Reserve Price Auctions in a Dynamic Heterogeneous Goods Market

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Abstract

This paper examines the steady state of a large market where the goods are differentiated but stochastically equivalent (e.g. houses, used books, oil drilling rights). The equilibrium selling mechanism is a standard auction with the optimal reserve price determined by not only the buyer value distribution in the static setting but also the market environment. In particular, the reserve price increases when there is more inflow of buyers and the buyers become more impatient, but the change in it is ambiguous if buyers exit the market more often or the length of a period changes. The general equilibrium welfare effects of the reserve price auctions are discussed as well.

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1 Introduction

Standard auctions with reserve prices prove to be revenue enhancing facing buyers with private values. The revenue maximizing (optimal) selling mechanism for a monopolist (Myerson, 1981) is an auction with the reserve price determined by equating marginal cost to marginal revenue, which derives from the buyer value distribution. Even when there are infinitely many sellers competing to attract buyers, an auction with reserve price equal to the seller's opportunity cost is the most desirable (McAfee, 1993).

In this paper, we first show that, in the steady state of a dynamic market in which buyers are matched randomly to each seller, the optimal mechanism, not surprisingly, is an auction with the reserve price determined by not only the buyer value distribution but also the market dynamics that affect the buyer's endogenous sequential outside option. With a positive reserve price, however, more buyers will stay in the market, raising the number of buyers a future seller will be matched with and causing a sequence of welfare changes.

The analyses of large bargaining and auction markets have often shied away from the possibility of sellers committing to a reserve price by making simplistic yet arguably troublesome restrictive assumptions on sellers' power, making the papers falling short of a general equilibrium approach. In Satterthwaite and Shneyerov (2007, 2008), sellers do not have the power to commit to a reserve price, yet their result that the dynamic competitive equilibrium with private information converges to the Walrasian equilibrium as if there is no imperfect information depends on this assumption. Said (2011) assumes that each sellers runs the efficient Vickrey (1961) auction partly for its analytical simplicity in his setting of sequential auctions with finite number of buyers, but nonetheless admits that a setting in which the seller runs a reserve price auction is worth pursuing.

It is particularly worth comparing the current paper with Wolinsky (1988). It explicitly models the buyer and seller entry and exit processes left unspecified in Wolinsky (1988).

This paper bestowing the sellers the power to commit to posting minimum bids each period. Although Wolinsky characterizes the market equilibrium with the sellers posting reserve prices towards the end of the paper, he only makes the casual remark that such extra power for the seller does not qualitatively alter the distinction between instantaneous and over-time bidding competition. This paper explicitly determines the effects of such power on the bidding competitions and the general welfare.

This paper, in spite of focusing on reserve price auctions, takes a different standpoint as those in the competing auctions literature which considers the equilibrium mechanism when the buyers can choose between sellers. The Vickrey auction with a reserve price lower than that of monopoly is optimal (Burguet and Sákovics, 1999; Pai, 2009). In the limit large market, the efficient Vickrey auction is optimal (McAfee, 1993; Peters and Severinov, 1997).

2 The Model

There is a countably infinite sequence of periods indexed by $t \in \mathbb{N}$ with each period lasting δ unit of time. All the traders have a common discount rate β so the traders discount the next period by the time discount factor $e^{-\delta\beta}$.

Once per period, infinitely many active sellers and active buyers present in the market meet to trade an indivisible good. Each of the sellers owns a unit of the good and has value 0 for holding onto his own good. Each of the buyers, however, has a different private value $v \in [0,1]$ for each of the good. Each good is assumed to be stochastically equivalent in the sense that each buyer has an independent private value distributed according to the distribution F with positive, differentiable density f, and realized when she observes the good.

