Strategy-proof and efficient location of public facilities*

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Abstract

Agents frequently have different opinions on the decision of where to locate a public facility: while some agents may prefer to have it closer to them, others may prefer to have it far away. To aggregate agents' preferences in these cases, we propose a new domain of preferences in which agents may have single-peaked or single-dipped preferences on the location of the facility, but such that the peak or dip is situated in the location of the agent. We characterize all strategy-proof rules in this domain and we show that all these rules are also group strategy-proof. We show that this family allows us to escape from the classical impossibility result of Gibbard and Satterthwaite with meaningful rules in almost all cases. Additionally, we characterize the subfamilies of rules that are also Pareto efficient in some focal cases.

Keywords: Single-peaked, single-dipped, social choice rule, strategy-proofness, Pareto efficiency.

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1 Introduction

Governments have to decide where to locate public facilities like schools, hospitals, prisons, nuclear plants, or industrial parks. To select a location for a particular facility, the public decision makers have to take into account not only technical constraints but also the preferences of the different agents in the society.

Since preferences are private information and agents have incentives to reveal them truthfully or not depending on how this information is incorporated in the selection of the final location, the objective is to construct social choice rules that, apart from having other good properties, induce agents to reveal their true preferences. This property is known as *strategy-proofness*. Unfortunately, Gibbard [8] and Satterthwaite [11] have shown in their famous impossibility result that no *relevant* strategy-proof rules can be constructed when preferences are unrestricted:

Result 1 *If all ordinal preferences over the feasible locations are admissible, all strategy-proof social choice rules with range greater than 2 are dictatorial.*[1]

Result 1 highlights the impossibility of combining strategy-proofness and a range greater than 3 without arriving at a dictatorship. The technical condition of having a range greater than 3 can be substituted by the more intuitive notion of Pareto efficiency without altering the impossibility result.

Result 2 *If all ordinal preferences over the feasible locations are admissible, all Pareto efficient and strategy-proof social choice rules are dictorial.*[2]

Given these negative results, the literature has concentrated on constructing meaningful strategy-proof social choice rules when some preferences over the set of alternatives are not admissible. In

[1]A dictatorial social choice rule is one that always selects a maximal alternative of a particular agent (the dictator), independently of the preferences of the other agents.

[2]There are two implicit assumptions in this result. On the one hand, there should be at least three feasible alternatives because if there are only two, there exist rules that satisfy both criteria. On the other hand, it is needed that the set of maximal alternatives for each admissible preference is well–defined because in other case, Pareto efficiency can not be attained even without requiring strategy-proofness.

fact, if the preference domain is restricted in a natural way, then we can find strategy-proof social choice rules that escape from the impossibilities of Results 1 and 2. See Barberà [1] for a survey. The most well–known preference restriction for the location of public facilities is the one of single-peaked preferences (see Moulin [10]) that appears naturally when the facility to be located is a public good. Formally, an agent has single-peaked preferences on the real line if she has an optimal point and the further one moves away from this optimal point (maintaining the direction), the worse the locations get for this agent. For the domain of all single-peaked preferences, Moulin [10] and Barberà and Jackson [5] show that all strategy-proof rules are generalized median voter rules. So, in this case one can escape from Results 1 and 2.

The dual of single-peaked preferences are the single-dipped preferences. Formally, an agent has single-dipped preferences on the real line if she has a worst possible location and the further one moves away from this worst location (maintaining the direction) the better the locations get for this agent. These preferences appear naturally when the facility to be located is a public bad. For the domain of single-dipped preferences, Manjunath [9] has shown that all strategy-proof and unanimous rules necessarily have to select an extreme element of the set of alternatives (see also Barberà et al. [3]). Given that all these rules are also Pareto efficient, it is possible to escape from Result 2 in this domain, but the range of these rules is still equal to 2.

Although the before mentioned frameworks are interesting ones, many public facilities cannot be accommodated by either of them because agents differ in their evaluation: some may consider that a particular facility is a good and want to have it built as close as possible to their own locations, but others prefer to have it as far away as possible (they consider it a bad). Some examples are: (i) dog parks that people could prefer to have closer to their homes or further away depending on whether or not they own a dog; (ii) cell towers that people could prefer to have closer or further away depending on whether they believe, contrary to the scientific evidence, that these facilities can produce cancer or not; (iii) industrial parks that municipalities could prefer to have closer to the city/town/village or not depending on whether they consider that this facility has positive/negative externalities to its model of economic development; (iv) soccer stadiums that people could prefer to have closer to them or further away depending on whether they are season ticket holders of the team that plays there or they do not like soccer.

Given that some agents may like the facility but others not, the set of admissible preferences to treat these cases has to include both single-peaked and single-dipped preferences. However, unfortunately, it is easy to show that the impossibilities of Results 1 and 2 are maintained if we allow agents to express any single-peaked or any single-dipped preference; i.e. if the domain considered is the union of the domains of single-peaked and single-dipped preferences.[3] However, the examples above are situations in which agents typically have single-peaked or single-dipped preferences with the peak or dip situated in their locations (their homes, their municipalities, ...), and this fact can then be used to construct a domain in which we can escape from the impossibilities with interesting strategy-proof social choice rules. If the location of the agents is known by the social planner, something that frequently occurs (for example, the regional government knows where each municipality is located or local governments know the residence of each person), we propose to consider a domain in which the set of admissible preferences of each agent corresponds to all single-peaked preferences with the peak in her location and all single-dipped preferences with the dip in her location. We will characterize all strategy proof rules and show that this new domain allows us to escape from the impossibilities of both Results 1 and 2.

In particular, we will show that all strategy-proof rules share a common structure whose main aspects are: (*i*) in a first step, all agents are asked only about their type of preferences (single-peaked or single-dipped) and, depending on the set of agents that have single-peaked preferences, at most two locations are preselected; (*ii*) if only one alternative has been preselected, then this alternative is finally chosen; (*iii*) finally, if two alternatives has been preselected, then all agents that are situated strictly between the two preselected alternatives have to indicate their ordinal preference over them (and, in case of indifference, their entire preferences) and, depending on the

[3]To see why, observe that an agent with single-dipped preferences will manipulate any generalized median voter rule in some instances. See Berga and Serizawa [7] for additional results showing the difficulty of expanding the domain of single-peaked preferences without arriving at the impossibilities.

answers, it is decided which of the two preselected alternatives is finally chosen.

The particular ways to pick the preselected locations that pass to the second phase and the form in which the final location is chosen between them have to satisfy some monotonicity conditions. These conditions are generally not too restrictive and allows to many possibilities, which constitutes the differences in the characterized rules. This allows the social planner to incorporate other aspects of the problem (technical, economic, ecological, ...) to the social choice rule. We will also show that all rules that cannot be manipulated by isolated agents cannot be manipulated by groups of coordinated agents either; i.e., we will show that all these rules are also *group strategy-proof*.

The remainder of the paper is organized as follows. Section 2 introduces the necessary notation and definitions. Section 3 shows some conditions that defines the main structure of the strategy-proof social choice rules. Section 4 provides a complete characterization of all strategy-proof rules in this new domain and shows that group strategy-proofness is implied in this case by strategy-proofness. Section 5 explains when this characterized family allows us to escape from the impossibilities of Results 1 and 2. Section 6 characterizes the rules that satisfy strategy-proofness and Pareto efficiency for some focal cases. Section 7 concludes. All proofs are relegated to the Appendix.

2 Notation and definitions

Consider a social planner who wants to locate a public facility in a point on a set $T \subseteq \mathbb{R}$ of feasible locations. There is a finite group of agents N (with size at least 2), and each agent belonging to N is located at a point on the real line. We indicate the agent located at $i \in \mathbb{R}$ by $i \in N$.[4] For the moment, we do not impose any restriction on N, on T or on the relation between them. For example, T can be finite or infinite, and we could have that all agents are located at feasible locations ($N \subseteq T$) or at unfeasible ones ($N \cap T = \emptyset$).

[4]This notation assumes implicitly that there is at most one agent at any point of the real line. As it can be seen from the proofs, the results of the paper still hold when allowing for multiple agents at a single point. We opted for the simple notation, because it is intuitive to follow.

Let R_i be the weak preference relation of agent $i \in N$ on T. Formally, R_i is a complete and transitive binary relation. P_i and I_i are the strict and the indifference preference relation induced by R_i. We then say that R_i is a single-peaked preference with peak i if for all $x, y \in T$ such that $i \geq x > y$ or $i \leq x < y$, we have that $x \, P_i \, y$. Similarly, R_i is a single-dipped preference with dip i if for all $x, y \in T$ such that $i \geq x > y$ or $i \leq x < y$, we have that $y \, P_i \, x$.[5] The preference domain of agent i is $\mathcal{R}_i = \mathcal{R}_i^+ \cup \mathcal{R}_i^-$, where $\mathcal{R}_i^+$ ($\mathcal{R}_i^-$) is the set of all single-peaked (single-dipped) preferences with peak (dip) in i. Observe that the intersection between the preference domains of two agents is empty.

A preference profile is a set of preferences $R = (R_i)_{i \in N}$. The domain of all admissible preference profiles is denoted by $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Let $\mathcal{R}^A$ be the set of preference profiles such that only the agents in $A \subseteq N$ have single-peaked preferences. Sometimes we will write $\mathcal{R}_i^A$ to indicate the domain of single-peaked preferences for agent i if $i \in A$, or the domain of single-dipped preferences for agent i if $i \notin A$. Similarly, R_S and R_{-S} are the restrictions of R to the agents in $S \subset N$ and $(N \setminus S)$, respectively. We will write R_{-i} instead of $R_{-\{i\}}$.

The following concepts will be useful in the course of our analysis. A non-ordered pair of alternatives $\{x, y\}$ is a fixed pair for agent i if for all $a, b \in \{x, y\}$ such that $a < b$, we cannot have that $a < i < b$. Observe that given any type of preferences of agent i (single-peaked or single-dipped), if $\{x, y\}$ is a fixed pair for i, then this agent has always the same ordinal preferences over $\{x, y\}$. Similarly, the ordered pair of alternatives (x, y) is said to be a fixed pair for agent i at $\mathcal{R}_i^A$ if for all $R_i \in \mathcal{R}_i^A$, $x \, P_i \, y$. Or, to say it differently, (x, y) is a fixed pair for i at $\mathcal{R}_i^A$ if $i \in A$ and $[y < x \leq i$ or $y > x \geq i]$, or if $i \notin A$ and $[i \leq y < x$ or $i \geq y > x]$. Finally, we can see that if $\{x, y\}$ is a fixed pair for agent i, then (x, y) is a fixed pair for one type of preferences of i and (y, x) is a fixed pair for the other type.

The solution concept is a *social choice rule*, a function $f : \mathcal{R} \to T$ that selects for each preference

[5] Technically speaking, these preferences only have a maximal/minimal alternative at i if $i \in T$. Otherwise, its most/less preferred feasible alternative would be the closest one situated at its left or the closest one situated at its right. This difference will play an important role in Sections 5 and 6.

profile $R \in \mathcal{R}$ a point $f(R) \in T$. We denote the range of f in the domain $\mathcal{R}^A$ by $R_f(A)$. We now introduce some properties on social choice rules. We say that f is *manipulable by agent* $i \in N$ if she can benefit from misrepresenting her preferences; that is, if there is a preference profile $R \in \mathcal{R}$ and some alternative preference $R'_i \in \mathcal{R}_i$ such that $f(R'_i, R_{-i})\, P_i\, f(R)$. Then, f is *strategy-proof* if it is not manipulable by any agent. Similarly, f is *group strategy-proof* if for all $R \in \mathcal{R}$, there is no $S \subseteq N$ and $R'_S \in \mathcal{R}_S$ such that $f(R'_S, R_{-S})\, P_i\, f(R)$ for all $i \in S$. The social choice rule f is *Pareto efficient* if for all $R \in \mathcal{R}$, there is no $x \in T$ such that $x\, R_i\, f(R)$ for all $i \in N$ and $x\, P_j\, f(R)$ for some $j \in N$. Finally, f is *dictatorial* if there exists an agent $i \in N$ (called the dictator) such that for all $R \in \mathcal{R}$, $f(R)\, R_i\, x$ for all $x \in \bigcup_{A \subseteq N} R_f(A)$.

3 The main structure of the strategy-proof rules

In this section, we analyze some conditions that a strategy-proof social choice rule has to satisfy. The union of these conditions will give us the general structure of the strategy-proof rules. In Section 4, we will explore this structue further to obtain additional conditions that then lead to a complete characterization.

A necessary condition on the range of f

Our first result is a necessary condition on the range of f that facilitates the further analysis. The condition states that if a social choice rule is strategy-proof, its range can include at most two different alternatives for a given set of agents A with single-peaked preferences.

Proposition 1 *If f is strategy-proof, $|R_f(A)| \leq 2$ for all $A \subseteq N$.*

Proposition 1 shows that any strategy-proof rule can be divided into two steps. In the first step, agents have to declare their types of preferences and depending on the set of agents with single-peaked preferences A, one or two locations are preselected. If only one alternative is preselected, this alternative is finally implemented. If two alternatives are preselected, the alternative that is finally implemented has to be determined in the second step of the procedure. The following example illustrates that there are strategy-proof social choice rules that select two different alternatives for a given set of agents with single-peaked preferences.

Example 1 *Suppose that $T = \{0, 1, 3\}$ and $N = \{2, 3\}$. Let the social choice rule f be such that for all preference profiles $R \in \mathcal{R}^A$, $f(R) = 0$ whenever $3 \notin A$, $f(R) = 1$ whenever $3 \in A$ and $1\, P_2\, 3$, and $f(R) = 3$ whenever $3 \in A$ and $3\, R_2\, 1$. It is easy to see that f is strategy-proof and that $|R_f(A)| = 2$ whenever $3 \in A$.*

Since $R_f(A)$ contains at most two preselected locations, we indicate by $l_f(A)$ and $r_f(A)$ the elements of $R_f(A)$ such that $l_f(A) \leq r_f(A)$. Then, $S_f(A) = N \cap (l_f(A), r_f(A))$ corresponds to the set of agents that are located strictly between the preselected alternatives.

Decisive sets

Obviously, if $l_f(A) = r_f(A)$, this alternative will be finally selected. Thus, we focus throughout this subsection on the cases when $l_f(A) < r_f(A)$ in order to derive several conditions the second step of a strategy-proof rule has to satisfy.[6] To do so, let $f_A : \mathcal{R}^A \to R_f(A)$ be the binary decision function that chooses between $l_f(A)$ and $r_f(A)$ if the set of agents that declared to have single-peaked preferences is equal to A. The next proposition establishes that only the preferences of the agents belonging to $S_f(A)$ can affect the outcome of f_A. The intuition of this result is as follows: since the two preselected locations form an ordered pair for all agents located to the left or to the right of both $l_f(A)$ and $r_f(A)$, the binary decision function f_A must be independent of these preferences in order to guarantee the strategy-proofness of f.

Proposition 2 *If f is strategy-proof, then for all preference profiles $R, R' \in \mathcal{R}^A$ such that $R_{S_f(A)} = R'_{S_f(A)}$, $f_A(R) = f_A(R')$.*

As a corollary of Proposition 2, we can reduce the domain of f_A from $\mathcal{R}^A$ to $\mathcal{R}_{S_f(A)}$. Also, if $l_f(A) \neq r_f(A)$, then at least one agent has to be situated between the two preselected alternatives. Then, we can partition $S_f(A)$ for a given profile $R \in \mathcal{R}^A$ into the following sets: $S^l_f(R) = \{i \in$

[6]Note that these conditions apply also straightforwardly to $l_f(A) = r_f(A)$.

