Incomplete Imperfect Information and Backward Induction*

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[PRELIMINARY!!]

Abstract

We consider finite extensive-form games in which the information structure of the game –the information and choice partitions, is not common knowledge once the game is endowed with an epistemic framework similar to the ones in Aumann (1995, 1998), and specially, Samet (2011). This approach allows for generalizing the result in Samet (2011) concerning how common belief of rationality implies backward induction to situations in which there is not common knowledge of perfect information. Instead, we consider the weaker doxastic event that there is common belief that all players believe that the rest of players have perfect information. In particular, it might be the case, not because of chance but rather as a consequence of common belief of rationality, that players behave inductively even when none of them know at which vertex they are when it is their turn to choose. Additionally, we prove that for any given tree and information structure, there exists an epistemic framework as defined by us such that the event that rationality and belief in others' perfect information are common belief is non-empty.

Keywords: Games with Perfect Information, Games with Incomplete Information, Backwards Induction, Rationality. *JEL Classification*: C72, D82, D83.

1 Introduction

1.1 A motivational example

The aim of the present section is to provide some intuition on the usual technicalities on the literature in the topic and on those we are presenting later, so even though all the reasoning here is not conclusive at all, it sheds some light on the ideas that find support when the formal weaponry developed later is called up for duty. Let's analyse a situation as the one represented in Figure 1.

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This is a simple example of the games considered in Samet (2011); we have two players, Alexei Ivanovich and Polina Alexandrovna (A and P in the figure, respectively), both of whom choose between two actions. The payoffs conditional on the profile of actions chosen are represented down below in the figure. Alexei chooses first, and Polina, who before making any choice observes Alexei's action, moves second. The game is played just one time, so that punishment and reinforcement take no place here. This description of the game is common knowledge among the players, *i.e.*, they know it, both know that they know it, both know that both know that they know it, and so on... It is additionally common belief among the players –understood in analogous way to common knowledge, that they are both rational, that is: that none of them will make a choice that they believe yields a strictly lower payoff than the one they do not make. It seems then reasonable to predict that players' choices will lead to node $(2, 1)$: since Polina is rational, if Alexei moves *left* (yours, reader), Polina will reply *left*, while if Alexei moves *right*, Polina will reply *right*. Alexei believes this, so since he himself is rational, he will move *left*.

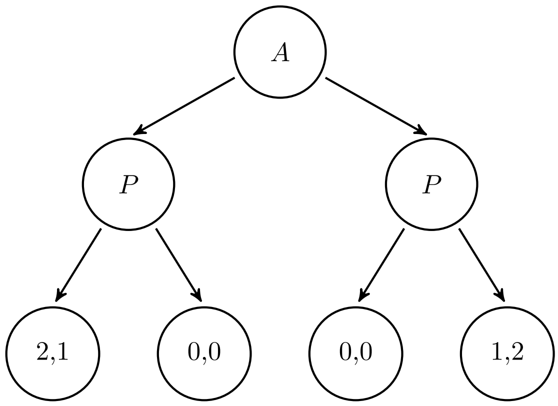

Figure 1: A game with perfect information.

Consider now a situation as the one in Figure 2, where the only variation with respect to the previous situation is that when it is her time to choose, Polina has not observed Alexei's previous move, so that she is not certain of where her choice will lead. It is easy to see that the argument above justifying outcome $(2, 1)$ is hard to defend this time.

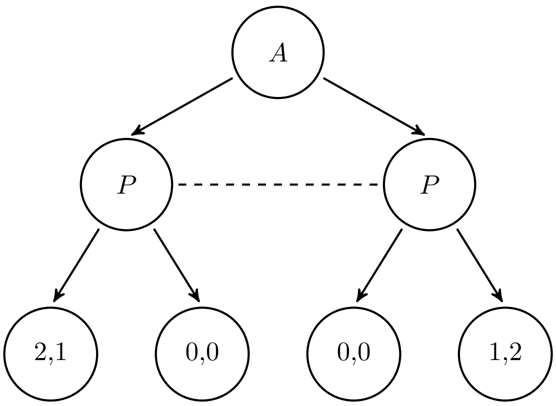

Figure 2: A game w/o perfect information.