In period $t = 0$, a measure S_0 continuum of active sellers and a measure B_0 continuum of active buyers have just entered the market. Every active buyer is uniformly and randomly matched with one active seller. Since there are continua of traders, the distribution of the number of buyers a seller is matched with is $\mathcal{P}(\kappa_0)$, Poisson with parameter $\kappa_0 = B_0/S_0$, which we refer to as the period 0 buyer-seller ratio; specifically, the realized number of buyers a seller is matched with is

$$p_n(\kappa_0) = \frac{\kappa_0^n}{n! e^{\kappa_0}} \quad \forall n = 0, 1, \cdots$$

Neither the sellers nor the buyers know the realized number of matches when choosing their strategies. Each seller first chooses a direct selling mechanism. Each buyer, observing the mechanism chosen by the seller she is matched with, chooses a report of her value. The mechanisms are then executed with according to predetermined rules, and at most one buyer receives the good.

The sellers who have successfully sold his good and the buyers who have obtained a good exit the market. The sellers who have not sold exit with exogenous instantaneous rate ν so survive with probability $e^{-\delta\nu}$ to the next period, while the buyers who have not did not bought exit with rate μ so survive with probability $e^{-\delta\mu}$ to the next period.

At the beginning of each period $t \geq 1$, a measure δ of new sellers and a measure $\delta\iota$ of new buyers whose values are drawn from F enter the market: that is, there is a constant inflow of measure ι of buyers and inflow of measure 1 of sellers per unit time. Together with the surviving traders, there are measure S_t of sellers and measure B_t of buyers. The processes in which the (active) traders match, the sellers choose among selling mechanisms, the buyers make purchasing decisions, the traders exit, and new traders arrive, are repeated in each period.

We will focus on studying the steady state market behavior, but to rigorously define the steady state equilibrium, we need to cast this dynamic game as a stochastic game with uncountably many possible states and actions.

2.1 The Markov Perfect Equilibrium

We rigorously cast this setting as a stochastic game with uncountably many states and actions and define the Markov perfect equilibrium in this setting. In each period t, the commonly known state is (S_t, B_t) with $\kappa_t = B_t/S_t$ denoting the buyer-seller ratio. For each trader, he has a state a_t denoting whether he or she is active ($=1$) or not ($=0$) in period t. Each buyer has a private observed state v_t that denotes her value which is commonly known to be drawn from F. In summary, each seller observes the state (B_t, S_t, a_t) and each buyer observes the state (B_t, S_t, v_t, a_t).

The action space includes the set of mechanisms the sellers can choose and the set of reports the buyers report in each mechanism; formally, each seller chooses m_t among the set of direct selling mechanisms, M and each buyer specifies her feasible report in each of the possible mechanism $\sigma_t = (\sigma_t^m)_{m\in M}$ with $\sigma_t^m \in [0,1]$ for each m. From now on, let us focus on the symmetric behavior where each active seller chooses the same mechanism m_t and each buyer of the same value v chooses the same report function $\sigma_t(v)$ in each mechanism in a period t.

The states evolve according to the previous states and the actions played. In particular, the expected measures of buyers and of sellers are

$$\begin{aligned}\mathbb{E}\left[B_{t+1}|B_t, S_t, m_t, b_t\right] &= \delta\iota + e^{-\delta\mu} L_t^b\left(B_t, S_t, m_t, \sigma_t\right)\\ \mathbb{E}\left[S_{t+1}|S_t, S_t, m_t, b_t\right] &= \delta + e^{-\delta\nu} L_t^s\left(B_t, S_t, m_t, \sigma_t\right)\end{aligned}$$

where $L_t^b(B_t, S_t, m_t, b_t)$ $(L_t^s(B_t, S_t, m_t, b_t))$ is the expected measure of buyers (sellers) who do not obtain (sell) a good in period t given the actions and states. $\Pr\left(a_{t+1}(v_t, m_t, b_t) = 1\right)$ is the probability that an active buyer in period t of value v_t is still active in $t+1$,

$$\Pr\left(a_{t+1}\left(v_t, m_t, \sigma_t\right) = 1\right) = e^{-\delta\mu} l_t^b\left(v_t, m_t, \sigma_t\right)$$

where $l_t^b(v_t, m_t, b_t)$ is the expected probability of losing in mechanism m_t by playing $\sigma_t^{m_t}(v_t)$ while others play $b_t^{m_t}$, and let $\Pr\left(a_{t+1}(m_t, b_t) = 1\right)$ be the probability that a seller is still active in $t+1$ after he plays m_t and all the buyers play b_t.