$S_f(A) : l_f(A)\, P_i\, r_f(A)\}$, $S^r_f(R) = \{i \in S_f(A) : r_f(A)\, P_i\, l_f(A)\}$, and $S^i_f(R) = \{i \in S_f(A) : r_f(A)\, I_i\, l_f(A)\}$. This partition separates the agents that can affect the outcome of f_A into three groups depending on their ordinal preferences over the two locations preselected in the first step. Now, we are going to describe some particular binary decision functions that are defined by the sets of agents that are able to impose one of the preselected alternatives over the other. These decisive sets depend on $R_{S^i_f(R)}$, the preferences of the agents belonging to $S_f(A)$ that are indifferent between $l_f(A)$ and $r_f(A)$ at R, and take the following structure.

Definition 1 *Given the subprofile $R_{S^i_f(R)}$ of a preference profile $R \in \mathcal{R}^A$, a family $\mathcal{G}(R_{S^i_f(R)})$ of $l_f(A)$–decisive sets over $R_f(A)$ is a family of subsets of $(S_f(A) \setminus S^i_f(R))$ such that:*

- *If $B \in \mathcal{G}(R_{S^i_f(R)})$ and $C \supset B$, then $C \in \mathcal{G}(R_{S^i_f(R)})$.*
- *If $j \notin B$ and $(B \cup \{j\}) \notin \mathcal{G}(R_{S^i_f(R)})$, then $B \notin \mathcal{G}(R'_{S^i_f(R')})$, where $R' \in \mathcal{R}^A$ is such that $R'_{S^i_f(R)} = R_{S^i_f(R)}$ and $S^i_f(R') = S^i_f(R) \cup \{j\}$.*
- *If $B \in \mathcal{G}(R_{S^i_f(R)})$ and $j \notin B \cup S^i_f(R)$, then $B \in \mathcal{G}(R'_{S^i_f(R')})$, where $R' \in \mathcal{R}^A$ is such that $R'_{S^i_f(R)} = R_{S^i_f(R)}$ and $S^i_f(R') = S^i_f(R) \cup \{j\}$.*
- *If $S^i_f(R) = \emptyset$, then $\emptyset \notin \mathcal{G}(R_{S^i_f(R)}) \neq \{S_f(A)\}$.*

We already indicated that the decisive sets that are able to impose $l_f(A)$ do not only depend on $S^i_f(R)$, the agents that are indifferent between $l_f(A)$ and $r_f(A)$, but also on their particular preferences $R_{S^i_f(R)}$. We can also see from the definition that the decisive sets have to satisfy three intuitive monotonicity properties and a non-emptiness condition. First, all supersets of a decisive set are also decisive. Second, if a set of agents $B \cup \{j\}$ cannot impose $l_f(A)$ when all agents of $B \cup \{j\}$ prefer $l_f(A)$ to $r_f(A)$, then the set B is also not able to impose $l_f(A)$ when agent j switches her preferences and becomes indifferent between the two preselected alternatives. Third, if a set of agents B can impose $l_f(A)$ when all agents of B prefer $l_f(A)$ to $r_f(A)$ and agent j prefers $r_f(A)$ to $l_f(A)$, then B is also able to impose $l_f(A)$ when agent j becomes indifferent between the two preselected alternatives. Finally, the non-emptiness condition guarantees that each of the preselected alternatives is implemented at least once if all agents have strict preferences over $R_f(A)$.

However, a description of a family of $l_f(A)$–decisive sets is not sufficient to define a binary decision function f_A, because it does not provide a solution when all agents of $S_f(A)$ are indifferent between $l_f(A)$ and $r_f(A)$. In this case, we have to apply a tie-breaking rule $t_A : \mathcal{R}_{S_f(A)} \to R_f(A)$.

Definition 2 *A binary decision function $f_A : \mathcal{R}_{S_f(A)} \to R_f(A)$ is called a* voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A *if for each subprofile $R_{S_f^i(R)}$ of a preference profile $R \in \mathcal{R}^A$, there exists a family of $l_f(A)$–decisive sets over $R_f(A)$, $\mathcal{G}(R_{S_f^i(R)})$, together with a tie-breaking rule t_A such that:*

$$f_A(R_{S_f(A)}) = \begin{cases} t_A(R_{S_f(A)}) & \text{if } S_f^i(R) = S_f(A) \\ l_f(A) & \text{if } S_f^l(R) \in \mathcal{G}(R_{S_f^i(R)}) \\ r_f(A) & \text{otherwise.} \end{cases}$$

A voting by collection of $l_f(A)$–decisive sets with tie-breakers t_A is, although formally complicated to define, a relatively simple binary decision function. First, a collection of $l_f(A)$–decisive sets over $R_f(A)$ is defined for each subprofile $R_{S_f^i(R)}$ of agents belonging to $S_f(A)$ that are indifferent between the preselected alternatives. The outcome of f_A for a subprofile $R_{S_f(A)}$ is then $l_f(A)$ if the set of agents of $(S_f(A) \setminus S_f^i(R))$ that prefer $l_f(A)$ is a $l_f(A)$-decisive set for $R_{S_f^i(R)}$, and $r_f(A)$ otherwise. If $S_f(A) = S_f^i(R)$, the tie-breaking rule t_A is used to determine the alternative that is ultimately chosen.

We note that our family of binary decision functions is almost identical to the family introduced under the same name in Manjunath [9], where it is shown that these are the unique type of rules that are strategy-proof and unanimous in the domain of single-dipped preferences when the two alternatives to choose from are $\min T$ and $\max T$. The only difference is that the non-emptiness requirement of the family of decisive coalitions $\mathcal{G}(R_{S_f^i(R)})$ is always needed in Manjunath [9] and not only when $S_f^i(R) = \emptyset$.

Our next result shows that the binary decision function f_A associated to a strategy-proof social choice rule f has to be a voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A.

Proposition 3 *If f is strategy-proof, the family of binary decision functions $\{f_A : \mathcal{R}_{S_f(A)} \to R_f(A)\}_{A \subseteq N}$ is a family of voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A for all $A \subseteq N$.*

A summary of the main structure

Propositions 1 and 3 describe the basic structure any strategy-proof social choice rule has to satisfy. First, agents have to indicate their type of preferences (single-peaked or single-dipped). Depending on the set of agents who declare to have single-peaked preferences, one or two alternatives are preselected. If only one alternative is preselected, this alternative is chosen. If two alternatives are preselected, the agents who are located strictly between these two alternatives have to indicate their ordinal preferences over the preselected alternatives. Still, we have to ask the agents that are indifferent between the two preselected alternatives for their entire preferences. This general structure can be summarized in the following corollary.

Corollary 1 *A strategy-proof social choice rule f can be decomposed into a function $R_f : 2^N \to T^2$ and a family $\{f_A : \mathcal{R}_{S_f(A)} \to R_f(A)\}_{A \subseteq N}$ of voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A such that for all preference profiles $R \in \mathcal{R}^A$, $f(R) = f_A(R_{S_f(A)})$.*

4 A complete characterization of the strategy-proof rules

So far, we have derived the main structure of all strategy-proof rules. Next, we derive additional necessary conditions that allow us to obtain a complete characterization.

The structure of R_f

We start by analyzing the function R_f used in the first step of the strategy-proof rules and, in particular, we are going to explain how the preselected alternatives could change as more agents declare to have single-peaked preferences. To do that, we introduce the following notation: $R_f(A) \neq R_f(A \cup \{i\})$ when $l_f(A) \neq l_f(A \cup \{i\})$ and $r_f(A) \neq r_f(A \cup \{i\})$. Observe that this condition is different from $R_f(A) \cap R_f(A \cup \{i\}) = \emptyset$, as it could be the case that $l_f(A \cup \{i\}) = r_f(A)$ or $r_f(A \cup \{i\}) = l_f(A)$.

Proposition 4 *If f is strategy-proof, for all $A \subset N$ and all $i \in N$:*

- *If $r_f(A) \leq i$, then $l_f(A \cup \{i\}) \in [l_f(A), i]$ and $r_f(A \cup \{i\}) \in [r_f(A), i]$. Moreover, if $R_f(A) \neq R_f(A \cup \{i\})$, then $l_f(A \cup \{i\}) \geq r_f(A)$.*
- *If $l_f(A) \geq i$, then $l_f(A \cup \{i\}) \in [i, l_f(A)]$ and $r_f(A \cup \{i\}) \in [i, r_f(A)]$. Moreover, if $R_f(A) \neq R_f(A \cup \{i\})$, then $r_f(A \cup \{i\}) \leq l_f(A)$.*
- *If $i \in (l_f(A), r_f(A))$, $i \in [l_f(A \cup \{i\}), r_f(A \cup \{i\})] \subseteq [l_f(A), r_f(A)]$. Additionally, if $i \in R_f(A \cup \{i\})$, then $R_f(A \cup \{i\}) = \{i\}$ or $R_f(A \cup \{i\}) \cap R_f(A) \neq \emptyset$.*

Proposition 4 establishes that if agent i passes from having single-dipped to single-peaked preferences, the preselected alternatives in $R_f(A \cup \{i\})$ have to be *closer* to i than those in $R_f(A)$. In fact, if agent i is located strictly in between the two preselected alternatives of $R_f(A)$, then $l_f(A) \leq l_f(A \cup \{i\}) \leq i$ and $r_f(A) \geq r_f(A \cup \{i\}) \geq i$, that is, agent i is also located weakly between the preselected alternatives of $R_f(A \cup \{i\})$, and the preselected alternatives weakly move into the direction of the location of agent i. Additionally, if this location i belongs to $R_f(A \cup \{i\})$, then either i is the unique preselected location or the second preselected alternative also belongs to $R_f(A)$. If, on the other hand, agent i is located to the left or to the right of the preselected alternatives of $R_f(A)$, then each of the preselected alternatives of $R_f(A \cup \{i\})$ has to be between the location of agent i and the corresponding preselected location of $R_f(A)$. Finally, if it turns out that both preselected alternatives change, then $l_f(A \cup \{i\}) \geq r_f(A)$ whenever $r_f(A) \leq i$ and $r_f(A \cup \{i\}) \leq l_f(A)$ whenever $i \leq l_f(A)$.

The relation between decisive sets

Next, we study additional necessary conditions that arise in the second step of the rules. In particular, we analyze how the decisive sets change as an agent passes from having single-dipped to single-peaked preferences. Our first result refers to the cases in which i is located strictly between the two preselected alternatives when she declares to have single-dipped preferences and the preselected alternatives change when i announces to have single-peaked preferences in such a way that agent i is also located strictly between $l_f(A \cup \{i\})$ and $r_f(A \cup \{i\})$. Then, i is a dictator in the second step of the social choice rule; that is, i belongs to all decisive coalitions that can impose $l_f(A)$ or $l_f(A \cup \{i\})$ in any profile of $\mathcal{R}^A$ or $\mathcal{R}^{A\cup\{i\}}$ in which she is not indifferent.

Proposition 5 *Suppose that f is strategy-proof. Then, for all $A \subset N$ and all agents $i \notin A$ such that $i \in S_f(A) \cap S_f(A \cup \{i\})$, $|R_f(A) \cap R_f(A \cup \{i\})| < 2$, and all preference profiles $R \in \mathcal{R}^{A\cup\{i\}}$ and $R' \in \mathcal{R}^A$ such that $i \notin S_f^i(R) \cup S_f^i(R')$,*

$$B \in \mathcal{G}(R_{S_f^i(R)}) \text{ if and only if } i \in B$$

$$B \in \mathcal{G}(R'_{S_f^i(R')}) \text{ if and only if } i \in B.$$

The next proposition considers the case when $R_f(A \cup \{i\}) = R_f(A)$. It turns then out that if agent i is situated to the left (right) of both preselected locations of $R_f(A)$, it has to be *easier* (more difficult) to select the left point of $R_f(A)$ when agent i changes her type of preferences from single-dipped to single-peaked. Here, easier (more difficult) means that the set of coalitions that can impose the left point of $R_f(A)$ when agent i declares to have single-peaked preferences has to be a superset (subset) of the set when she declares to have single-dipped preferences. Finally, if agent i is located between the two preselected alternatives, the set of coalitions that can impose the left point of the range has to be invariant to the type of preferences of agent i.

Proposition 6 *Suppose that f is strategy-proof. Then, for all $A \subset N$ and all agents $i \notin A$ such that $R_f(A) = R_f(A \cup \{i\})$ and all preference profiles $R \in \mathcal{R}^A$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$ such that $i \notin S_f^i(R) \cup S_f^i(R')$, $\mathcal{G}(R_{S_f^i(R)}) \subseteq \mathcal{G}(R'_{S_f^i(R')})$ whenever $i \leq r_f(A)$, and $\mathcal{G}(R'_{S_f^i(R')}) \subseteq \mathcal{G}(R_{S_f^i(R)})$ whenever $i \geq l_f(A)$.*

The next proposition refers to the cases when i is located to the left or to the right of the alternatives in $R_f(A)$ and only one of these locations changes when agent i changes her type of preferences from single-dipped to single-peaked. To be exact, the result states that if agent i is located to the right (left) of $R_f(A)$ and a coalition B is minimally decisive when agent i declares to have single-peaked (single-dipped) preferences, then the intersection between B and all agents with single-dipped preferences that are located between the two alternatives is a decisive set when agent i changes her type of preferences. To describe the result, we need to define $\mathcal{G}^*(R_{S_f^i(R)})$ for $R \in \mathcal{R}^A$ as the set of all minimal $l_f(A)$–decisive coalitions. Formally, $B \in \mathcal{G}^*(R_{S_f^i(R)})$ if $B \in \mathcal{G}(R_{S_f^i(R)})$ and there is no $C \subset B$ such that $C \in \mathcal{G}(R_{S_f^i(R)})$.

Proposition 7 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$, all agents $i \notin A$, and all preference profiles $R \in \mathcal{R}^{A\cup\{i\}}$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^A$ such that $l_f(A \cup \{i\}) = l_f(A)$ or $r_f(A \cup \{i\}) = r_f(A)$ but not both:*

- *If $i \geq r_f(A)$, then for all $B \in \mathcal{G}^*(R_{S_f^i(R)})$, $(B \cap (S_f(A) \setminus A)) \in \mathcal{G}(R'_{S_f^i(R')})$.*
- *If $i \leq l_f(A)$, then for all $B \in \mathcal{G}^*(R'_{S_f^i(R')})$, $(B \cap (S_f(A \cup \{i\}) \setminus A)) \in \mathcal{G}(R_{S_f^i(R)})$.*

The last proposition studies what happens if i is situated strictly between the two preselected alternatives of $R_f(A)$ and i belongs to $R_f(A\cup\{i\})$. Then, by Proposition 4, the location of agent i is the unique preselected location or it is accompanied by $l_f(A)$ or $r_f(A)$. In the first case, there is no room for additional conditions in the second step of the rule. In the latter case, it turns out that if a coalition is able to impose the point different from i when she has single-peaked preferences, the same coalition also has to be a winning coalition to impose the very same point if agent i has single-dipped preferences.

Proposition 8 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$, all agents $i \notin A$ such that $i \in (S_f(A) \setminus S_f(A \cup \{i\}))$, and all preference profiles $R \in \mathcal{R}^{A\cup\{i\}}$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^A$ such that $i \notin S_f^i(R')$:*

- *If $l_f(A) = l_f(A \cup \{i\})$, then for all $B \in \mathcal{G}^*(R_{S_f^i(R)})$, $(B \cap A) \in \mathcal{G}(R'_{S_f^i(R')})$.*

- *If* $r_f(A) = r_f(A \cup \{i\})$, *then for all* $B \in \mathcal{G}^*(R'_{S^i_f(R')})$, $(B \cap S_f(A \cup \{i\}) \cap A) \in \mathcal{G}(R_{S^i_f(R)})$.