Consider finally a situation as the one in Figure 2, but in which Alexei believes to be in a situation as the one Figure 1, and Polina believes that Alexei believes all the above. When it is her time to choose, despite Polina has not observed Alexei's previous move, she can infer that since Alexei believes to be in a situation with perfect information, he also believes *left* to be followed by *left* and *right* by *right* and will therefore, choose *left*. Hence, despite not observing Alexei' previous move, Polina believes Alexei has chosen *left* and chooses consequently, *left*.

2 Formalization

The present section is devoted to the formalization of both the class of underlying games we are considering, and the epistemic framework we will make use of. In the first subsection we present a formalization of extensive-form games similar to the one in Selten (1975) which only deals with perfect information, in the second, we extend some concepts of this formalization to treat situations where there is incomplete information regarding the information structure of the game. Finally, the third section introduces belief systems based on the work in Aumann (1995, 1998) and Samet (2011).

2.1 Finite game trees

A (reduced) *finite game tree* is a tuple $\mathcal{T} = \left\langle I, V, E, (V_i)_{i\in I}, (h_i)_{i\in I}, (A_i)_{i\in I}\right\rangle$, where I is a finite set of *players* and (V, E) is a finite tree with terminal nodes Z and root v^0, and for each $i \in I$, V_i represents the set of vertices corresponding to player i, and $h_i : Z \to \mathbb{R}$, player i's *payoff function*. For any two vertices v and w we denote $v < w$ when w is a vertex that follows v (i.e., v precedes w), that is, when w is a vertex in the subtree the root of which is v, and by $v \wedge w$, the root of the minimal subtree that contains v and w. For any $v \in V$, By v^+ and v^- we represent the set of vertices immediately following v and the one immediately preceding v, respectively, and for $W \subseteq V$, we denote $W^+ = \bigcup_{w\in W} w^+$ and $W^- = \bigcup_{w\in W} w^-$. Player i's actions are represented by A_i, a partition of V_i^+ such that (i) for any different $v, w \in V^+$, if $w \in A_i(v)$ then $v \wedge w \notin \{v, v^-\}$, and (ii) for any $v_i, w_i \in V_i$, if $A_i(v) \cap w_i^+ \neq \emptyset$ for some $v \in v_i^+$, then $A_i(w) \cap v_i^+ \neq \emptyset$ for any $w \in w_i^+$. For any $u \subseteq V_i$, we denote $A_u = \bigcup_{v\in u^+} \{A_i(v)\}$. A *strategy* is an element $t_i \in S_i = \prod_{v_i\in V_i} A_{v_i}$. We denote $S = \prod_{i\in I} S_i$.

For any $v \in V$, $p(v) = \{w \in V \mid w \leq v\}$ is the *path until* v, and we say that $p \subseteq V$ is a *path* if there is some $z \in Z$ such that $p = p(z)$. Note that for any $v \in V$, any profile of strategies $t \in S = \prod_{i\in I} S_i$ induces a unique path $p(v, t) \in p(\mathcal{T})$ that crosses v and corresponds to the choices described by t after v. For any $t \in S$ we denote $p(t) = p\left(v^0, t\right)$.

For $i \in I$, $v_i \in V_i$ and $t \in S$, we define *player i's conditional payoff at v_i induced by t* as $h_{v_i}(t) = h_i(z)$ where $\{z\} = p(v, t) \cap Z$. Then, a *inductive strategy* is a profile of strategies t such that for any $i \in I$ and any $v_i \in V_i$.

$$t_{v_i} \in \underset{a_{v_i}\in A_{v_i}}{\operatorname{argmax}}\, h_{v_i}\left(t_{-i}; (t_i, a_{v_i})\right),$$

that is, one in which at every vertex, the corresponding player is choosing an action that maximizes her conditional payoff at that vertex given that the inductive choice is made at any following vertex. For $i \in I$ and $v_i \in V_i$, we denote by $b_{\mathcal{I},v_i}$ player i's *inductive choice at* v_i. In the following, we assume that $\mathcal{T}$ is such that there exists a unique inductive strategy profile, and denote it by b. We define the *inductive outcome*, $z_{\mathcal{I}}$, as $z \in Z$ such that $\{z\} = p(b) \cap Z$.

2.2 Information sets and generalized strategies

Now, the concept of strategy as defined above correspopnds to the case of perfect information, that is, to the one in which it is possible to player to prescribe as an action to each vertex. The assumption that players know when a vertex of theirs is reached is implicit in this definition. We are interested in situations in which players do not know the information structure of the game; for example: let $i \in I$ and $v_i \in V_i$; if v_i is reached, it might the case that player i knows that if so, she knows that v_i has been reached... or it might be the case, that if so, player i knows that say, set $\{v_i, w_i\}$ has been reached, but not exactly which element of it... or it might be the case that player i does not know *ex ante* what she will know in case v_i is reached. What kind of prescriptions can treat these kind of contingencies?