Buyer's and seller's sums of discounted expected payoffs are defined as follows. A seller's period 0 discounted payoff is

$$\Pi = \sum_{t=0}^{\infty} a_t e^{-\delta\beta t} \pi \left(m_t, \sigma_t\right)$$

where $\pi\left(m_t, \sigma_t\right)$ is the expected payoff of seller employing mechanism m_t and buyers play σ_t, and a buyer's period 0 discounted payoff is

$$U = \sum_{t=0}^{\infty} a_t e^{-\delta\beta t} u \left(v_t, m_t, \sigma_t\right)$$

$u\left(v_t, m_t, \sigma_t\right)$ is buyer v_t's expected utility in mechanism m_t playing σ_t. More specifically, sum of the expected payoffs of an active seller in period τ discounted to period 0 is

$$\Pi_\tau \left(\left(m_t\right)_{t=\tau}^{\infty}\right) = e^{-\delta\beta\tau} \pi \left(m_\tau, \sigma_\tau\right) + \sum_{t=\tau+1}^{\infty} a_t \left(h_{t-1}\right) e^{-\delta\beta t} \pi \left(m_t, \sigma_t\right)$$

where h_{t-1} is the sequence of histories of actions of the seller and buyers up to period $t-1$.

Definition 1. S_*, B_*, m_*, and σ_* constitute the steady state Markov perfect equilibrium, or we call the **market equilibrium**, if

$$m_* = m^* \left(S_*, B_*, 1\right)$$

$$\sigma_* \left(v\right) = \sigma^* \left(S_*, B_*, v, 1\right)$$

$$B_* = \mathbb{E}\left[B_{t+1} | B_{t+1} = B_*, S_{t+1} = S_*, m_*, \sigma_*\right]$$

$$S_* = \mathbb{E}\left[S_{t+1} | B_{t+1} = B_*, S_{t+1} = S_*, m_*, \sigma_*\right]$$

3 The Market Equilibrium

As in Wolinsky (1988), we focus on the steady state where two key conditions are satisfied,

1. The inflow of the traders balances the outflow of the traders on both sides of the market.

2. Each trader's strategy is optimal to maximize his or her expected utility after any personal history, given the steady state distributions.

which are characterized by the mass balance equations and the equilibrium optimal mechanism, both derived in this section.

In the steady state the measures of buyers B_* and sellers S_* are Markov given the sellers' and buyers' equilibrium strategies. By symmetry, each seller runs the same mechanism m_*, and each buyer of the same value v uses the same constant strategy $b_*(v)$.

A seller's discounted expected payoff by choosing a mechanism m in a period is

$$\Pi(m, m_*) = \pi(m) + \Pr(\text{unsold}|m)\, e^{-\delta(\nu+\beta)}\Pi_*$$

where $\Pi_* = \Pi(m_*, m_*)$ is a seller's equilibrium expected continuation payoff.

Lemma 1. *The equilibrium selling mechanism m_* is the Vickrey auction with the reserve price $\rho_* = r_* - e^{-\delta(\mu+\beta)}U_*$, where r_* is determined by,*

$$r_* - e^{-\delta(\mu+\beta)}U_* - \frac{1-F(r_*)}{f(r_*)} = e^{-\delta(\nu+\beta)}\Pi_* \tag{1}$$

Each buyer's willingness to pay is reduced by the sequential outside opportunities, but discounted by time and her possible exit. The seller chooses his per period mechanism while facing the set of willingness-to-pay distribution, but the reserve price determination of Myerson (1981) still holds. The virtual utility is reduced by the buyer discounted continuation payoff, and it is set to equate to the seller's opportunity cost - his discounted continuation payoff. For a more rigorous and general treatment, refer to the Section 3 and Appendix of Zhang (2013).