The characterization result

So far, we have established a set of necessary conditions for a social choice rule f to be strategy-proof in addition to the main structure summarized in Corollary 1. The following theorem shows that the union of all these necessary conditions constitutes a sufficient condition and, therefore, we have characterized all strategy-proof social choice rules. This theorem also shows that the characterized rules are group strategy-proof. So, this stronger property is obtained for free on the domain $\mathcal{R}$ when strategy-proofness is imposed.[7]

Theorem 1 *The following statements are equivalent:*

1. *The social choice rule* $f : \mathcal{R} \to T$ *is strategy-proof.*
2. *The social choice rule* $f : \mathcal{R} \to T$ *is group strategy-proof.*
3. *There is a function* $R_f : 2^N \to T^2$ *satisfying Proposition 4 and a family* $\{f_A : \mathcal{R}_{S_f(A)} \to R_f(A)\}_{A \subseteq N}$ *of voting by collections of* $l_f(A)$*–decisive sets with tie-breakers* t_A *satisfying Propositions 5, 6, 7 and 8 such that for all preference profiles* $R \in \mathcal{R}^A$, $f(R) = f_A(R_{S_f(A)})$.

5 Possibility results

Theorem 1 characterizes the family of all strategy-proof rules on $\mathcal{R}$. The structure of this family depends on the relation between N and T. In particular, it can happen that this family reduces to the dictatorial rules when the range of the rules is at least 3, thereby arriving again at the impossibility of Result 1. Fortunately, this is generally not the case. To characterize all situations

[7] Barberà et al. [2] show in their Theorem 1 that if a domain $\mathcal{D}$ satisfies a condition that they call *sequential inclusion*, then any strategy-proof social choice rule is also group strategy-proof on $\mathcal{D}$. It is also shown in their Theorem 4 that if a domain $\mathcal{D}$ allows for opposite preferences and a strategy-proof social choice rule on any subdomain of $\mathcal{D}$ is also group strategy-proof, then $\mathcal{D}$ satisfies sequential inclusion. Since our domain $\mathcal{R}$ allows for opposite preferences but does not satisfy sequential inclusion, their result implies that there exists a subdomain of $\mathcal{R}$ where strategy-proofness does not imply group strategy-proofness.

in which we can escape from Result 1, we need to define a new concept: a triple $\{x, y, z\} \subseteq T$ is *full at* N if $N \cap \{x, y, z\} = \emptyset$ and $N \subset (\min\{x, y, z\}, \max\{x, y, z\})$.

Theorem 2 *There are strategy proof social choice rules on* $\mathcal{R}$ *with range greater than or equal to 3 different from the dictatorial ones if and only if there is a triple of* T *that is not full at* N*.*

The condition included in Theorem 2 can be explained more intuitively depending on the size of T.

Corollary 2 *The following statements hold:*

- *We can escape from Result 1 if* $|T| \geq 5$*.*
- *If* $|T| = 4$*, we can escape from Result 1 if and only if* T *cannot be partitioned into* T_1 *and* T_2 *such that* $|T_1| = |T_2| = 2$*,* $\max T_1 < \min N$*, and* $\max N < \min T_2$*.*
- *If* $|T| = 3$*, we can escape from Result 1 if and only if* $N \cap T = \emptyset$ *or* $N \not\subset (\min T, \max T)$*.*

Theorem 2 and Corollary 2 provide an almost unanimous positive answer to the possibility of escaping at $\mathcal{R}$ from the impossibility result of Gibbard [8] and Satterthwaite [11]. If there are more than 4 alternatives, it is always possible to obtain rules that satisfy strategy-proofness (and group strategy-proofness) with range greater than or equal to 3. Otherwise, it is sufficient to have an agent that is located at a feasible location or at the left (or right) of at least three alternatives.[8]

However, the fulfillment of the condition in Theorem 2 does not necessarily imply that it is possible to construct *meaningful* rules, because a priori it is possible that the addition of Pareto efficiency leads again to dictatorial rules arriving at the impossibility of Result 2. Yet, Example 1 provides an example of a strategy-proof and Pareto efficient social choice rule that is different from the dictatorial rule in a particular case, showing that we can escape from this impossibility at least in some cases. The following proposition shows a first condition that T has to satisfy to make it possible to obtain a Pareto efficient rule.

[8]Observe that if $|T| = 3$ and none of these conditions are satisfied, $\mathcal{R}$ coincides with the universal domain.

Proposition 9 *Let T be a set of alternatives so that it is possible to find Pareto efficient rules on $\mathcal{R}$. Then,* $\min T$ *and* $\max T$ *exist.*

Proposition 9 concludes that, independently of the locations of the agents, a first necessary condition on the set of feasible locations T is that it must have a minimum and a maximum. So, we assume from now on that T has a minimum and a maximum. The following result characterizes the additional conditions the relation between N and T have to satisfy to escape from Result 2. To introduce it, we need to define $l_j = \max\{x \in T : x \leq j\}$ and $r_j = \min\{x \in T : x \geq j\}$.

Theorem 3 *There are Pareto efficient and strategy proof social choice rules on $\mathcal{R}$ different from the dictatorial ones if and only if there are two agents $i, j \in N$ such that each $k \in \{i, j\}$ satisfies one of the following conditions: (a) $k \in T$, (b) $k \notin (\min T, \max T)$ or (c) l_k and r_k exist; with at least one of them satisfying (a) or (b).*

According to Theorem 3 the condition to obtain Pareto efficiency additionally to strategy-proofness without arriving at the dictatorial rules is that there is one agent located at a feasible point or to the left (or right) of all feasible points; and another agent that satisfies any of these characteristics or is located at a point that has a defined nearest point to both its left and right.

6 Characterizations in some particular domains

Since a general characterization of all Pareto efficient and strategy-proof rules for all sets N and T that satisfy the conditions of Theorem 3 is quite difficult to define in an intuitive way, we consider now two extreme cases that reflect the two conditions that are sufficient to guarantee possibility results. First, we assume that all agents are located at the left or right of all feasible locations (that is, the cases in which all agents satisfy condition (b) of Theorem 3). Afterwards, we consider that all agents are located at feasible locations (that is, the cases in which all agents satisfy condition (a) of Theorem 3).

Agents outside the interval of feasible locations

The first condition that guarantees on its own the possibility of combining Pareto efficiency and strategy-proofness without arriving at dictatorial rules is the presence of at least two agents at the left or at the right of all feasible locations for the facility. We are going to provide the characterization for the extreme case when $N \cap (\min T, \max T) = \emptyset$. Then, we define $N_l = \{i \in N : i < \min T\}$ and $N_r = \{i \in N : i > \max T\}$ as the set of agents that are located to the left (right) of T. Obviously, $N_l \cup N_r = N$.

Since no agent is located in the interior of T, only one alternative gets preselected in the first step of the rule. Thus, the second step of the two-step functions characterized in Theorem 1 does not apply and the range of R_f is T instead of T^2. This implies that the characterized rules are types-only; i.e., they only depend on the type of preferences (single-peaked or single-dipped) of each agent, but not on its particular structure. Formally, a social choice rule is *types-only* if for all sets of agents $A \subseteq N$ and all preference profiles $R, R' \in \mathcal{R}^A$, we have that $f(R) = f(R')$. It follows then directly from Proposition 4 (first and second cases) that R_f is *monotone*: $R_f(A \cup \{i\}) \leq R_f(A)$ whenever $i \in N_l$ and $R_f(A \cup \{i\}) \geq R_f(A)$ whenever $i \in N_r$.

Theorem 4 *Let $N \cap (\min T, \max T) = \emptyset$. Then, a social choice rule $f : \mathcal{R} \to T$ is strategy-proof and Pareto efficient if and only if there is a monotone function $R_f : 2^N \to T$ satisfying that $R_f(N_l) = \min T$ and $R_f(N_r) = \max T$ such that for all $R \in \mathcal{R}^A$ for some $A \subseteq N$, $f(R) = R_f(A)$.*

Theorem 4 provides a characterization of all the rules that satisfy these two natural conditions in these situations in which all agents are at the left or at the right of all feasible locations. These rules only ask agents about their type of preferences and the result is monotone with respect to these preferences: if an agent at the left (respectively, at the right) of all feasible alternatives passes from having single-dipped to single-peaked preferences, the chosen location does not move to the right (respectively, to the left). Additionally, if all agents prefer the alternative situated most to the left (respectively, most to the right) to the remaining ones, this location must be chosen.

All agents located at feasible locations

The second condition that guarantees on its own the possibility of combining Pareto efficiency and strategy-proofness without arriving at dictatorial rules is the presence of at least two agents at feasible locations. We are thus interested in the extreme case when $N \subset T$ and $N \subset (\min T, \max T)$. The reason to impose additionally that $N \cap \{\min T, \max T\} = \emptyset$ is that agents located at the extremes have exactly the same set of admissible preferences over T as the agents outside of T and the effect of these agents is the same as in the previous subsection. Since the description of the Pareto efficient and strategy-proof rules is quite complex, we impose tops-onliness as an additional condition. Formally, a social choice rule f is *tops-only* if for all $i \in N$ and all preference profiles $R, R' \in \mathcal{R}^A$ such that $\{x \in T : x\, R_i\, y \text{ for all } y \in T\} = \{x \in T : x\, R'_i\, y \text{ for all } y \in T\}$, then $f(R) = f(R')$.[9] It is easy to see that the possible tops of any agent $i \in N$ are $t(R_i) \in \{i, \min T, \max T\}$. For the sake of simplicity, we assume that the top of each agent is unique.

Before providing a formal definition of the rules, we are going to describe some of the implications tops-onliness has in this setting. First, and foremost, the second step of the characterized rules in Theorem 1 only applies if the two preselected alternatives in the first step are the two extremes: $\min T$ and $\max T$. Otherwise, only one alternative will be preselected and, therefore, chosen. Moreover, the absence of indifferent agents implies that the set of decisive coalitions when the two extreme points are preselected is independent of other aspects of the problem. Thus, we can denote the decisive sets that can implement $\min T$ by $\mathcal{G}_f$.

Let us now consider the situation when all agents have single-dipped preferences. Since both extreme locations can Pareto dominate interior points, we must have that $R_f(\emptyset) = \{\min T, \max T\}$. This allows us then to ask how the set of preselected alternatives changes when a single agent i declares to have single-peaked instead of single-dipped preferences. One possibility is that $R_f(\{i\}) = R_f(\emptyset)$. But another possibility is that an interior point is selected. And strategy-proofness requires then that this interior point is i. This insight allows us to classify agents into

[9] Tops-onliness is a weaker property than types-onliness. Moreover, types-onliness is incompatible with Pareto efficiency in this case.

two groups: *decisive* agents, the ones that can force the outcome to be exactly her location when the rest of agents have single-dipped preferences and they have single-peaked ones; and *indecisive* agents. Let $D_f \subseteq N$ be the set of decisive agents. We are now ready to introduce the family of *conditional two-step rules.*

Definition 3 *The social choice rule f is said to be a* conditional two-step rule *if there is a non-empty $D_f \subseteq N$, a function $f_1 : 2^N \to (\min T, \max T)$, and a set of decisive sets $\mathcal{G}_f$ such that for all $A \subseteq N$ and all preference profiles $R \in \mathcal{R}^A$,*

$$f(R) = \begin{cases} f_1(A) \in [\min(A \cap D_f), \max(A \cap D_f)] & \text{if } A \cap D_f \neq \emptyset \\ \min T & \text{if } A \cap D_f = \emptyset \text{ and } \{i \in N : t(R_i) = \min T\} \in \mathcal{G}_f \\ \max T & \text{otherwise.} \end{cases}$$

The conditional two-step rules work as follows: first, it establishes a partition of the set of agents into the decisive ones D_f and the rest. Then, if all decisive agents have single-dipped preferences, the result is one of the extreme points, otherwise an interior point situated between the location of the decisive agents with single-peaked preferences is chosen. If the decision is between the two extremes, the exact result is determined depending on whether the set of single-dipped agents with top $\min T$ belongs to $\mathcal{G}_f$ or not. On the other hand, if an interior point has to be chosen, an aggregator f_1 of the peaks of the single-peaked agents is used to choose the exact point.

Not all conditional two-step rules are strategy-proof and Pareto efficient. In particular, it follows from Proposition 4 (first and second case) that if an interior point is chosen for some particular set of agents A with single-peaked preferences and agent $i \notin A$ changes her preferences to single-peaked, then the selected location moves weakly into direction of agent i. Formally, we say that the conditional two-step rule f is *monotone* if for all $A \subseteq N$ and $i \in N$, $f_1(A) \geq f_1(A \cup \{i\}) \geq i$ (respectively, $f_1(A) \leq f_1(A \cup \{i\}) \leq i$) whenever $f_1(A) \geq i$ (respectively, $f_1(A) \leq i$). Finally, there is also a restriction on the set of decisive agents. We say that the conditional two-step rule f satisfies *conditional decisiveness* if for all $i \in C$ for some $C \in \mathcal{G}_f$, then $i \in D_f$. The following result introduces the characterization for these cases.

Theorem 5 *Let $N \subset T$ and $N \subset (\min T, \max T)$. A social choice rule f is strategy-proof, Pareto efficient, and tops-only if and only if it is a monotone conditional two-step rule satisfying conditional decisiveness.*

7 Conclusions

In this paper, we considered the situation when a social planner has to locate a public facility on a subset of the real line. Agents are assumed to have single-dipped (the facility is a bad for them) or single-peaked (the facility is a good for them) preferences on the set of alternatives. The type and the structure of the preferences is only known to the agents, yet the public decision maker knows the agents' location on the real line and that the dip or peak of the preferences of an agent corresponds to her location.

In this setting, we show that the set of all strategy-proof social choice rules on this domain follow a two-step procedure. In the first step, all agents have to indicate their type of preferences (single-peaked or single-dipped) and depending on the set of agents that declare to have single-peaked preferences, at most two alternatives are preselected. If the rule preselects only one alternative, then this alternative is finally chosen. If two alternatives are preselected, then all agents that are situated strictly between the two preselected alternatives have to indicate which of them they prefer. Then, depending on the preference structure of the agents that are indifferent between the two preselected alternatives, winning coalitions determine which of the two preselected alternatives is selected. Additionally, we show that all strategy-proof social choice rules are also group strategy-proof. Finally, we have shown the conditions that the sets of agents and alternatives have to satisfy to escape from the classical Gibbard-Satterthwaite impossibility result and we have characterized the rules that satisfy additionally Pareto efficiency in some of these domains.

These results can improve the way in which the decisions of where to locate some facilities are taken. It complements the classical domains of single-peaked and single-dipped preferences in such a way that the decision of which is the best domain restriction and, as a consequence, which are the most appropriate rules to implement in each particular problem depends on the structure of the particular facility. To see this, consider the following two alternative assumptions: (i) the social planner assumes that the facility is unanimously considered a good or a bad; or (ii) the social planner allows that the facility can be considered a good or a bad by each agent, but the peak or dip is assumed to be situated in the point in which this agent is located. If, for a particular facility, the first assumption seems stronger (weaker), then this new framework is more (less) appropriate than the uniform single-peaked or single-dipped one.

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Appendix

Proof of Proposition 1

Let $O_i(A, R_{-i})$ be the option set of agent i given the preferences R_{-i} of the other agents and given that the set of agents with single-peaked preferences is equal to $A \subseteq N$. So, alternative $x \in T$ belongs to $O_i(A, R_{-i})$ if there is a preference profile $R = (R_i, R_{-i}) \in \mathcal{R}^A$ such that $f(R) = x$. Our first lemma shows that if x and y belong to the option set of agent i, then i is located between x and y.