In order to answer this question, first, following Selten (1975), we say that family $U = (U_i)_{i \in I}$ is an *information structure*, if for any $i \in I$, U_i is an *eligible* a partition of V_i, that is, a partition such that (i) for any $u \in U_i$ and any $v_i, w_i \in u$, $A_{v_i} = A_{w_i}$ and $v_i \not< w_i$, and (ii) for any $v_i, v_i', w_i, w_i' \in V_i$ such that $v_i' \gneq v_i$ and $w_i' \gneq w_i$, if $w_i \notin U_i(v_i)$, then $w_i' \notin U_i(v_i')$. Let's go on with some notation:

- We denote the set of eligible partitions of V_i by $\mathcal{P}^*(V_i)$, and by $\mathcal{P}^*(V)$, the set of information structures for $\mathcal{T}$, $\prod_{i \in I} \mathcal{P}^*(V_i)$. By U_i^* we represent *player i's set of eligible sets*, that is, $\bigcup \mathcal{P}^*(V_i)$. We denote $U^* = \prod_{i \in I} U_i^*$.
- Then, for any $i \in I$, a *generalized strategy* is a prescription for any possible information set player i might find herself in, i.e., a list $t_i^* = (t_u)_{u \in U_i^*} \in S_i^* = \prod_{u \in U_i^*} A_u$. We denote $S^* = \prod_{i \in I} S_i^*$.
- Note that a profile of generalized strategies does not induce a strategy profile *per se*, but so does any pair $(U, t^*) \in \mathcal{P}^*(V) \times S^*$: let $t_U = (t_{U_i})_{i \in I}$, where for any $i \in I$, $t_{U_i} = \left(t_{U_i(v_i)}\right)_{v_i \in V_i}$.

Again, a profile of generalized strategies does not induce payoffs or conditional payoffs, since the way it is going to realize depends in the information structure. Thus, for any $i \in I$, any $v \in V \setminus Z$, any $U \in \mathcal{P}^*(V)$ and any $t^* \in S^*$ we define *player i's conditional payoff on* v when the information set structure is U and t^* is played, as,

$$h_{v,i}(U, t^*) = h_i(z) \text{ where } \{z\} = p(v, t(U, t^*)) \cap Z,$$

and for any $u \in U_i^*$, any $U \in \mathcal{P}^*(V)$. Finally, we would like to define conditional payoffs, not only for vertices, but also for information sets. For any $i \in I$, any $v \in V \setminus Z$ and any $U_i \in \mathcal{P}^*(V_i)$, we define the following partition for $\{v_i \in V_i \mid v_i \geq v\}$,

$$[U_i, v](v_i) = U_i(v_i) \cap \{v_i \in V_i \mid v_i \geq v\} \text{ for any } v_i \in \{v_i \in V_i \mid v_i \geq v\}.$$

Then, for any $u \in U_i^*$, $[U_i, \wedge u]$ represents the information structure for player i corresponding to the minimal subtree that contains u when her information structure for the whole tree is U_i.[1]. For $U \in \mathcal{P}^*(V_i)$, we denote

[1]That is, player i's hypothetical information update had information set u been reached.

$[U, u] = ([U_i, u])_{i \in I}$. Thus, for any $t^* \in S^*$, *player i's conditional payoff in u* when the information structure is U and t is played is,

$$h_u (U, t^*) = h_{\wedge u, i} (([U, \wedge u]), t^*) .$$

Note that despite its irrelevant indetermination w.r.t. vertices off the minimal subtree containing u, both $[U, \wedge u]$ and h_u are well defined, and that for any $v_i \in V_i$, $h_{v_i} \equiv h_{\{v_i\},i}$. For any $i \in I$ and any $u \in U_i^*$, we denote and define the inductive choice in u as $b_u = b_{\mathcal{I}, v_i}$ where $\{v_i\} = p(\wedge u, b) \cap u$, and denote $b^* = (b_i^*)_{i \in I}$, where for any $i \in I$, $b_i^* = (b_u)_{u \in U_i^*}$.