We refer to r_* as the reserve type. In the auction with reserve price $\rho_* = r_* - e^{-\delta(\mu+\beta)}U_*$, all the buyers with values above r_* will participate in the auction. (1) can be rearranged as

$$\mathrm{MR}(r_*) = e^{-\delta(\nu+\beta)}\Pi_* + e^{-\delta(\mu+\beta)}U_* \tag{2}$$

where $\mathrm{MR}(\cdot)$ is the increasing and continuous marginal revenue function. The equilibrium reserve price is

$$\rho_* = \frac{1-F(r_*)}{f(r_*)} + e^{-\delta(\nu+\beta)}\Pi_*$$

3.1 Mass Balance Equation

In each period in the steady state, inflow equals outflow on both sides of the market. The measure of winning buyers, which is the same as the measure of sellers who successfully sell and quit the market, is

$$S_* \sum_{n=0}^{\infty} \left(1 - F^n(r_*)\right) \frac{\kappa_*^n}{n! e^{\kappa_*}} = S_* \left[1 - e^{-\kappa_*(1-F(r_*))}\right]$$

where the equality follows from an algebraic simplification that we will use throughout,

$$\sum_{n=0}^{\infty} p_n(\kappa) x^n = \sum_{n=0}^{\infty} \frac{\kappa^n}{n! e^{\kappa}} x^n = \sum_{n=0}^{\infty} \frac{(\kappa x)^n}{n! e^{\kappa x}} \frac{e^{\kappa x}}{e^{\kappa}} = e^{-\kappa(1-x)}$$

The measure of losing buyers is

$$B_* - S_* \left[1 - e^{-\kappa_*(1-F(r_*))}\right] = S_* \left[\kappa_* - \left(1 - e^{-\kappa_*(1-F(r_*))}\right)\right]$$

and the measure of unsold sellers is

$$S_* e^{-\kappa_*(1-F(r_*))}$$

The inflow-outflow balance equation on the buyers' side is then

$$\begin{aligned} \delta\iota &= S_* \left[1 - e^{-\kappa_*(1-F(r_*))}\right] + \left(1 - e^{-\delta\mu}\right) S_* \left[\kappa_* - \left(1 - e^{-\kappa_*(1-F(r_*))}\right)\right] \\ &= S_* \left[e^{-\delta\mu} \left(1 - e^{-\kappa_*(1-F(r_*))}\right) + \left(1 - e^{-\delta\mu}\right) \kappa_*\right] \end{aligned} \tag{3}$$

And the inflow-outflow balance equation on the sellers' side is

$$\begin{aligned} \delta &= S_* \left[1 - e^{-\kappa_*(1-F(r_*))} + \left(1 - e^{-\delta\nu}\right) e^{-\kappa_*(1-F(r_*))}\right] \\ &= S_* \left[1 - e^{-\delta\nu} e^{-\kappa_*(1-F(r_*))}\right] \end{aligned} \tag{4}$$

The steady-state balance equation, obtained by dividing (3) by (4), is

$$\iota = \frac{e^{-\delta\mu} \left(1 - e^{-\kappa_*(1-F(r_*))}\right) + \left(1 - e^{-\delta\mu}\right) \kappa_*}{1 - e^{-\delta\nu} e^{-\kappa_*(1-F(r_*))}} \tag{5}$$

3.2 Continuation Payoffs

3.2.1 Buyer's Continuation Payoff

A buyer's equilibrium continuation payoff is her total discounted expected payoff before knowing her own private value,

$$U_* = \int_0^1 U_*(v)\, dF(v)$$

$$U_* = \frac{1}{1 - e^{-\delta(\mu+\beta)}} \int_{r_*}^1 (1 - F(v))\, w(v)\, dv$$

where $w(v) = e^{-\kappa_*(1-F(v))}$ is a value v buyer's probability of winning the auction when the number of opponents is distributed $\mathcal{P}(\kappa_*)$. She wins when her value v is higher than all the other buyers

$$\lim_{S\to\infty} \sum_{n=0}^{\kappa S-1} \binom{\kappa S-1}{n} \left(\frac{1}{S}\right)^n \left(1-\frac{1}{S}\right)^{\kappa S-1-n} F^n(v)$$

taking the large market limit of the probability that v beats all the opponents in a finite market of S sellers and κS buyers.