Lemma 1 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$, all agents $i \in N$, all profiles $R \in \mathcal{R}$, and all alternatives $x, y \in T$ such that $x < y$, if $x, y \in O_i(A, R_{-i})$, then $x < i < y$.*

Proof: Since $x, y \in O_i(A, R_{-i})$, there are two preferences $R_i, R'_i \in \mathcal{R}_i^A$ for agent i such that $f(R) = x$ and $f(R'_i, R_{-i}) = y$. If $i \leq x$, then agent i can manipulate f at (R'_i, R_{-i}) via R_i whenever $i \in A$ and at R via R'_i whenever $i \notin A$. Similarly, if $y \leq i$, then agent i can manipulate f at R via R'_i whenever $i \in A$ and at (R'_i, R_{-i}) via R_i whenever $i \notin A$. Hence, $x < i < y$. $\square$

Lemma 1 directly implies that the option set of any agent contains at most two alternatives.

Corollary 3 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$, all agents $i \in N$, and all profiles $R \in \mathcal{R}$, $|O_i(A, R_{-i})| \leq 2$.*

The next lemma shows that f always selects a maximal alternative of an agent's option set.

Lemma 2 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$, all agents $i \in N$, and all profiles $R \in \mathcal{R}^A$, if $O_i(A, R_{-i}) = \{x, y\}$ and $f(R) = x$, then $x\, R_i\, y$.*

Proof: Suppose otherwise; that is, there is a preference profile R such that $f(R) = x$ and $y\, P_i\, x$. Since $y \in O_i(A, R_{-i})$ by assumption, there is a preference $R_i' \in \mathcal{R}_i^A$ such that $f(R_i', R_{-i}) = y$. Then, agent i manipulates f at R via R_i'. Hence, xR_iy. □

The next lemma is crucial to prove Proposition 1 by contradiction. It shows that if the proposition would not be correct (i.e. f is strategy-proof and $|R_f(A)| > 2$ for some $A \subseteq N$), there would be a profile R and two preferences $R_i' \in \mathcal{R}_i^A$ and $R_j' \in \mathcal{R}_j^A$ so that the outcomes at R, (R_i', R_{-i}), and (R_j', R_{-j}) differ.

Lemma 3 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$ such that $|R_f(A)| > 2$, there are two agents $i, j \in N$ and three profiles $R, (R_i', R_{-i}), (R_j', R_{-j}) \in \mathcal{R}^A$ such that $f(R) \neq f(R_i', R_{-i}) \neq f(R_j', R_{-j}) \neq f(R)$.*

Proof: Consider any $A \subseteq N$ and suppose that $|R_f(A)| > 2$. Then, there are three preference profiles $D, D', D'' \in \mathcal{R}^A$ such that $f(D) \neq f(D') \neq f(D'') \neq f(D)$. Suppose without loss of generality that $f(D) = x$. Starting at D, change the preferences of all agents one-by-one so that we end up in profile D'. Since $f(D) \neq f(D')$, the function must have changed during this process. Let B be the profile where the outcome is still x and let (B_i', B_{-i}) be the profile when the outcome switches the first time to another alternative, say y.

Next, construct the preference $\hat{D}_i \in \mathcal{R}_i^A$ in the following way: if $f(D'') \neq y$, $\hat{D}_i$ is set equal to D_i'', otherwise $\hat{D}_i$ is set equal to D_i'. We can see that if $f(\hat{D}_i, B_{-i}) \notin \{x, y\}$, then $|O_i(A, B_{-i})| > 2$ contradicting Corollary 1. So, $f(\hat{D}_i, B_{-i}) \in \{x, y\}$. Next, consider the profile $(\hat{D}_i, B_{-i})$ together with the profile from B and (B_i', B_{-i}) that has not the same outcome as $(\hat{D}_i, B_{-i})$. Suppose without loss of generality that the alternative selected at B is not equal to the one selected at $(\hat{D}_i, B_{-i})$. Then, starting at B and $(\hat{D}_i, B_{-i})$, change the preferences of all agents except i one-by-one so that we end up in profile D'' if $f(D'') \neq y$ and in profile D' if $f(D'') = y$. The outcome at either of the two profiles has to change at some point of the process. Let C and (C'_i, C_{-i}) be the profiles where the outcomes are still x and y, respectively, and let (C'_S, C_{-S}) and $(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})})$ be the first time in which one (or both) profiles have an outcome $z \notin \{x, y\}$. Assume without loss of generality that $f(C'_S, C_{-S}) = z$. Let $S = \{j_1, \ldots, j_s\}$ and observe that $s+1$ agents have changed their preferences in the sequence to produce the three different outcomes: first, agent i changes preferences from C'_i to C_i; then, s agents change preferences (from the ones in C_S to the ones in C'_S) to arrive at z. We complete the proof by induction on s.

- If $s = 1$, denote profile C by R, preference C'_i by R'_i, agent j_1 by j, and preference C'_{j_1} by R'_j. Then, we have established the result.
- Suppose that the statement of the proposition is correct for all $s < k$.
- We now prove the statement of the proposition for $s = k$. Consider any $V \subset S \cup \{i\}$ with $V \notin \{S, \{i\}\}$. If $f(C'_V, C_{-V}) \notin \{x, y\}$, consider the sequence that starts with (C'_i, C_{-i}), passes through C, and ends at (C'_V, C_{-V}) whenever $i \notin V$, and the sequence that starts with C, passes through (C'_i, C_{-i}), and ends at (C'_V, C_{-V}) whenever $i \in V$. We can see that the number of agents that have changed their preferences to produce three different results in these sequences is smaller than $k+1$. So, the result follows from the induction hypothesis. If, on the other hand, $f(C'_V, C_{-V}) = y$, then consider the sequence that starts at C, passes through (C'_V, C_{-V}), and ends at (C'_S, C_{-S}). In this sequence, the number of agents that have changed their preferences to produce the three different outcomes is $k < k + 1$. The result follows then again from the induction argument. Consequently, we assume from now on that $f(C'_V, C_{-V}) = x$ for all $V \subset S \cup \{i\}$ with $V \notin \{S, \{i\}\}$.

  Next, we concentrate on $f(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})})$. If $f(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})}) \notin \{x, z\}$, consider the sequence that starts at $(C'_{S\cup\{i\}\setminus\{j\}}, C_{-(S\cup\{i\}\setminus\{j\})})$ for any $j \in S$, passes through $(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})})$, and ends at (C'_S, C_{-S}). In this sequence, only two individuals have changed their preferences to produce the three different outcomes and, then, the induction argument applies. If, however, $f(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})}) = z$, consider the sequence that starts at (C'_i, C_{-i}) and then changes the preferences of all agents $j \in S$ one-by-one in order to end up at $(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})})$. In this sequence, the number of agents that have changed their preferences to produce the three different outcomes is $k < k+1$. So, the induction argument applies again. Thus, we conclude that $f(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})}) = x$.

Since $f(C'_i, C_{-i}) = y$ and $f(C'_S, C_{-S}) = z$, we have that $O_j(A, (C'_i, C_{-\{i,j\}})) = \{x, y\}$ and $O_j(A, (C'_{S\setminus\{j\}}, C_{-S})) = \{x, z\}$ for all $j \in S$. Similarly, we obtain that $O_i(A, C_{-i}) = \{x, y\}$ and $O_i(A, (C'_S, C_{-S})) = \{x, z\}$. So, it follows from Lemma 1 that $\{y, z\}$ is a fixed pair of alternatives for all $l \in S \cup \{i\}$. Consider now any agent $m \in S$ and any preference $R''_m \in \mathcal{R}^A_m$ such that $y\, P''_m\, x$ and $z\, P''_m\, x$, which exists given the previous findings. It follows then from Lemma 2 together with $O_m(A, (C'_i, C_{-\{i,m\}})) = \{x, y\}$ and $O_m(A, (C'_{S\setminus\{m\}}, C_{-S})) = \{x, z\}$ that $f(R''_m, C'_i, C_{-\{i,m\}}) = y$ and $f(R''_m, C'_{S\setminus\{m\}}, C_{-S}) = z$, respectively.

Finally, take the sequence of profiles that starts with $(R''_m, C'_i, C_{-\{i,m\}})$ and change the preferences of all agents $l \in S \cup \{i\} \setminus \{m\}$ so that the sequence ends at $(R''_m, C'_{S\setminus\{m\}}, C_{-S})$. If only y and z are chosen along this sequence, then the agent at whom the outcome changes can manipulate f given that $\{y, z\}$ is a fixed pair for all agents belonging to $S \cup \{i\}$. So, there have to be at least three different outcomes along this sequence. However, observe that the number of agents that have changed their preferences to produce the three different outcomes is $k < k+1$. So, the result follows from the induction hypothesis.

This concludes the proof. $\square$

Now, we are ready to prove Proposition 1.

Proof of Proposition 1: Suppose that $|R_f(A)| > 2$. Then, by Lemma 3, there are three profiles $R, (R'_i, R_{-i}), (R'_j, R_{-j}) \in \mathcal{R}^A$ such that $f(R) = x$, $f(R'_i, R_{-i}) = y$, and $f(R'_j, R_{-j}) = z$. Let $f(R'_i, R'_j, R_{-\{i,j\}}) = w$ and observe that although x, y and z are different, it could be that

$w \in \{x, y, z\}$. Also, assume without loss of generality that $x < y$. Next, we study the implications of the different option sets.

(1) Since $O_i(A, R_{-i}) = \{x, y\}$ and $x < y$ by assumption, Lemma 1 implies that $x < i < y$.

(2) Observe that $\{z, w\} \subseteq O_i(A, (R'_j, R_{-\{i,j\}}))$. So, if $w \neq z$, then $w < i < z$ or $z < i < w$ by Lemma 1.

(3) Since $O_j(A, R_{-j}) = \{x, z\}$, Lemma 1 implies that $x < j < z$ or $z < j < x$.

(4) Observe that $\{w, y\} \subseteq O_j(A, (R'_i, R_{-\{i,j\}}))$. So, if $w \neq y$, then $w < j < y$ or $y < j < w$ by Lemma 1.

We show that f is not strategy-proof by constructing manipulations depending on how w relates to the other alternatives.

<u>Case 1:</u> Suppose that $w = y$. Then, $y < i < z$ or $z < i < y$ by (2) and $x < j < z$ or $z < j < x$ by (3). Observe that if $y < i < z$ by (2), the fact that $x < i < y$ by (1) implies that $i < y < i$, which is impossible. So, we must have that $z < i < y$ by (2). This, together with $x < i < y$ by (1), implies that $\{x, z\}$ is a fixed pair for agent i. Since $x < i$, $z < i$, and j is between x and z by (3), we also have that either $\{x, y\}$ or $\{y, z\}$ is a fixed pair for agent j. Suppose that $\{x, y\}$ is a fixed pair for j (the other case is similar and thus omitted). We can then conclude that $z < j < x < i < y$. The remainder of this case is divided into two parts.

- If $i \in A$, consider any preference $R''_i \in \mathcal{R}^+_i$ such that $x \, P''_i \, y \, P''_i \, z$. Since $O_i(A, R_{-i}) = \{x, y\}$ and $O_i(A, (R'_j, R_{-\{i,j\}})) = \{y, z\}$, it follows from Lemma 2 that $f(R''_i, R_{-i}) = x$ and $f(R''_i, R'_j, R_{-\{i,j\}}) = y$. Thus, $O_j(A, (R''_i, R_{-\{i,j\}})) = \{x, y\}$ and, by Lemma 1, j lies between x and y. This is a contradiction because we have already seen before that $j < x < y$.

- If $i \notin A$, consider any preference $R''_i \in \mathcal{R}^-_i$ such that $z \, P''_i \, y \, P''_i \, x$. Since $O_i(A, R_{-i}) = \{x, y\}$ and $O_i(A, (R'_j, R_{-\{i,j\}})) = \{y, z\}$, it follows from Lemma 2 that $f(R''_i, R_{-i}) = y$ and $f(R''_i, R'_j, R_{-\{i,j\}}) = z$. Thus, $O_j(A, (R''_i, R_{-\{i,j\}})) = \{y, z\}$. Now, we separate the proof depending whether j belongs or not to A.

If $j \notin A$, consider any preference $R''_j \in \mathcal{R}^-_j$ such that $y \, P''_j \, z \, P''_j \, x$. Since $O_j(A, R_{-j}) = \{x, z\}$, Lemma 2 implies that $f(R''_j, R_{-j}) = z$. We know that $O_j(A, (R''_i, R_{\{i,j\}})) = \{y, z\}$, so Lemma 2 also implies that $f(R''_i, R''_j, R_{-\{i,j\}}) = y$. Individual i will then manipulate f at this profile via R_i to obtain z. If $j \in A$, consider any preference $R''_j \in \mathcal{R}^+_j$ such that $x \, P''_j \, z \, P''_j \, y$. Since $O_j(A, R_{-j}) = \{x, z\}$, Lemma 2 implies that $f(R''_j, R_{-j}) = x$. Given that $O_j(A, (R''_i, R_{\{i,j\}})) = \{y, z\}$, Lemma 2 also implies that $f(R''_i, R''_j, R_{-\{i,j\}}) = z$. Agent i will then manipulate f at (R''_j, R_{-j}) via R''_i (observe that (x, z) is a fixed pair for i at $\mathcal{R}^A_i$).

<u>Case 2:</u> Suppose that $w = z$. The proof is similar to the one above and is thus omitted.

<u>Case 3:</u> Suppose that $w \notin \{y, z\}$. By (2), we have that $z < i < w$ or $w < i < z$. Let $z < i < w$. Then, since j lies between x and z by (3) and both x and z are smaller than i by (1), we conclude that $j < i$. Also observe that j lies between y and w by (4) and that both y and w are greater than i by (1). Consequently, $j > i$, which cannot be. Hence, we must have $w < i < z$. Next, by (4), we have that $y < j < w$ or $w < j < y$. Let $y < j < w$. Then, (1) and (4) imply that $x < i < y < j < w$. So, $i < w$, which contradicts that $w < i$. Hence, we must have that $w < j < y$. Similarly we have by (3) that $z < j < x$ or $x < j < z$. Let $z < j < x$. Then, $w < i < z < j < x$ by (2) and (3), which contradicts that $x < i$ by (1). So, we must have that $x < j < z$.

At this point, we can see that that the four conditions $x < i < y$, $w < i < z$, $w < j < y$, and $x < j < z$ are indeed compatible for the moment. In fact, it turns out that x and w are both smaller than each i and j, which are in turn both smaller than each y and z. This also implies that both $\{w, x\}$ and $\{y, z\}$ are fixed pairs for both agents i and j. Finally, consider any preference $\hat{R}_j \in \mathcal{R}^A_j$ such that $z \, \hat{P}_j \, x$ and $y \, \hat{P}_j \, w$. Since $O_j(A, R_{-j}) = \{x, z\}$ and $O_j(A, (R'_i, R_{-\{i,j\}})) = \{w, y\}$, $f(\hat{R}_j, R_{-j}) = z$ and $f(R'_i, \hat{R}_j, R_{-\{i,j\}}) = y$. Then, agent i will manipulate f at $(\hat{R}_j, R_{-j})$ via R'_i when her fixed pair at $\mathcal{R}^A_i$ is (y, z) and at $(R'_i, \hat{R}_j, R_{-\{i,j\}})$ via R_i when her fixed pair at $\mathcal{R}^A_i$ is (z, y). $\square$

Proof of Proposition 2

We first establish the following lemma.