2.3 Epistemic framework

For the epistemic modelling, we first consider $\langle \Omega, (\Pi_i)_{i \in I}, (b_i)_{i \in I} \rangle$, a *belief structure* as defined in Samet (2011),[2] i.e., a list consisting on a finite set of states Ω, and for each $i \in I$, a partition Π_i of Ω where for any $\omega \in \Omega$ the element of the partition containing ω is denoted by $\Pi_i (\omega)$, and a belief map $b_i : \Omega \to 2^\Omega \setminus \{\emptyset\}$ measurable w.r.t. Π_i and such that for any $\omega \in \Omega$, $b_i (\omega) \subseteq \Pi_i (\omega)$. As usual, for any $\omega \in \Omega$ and any $E \subseteq \Omega$, we say that *player i knows (resp. believes) E at ω* if $\Pi_i (\omega) \subseteq E$ (resp. $b_i (\omega) \subseteq E$). For each $i \in I$ we introduce $\mathcal{U}_i$ and σ_i,

Information set maps. $\mathcal{U}_i$ is a map from Ω into $\mathcal{P}^* (V_i)$. For each $\omega \in \Omega$, $\mathcal{U}_i$ specifies the information set of player i corresponding to ω, which we denote by $\mathcal{U}_{i,\omega}$, and for each $v_i \in V_i$ we denote the element of $\mathcal{U}_{i,\omega}$ containing v_i, by $\mathcal{U}_{i,\omega} (v_i)$, and $\mathcal{U} = (\mathcal{U}_i)_{i \in I}$.

Generalized strategy maps. $\sigma_i : \Omega \to S_i^*$ is a map measurable w.r.t. Π_i. We denote $\sigma = (\sigma_i)_{i \in I}$. Note that a *strategy map* $s_i : \Omega \to S_i$ is then induced, where for any $\omega \in \Omega$, $s_i (\omega) = \sigma_{\mathcal{U}_{i,\omega}} (\omega)$. We denote $\sigma = (\sigma_i)_{i \in I}$ and $s = (s_i)_{i \in I}$.

Each $\omega \in \Omega$ induces then a unique path $p(\omega)$ *via* s,

$$p(\omega) = \{v \in V \mid \text{ for any } w \leq v \text{ there exist } i \in I, \text{ and } v_i \in V_i, \text{ such that } w \in s_{v_i} (\omega) \} .$$

We denote by $z(\omega)$ the outcome corresponding to $p(\omega)$. For any $i \in I$ and any $v_i \in V_i$, the event that *v_i is reached* is defined as $[v_i] = \{\omega \in \Omega \, | v_i \in p(\omega) \}$, and the event that *an information set containing v_i is reached*, as $\Omega_{v_i} = \{\omega \in \Omega \mid U_{i,\omega} (v_i) \cap p(\omega) \neq \emptyset\}$. A *generalized belief model* is then a tuple $\mathcal{B} = \langle \Omega, (\Pi_i)_{i \in I}, (b_i)_{i \in I}, (\mathcal{U}_i)_{i \in I}, (\sigma_i)_{i \in I} \rangle$ consisting on the elements just presented above, and such that the following two assumptions are satisfied,

[2] An extension of the knowledge model by Aumann (1995) that allows for playing with not only knowledge but also with beliefs.

- **Knowledge of the information sets reached.** For any $\omega \in \Omega$,

$$\Pi_i(\omega) \subseteq \bigcap_{v_i \in V_i} \left(\neg\Omega_{v_i} \cup [\mathcal{U}_i(v_i) = \mathcal{U}_{i,\omega}(v_i)]\right).$$

  That is, whenever an information set of hers is reached, player i knows it.

- **Consistency of knowledge partitions and information maps.** For any $\omega \in \Omega$, and any $v_i, w_i \in V_i$ such that $[v_i] \cap \Pi_i(\omega) \neq \emptyset$ and $w_i \in \mathcal{U}_{i,\omega}(v_i)$, it holds that $[w_i] \cap \Pi_i(\omega) \neq \emptyset$. That is, at any state, if player i considers possible a certain vertex v_i to be reached, she must considered all the vertices of the information set corresponding to v_i possible to be reached.

It is pertinent to wonder whether this kind of structure exist for any given game tree; or at least if does in a non trivial way, since it is obvious that for information set maps that assign to every state a finest partition of the set of vertices, it does.[3] The answer is positive and is proved later in Theorem 2 of Section 3.