It is a sum of discounted per-period expected payoff, with a composite discount factor Δ composed of the time discount rate β and the buyer exit rate μ, adjusted for the length of a period,

$$U_* = \frac{1}{1-\Delta} u_*$$

Explicitly, the buyer continuation payoff depends on the primitive parameters of the model, δ, μ, β, F as well as the equilibrium variables κ_* and r_* (which of course depend on the primitives), and it decreases in both κ_* and r_*.

3.2.2 Seller's Continuation Payoff

Just as the buyer's continuation payoff, seller's is the sum of a stream of per-period revenue, discounted by the composite discount factor as well as the contingent probability of selling the good in the previous period.

$$\Pi_* = \pi_* + e^{-\kappa_*(1-F(r_*))} e^{-\delta(\nu+\beta)} \Pi_*$$

In particular, the per period revenue is

$$\pi_* = \int_{r_*}^{1} \left[v - \Delta U_* - \frac{1-F(v)}{f(v)} \right] de^{-\kappa_*(1-F(v))}$$

3.3 Existence and Uniqueness of the Equilibrium

Theorem 1. *There exists a unique market equilibrium.*

In general, (5) and (2) pin down κ_* and r_*, which characterize the steady state relative demand and equilibrium reserve type. To show the existence of the market equilibrium is to show the existence of solutions to the system of equations, and to show uniqueness to show that there is only one solution to the system.

4 Comparative Statics

Let us first focus on the case when sellers can only live for one period, i.e. $\nu = \infty$. Then the equilibrium conditions are simplified to be

$$\begin{aligned} \iota &= e^{-\delta\mu}\left(1 - e^{-\kappa_*(1-F(r_*))}\right) + \left(1 - e^{-\delta\mu}\right)\kappa_* && (6) \\ \mathrm{MR}\left(r_*\right) &= \frac{e^{-\delta(\mu+\beta)}}{1 - e^{-\delta(\mu+\beta)}} \int_{r_*}^{1} \left(1 - F\left(v\right)\right) e^{-\kappa_*(1-F(v))} dv && (7) \end{aligned}$$

We can investigate effects of ι, μ, β, and δ on κ_* and r_* by differentiating the equilibrium conditions. The Implicit Function Theorem can be used as the functions are all continuous and differentiable.

Differentiating the two equations with respect to a parameter α yields respectively,

$$\begin{aligned} &\frac{dr_*}{d\alpha} \underbrace{\left[-e^{-\delta\mu}e^{-\kappa_*(1-F(r_*))}\kappa_* f\left(r_*\right)\right]}_{\equiv \mathbf{A},(-)} + \frac{d\kappa_*}{d\alpha} \underbrace{\left[1 - e^{-\delta\mu} + e^{-\delta\mu}e^{-\kappa_*(1-F(r_*))}\left(1 - F\left(r_*\right)\right)\right]}_{\equiv \mathbf{B},(+)} \\ = \;& \underbrace{\frac{d\iota}{d\alpha} - \frac{d\left(\delta\mu\right)}{d\alpha} \underbrace{e^{-\delta\mu}\left[\kappa_* - \left(1 - e^{-\kappa_*(1-F(r_*))}\right)\right]}_{(+)}}_{\equiv \mathbf{E}} \qquad (8) \end{aligned}$$