Lemma 4 *Suppose that f is strategy-proof. Then, for any two profiles $R, (R'_S, R_{-S}) \in \mathcal{R}^A$ for some $A, S \subseteq N$ such that $f(R) \neq f(R'_S, R_{-S})$ there exists a set $B \subset S$ and an agent $i \in (S \setminus B)$ such that $f(R) = f(R'_B, R_{-B}) \neq f(R'_{B\cup\{i\}}, R_{-(B\cup\{i\})}) = f(R'_S, R_{-S})$.*

Proof: Starting at R, construct the sequence of profiles in which we change the preferences of all agents $j \in S$ one-by-one from R_j to R'_j. Since $f(R) \neq f(R'_S, R_{-S})$ by assumption, there is subset B of S and an agent $i \in (S \setminus B)$ such that $f(R'_B, R_{-B}) \neq f(R'_{B\cup\{i\}}, R_{-(B\cup\{i\})})$. Since $|R_f(A)| \leq 2$ by Proposition 1, it follows that $f(R) = f(R'_B, R_{-B})$ and $f(R'_{B\cup\{i\}}, R_{-(B\cup\{i\})}) = f(R'_S, R_{-S})$. $\square$

Now, we are ready to prove the proposition.

Proof of Proposition 2: Suppose there are two profiles $R, R' \in \mathcal{R}^A$ such that $R_{S_f(A)} = R'_{S_f(A)}$ but $f(R) \neq f(R')$. By Lemma 4, there is a set of agents $B \subset N \backslash S_f(A)$ and an agent $i \in N \backslash (S_f(A) \cup B)$ such that $f(R) = f(R'_B, R_{-B}) \neq f(R'_{B\cup\{i\}}, R_{-(B\cup\{i\})}) = f(R')$. Thus, $O_i(A, (R'_B, R_{-(B\cup\{i\})})) = R_f(A)$ by Corollary 3. Since $i \notin S_f(A)$ by construction, this contradicts Lemma 1. $\square$

Proof of Proposition 3

Consider any $A \subseteq N$ and define for each subprofile $R_{S^i_f(R)}$ of a profile $R \in \mathcal{R}^A$ a set $\mathcal{G}(R_{S^i_f(R)}) \subseteq 2^N$ of $l_f(A)$–decisive coalitions in the following way: the set B belongs to $\mathcal{G}(R_{S^i_f(R)})$ if there is a profile $R' \in \mathcal{R}^A$ such that $R_{S^i_f(R)} = R'_{S^i_f(R')}$, $S^l_f(R') = B$ and $f(R') = l_f(A)$.

We show first that a voting by these collections of $l_f(A)$-decisive sets is a well–defined binary decision function. That is, if $B \in \mathcal{G}(R_{S^i_f(R)})$, then for all profiles $\bar{R} \in \mathcal{R}^A$ such that $B = S^l_f(\bar{R})$ and $\bar{R}_{S^i_f(\bar{R})} = R_{S^i_f(R)}$, $f(\bar{R}) = l_f(A)$. Suppose to the contrary that $f(\bar{R}) = r_f(A)$. By definition, there is a profile $R' \in \mathcal{R}^A$ such that $R_{S^i_f(R)} = R'_{S^i_f(R')}$, $S^l_f(R') = B$, and $f(R') = l_f(A)$. Then, by Lemma 4, there is a set $B \subset S_f(A)$ and an agent $j \in (S_f(A) \backslash B)$ such that $f(R') = f(\bar{R}_B, R'_{-B}) = l_f(A) \neq r_f(A) = f(\bar{R}_{B\cup\{j\}}, R'_{-(B\cup\{j\})}) = f(\bar{R})$. If $j \in S^l_f(\bar{R})$ (respectively, $j \in S^r_f(\bar{R})$), we have that $j \in S^l_f(R')$ (respectively, $j \in S^r_f(R')$) by assumption and agent j manipulates f at $(\bar{R}_{B\cup\{j\}}, R'_{-(B\cup\{j\})})$ (respectively, at $(\bar{R}_B, R'_{-B})$) via R'_j (respectively, via $\bar{R}_j$).

Now we show that if $B \in \mathcal{G}(R_{S^i_f(R)})$, $B \cup \{j\} \in \mathcal{G}(R_{S^i_f(R)})$ for all $j \in (S_f(A) \setminus S^i_f(R))$. Suppose otherwise. Consider a profile $\bar{R} \in \mathcal{R}^A$ such that $R_{S^i_f(R)} = \bar{R}_{S^i_f(\bar{R})}$, $S^l_f(\bar{R}) = B \cup \{j\}$ and $f(\bar{R}) = r_f(A)$. Then, agent j manipulates f at this profile via any $R'_j \in \mathcal{R}^A_j$ such that $r_f(A)P'_j l_f(A)$ to obtain $l_f(A)$.

Next, we show that if $S^i_f(R) = \emptyset$, then $\mathcal{G}(R_{S^i_f(R)}) \neq \{S_f(A)\}$ or, what is the same, that if $S^l_f(R) = S_f(A)$, then $f(R) = l_f(A)$. Suppose to the contrary that $f(R) = r_f(A)$. Since $l_f(A) \in R_f(A)$, there is a profile $R' \in \mathcal{R}^A$ such that $f(R') = l_f(A)$. It follows from Proposition 2 that $f(R) = f(R_{S_f(A)}, R'_{-S_f(A)}) = r_f(A)$. Then, considering the profiles R' and $(R_{S_f(A)}, R'_{-S_f(A)})$ and applying Lemma 4, there is a set $B \subset S_f(A)$ and an agent $i \in (S_f(A) \setminus B)$ such that $f(R'_B, R_{-B}) = r_f(A) \neq l_f(A) = f(R'_{B\cup\{i\}}, R_{-(B\cup\{i\})})$. Since $l_f(A)\, P_i\, r_f(A)$ by assumption, agent i manipulates f at (R'_B, R_{-B}) via R'_i. It is possible to show in a similar way that if $S^i_f(R) = \emptyset$, then $\emptyset \notin \mathcal{G}(R_{S^i_f(R)})$ or, what is the same, that if $S^r_f(R) = S_f(A)$, then $f(R) = r_f(A)$.

Now, we establish that if $B \in \mathcal{G}(R_{S^i_f(R)})$ and $j \notin B \cup S^i_f(R)$, then $B \in \mathcal{G}(R'_{S^i_f(R')})$, where $R' \in \mathcal{R}^A$ is such that $R_{S^i_f(R)} = R'_{S^i_f(R)}$ and $S^i_f(R') = S^i_f(R) \cup \{j\}$. Suppose to the contrary that this is not the case and that for some $R, R' \in \mathcal{R}^A$ with $R_{S^i_f(R)} = R'_{S^i_f(R)}$ and $S^i_f(R') = S^i_f(R) \cup \{j\}$ such that $S^l_f(R) = S^l_f(R') = B$, although $f(R) = l_f(A)$, we have that $f(R') = r_f(A)$. Then, by Lemma 4, there is a set $B \subset S_f(A)$ and an agent $k \in (S_f(A) \setminus B)$ such that $f(R) = f(R'_B, R_{-B}) = l_f(A) \neq r_f(A) = f(R'_{B\cup\{k\}}, R_{-(B\cup\{k\})}) = f(R')$. If $k \in S^i_f(R)$, $R_k = R'_k$ by construction and, then, $f(R'_B, R_{-B}) = f(R'_{B\cup\{k\}}, R_{-(B\cup\{k\})})$, which contradicts the fact that they are different. If $k \in S^r_f(R)$, agent k manipulates f at (R'_B, R_{-B}) via R'_k. If, however, $k \in S^l_f(R)$, then $k \in S^l_f(R')$ by construction. So, agent k manipulates f at $(R'_{B\cup\{k\}}, R_{-(B\cup\{k\})})$ via R_k. Finally, the proof that if $(B \cup \{j\}) \notin \mathcal{G}(R_{S^i_f(R)})$, then $B \notin \mathcal{G}(R'_{S^i_f(R')})$, where $R' \in \mathcal{R}^A$ is such that $R_{S^i_f(R)} = R'_{S^i_f(R)}$ and $S^i_f(R') = S^i_f(R) \cup \{j\}$ follows a similar argument and is thus omitted.

Proof of Proposition 4

Since the first two statements are dual, we only consider the case $r_f(A) \leq i$. First, to see that $r_f(A \cup \{i\}) \leq i$ suppose otherwise. If $r_f(A) < i$, consider a profile $R \in \mathcal{R}^A$ such that $r_f(A \cup \{i\})\, P_j\, l_f(A \cup \{i\})$ for all $j \in S_f(A \cup \{i\})$ and $r_f(A \cup \{i\})\, P_i\, l_f(A)$. Agent i then manipulates f at R via any $R'_i \in \mathcal{R}_i^+$ such that $r_f(A \cup \{i\})\, P'_i\, l_f(A \cup \{i\})$ in order to obtain $r_f(A \cup \{i\})$ by Proposition 3. If, on the other hand, $r_f(A) = i$, consider a profile $R \in \mathcal{R}^{A \cup \{i\}}$ such that $r_f(A)\, P_j\, l_f(A)$ for all $j \in S_f(A)$ and $r_f(A \cup \{i\})\, P_k\, l_f(A \cup \{i\})$ for all $k \in S_f(A \cup \{i\})$. Then, $f(R) = r_f(A \cup \{i\})$ by Proposition 3, and agent i manipulates f at this profile via any $R'_i \in \mathcal{R}_i^-$. Next, we show that $r_f(A \cup \{i\}) \geq r_f(A)$. Suppose that $r_f(A \cup \{i\}) < r_f(A)$ and consider the profile $R \in \mathcal{R}^{A \cup \{i\}}$ such that $r_f(A)\, P_j\, l_f(A)$ for all $j \in S_f(A)$ and $r_f(A \cup \{i\})\, P_k\, l_f(A \cup \{i\})$ for all $k \in S_f(A \cup \{i\})$. It follows from Proposition 3 that $f(R) = r_f(A \cup \{i\})$ and $f(R'_i, R_{-i}) = r_f(A)$ for all $R'_i \in \mathcal{R}_i^-$. Then, agent i can manipulate f at R via any $R'_i \in \mathcal{R}_i^-$. Observe that a similar argument can be used to show that $l_f(A \cup \{i\}) \geq l_f(A)$.

To complete this part of the proposition, we only need to show that $l_f(A \cup \{i\}) \geq r_f(A)$ whenever $R_f(A) \neq R_f(A \cup \{i\})$. Suppose otherwise; that is, $l_f(A \cup \{i\}) < r_f(A)$ and consider the profile $R \in \mathcal{R}^A$ such that $r_f(A)\, P_j\, l_f(A)$ for all $j \in S_f(A)$ and $l_f(A \cup \{i\})\, P_k\, r_f(A \cup \{i\})$ for all $k \in S_f(A \cup \{i\})$. It is possible to construct such a profile because by assumption and the reasoning of the previous paragraphs, $l_f(A \cup \{i\}) > l_f(A)$, and $r_f(A \cup \{i\}) > r_f(A)$. Then, by Proposition 3, $f(R) = r_f(A)$ and $f(R'_i, R_{-i}) = l_f(A \cup \{i\})$ for any $R'_i \in \mathcal{R}_i^+$. Thus, agent i manipulates f at R via R'_i.

Finally, we consider the cases when $i \in (l_f(A), r_f(A))$. In order to see that $i \leq r_f(A \cup \{i\})$, assume by contradiction that $i > r_f(A \cup \{i\})$ and take a profile $R \in \mathcal{R}^{A \cup \{i\}}$ such that $r_f(A)\, P_j\, l_f(A)$ for all $j \in S_f(A)$ and $r_f(A)\, P_i\, r_f(A \cup \{i\})$. Then, agent i manipulates f at R via any $R'_i \in \mathcal{R}_i^-$ such that $r_f(A)\, P'_i\, l_f(A)$ in order to obtain $r_f(A)$ instead of any element of $R_f(A \cup \{i\})$. One can show in a similar way that $i \geq l_f(A \cup \{i\})$. Therefore, $i \in [l_f(A \cup \{i\}), r_f(A \cup \{i\})]$. We establish next that $l_f(A \cup \{i\}) \geq l_f(A)$ (the proof that $r_f(A \cup \{i\}) \leq r_f(A)$ is dual). Suppose otherwise; that is, $l_f(A \cup \{i\}) < l_f(A)$. Consider a profile $R \in \mathcal{R}^{A \cup \{i\}}$ such that $l_f(A \cup \{i\})\, P_j\, r_f(A \cup \{i\})$ for all

$j \in S_f(A\cup\{i\})$ and $l_f(A)\, P_k\, r_f(A)$ for all $k \in S_f(A)$. Then, by Proposition 3, $f(R) = l_f(A\cup\{i\})$. Thus, agent i manipulates it via any $R'_i \in \mathcal{R}^-_i$ such that $l_f(A)\, P'_i\, r_f(A)$ in order to obtain $l_f(A)$ instead of $l_f(A \cup \{i\})$. We show next that if $l_f(A \cup \{i\}) = i$, then $r_f(A \cup \{i\}) \in \{i, r_f(A)\}$. Suppose otherwise; that is, $l_f(A\cup\{i\}) = i$, but $r_f(A\cup\{i\}) \in (i, r_f(A))$. Then, consider a profile $\bar{R} \in \mathcal{R}^{A\cup\{i\}}$ such that $r_f(A\cup\{i\})\, \bar{P}_j\, i$ for all $j \in S_f(A\cup\{i\})$, $l_f(A)\, \bar{P}_k\, r_f(A)$ for all $k \in S_f(A)$ and $l_f(A)\bar{P}_i r_f(A\cup\{i\})$. Then, by Proposition 3, we have that $f(R) = r_f(A\cup\{i\})$. However, agent i manipulates it via any $\hat{R}_i \in \mathcal{R}^-_i$ such that $l_f(A)\hat{P}_i r_f(A)$ to obtain $l_f(A)$ by Proposition 3. It is similar to show that if $r_f(A\cup\{i\}) = i$, then $l_f(A\cup\{i\}) \in \{l_f(A), i\}$ and thus we omit it.

Proof of Proposition 5

Consider first the case when $R_f(A)\cap R_f(A\cup\{i\}) = \emptyset$. We will only show that $B \in \mathcal{G}(R'_{S^i_f(R')})$ if and only if $i \in B$ for all $R' \in \mathcal{R}^A$ such that $i \notin S^i_f(R')$. The proof that $B \in \mathcal{G}(R_{S^i_f(R)})$ iff $i \in B$ for all $R \in \mathcal{R}^{A\cup\{i\}}$ such that $i \notin S^i_f(R)$ is similar. Suppose otherwise; that is, there is some profile $\bar{R} \in \mathcal{R}^A$ such that $f(\bar{R}) = x$ and $y\, \bar{P}_i\, x$, where $x, y \in R_f(A)$. Assume without loss of generality that $x = r_f(A)$. Then, consider a profile $\hat{R} \in \mathcal{R}^A$ with $r_f(A)\, \hat{P}_j\, l_f(A)$ for all $j \in (S_f(A)\setminus\{i\})$, $l_f(A\cup\{i\})\, \hat{P}_k\, r_f(A\cup\{i\})$ for all $k \in S_f(A\cup\{i\})$, and $l_f(A\cup\{i\})\, \hat{P}_i\, r_f(A)$. Since $f(\bar{R}) = r_f(A)$, it follows from Proposition 3 that $f(\hat{R}) = r_f(A)$. However, agent i then manipulates f at $\hat{R}$ via any $R''_i \in \mathcal{R}^+_i$ such that $l_f(A\cup\{i\})\, P''_i\, r_f(A\cup\{i\})$ in order to obtain $l_f(A\cup\{i\})$ by Proposition 3.