Knowledge and belief operators are defined in the usual way. For $i \in I$, player i's knowledge and belief operators are respectively defined as,

$$K_i(E) = \{\omega \in \Omega \mid \Pi_i(\omega) \subseteq E\}, \text{ and } B_i(E) = \{\omega \in \Omega \mid b_i(\omega) \subseteq E\},$$

for any $E \subseteq \Omega$. Regarding reciprocal information, the present work does not rely in common knowledge,[4] but rather in common belief, so the latter is the only notion we will define; following Monderer and Samet (1987), for any $E \subseteq \Omega$, the event that *there is common belief of E* is defined as,

$$CB(E) = \left\{\omega \in \Omega \;\middle|\; \text{there exists } C \subseteq \Omega \text{ where } \omega \in C \subseteq \bigcap_{i \in I} B_i(E) \cap \bigcap_{i \in I} B_i(C)\right\}.$$

3 Rationality, perfect information and backward induction

Following Samet (2011), we define rational behaviour in terms of beliefs rather than knowledge, as done in Aumann (1995), and, as in both works, following a very weak notion that needs to be adapted to our framework, where there may not be knowledge of the vertex in which a certain action will apply. For any $i \in I$, any $t_i^* \in S^*$ and any $u \in U_i^*$ we can define the event that *generalized strategy would have yielded player i a higher conditional payoff at u* as $[h_u(s) < h_u((s_{-i}; t_{\mathcal{U}_i}))]$. The definition of this set stablishes a clear analogy between our model, and those by Aumann (1995) and Samet (2011), and leads to the following definition of substantive rationality in terms of beliefs,

[3]Since the model becomes the one in Samet (2011).
[4]Beyond common knowledge of the model itself, which is taken for granted.

Definition 1 (Substantive rationality) *Let finite game tree $\mathcal{T}$ and $\mathcal{B}$, a generalized belief model for $\mathcal{T}$. Let $i \in I$ and $u \in U_i^*$. The event that i is rational at u is defined as,*

$$R_u = \bigcap_{t_i^* \in S_i^*} \neg B_i \left(h_u \left(s \right) < h_u \left(\left(s_{-i}; t_{\mathcal{U}_i} \right) \right) \right),$$

and the event that i is substantive rational as $R_i = \bigcap_{u \in U_i^*} R_u$. The event that players are substantive rational is as then, as usual, $R = \bigcap_{i \in I} R_i$.

This notion of rationality generalizes that by Samet (2011) for the case in which there is perfect information, or in other words, common knowledge of observability.

Definition 2 (Observability, belief in others' perfect information) *Let $\mathcal{T}$ a finite game tree and $\mathcal{B}$, a generalized belief model for $\mathcal{T}$. For $i \in I$ and $v_i \in V_i$, the event that* player i has perfect information at vertex *v_i is defined as $PI_{v_i} = \left[\mathcal{U}_i \left(v_i \right) = \{ v_i \} \right]$, the event that* player i has perfect information*, as $PI_i = \bigcap_{v_i \in V_i} PI_{v_i}$, and* belief in others' perfect information*, as $PI^- = \bigcap_{i \in I} B_i \left(\bigcap_{j \neq i} PI_j \right)$.*

That is, belief in others' perfect information is just the event that every player i believes that every player $j \neq i$ has perfect information. Recall that in paragraph 2.1 we denoted the only inductive outcome of $\mathcal{T}$ by $z_{\mathcal{I}}$. Then, the event that *players follow the backward induction path* is defined as $\left[z_{\mathcal{I}} \right] = \{ \omega \in \Omega \, | z \left(\omega \right) = z_{\mathcal{I}} \}$. Note that this event by no means implies that players are choosing inductively, but rather that they are just choosing the same action as they would if they were choosing inductively.

Theorem 1 *Let $\mathcal{T}$ a finite game tree and $\mathcal{B}$ a belief system. Then, if there is common belief of substantive rationality and there is common belief of belief in others' perfect information,, players follow the backward induction path; i.e.,*

$$CB \left(R \right) \cap CB \left(PI^- \right) \subseteq \left[z_{\mathcal{I}} \right].$$

Note that we say there is perfect information, when there is common knowledge of the vent that all players observe. Hence, from Theorem 1 we obtain:

Corollary 1 (Samet (2012)) *Let $\mathcal{T}$ a finite game tree and $\mathcal{B}$ a belief system that guarantees perfect information; that is, such that $\mathrm{Im} \left(U_i \right) = \{ \{ v_i \} \, | v_i \in V_i \}$ for any $i \in I$. Then, common belief of substantive rationality implies the inductive outcome; i.e.,*

$$CB \left(R \right) \subseteq \left[z_{\mathcal{I}} \right].$$

Finally, we positively answer to the question concerning existence of belief systems as defined in paragraph 2.3. We prove the stronger result, in the spirit of Theorem B by Aumann (1995) that is is always possible to construct a belief system such that for any information structure $\left(U_i \right)_{i \in I}$, the intersection of the event that both rationality and belief in co-observability are common belief and that the information structure of the game is $(U_i)_{i\in I}$ is non-empty:

Theorem 2 *Let $\mathcal{T}$ a finite game tree. For any possible information structure U, there exists some generalized belief model $\mathcal{B}$ such that,*

$$CB(R) \cap CB\left(PI^{-}\right) \cap [\mathcal{U} = U] \neq \emptyset.$$

References

Aumann, Robert J. (1995). "Backwards Induction and Common Knowledge of Rationality". *Games and Economic Behavior* **8**, 6–19.

Aumann, Robert J. (1998). "On the Centipede Game". *Games and Economic Behavior* **23**, 97–105.

Monderer, Dov and Dov Samet (1987). "Approximating Common Knowledge with Common Beliefs". *Games and Economic Behavior* **1**, 170–190.

Samet, Dov (2011). "Common Belief of Rationality in Games with Perfect Information". Working Paper.

Selten, Reinhard (1975). "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games". *International Journal of Game Theory* **4**(1), 25–55.

A Proof of Theorem 1

Let's begin with an auxiliary result that will eventually become recurring during the proof:

Lemma 1 (The small lemma) *For any $i \in I$ and any $v_i \in V_i$,*

$$R_i \cap B_i \left(PI_{-i} \cap \bigcap_{v > v_i, v \notin V_i} \left[\sigma_{\{v\}} = b_v \right] \right) \subseteq \left[\sigma_{\{v_i\}} = b_{v_i} \right].$$

Proof. First, $B_i \left(PI_{-i} \cap \bigcap_{v > \wedge u, v \notin V_i} \left[\sigma_{\{v\}} = b_v \right] \right) \subseteq B_i \left(\bigcap_{v > v_i, v \notin V_i} \left[s_v = b_v \right] \right) \subseteq B_i \left(h_{\{v_i\}} (s) = h_{\{v_i\}} \left(t_{-i} \left(\mathcal{U}, b^*_{-i} \right); s_i \right) \right)$. Remember that by definition, we have that $R_i \subseteq \neg B_i \left(h_{\{v_i\}} (s) < h_{\{v_i\}} \left(s_{-i}; t_i \left(\mathcal{U}, b^*_i \right) \right) \right)$. Now, note that:

$$B_i \left(h_{\{v_i\}} (s) = h_{\{v_i\}} \left(t_{-i} \left(\mathcal{U}, b^*_{-i} \right); s_i \right) \right) \cap B_i \left(h_{\{v_i\}} (s) < h_{\{v_i\}} \left(s_{-i}; t_i \left(\mathcal{U}, b^*_i \right) \right) \right) =$$

$$= B_i \left(h_{\{v_i\}} (s) = h_{\{v_i\}} \left(t_{-i} \left(\mathcal{U}, b^*_{-i} \right); s_i \right) \right) \cap B_i \left(h_{v_i} \left(t_{-i} \left(\mathcal{U}, b^*_{-i} \right); s_i \right) < h_{v_i} (b) \right),$$

and therefore,

$$B_i \left(h_{\{v_i\}} (s) = h_{\{v_i\}} \left(t_{-i} \left(\mathcal{U}, b^*_{-i} \right); s_i \right) \right) \cap \neg B_i \left(h_{v_i} \left(t_{-i} \left(\mathcal{U}, b^*_{-i} \right); s_i \right) < h_{v_i} (b) \right) \subseteq$$

$$\subseteq \neg B_i \left(\sigma_u \neq b_u \right) \subseteq \left[\sigma_u = b_u \right],$$

and the proof is complete. ■

So, let's go on with the proof then:

A backward flow. Let $i \in I$ and $v_i \in V_i$ such that $v_i^+ \subseteq Z$. Then, we can write, with some abuse of nation, $h_{\{v_i\}} (s(\omega)) = h_{v_i} \left(\sigma_{\{v_i\}} (\omega) \right)$, and therefore, $R_i \subseteq \left[\sigma_{\{v_i\}} = b_{v_i} \right]$. Thus, $CB(R) \cap CB(PI^-) \subseteq CB \left(\bigcap_{v \in V, v^+ \subseteq Z} \left[\sigma_{\{v\}} = b_v \right] \right)$. Now, suppose that we have $i \in I$ and $v_i \in V_i$ such that $CB(R) \cap CB(PI^-) \subseteq CB \left(\bigcap_{v > v_i} \left[\sigma_{\{v\}} = b_v \right] \right)$, then,

$$CB(R) \cap CB(PI^-) \subseteq CB \left(R_i \cap B_i \left(PI_{-i} \cap \bigcap_{v > v_i} \left[\sigma_{\{v\}} = b_v \right] \right) \right) \subseteq CB \left(\sigma_{\{v_i\}} = b_{v_i} \right),$$

being the last inclusion a consequence of the small lemma. This way, we conclude that $CB(R) \cap CB(PI^-) \subseteq CB \left(\bigcap_{v \in V \setminus Z} \left[\sigma_{\{v\}} = b_v \right] \right)$.

A forward flow. In the following we denote $p_{\mathcal{I}} = \left\{ v^k \right\}_{k=0}^n$, and suppose that $v^k \in V_{i_k}$. From the backward flow we obtain that $CB(R) \cap CB(PI^-) \subseteq \left[v^1 \right]$. Now, suppose that $CB(R) \cap CB(PI^-) \subseteq \left[v^k \right]$ for some $k \geq 1$. Let $u \in U^*_{i_k}$ such that $v_{i_k} \in u$. Then, from the backward flow we know that $CB(R) \cap CB(PI^-) \subseteq R_{i_k} \cap B_{i_k} \left(\left[v^k \right] \cap PI_{-i_k} \cap \bigcap_{v > \wedge u} \left[\sigma_{\{v\}} = b_v \right] \right)$. Now, note that it holds in general that $\left[v^k\right] \cap PI_{-i_k} \cap \bigcap_{v > \wedge u} \left[\sigma_{\{v\}} = b_v\right] \subseteq \left[h_u\left(s\right) = h_{\{v^k\}}\left(t_{-i}\left(\mathcal{U}, b^*_{-i}\right); s_i\right)\right]$, and since,

$$B_i\left(h_u\left(s\right) = h_{v^k}\left(t_{-i}\left(\mathcal{U}, b^*_{-i}\right); s_i\right)\right) \cap B_i\left(h_u\left(s\right) < h_u\left(s_{-i}; t_i\left(\mathcal{U}, b^*_i\right)\right)\right) =$$

$$= B_i\left(h_u\left(s\right) = h_{v^k}\left(t_{-i}\left(\mathcal{U}, b^*_{-i}\right); s_i\right)\right) \cap B_i\left(h_{\{v^k\}}\left(t_{-i}\left(\mathcal{U}, b^*_{-i}\right); s_i\right) < h_{v^k}\left(b\right)\right),$$

and therefore,

$$B_i\left(h_u\left(s\right) = h_{v^k}\left(t_{-i}\left(\mathcal{U}, b^*_{-i}\right); s_i\right)\right) \cap \neg B_i\left(h_{v^k}\left(t_{-i}\left(\mathcal{U}, b^*_{-i}\right); s_i\right) < h_{v^k}\left(b\right)\right),$$

we obtain that $CB\left(R\right) \cap CB\left(PI^-\right) \subseteq \neg B_{i_k}\left(h_u\left(t_{-i}\left(\mathcal{U}, b^*_{-i}\right); s_i\right) < h_{v^k}\left(b\right)\right) \subseteq \neg B_{i_k}\left(\sigma_u \neq b_u\right) \subseteq \left[\sigma_u = b_u\right]$.

