

If i choses $v_i = (s, 1, 0)$, the associated candidates' scores are $|v^a| = n + 1 + s(n - x)$, $|v^b| = n$ and $|v^c| = s(n + 1 + x)$. Hence, a wins the election as it has a higher score than b and c:

$$|v^a| - |v^c| = n + 1 - s - 2sx > n + 1 - s - 2s\left(\frac{n+1}{2s} - \frac{1}{2}\right) = 0.$$

If on the other hand, i chooses $v_i = (0, 1, s)$, a's score is weakly less than c's:

$$|v^a| - |v^c| = n + 1 - 3s - 2sx \leq n + 1 - 3s - 2s\left(\frac{(n+1)}{2s} - \frac{3}{2}\right) = 0.$$

As a tie would be broken in favour of c, and $|v^c| \geq |v^a| \geq n+1 > n = |v^b|$, c would win the election. □

Looking beyond scoring rules, the situation is not as bleak. In particular, the additional flexibility of generalized scoring rules, allows some of them to simultaneously satisfy unanimity and independence of least preferred alternative. A case in point is Approval Voting, for which Nunez and Courtin [2013] provide necessary and sufficient conditions for $\Gamma_{AV}(\succ_I, V)$ to be dominance solvable.

However, Approval Voting fails a desirable monotonicity property:

Example 3. Consider a prefernce profile $\succ_I$ where $I = I_{abc} \cup I_{bac} \cup I_{cba}$ and

$$|I_{abc}| = 4, \qquad |I_{bac}| = 2, \qquad |I_{cba}| = 3.$$

After eliminating dominated strategies, it is clear that a will have a score of at least 4, while the score of c is at most 3. This reduces the game further, to an election between a and b, which b wins with a score of $5:4$. Hence b is the unique dominance solvable winner of $\Gamma_{AV}(\succ_I, V)$.

But, if b increases in popularity, so that we now have $\succ'_I$ with $I = I'_{abc} \cup I'_{bac} \cup I'_{cba}$ and $|I'_{abc}| = |I'_{bac}| = |I'_{cba}| = 3$, candidate c is not sure to lose against a so that the game cannot be reduced to an election between a and b. No other candidate is sure to lose either, so that by results in Nunez and Courtin [2013], we know that $\Gamma_{AV}(\succ'_I, V)$ is not dominance solvable.

In contrast to Approval Voting, the Borda Count satisfies monotonicity:

Proposition 3. *If a candidate, say a, is the unique dominance solvable winner in a Borda Count voting game $\Gamma_{\frac{1}{2}}(\succ_I, V)$, then a is also the unique dominance solvable winner in any other Borda Count voting game $\Gamma_{\frac{1}{2}}(\succ'_I, V)$, where $\succ_I$ and $\succ'_I$ differ in that one $i \in I$ ranks a higher, while keeping the order of b and c fixed.*

Proof. As a is the unique dominance solvable winner in $\Gamma_{\frac{1}{2}}(\succ_I, V)$, we know from Proposition 1 that there is another candidate, say c, such that

$$\mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| > |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|.$$

As we move from $\succ_I$ to $\succ'_I$, one $i \in I$ ranks a higher. Denote by $I'_{xyz} = \{i \in I | x, y, z \in A, x \succ'_i y \succ'_i z\}$ the new subsets of voter, characterized by their preferences over candidates. We will show that the above condition also holds for the new subsets of voters (or rather their cardinalities), so that a remains the unique dominance solvable winner.

(i) Assume that in the profile $\succ_I$, $i \in I_{bac}$, while in the profile $\succ'_I$ she ranks a higher, i.e. $i \in I'_{abc}$. Then $|I'_{bac}| = |I_{bac}| - 1$ and $|I'_{abc}| = |I_{abc}| + 1$; for all other subsets of voters, we have $|I_{xyz}| = |I'_{xyz}|$. Hence,

$$\begin{aligned}\mathbb{1}_{\{|I'_{acb}|>0\}} + |I'_{abc}| = \mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| + 1 &> |I_{bac}| + 1 + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| \\ &> |I'_{bac}| + |I'_{acb}| + 2|I'_{bca}| + 2|I'_{cab}| + 2|I'_{cba}|.\end{aligned}$$

(ii) Now, $i \in I_{cab} \cap I'_{acb}$. Then $|I'_{cab}| = |I_{cab}| - 1$ and $|I'_{acb}| = |I_{acb}| + 1$; for all other subsets of voters, we have $|I_{xyz}| = |I'_{xyz}|$. Hence,

$$\begin{aligned}\mathbb{1}_{\{|I'_{acb}|>0\}} + |I'_{abc}| \geq \mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| &> |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| \\ &> |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2(|I_{cab}| - 1) + 2|I_{cba}| \\ &= |I'_{bac}| + |I'_{acb}| + 2|I'_{bca}| + 2|I'_{cab}| + 2|I'_{cba}|.\end{aligned}$$

(iii) $i \in I_{bca} \cap I'_{bac}$. Then $|I'_{bca}| = |I_{bca}| - 1$ and $|I'_{bac}| = |I_{bac}| + 1$, so that

$$\begin{aligned}\mathbb{1}_{\{|I'_{acb}|>0\}} + |I'_{abc}| = \mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| &> |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| \\ &> |I_{bac}| + 1 + |I_{acb}| + 2(|I_{bca}| - 1) + 2|I_{cab}| + 2|I_{cba}| \\ &= |I'_{bac}| + |I'_{acb}| + 2|I'_{bca}| + 2|I'_{cab}| + 2|I'_{cba}|.\end{aligned}$$

(iv) $i \in I_{bca} \cap I'_{abc}$. Then $|I'_{bca}| = |I_{bca}| - 1$ and $|I'_{abc}| = |I_{abc}| + 1$, so that

$$\begin{aligned}\mathbb{1}_{\{|I'_{acb}|>0\}} + |I'_{abc}| = \mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| + 1 &> |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| \\ &> 1 + |I_{bac}| + |I_{acb}| + 2(|I_{bca}| - 1) + 2|I_{cab}| + 2|I_{cba}| \\ &= |I'_{bac}| + |I'_{acb}| + 2|I'_{bca}| + 2|I'_{cab}| + 2|I'_{cba}|.\end{aligned}$$

(v) $i \in I_{cba} \cap I'_{cab}$. Then $|I'_{cba}| = |I_{cba}| - 1$ and $|I'_{cab}| = |I_{cab}| + 1$, so that

$$\begin{aligned}\mathbb{1}_{\{|I'_{acb}|>0\}} + |I'_{abc}| = \mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| &> |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| \\ &= |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2(|I_{cab}| + 1) + 2(|I_{cba}| - 1) \\ &= |I'_{bac}| + |I'_{acb}| + 2|I'_{bca}| + 2|I'_{cab}| + 2|I'_{cba}|.\end{aligned}$$

(vi) $i \in I_{cba} \cap I'_{acb}$. Then $|I'_{cba}| = |I_{cba}| - 1$ and $|I'_{acb}| = |I_{acb}| + 1$, so that

$$\begin{aligned}\mathbb{1}_{\{|I'_{acb}|>0\}} + |I'_{abc}| \geq \mathbb{1}_{\{|I_{acb}|>0\}} + |I_{abc}| &> |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| \\ &> |I_{bac}| + |I_{acb}| + 1 + 2|I_{bca}| + 2|I_{cab}| + 2(|I_{cba}| - 1) \\ &= |I'_{bac}| + |I'_{acb}| + 2|I'_{bca}| + 2|I'_{cab}| + 2|I'_{cba}|.\end{aligned}$$

□

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