Suppose now that $|R_f(A)\cap R_f(A\cup\{i\})| = 1$. It is assumed without loss of generality that $R_f(A)\cap R_f(A\cup\{i\}) = r_f(A) = r_f(A\cup\{i\})$. We first show that $[B \in \mathcal{G}(R_{S^i_f(R)})$ iff $i \in B$ for all $R \in \mathcal{R}^{A\cup\{i\}}$ with $i \notin S^i_f(R)]$ if and only if $[B \in \mathcal{G}(R'_{S^i_f(R')})$ iff $i \in B$ for all $R' \in \mathcal{R}^A$ with $i \notin S^i_f(R')]$. So, suppose first that $B \in \mathcal{G}(R_{S^i_f(R)})$ iff $i \in B$ for all $R \in \mathcal{R}^{A\cup\{i\}}$ with $i \notin S^i_f(R)$, but, by contradiction, that there is some $R' \in \mathcal{R}^A$ such that $f(R') = x$ and $y\, P'_i\, x$, where $x, y \in R_f(A)$. Let $x = r_f(A)$ –the case when $x = l_f(A)$ is similar and thus omitted– and consider a preference $R''_i \in \mathcal{R}^-_i$ such that $l_f(A\cup\{i\})\, P''_i\, r_f(A)$. If it was the case that $f(R''_i, R'_{-i}) = l_f(A)$, then individual i could manipulate f at R' via R''_i. Hence, $f(R''_i, R'_{-i}) = r_f(A)$. Since $B \in \mathcal{G}(R_{S^i_f(R)})$ iff $i \in B$ for all $R \in \mathcal{R}^{A\cup\{i\}}$ with $i \notin S^i_f(R)$ by assumption, we also have that for all $\bar{R}_i \in \mathcal{R}^+_i$ with

$l_f(A \cup \{i\})\, \bar{P}_i\, r_f(A)$, $f(\bar{R}_i, R'_{-i}) = l_f(A \cup \{i\})$. But agent i can then manipulate f at (R''_i, R'_{-i}) via any $\bar{R}_i \in \mathcal{R}_i^+$ with $l_f(A \cup \{i\})\, \bar{P}_i\, r_f(A)$. Observe finally that the proof of the other implication is similar and thus omitted. Consequently, the two implications are equivalent.

Consider now any profile $R \in \mathcal{R}^A$ with $i \notin S_f^i(R)$ such that $f(R) = l_f(A)$. If there is a preference $R'_i \in \mathcal{R}_i^+$ such that $f(R'_i, R_{-i}) = r_f(A \cup \{i\}) = r_f(A)$, then strategy-proofness implies that $O_i(A, R_{-i}) = R_f(A)$; in fact, if it was the case that $f(R''_i, R_{-i}) = l_f(A)$ for all $R''_i \in \mathcal{R}_i^-$, then agent i would be able to manipulate f at any profile $(\bar{R}_i, R_{-i}) \in \mathcal{R}^A$ such that $r_f(A)\, \bar{R}_i\, l_f(A)$ via R'_i. Similarly, if there is a preference $\bar{R}_i \in \mathcal{R}_i^-$ such that $f(\bar{R}_i, R_{-i}) = r_f(A)$, we can also conclude that $O_i(A, R_{-i}) = R_f(A)$. In any of these cases, given that $O_i(A, R_{-i}) = R_f(A)$, Lemma 2 implies that $B \in \mathcal{G}(R_{S_f^i(R)})$ iff $i \in B$. If this occurs with all profiles of $\mathcal{R}^A$, we would obtain that i is a dictator in all profiles of $\mathcal{R}^A$ and, by the equivalence obtained in the previous paragraph, i is also a dictator in all profiles of $\mathcal{R}^{A \cup \{i\}}$ and the proof is complete. So, suppose from now on that there is a profile $R \in \mathcal{R}^A$ with $i \notin S_f^i(R)$ such that $f(R) = l_f(A)$, $f(R'_i, R_{-i}) = l_f(A \cup \{i\})$ for all $R'_i \in \mathcal{R}_i^+$ and $f(\bar{R}_i, R_{-i}) = l_f(A)$ for all $\bar{R}_i \in \mathcal{R}_i^-$. Then, $\mathcal{G}(R_{S_f^i(R)}) = \mathcal{G}((R'_i, R_{-i})_{S_f^i(R'_i, R_{-i})})$.

We now use the definition of minimal coalitions introduced in the main text. We establish in the next step that all minimal coalitions $D \in \mathcal{G}^*(R_{S_f^i(R)})$ have to be subsets of $S_f(A) \cap A$, except agent i. Suppose otherwise; that is, there is some $D \in \mathcal{G}^*(R_{S_f^i(R)})$ such that $D \setminus (A \cap S_f(A)) \notin \{\emptyset, \{i\}\}$. Consider the profile $\bar{R} \in \mathcal{R}^A$ such that $\bar{R}_{S_f^i(\bar{R})} = R_{S_f^i(R)}$, $l_f(A)\, \bar{P}_j\, r_f(A)$ for all $j \in D \cap A \cap S_f(A)$, $l_f(A)\, \bar{P}_k\, r_f(A)\, \bar{P}_k\, l_f(A \cup \{i\})$ for all $k \in D \setminus (A \cap S_f(A))$, and both $r_f(A)\, \bar{P}_l\, l_f(A)$ and $r_f(A)\, \bar{P}_l\, l_f(A \cup \{i\})$ for all $l \in (S_f(A \cup \{i\}) \setminus D)$. Since $D \in \mathcal{G}^*(R_{S_f^i(R)}) = \mathcal{G}^*((R'_i, R_{-i})_{S_f^i(R'_i, R_{-i})})$ by assumption, $f(\bar{R}) = l_f(A)$ and $f(R'_i, \bar{R}_{-i}) = r_f(A \cup \{i\}) = r_f(A)$ for any $R'_i \in \mathcal{R}_i^+$. However, agent i will then manipulate f at $\bar{R}$ via R'_i. Consequently, all minimal coalitions D must be subsets of $(A \cup \{i\}) \cap S_f(A)$.

It can be shown in a very similar way that all minimal coalitions $D \in \mathcal{G}^*((R'_j, R_{-j})_{S_f^i(R'_j, R_j)})$ have to be subsets of $(S_f(A \cup \{i\}) \setminus A) \cup \{i\}$. Thus, $\mathcal{G}^*(R_{S_f^i(R)}) = \mathcal{G}^*((R'_i, R_{-i})_{S_f^i(R'_i, R_{-i})}) \in \{\emptyset, \{i\}\}$. This implies that the range of f is 1, which contradicts the assumptions of the proposition, or that $B \in \mathcal{G}(R_{S_f^i(R)}) = \mathcal{G}((R'_i, R_{-i})_{S_f^i(R'_i, R_{-i})})$ iff $i \in B$.

Proof of Proposition 6

We will only show that for all preference profiles $R \in \mathcal{R}^A$ and all $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$, $\mathcal{G}(R_{S^i_f(R)}) \subseteq \mathcal{G}(R'_{S^i_f(R')})$ whenever $i \leq r_f(A)$. Suppose there is a set C of agents that is a winning coalition when i has single-dipped preferences but not when i has single-peaked preferences; that is, $C \in (\mathcal{G}(R_{S^i_f(R)}) \setminus \mathcal{G}(R'_{S^i_f(R')}))$. If $i \notin C$, consider a profile $R'' \in \mathcal{R}^A$ such that $R''_{S^i_f(R'')} = R_{S^i_f(R)}$, $r_f(A)\, P''_i\, l_f(A)$, and for all agents $k \in (S_f(A) \setminus S^i_f(R''))$, $l_f(A)\, P''_k\, r_f(A)$ if and only if $k \in C$. Then, $f(R'') = l_f(A)$ and $f(R'_i, R''_{-i}) = r_f(A)$. Thus, agent i manipulates f at R'' via R'_i. If, on the other hand, $i \in C$, consider a profile $\bar{R} \in \mathcal{R}^{A\cup\{i\}}$ such that $\bar{R}_{S^i_f(\bar{R})} = R'_{S^i_f(R')}$ and for all $k \in (S_f(A) \setminus S^i_f(\bar{R}))$, $l_f(A)\, \bar{P}_k\, r_f(A)$ if and only if $k \in C$. Then, $f(\bar{R}) = r_f(A)$, but agent i can manipulate f at this profile via any $\hat{R}_i \in \mathcal{R}^-_i$ such that $l_f(A)\, \hat{P}_i\, r_f(A)$ to obtain $l_f(A)$.

Proof of Proposition 7

We will only consider the case when $i \geq r_f(A)$ and $l_f(A \cup \{i\}) = l_f(A)$, the other three situations are similar. So suppose that $B \in \mathcal{G}^*(R_{S^i_f(R)})$ for some $R \in \mathcal{R}^{A\cup\{i\}}$ and assume by contradiction that there is a preference $R'_i \in \mathcal{R}^-_i$ such that $(B \cap (S_f(A) \setminus A)) \notin \mathcal{G}((R'_i, R_{-i})_{S^i_f(R'_i, R_{-i})})$. Consider a profile $R'' \in \mathcal{R}^{A\cup\{i\}}$ such that $R''_{S^i_f(R'')} = R_{S^i_f(R)}$, $l_f(A)\, P''_k\, r_f(A \cup \{i\})$ for $k \in S_f(A \cup \{i\})$ if and only if $k \in B$, and $r_f(A)\, P''_l\, l_f(A)$ for all $l \in (B \cap S_f(A) \cap A) \cup (S_f(A) \setminus (B \cup A))$. Then, $f(R'') = l_f(A)$ and $f(R'_i, R''_{-i}) = r_f(A)$. Thus, agent i can manipulate f at R'' via R'_i.

Proof of Proposition 8

We will only consider the case when $l_f(A) = l_f(A \cup \{i\})$ because the other is similar. By contradiction, suppose that there is some $B \in \mathcal{G}^*(R_{S^i_f(R)})$ for some $R \in \mathcal{R}^{A\cup\{i\}}$, but $(B \cap A) \notin \mathcal{G}(R'_{S^i_f(R')})$ for some $R' = (R'_i, R_{-i}) \in \mathcal{R}^A$. Then, consider a profile $\bar{R} \in \mathcal{R}^{A\cup\{i\}}$ such that $\bar{R}_{S^i_f(\bar{R})} = R_{S^i_f(R)}$, $l_f(A \cup \{i\}) \bar{P}_j i$ for $j \in S_f(A \cup \{i\})$ if and only if $j \in B$ and $r_f(A) \bar{P}_k l_f(A)$ for all $k \in (S_f(A) \setminus (B \cap A))$. Then, $f(\bar{R}) = l_f(A)$, but agent i can manipulate it via any $\hat{R}_i \in \mathcal{R}^-_i$ to obtain $r_f(A)$.

Proof of Theorem 1

[1] $\Rightarrow$ [3]: It has been shown in the corresponding propositions.

[3] $\Rightarrow$ [1]: We are going to show that no agent i has incentives to manipulate any of these social choice rules. We concentrate on the case when agent i has single-peaked preferences; that is, we consider a profile $R \in \mathcal{R}^{A\cup\{i\}}$ for some $A \subseteq N$ and $i \notin A$ (the analysis of the situation when agent i has single-dipped preferences is similar and thus omitted). If $f(R) = i$, agent i cannot manipulate f at R. If $f(R) < i$ (we omit the situation when $f(R) > i$), consider the following cases:

- If $R_f(A \cup \{i\}) = \{f(R)\}$, agent i can only change the outcome of f at R by declaring a single-dipped preference. However, we know from Proposition 4(first case) that $r_f(A\cup\{i\}) \in [r_f(A), i]$ whenever $r_f(A \cup \{i\}) \leq i$. So, for all $x \in R_f(A)$, $x \leq f(R)$. This implies that i cannot manipulate f at R.

- If $R_f(A\cup\{i\}) = \{f(R), x\}$ and $x \leq i$, then $i \notin S_f(A\cup\{i\})$. So, by Proposition 2, $f(\bar{R}_i, R_{-i}) = f(R)$ for all $\bar{R}_i \in \mathcal{R}_i^+$. This implies that agent i can only change the outcome of f at R by declaring a single-dipped preference. If $x < f(R)$, then $l_f(A \cup \{i\}) = x$ and $r_f(A \cup \{i\}) = f(R)$. Hence, it follows from Proposition 4(first case) that $f(R) \in [r_f(A), i]$ and $x \in [l_f(A), i]$. So, for all $y \in R_f(A)$, $y \leq f(R)$ and i cannot manipulate. If, on the other hand, $x > f(R)$, then $l_f(A \cup \{i\}) = f(R)$ and $r_f(A \cup \{i\}) = x$. It follows from Proposition 4(first and third cases) that $l_f(A) \leq l_f(A \cup \{i\}) = f(R)$. Now, we have two possibilities. If $R_f(A) \neq R_f(A\cup\{i\})$, we have by Proposition 4 (first case) that for all $y \in R_f(A)$, $y \leq f(R)$ and i cannot manipulate. If, however, $l_f(A) = l_f(A \cup \{i\})$ or $r_f(A) = r_f(A \cup \{i\})$, it follows from Propositions 7 and 8 that $f(R'_i, R_{-i}) = l_f(A)$ for all $R'_i \in \mathcal{R}_i^-$. This guarantees that agent i cannot manipulate f at R.

- If $R_f(A) = \{f(R), x\}$ and $x > i$, then $i \in S_f(A \cup \{i\})$. By Proposition 3, we have that $f(R)\, R_i\, f(\bar{R}_i, R_{-i})$ for all $\bar{R}_i \in \mathcal{R}_i^+$. Then, suppose now that i try to improve its welfare at R by declaring a single-dipped preference $R'_i \in \mathcal{R}_i^-$. We know from Proposition 4 (third case) that $[l_f(A \cup \{i\}, r_f(A \cup \{i\})] \subseteq [l_f(A), r_f(A)]$. There are two possibilities. If $R_f(A) = R_f(A \cup \{i\})$, Proposition 6 implies that $\mathcal{G}(R_{S^i_f(R)}) = \mathcal{G}((R'_i, R_{-i})_{S^i_f(R'_i, R_{-i})})$. Hence, agent i cannot manipulate in this situation. If, however, $|R_f(A) \cap R_f(A \cup \{i\})| < 2$, we know by Proposition 5 that i is a dictator in both $R_f(A)$ and in $R_f(A \cup \{i\})$. Hence, agent i cannot manipulate f at R in this situation either.

Then, this statement is proved.

[3] $\Rightarrow$ [2]: Take any of these social choice rules f. Assume, by contradiction, that there is a profile $R \in \mathcal{R}^A$, a group of agents $S \subseteq N$ with $|S| > 1$ and a subprofile $R'_S \in \mathcal{R}_S$ such that $f(R'_S, R_{-S}) \, P_i \, f(R)$ for all $i \in S$. We can assume without loss of generality that $f(R'_S, R_{-S}) > f(R)$. We can then see that all agents $i \leq f(R)$ of S must have single-dipped preferences at R and all agents $i \geq f(R'_S, R_{-S})$ of S must have single-peaked preferences at R for the manipulation to be effective.