B Proof of Theorem 2

Let $P(\mathcal{T})$ the set of paths of $\mathcal{T}$, and $U \in \prod_{i\in I} \mathcal{P}^*(V_i)$. Let $\Omega = \{0,1\}^I \times P(\mathcal{T})$. For each $i \in I$, and $p \in p(\mathcal{T})$ we denote $u_i(p) = \{u \in U_i \,|p \cap u \neq \emptyset\}$ and define the following equivalence relation on $P(\mathcal{T})$,

$$p \sim_i p' \quad \Longleftrightarrow \quad u_i(p) \cap u_i(p') \neq \emptyset \text{ and } A_i\left(p \cap v_i^+\right) = A_i\left(p' \cap w_i^+\right),$$
$$\text{for any } v_i, w_i \in V_i \text{ such that } v_i \in p, w_i \in p' \text{ and } A_{v_i} = A_{w_i}.$$

And denote the equivalence class corresponding to p by $[p]_i$. Now, for each $i \in I$ we define:

- A knowledge partition,

$$\Pi_i(g,p) = \begin{cases} \{g(i)=0\} \times [p]_i & \text{if } g(i)=0 \\ \{g(i)=1\} \times [p]_i & \text{if } g(i)=1 \end{cases}, \text{ for any } (g,p) \in \Omega.$$

- A belief map,

$$b_i(g,p) = \begin{cases} \{(0, p(\wedge u_i(p), b))\} & \text{if } g(i)=0 \\ \{\left(1_{\{i\}}, p(\wedge u_i(p), b)\right)\} & \text{if } g(i)=1 \end{cases}, \text{ for any } (g,p) \in \Omega,$$

  where $\wedge u_i(p) = \wedge \{\wedge u \,|u \in u_i(p)\}$. It is measurable w.r.t. Π_i and satisfies that $b_i(\omega) \subseteq \Pi_i(\omega)$ for any $\omega \in \Omega$.

- An information set map,

$$\mathcal{U}_{i,(g,p)} = \begin{cases} \{\{v_i\} \,|v_i \in V_i\} & \text{if } g(i)=0 \\ U_i & \text{if } g(i)=1 \end{cases}, \text{ for any } (g,p) \in \Omega.$$

  Note that U_i is measurable w.r.t. Π_i, what implies that in particular, knowledge of the information sets reached holds.

- A generalized strategy map, where for any $u \in U_i^*$,

$$\sigma_u(g,p) = \begin{cases} A_i(w) \text{ where } w \in u^+ \cap \bigcup [p]_i & \text{if} \quad u \cap \bigcup [p]_i \neq \emptyset, \\ b_u & \text{if} \quad u \cap \bigcup [p]_i = \emptyset, \end{cases},$$

  for any $(g,p) \in \Omega$. Note that σ_i is well defined and is measurable w.r.t. Π_i, and that $p(g,p) = p$ for any $(g,p) \in \Omega$.

We still need to check that consistency holds. Let $(g,p) \in \Omega$, and $v_i \in V_i$. If $g(i) = 0$ or $g(i) = 1$ and $U_i(v_i) = \{v_i\}$, consistency holds trivially. If $g(i) = 1$ and $U_i(v_i) \neq \{v_i\}$, let $w_i \in U_i(v_i)$, $w_i \neq v_i$, and let then, $p' \in p(\mathcal{T})$ such that $w_i \in p'$ and, for any $w_i' \in p'$ such that $A_{w_i'} = A_{v_i'}$ for some $v_i' \in p$,

$A_i\left(p' \cap w_i'^{+}\right) = A_i\left(p \cap v_i'^{+}\right)$. Obviously, $p' \in [p]_i$ and therefore, $[w_i] \cap \Pi_i(g,p) \neq \emptyset$. Thus, we have checked that $\mathcal{B} = \left\langle \Omega, (\Pi_i)_{i\in I}, (b_i)_{i\in I}, (\mathcal{U}_i)_{i\in I}, (\sigma_i)_{i\in I} \right\rangle$ is a generalized belief model for $\mathcal{T}$. Now:

- **Regarding common belief of rationality...** Let $C = \{0,1\}^I \times \{p_{\mathcal{I}}\}$. It is immediate that $C \subseteq \bigcap_{i\in I} B_i(C)$. Now, let $i \in I$ and $(g,p) \in C$, we have two cases:

  1. $g(i) = 0$. Then, $b_i(g,p) = \{(0, p_{\mathcal{I}})\}$,
  2. $g(i) = 1$. Then, $b_i(g,p) = \left\{\left(1_{\{i\}}, p_{\mathcal{I}}\right)\right\}$,

- **Regarding common belief of co-observability...** For any $i \in I$, any $\omega \in \Omega$ and any $j \neq I$, $U_{j,\omega'} = \{\{\{v_j\} \,|v_j \in V_j\}\}$ for any $\omega' \in b_i(\omega)$, so $B_i(O_j) = \Omega$, and therefore, $CB(BO^-) = \Omega$.