Let $A \subseteq N$ be the set of agents with single-peaked preferences at profile R and let $C \subseteq N$ be the set of agents with single-peaked preferences at profile (R'_S, R_{-S}). Change the preferences of all agents $i \in S$ one-by-one from R_i to R'_i in the following order: (i) start with the agents belonging to $S_f(A)$ in any arbitrary order; at the end of these changes, we will have a set of agents with single-peaked preferences denoted by B; (ii) continue in any arbitrary order with all agents of $(S_f(B) \setminus S_f(A))$; (iii) continue with the agents located to the left of $l_f(B)$ taking in each step the agent located most to the right; (iv) continue with the single-peaked agents located to the right of $r_f(B)$ taking in each step the agent located most to the right; and (v) complete the process with the single-dipped agents located to the right of $r_f(B)$ taking in each step the agent located most to the left. It is easy to see that $[l_f(A), r_f(A)] \cap [l_f(B) \cap r_f(B)] \neq \emptyset$.

If $l_f(B) < l_f(C)$ or $r_f(B) < r_f(C)$, then there is at least one agent $i \in (S \setminus S_f(A))$ that moves one of the preselected locations to the right when changing her preferences from R_i to R'_i. We divide the analysis depending on the location of i:

- Consider first the case in which $i \leq l_f(B)$. Given that $i \notin S_f(A)$ and $[l_f(A), r_f(A)] \cap [l_f(B) \cap$

$r_f(B)] \neq \emptyset$, we can also deduce that $i \leq l_f(A)$ and, therefore, $i \leq f(R)$. Then, $R_i \in \mathcal{R}_i^+$ and $R_i' \in \mathcal{R}_i^-$ by Proposition 4 (second case). This is a contradiction with the fact that all agents of S at the left of $f(R)$ have single-dipped preferences at R.

- Consider the case in which $i \geq r_f(B)$. Then, $R_i \in \mathcal{R}_i^-$ and $R_i' \in \mathcal{R}_i^+$ by Proposition 4 (first case). Let $S_r^d = \{i \in S \,|\, i \geq r_f(B) \text{ and } R_i \in \mathcal{R}_i^-\}$. Given the order of changes that we have established, it follows from the iterated application of Proposition 4 (all cases) that $\max S_r^d \geq r_f(C)$ and, therefore, $f(R) \, P_{\max S_r^d} \, f(R_S', R_{-S})$. This contradicts that S can manipulate f at R via R_S'.

- Consider finally the case in which $i \in (S_f(B) \setminus S_f(A))$. The fact that $(S_f(B) \setminus S_f(A)) \neq \emptyset$ implies that there is at least one agent $j \in S_f(A) \cap S \cap A$ such that $[l_f(A), r_f(A)] \subset [l_f(A \setminus \{j\}), r_f(A \setminus \{j\})]$ and $R_j' \in \mathcal{R}_j^-$. Then, by Proposition 5 we have that j is a dictator in $R_f(A)$ and in $R_f(A \setminus \{j\})$. When the rest of the single-peaked agents (that is, the agents of $S_f(A) \cap S \cap A$) change their preferences, Proposition 5 implies that $R_f(A \setminus \{j\}) = R_f(E)$, where E is the set of agents with single-peaked preferences in that moment. Then, by Proposition 6, we have that j is a dictator in $R_f(E)$. Now, consider any agent $k \in (S_f(A) \cap (S \setminus A))$ such that $R_k' \in \mathcal{R}_k^+$. Then, we would like to show that $R_f(E \cup \{k\}) = R_f(E)$. Suppose by contradiction that this is not the case. Then, Proposition 4 (third case) gives two possibilities: $R_f(E \cup \{k\}) = \{k\}$ or $R_f(E \cup \{k\}) \in \{\{k, l_f(E)\}, \{k, r_f(E)\}\}$. The former is not possible because if we continue changing the preferences of the rest of agents of $(S_f(A) \cap (S \setminus A))$, Proposition 4 (first and second cases) would imply that $S_f(B) \subseteq S_f(A)$ contradicting that $i \in (S_f(B) \setminus S_f(A))$. The later case is also impossible because Proposition 8 would imply that the minimal coalitions in any profile of $\mathcal{R}^E$ have to be formed by single-peaked agents and j has single-dipped preferences in these profiles in which she is a dictator. Therefore, $R_f(E \cup \{k\}) = R_f(E)$ and, by Proposition 6 we also have that j is a dictator in $R_f(E \cup \{k\})$. Repeating the same arguments we arrive at $R_f(A \setminus \{j\}) = R_f(B)$ and j being a dictator in $R_f(B)$. Then, $l_f(B) \leq l_f(A)$ and $r_f(A) \leq r_f(B)$.

Given that $j \in A$ and that she was a dictator in $R_f(A)$, we necessarily need that $f(R'_S, R_{-S}) \in (l_f(A), r_f(A))$ for the manipulation to be effective. Given that $f(R) < f(R'_S, R_{-S})$, we have that $f(R) = l_f(A)$.

Let $r_f(C) > r_f(B)$. Then, agent $i \in (S_f(B) \setminus S_f(A))$ moves the right preselected alternative to the right in some moment. By Proposition 4 (third case), $R_i \in \mathcal{R}_i^+$ and $R'_i \in \mathcal{R}_i^-$. Then, given that all agents of S at the left of $f(R)$ have single-dipped preferences at R, we must have that $i \in (r_f(A), r_f(B))$. Then, by Proposition 5, i is a dictator for $R_f(B)$. This contradicts that j is a dictator for the same set.

Let $l_f(C) > l_f(B)$. Then, agent $i \in (S_f(B) \setminus S_f(A))$ moves the left preselected alternative to the right in some moment. By Proposition 4 (third case), $R_i \in \mathcal{R}_i^-$ and $R'_i \in \mathcal{R}_i^+$. Then, given that all agents of S at the right of $r_f(A)$ have single-peaked preferences at R, we must have that $i \in (l_f(B), l_f(A))$. By Proposition 4 (third case), no agent $k \in (l_f(B), l_f(A))$ can move the left preselected alternative to the right of $l_f(A)$ and, therefore, $f(R'_S, R_{-S}) = r_f(C) < r_f(A) \leq r_f(B)$. Then, by Proposition 5, i is a dictator for $R_f(B)$, contradicting that j is a dictator for the same set.

Consequently, we assume from now on that $l_f(B) \geq l_f(C)$ and $r_f(B) \geq r_f(C)$. Then, there are no agents of S at the right of $r_f(B)$ with single-dipped preferences and step (v) of the chain of changes is empty. The remainder of the proof is divided into two cases.

1. Let $f(R'_S, R_{-S}) = l_f(C)$. Since $l_f(B) \geq l_f(C) = f(R'_S, R_{-S})$ and $f(R'_S, R_{-S}) > f(R)$, we must have that $l_f(B) > l_f(A)$. By Proposition 4, there exists an agent $i \in S_f(A) \cap S$ such that $i \geq l_f(B)$ that changes her preferences from $R_i \in \mathcal{R}_i^-$ to $R'_i \in \mathcal{R}_i^+$. It follows then that $i \geq f(R'_S, R_{-S})$. Given that i has single-dipped preferences at R, we have a contradiction.

2. Let $f(R'_S, R_{-S}) = r_f(C)$. Assume first that $f(R) = r_f(A)$. Since $r_f(B) \geq r_f(C) = f(R'_S, R_{-S})$ and $f(R'_S, R_{-S}) > f(R)$, we have that $r_f(B) > r_f(A)$. By Proposition 4, there exists an agent $i \in S_f(A) \cap S$ that changes her preferences from $R_i \in \mathcal{R}_i^+$ to $R'_i \in \mathcal{R}_i^-$. Given that $i \leq f(R)$ and she has single-peaked preferences at R, we have a contradiction. So, we

assume from now on that $f(R) = l_f(A)$. We divide the analysis into three cases:

- Suppose that $r_f(B) < r_f(A)$; the case when $l_f(B) > l_f(A)$ is similar and thus omitted. Since $r_f(C) \leq r_f(B)$, we have that $f(R'_S, R_{-S}) = r_f(C) \in (l_f(A), r_f(B)]$. By Proposition 4, there exists an agent $i \in S_f(A) \cap S$ such that $i \leq r_f(B)$ that changes her preferences from $R_i \in \mathcal{R}_i^-$ to $R'_i \in \mathcal{R}_i^+$. Suppose first that $i < r_f(B)$. Then, Proposition 5 implies that i is a dictator at $R_f(A)$ or, what is the same, that $l_f(A) R_i r_f(A)$. Then, given that $R_i \in \mathcal{R}_i^-$, we have that $l_f(A) P_i x$ for all $x \in (l_f(A), r_f(A))$ and, in particular, $f(R) P_i f(R'_S, R_{-S})$, contradicting the fact that S can manipulate f. Suppose finally that $i = r_f(B)$. Then, $i \geq f(R'_S, R_{-S})$ and has single-dipped preferences at R, which is not possible.

- Suppose that $r_f(B) > r_f(A)$; the case when $l_f(A) > l_f(B)$ is similar and thus omitted. By Proposition 4 (third case), the right preselected alternative changes from $r_f(A)$ to its left when some agent $j \in S_f(A) \cap S$ changes her preferences from $R_j \in \mathcal{R}_j^+$ to $R_j \in \mathcal{R}_j^-$. By similar arguments to ones used previously in the proof, we have that $R_f(A \setminus \{j\}) = R_f(B)$, $l_f(B) \leq l_f(A)$ and that j is a dictator for both $R_f(A)$ and $R_f(B)$. Thus, $f(R) = l_f(A)\, R_j\, r_f(A)$. If $r_f(C) \geq r_f(A)$, then $f(R)\, R_j\, f(R'_S, R_{-S})$. So, this cannot be.

  If $r_f(C) < r_f(A)$, we have that $r_f(C) < r_f(A) < r_f(B)$ and $l_f(B) \leq l_f(A)$. This is only possible if some agent $i \leq l_f(B)$ with $R_i \in \mathcal{R}_i^-$ and/or some agent $i \geq r_f(B)$ with $R_i \in \mathcal{R}_i^+$ change their type of preferences. Consider the case of an agent $i \leq l_f(B)$ (the analysis of agents $i \geq r_f(B)$ is dual and thus omitted). If both preselected alternatives changes when agent i changes from $R_i \in \mathcal{R}_i^-$ to $R'_i \in \mathcal{R}_i^+$, then it follows from Proposition 4 (second case) that $r_f(B \cup \{i\}) \leq l_f(B)$ and, then, $r_f(B \cup \{i\}) \leq l_f(A)$. By the iterated application of this reasoning, we obtain that $r_f(C) \leq l_f(A)$, which contradicts that $f(R'_S, R_{-S}) > f(R)$. If only one preselected alternative changes when agent i changes from $R_i \in \mathcal{R}_i^-$ to $R'_i \in \mathcal{R}_i^+$, then it follows from Proposition 4 (second case) that $l_f(B \cup \{i\}) = l_f(B)$. By the iterated application of this reasoning, we deduce that $l_f(C) = l_f(B)$. Then, by Proposition 7, j is also a dictator for $R_f(C)$. Thus, $j \leq r_f(C)$.

Consider now the profile $(R'_{S\setminus\{j\}}, R_{-(S\setminus\{j\})}) \in \mathcal{R}^{C\cup\{j\}}$; that is, all agents $i \in S$ apart from j have changed their preferences from R_i to R'_i. By Proposition 4 (second case), $r_f(C \cup \{j\}) \in [j, r_f(C)]$. Thus, there is an agent $i \leq l_f(A) = f(R)$ of S with $R_i \in \mathcal{R}_i^-$ and $R'_i \in \mathcal{R}_i^+$ and/or an agent $i \geq r_f(A) > f(R'_S, R_{-S})$ of S with $R_i \in \mathcal{R}_i^+$ and $R'_i \in \mathcal{R}_i^-$. However, this is not possible.

- Suppose that $r_f(B) = r_f(A)$ and $l_f(B) = l_f(A)$. Then, step (ii) of the chain of changes is empty and all agents of $S \setminus S_f(A)$ (steps (iii) and (iv)) move these preselected alternatives to the right. Additionally, we have that $r_f(C) = f(R'_S, R_{-S}) \in (l_f(A), r_f(A)]$. Then, by Proposition 4 (second case), we have three possibilities: $R_f(C) = R_f(A)$, $R_f(C) = \{l_f(C), r_f(A)\}$ or $R_f(C) = \{l_f(A), r_f(C)\}$. If $R_f(C) = R_f(A)$, by the iterated application of Proposition 6, we have that the set of decisive sets of agents for $R_f(C)$ is equal to the set of decisive sets of agents for $R_f(A)$. It follows then from $f(R) = l_f(A)$ and $f(R'_S, R_{-S}) = r_f(C)$ that there is some agent $i \in S \cap S_f(A)$ that weakly prefers $l_f(A)$ to $r_f(A)$ at R but the other way around at (R'_S, R_{-S}). Then, $f(R)\, R_i\, f(R'_S, R_{-S})$. If either $R_f(C) = \{l_f(C), r_f(A)\}$ or $R_f(C) = \{l_f(A), r_f(C)\}$, then there is an agent $i \in S \setminus S_f(A)$ that changes her type of preferences. Suppose without loss of generality that $R_f(C) = \{l_f(A), r_f(C)\}$. Then, it follows then from the iterated application of Proposition 7 that for all $D \in \mathcal{G}((R'_{S_f(A)\cap S}, R_{-(S_f(A)\cap S)})_{S_f^i(R'_{S_f(A)\cap S}, R_{-(S_f(A)\cap S)})})$, $D \cap (S_f(A) \setminus A) \in \mathcal{G}((R'_S, R_{-S})_{S_f^i(R'_S, R_{-S})})$. Since $f(R) = l_f(A)$, there is some group $D \in \mathcal{G}(R_{S_f^i(R)})$ that strictly prefers $l_f(A)$ to $r_f(A)$ at R. So, $l_f(A)\, P_i\, r_f(A)\, R_i\, r_f(C)$ for all $i \in (D \cap (S_f(C) \setminus C))$. Therefore, $f(R'_S, R_{-S}) = l_f(A)$, leading to a contradiction.

Proof of Theorem 2

$\Leftarrow$]: Suppose that there is a triple $\{x, y, z\} \subseteq T$ that is not full at N. Consider first the case in which $N \cap \{x, y, z\} \neq \emptyset$. Suppose w.l.o.g. that $x \in N$ and $y < z$. If $(N \setminus \{x\}) \not\subset (y, z)$, consider an agent $j \notin (y,z)$. Then, a rule f such that $R_f(A) = x$ if $x \in A$, $R_f(A) = y$ if $[A \cap \{x,j\} = \{j\}$ and $j < y]$ or $[A \cap \{x,j\} = \emptyset$ and $j > z]$ and $R_f(A) = z$ otherwise is an strategy-proof rule with range 3, but is not dictatorial. If, however, $(N \setminus \{x\}) \subset (y,z)$, a rule f such that $R_f(A) = x$ if $x \in A$, $R_f(A) = \{y,z\}$ otherwise and $\mathcal{G}(R_{S^i_f(R)})$ containing all non-empty coalitions for all $R \in \mathcal{R}^A$ with $x \notin A$ is strategy-proof and has range 3, but is not dictatorial.

Consider now the cases in which, although $N \cap \{x,y,z\} = \emptyset$, $N \not\subset (\min\{x,y,z\}, \max\{x,y,z\})$. Assume w.l.o.g. that $x < y < z$ and there is one agent $i \in N$ with $i < x$. If $(N \setminus \{i\}) \not\subset (y,z)$, consider an agent $j \notin (y,z)$. Then, a rule f such that $R_f(A) = x$ if $i \in A$, $R_f(A) = y$ if $[A \cap \{i,j\} = \{j\}$ and $j < y]$ or $[A \cap \{i,j\} = \emptyset$ and $j > z]$ and $R_f(A) = z$ otherwise is an strategy-proof rule with range 3, but is not dictatorial. If, however, $(N \setminus \{i\}) \subset (y,z)$, a rule f such that $R_f(A) = x$ if $i \in A$, $R_f(A) = \{y,z\}$ otherwise and $\mathcal{G}(R_{S^i_f(R)})$ containing all non-empty coalitions for all $R \in \mathcal{R}^A$ with $i \notin A$ is strategy-proof and has range 3, but is not dictatorial.

$\Rightarrow$]: Suppose that all triples are full at N and that, by contradiction, there is a strategy-proof rule f with range greater than or equal to 3 that is not dictatorial. Consider a triple $\{x,y,z\}$ with $x < y < z$ contained in the range of f. Then, $N \subset ((x,z) \setminus \{y\})$. We distinguish two cases:

Suppose first that $|R_f(\emptyset)| = 1$. If $R_f(\emptyset) < z$, we have by iterated application of Proposition 4 that $z \notin R_f(A)$ for any $A \subseteq N$, contradicting the fact that z belongs to the range of f. Similarly, if $R_f(\emptyset) > x$, we have by iterated application of Proposition 4 that $x \notin R_f(A)$ for any $A \subseteq N$, contradicting the fact that x belongs to the range of f.

Suppose now that $|R_f(\emptyset)| = 2$. If $r_f(\emptyset) < z$ (respectively, $l_f(\emptyset) > x$), we have by iterated application of Proposition 4 that $z \notin R_f(A)$ (respectively, $x \notin R_f(A)$) for any $A \subseteq N$, contradicting the fact that z (respectively, x) belongs to the range of f. Then, $l_f(\emptyset) \leq x$ and $r_f(\emptyset) \geq z$. Given that no agent is located in a feasible point, we have by Proposition 4 (third case) that $R_f(\{i\}) = R_f(\emptyset)$ (in this case, we will say that i is *not decisive*) or $i \in (l_f(\{i\}), r_f(\{i\})) \subset [l_f(\emptyset), r_f(\emptyset)]$ (in this case, we will say that i is *decisive*). By Proposition 5, we have that if an agent i is decisive, she is a dictator in $R_f(\{i\})$ and in $R_f(\emptyset)$. Given that it is not possible to have more than one dictator at $R_f(\emptyset)$, we obtain that there is at most one decisive agent that we will denote, when exists, by $\hat{\imath}$. Consider now any set A with $|A| > 1$. Suppose first that $\hat{\imath} \notin A$ and we are going to show that $R_f(A) = R_f(\emptyset)$. Suppose that this occurs for all subsets $D \subset A$. Take now any profile $R \in \mathcal{R}^A$ and suppose that $f(R) \notin \{l_f(\emptyset), r_f(\emptyset)\}$, contrary to what we intend to show. By construction, $f(R) \in (l_f(\emptyset), r_f(\emptyset))$. By assumption, $f(R) \neq j$ for all $j \in A$. Then, $f(R) \in R_f(A)$ and $|R_f(A)| = 2$. Given that $R_f(D) = \{l_f(\emptyset), r_f(\emptyset)\}$ for all $D \subset A$, we have by Proposition 5 that all agents of A are dictators in $R_f(A)$, which obviously is not possible. Suppose now that $\hat{\imath} \in A$ and we are going to show that $R_f(A) = R_f(\{\hat{\imath}\})$. Suppose that this occurs for all subsets $D \subset A$ with $\hat{\imath} \in D$, but $R_f(A) \neq R_f(\{\hat{\imath}\})$. Then, by Proposition 6, we have that $\hat{\imath}$ is a dictator at all $R_f(D)$. Suppose first that $S_f(A) \neq \emptyset$. Then, we have by Proposition 5 that any agent $j \in S_f(A)$ is a dictator at $R_f(A \setminus \{j\})$, which is a contradiction with the fact that $\hat{\imath}$ is also a dictator there. Suppose now that $S_f(A) = \emptyset$. We have that $\hat{\imath} \in S_f(A \setminus \{\hat{\imath}\})$, but $\hat{\imath} \notin S_f(A)$. Given that $\hat{\imath} \notin R_f(A)$ by construction, we have a contradiction with Proposition 4 (third case).

Then, $R_f(A) = R_f(\{\hat{\imath}\})$ if $\hat{\imath} \in A$ and $R_f(A) = R_f(\emptyset)$ otherwise. If $\hat{\imath}$ does not exist, we have that the range of f only contains $\min T$ and $\max T$. Suppose then that $\hat{\imath}$ exists. Then, by Proposition 6, we have that $\hat{\imath}$ is a dictator at all $R_f(A)$. Given that $l_f(\{\hat{\imath}\}) \geq l_f(\emptyset)$ and $r_f(\{\hat{\imath}\}) \leq r_f(\emptyset)$, this rule is dictatorial, being $\hat{\imath}$ the dictator.

Proof of Proposition 9

We will prove that if $\min T$ or $\max T$ does not exist, then there are no Pareto efficient rules. To do that we need the following notation: $N_l = \{i \in N \,|\, i \leq \inf T\}$, $N_r = \{i \in N \,|\, i \geq \sup T\}$, and $N_c = N \setminus (N_l \cup N_r)$.[10]

Suppose that $\min T$ does not exist (the proof when $\max T$ does not exist is similar). Consider a profile $R \in \mathcal{R}^{N_l}$ such that for all $i \in N_c$ and all $y > i$, there exists $x \leq i$ with xP_iy. Then, there is no Pareto efficient alternative in this profile and, therefore, it is not possible to construct a Pareto efficient social choice rule.

[10] If $\inf T$ or $\sup T$ does not exist, the corresponding sets are empty.

Proof of Theorem 3

$\Leftarrow$]: Consider first the case in which $i \in N \cap T$ and l_j and r_j exist (this allows j to satisfy (a) or (c)). Consider the social choice rule f given by $R_f(A) = \{i\}$ if $i \in A$, $R_f(A) = \{l_j, r_j\}$ if $A \cap \{i,j\} = \{j\}$ and $R_f(A) = \{\min T, \max T\}$, otherwise, with j being a dictator in all $R_f(A)$ in which $i \notin A$ and $l_j \neq r_j$. This rule is strategy-proof and Pareto efficient.

Consider now the case in which $i \notin (\min T, \max T)$ and l_j and r_j exist (this allows j to satisfy (a) or (c)). Assume w.l.o.g. that $i < \min T$ and consider the social choice rule f given by $R_f(A) = \{\min T\}$ if $i \in A$, $R_f(A) = \{l_j, r_j\}$ if $A \cap \{i,j\} = \{j\}$ and $R_f(A) = \{\min T, \max T\}$, otherwise, with j being a dictator in all $R_f(A)$ in which $i \notin A$ and $l_j \neq r_j$. This rule is strategy-proof and Pareto efficient.

Consider finally the case in which $i, j \notin (\min T, \max T)$. Then, consider any social choice rule f such that $R_f(A) = \{\min T\}$ if for all $k \in (A \cap \{i,j\})$, $k < \min T$ and $R_f(A) = \{\max T\}$ otherwise. Again, any of these social choice rules are strategy-proof and Pareto-efficient.

$\Rightarrow$]: Suppose that $N \cap T = \emptyset$ and $N \subseteq (\min T, \max T)$. Then, using similar arguments as in the proof of Proposition 9 we have that $R_f(\emptyset) = \{\min T, \max T\}$. By similar arguments as in the proof of Theorem 2, we have that $R_f(\{i\}) \neq R_f(\emptyset)$ for at most one agent $i \in N$.

If $R_f(\{i\}) = R_f(\emptyset)$ for all $i \in N$, we can deduce applying the same arguments as in the proof of Theorem 2 that $R_f(A) = R_f(\emptyset)$ for all $A \subseteq N$. Then, consider a profile $R \in \mathcal{R}^N$ such that there is some $x \in (T \setminus \{\min T, \max T\})$ with $x P_i \min T P_i \max T$ for all $i \in N$, which exists given the assumptions. Then, $f(R) = \min T$, but this location is Pareto dominated by x.

If $R_f(\{j\}) \neq R_f(\emptyset)$ for only one agent $j \in N$, applying the same arguments as in the proof of Theorem 2 we obtain that $R_f(A) = R_f(\{j\})$ if $j \in A$ and $R_f(A) = \{\min T, \max T\}$ otherwise. By Propositions 5 and 6, we have that j is a dictator at all $R_f(A)$. Given that $l_f(\{j\}) \geq \min T$ and $r_f(\{j\}) \leq \max T$, this rule is dictatorial, being j the dictator.

Proof of Theorem 4

It is straightforward to see that $N \cap (\min T, \max T)$ implies that the range of R_f is T instead of T^2. Then, Propositions 5 to 8 are obviously satisfied. Similarly, the condition that R_f is monotone guarantees that Proposition 4 is satisfied. Then, strategy-proofness is guaranteed by Theorem 1. To show that these rules are Pareto efficient, take any profile $R \in \mathcal{R}^A$. If $A = N_l$ (respectively, $A = N_r$), all agents prefer $\min T$ (respectively, $\max T$) to any other alternative, and precisely this unanimously maximal alternative is chosen. In other case, observe that for all $x, y \in T$, there are two agents $i, j \in N$ such that xP_iy and yP_jx, guaranteeing than any choice in these cases is Pareto efficient.

Consider now any strategy-proof and Pareto efficient social choice rule. Then, the rule belongs to the one characterized in Theorem 1. Given that the range of R_f is T by the structure of N and T, we can concentrate directly on this function. This function is obviously monotone given that Proposition 4 is implied by strategy-proofness. Similarly, we can derive that $R_f(N_l) = \min T$ and $R_f(N_r) = \max T$ by Pareto efficiency using the same arguments as in the previous paragraph.

Proof of Theorem 5

It can be checked easily that all conditional two-step rules f satisfying conditional decisiveness such that f_1 is monotote are strategy-proof, Pareto efficient and tops-only. To see the other implication, we start with the strategy-proof rules characterized in Theorem 1 and we investigate the possible values of R_f in some cases.

Lemma 5 *Let $N \subset T$ and $N \subset (\min T, \max T)$, and consider any f that is strategy-proof, Pareto efficient, and tops-only. Then, $R_f(\emptyset) = \{\min T, \max T\}$ and for all $i \in N$, $R_f(\{i\}) \in \{\{i\}, R_f(\emptyset)\}$.*

Proof: Using similar arguments as in the proof of Proposition 9, we can deduce that $R_f(\emptyset) = \{\min T, \max T\}$. Consider now any $R_f(\{i\})$ and suppose that $R_f(\{i\}) \neq R_f(\emptyset)$. Then, $|R_f(\{i\})| = 1$ by tops-onliness. Then, by Proposition 4 (third case), we have that $R_f(\{i\}) = i$. □

Lemma 5 allows us to divide N into the group of decisive agents D_f and the rest. The next lemma shows that there is at least one decisive agent.

Lemma 6 *Let $N \subset T$ and $N \subset (\min T, \max T)$, and consider any f that is strategy-proof, Pareto efficient, and tops-only. Then, D_f is non-empty.*

Proof: Suppose that $D_f = \emptyset$. Applying the same arguments as in the proof of Theorem 2, we have that $R_f(A) = R_f(\emptyset)$ for all $A \subseteq N$. Then, it is easy to see that for all $R \in \mathcal{R}^N$, $\min N$ (respectively, $\max N$) Pareto dominates $\min T$ (respectively, $\max T$). Since $R_f(\emptyset) = \{\min T, \max T\}$ by Lemma 5, this contradicts Pareto efficiency. Hence, $D_f \neq \emptyset$. $\square$

Using Lemmas 5 and 6, we can now determine the selected alternative. It follows from similar arguments as those in the proof of Theorem 2 that if $A \cap D_f = \emptyset$, then $R_f(A) = R_f(\emptyset) = \{\min T, \max T\}$. For these cases, tops-onliness implies that the minimal decisive coalitions can only be formed by the tops of the single-dipped agents and, then, we denote them by $\mathcal{G}_f$.

So, we can now concentrate on the case when $A \cap D_f \neq \emptyset$. The proof proceeds by double induction.

1. We first consider the case when only one decisive agent i has single-peaked preferences. We have to show that for all preference profiles $R \in \mathcal{R}^A$ such that $A \cap D_f = \{i\}$, $f(R) = i$.

    (a) Let $A = \{i\}$ and $i \in D_f$. Since i is a decisive agent, $f(R) = i$ by definition.

    (b) Suppose that for all preference profiles $R \in \mathcal{R}^B$ such that $B \subset A$ and $B \cap D_f = \{i\}$, $f(R) = i$. Since $A \cap D_f = \{i\}$ by assumption and the set of preselected alternatives is equal to $\{\min T, \max T\}$ whenever all decisive agents have single-dipped preferences, $R_f(A \setminus \{i\}) = \{\min T, \max T\}$. Thus, by Proposition 4 (third case), $R_f(A) \in \{\{i\}, \{\min T, \max T\}\}$. Moreover, by the induction hypothesis, $R_f(A \setminus \{j\}) = \{i\}$ for all $j \in (A \setminus \{i\})$. Then, by Proposition 4 (third case) it is not possible that $R_f(A)$ is equal to $\{\min T, \max T\}$. We can therefore conclude that $R_f(A) = \{\{i\}\}$; that is, we have shown that for all $R \in \mathcal{R}^A$ such that $A \cap D_f = \{i\}$, $f(R) = i$.

2. We now move to the case when a subset C of decisive agents of size greater than one has

single-peaked preferences. So, we consider a preference profile $R \in \mathcal{R}^A$ such that $A \cap D_f = C$, $|C| > 1$, and we have to show that $f(R) \in [\min C, \max C]$. The induction hypothesis states that for all preference profiles $R' \in \mathcal{R}^B$ such that $B \subset A$, $f(R') \in [\min(B \cap D_f), \max(B \cap D_f)]$. Suppose by contradiction that $f(R) < \min C$ (the proof when $f(R) > \max C$ is similar and thus omitted). Given that $(B \cap D_f) \subset C$, by the induction hypothsesis, we obtain that for all $\hat{R} \in \mathcal{R}^{A \setminus \{\min C\}}$, $f(\hat{R}) \in [\min C, \max C]$. Then, by Proposition 4, we have that $f(R) \in [\min C, \max C]$.

Given tops-onliness, for all $\bar{R} \in \mathcal{R}^A$, $f(R) = f(R')$ and we will consider function $f_1 : 2^N \to (\min T, \max T)$ the function such that $f_1(A) = f(R)$ for all $R \in \mathcal{R}^A$ such that $A \cap D_f \neq \emptyset$. The fact that f_1 is monotone can be deduced from Proposition 4. To deduce that f satisfies

Finally, we only have to show that f satisfies conditional decisiveness. Take any $A \in \mathcal{G}_f$. We have to show that $i \in D_f$ for all $i \in \mathcal{G}_f$. Suppose by contradiction that $f(R'_i, R_{-i}) \neq i$ for all $R'_i \in \mathcal{R}_i^+$. If $f(R'_i, R_{-i}) > i$, consider a preference profile $R \in \mathcal{R}^\emptyset$ such that $t(R_j) = \min T$ if and only if $j \in A$. Then, consider $\hat{R}_i \in \mathcal{R}_i^+$ such that $\min T \, \hat{P}_i \, f(R'_i, R_{-i})$. Then, $f(\hat{R}_i, R_{-i}) = f(R'_i, R_{-i})$, but agent i can manipulate f via R_i to obtain $\min T$. The case $f(R'_i, R_{-i}) < i$ is similar and thus omitted.