and

$$\begin{aligned} &\frac{dr_*}{d\alpha} \underbrace{\left[\mathrm{MR}'\left(r_*\right) + \frac{\Delta}{1-\Delta}\left(1 - F\left(r_*\right)\right) e^{-\kappa_*(1-F(r_*))}\right]}_{\equiv \mathbf{C},(+)} + \frac{d\kappa_*}{d\alpha} \underbrace{\left[\frac{\Delta}{1-\Delta}\int_{r_*}^{1} \left(1 - F\left(v\right)\right)^2 e^{-\kappa_*(1-F(v))} dv\right]}_{\equiv \mathbf{D},(+)} \\ = \;& \underbrace{u_* \frac{1}{\left(1-\Delta\right)^2} \frac{d\Delta}{d\alpha}}_{\equiv \mathbf{F}} \qquad (9) \end{aligned}$$

which comes from rearranging and expanding

$$\mathrm{MR}'\left(r_*\right)\frac{dr_*}{d\alpha} = \frac{\Delta}{1-\Delta}\frac{\partial u_*}{\partial r_*}\frac{dr_*}{d\alpha} + \frac{\Delta}{1-\Delta}\frac{\partial u_*}{\partial \kappa_*}\frac{d\kappa_*}{d\alpha} + u_* \frac{d\left(\frac{\Delta}{1-\Delta}\right)}{d\alpha}.$$

In the following subsections by utilizing the two equations, we investigate the changes of r_* and κ_* with respect to each of the parameters, ι, β, μ and δ. By Cramer's Rule,

$$\frac{d\kappa_*}{d\mu} = \frac{\overset{-}{\mathbf{A}}\mathbf{F} - \mathbf{E}\overset{+}{\mathbf{C}}}{\underset{-}{\mathbf{A}}\underset{+}{\mathbf{D}} - \underset{+}{\mathbf{B}}\underset{+}{\mathbf{C}}} \qquad (10)$$

and

$$\frac{dr_*}{d\mu} = \frac{\overset{}{\mathbf{E}}\overset{+}{\mathbf{D}} - \overset{+}{\mathbf{B}}\mathbf{F}}{\underset{-}{\mathbf{A}}\underset{+}{\mathbf{D}} - \underset{+}{\mathbf{B}}\underset{+}{\mathbf{C}}} \tag{11}$$

The equilibrium reserve type is important because it is inversely related to the participation rate, percentage of population actively bidding in respective auctions. Since all the buyers with values higher than the reserve type bid, the participation rate is $1 - F(r_*)$,

$$\frac{d(1 - F(r_*))}{d\alpha} = \underbrace{-f(r_*)}_{(-)} \frac{dr_*}{d\alpha}$$

Changes in reserve price, buyer's continuation payoff, and seller's revenue all depend on the changes in equilibrium steady-state demand and equilibrium reserve type. In particular, change in equilibrium reserve price has the opposite sign as change in the equilibrium reserve type,

$$\frac{d\rho_*}{d\alpha} = \frac{d(r_* - \Delta U_*)}{d\alpha} = \underbrace{\left(\frac{1 - F(r_*)}{f(r_*)}\right)'}_{(-)} \frac{dr_*}{d\alpha}, \tag{12}$$

change in the total discounted continuation payoff has the same sign as change in the equilibrium reserve type

$$\frac{d(\Delta U_*)}{d\alpha} = \underbrace{\mathrm{MR}'(r_*)}_{(+)} \frac{dr_*}{d\alpha}, \tag{13}$$

and change in per period revenue is

$$\begin{aligned} \frac{d\pi_*}{d\alpha} &= \underbrace{\frac{\partial \pi_*}{\partial \kappa_*}}_{(+)} \frac{d\kappa_*}{d\alpha} - \int_{r_*}^{1} \frac{d(\Delta \Pi_*)}{d\alpha} de^{-\kappa_*(1-F(v))} \\ &= \underbrace{\frac{\partial \pi_*}{\partial \kappa_*}}_{(+)} \frac{d\kappa_*}{d\alpha} - \underbrace{\left[1 - e^{-\kappa_*(1-F(v))}\right] \mathrm{MR}'(r_*)}_{(+)} \frac{dr_*}{d\alpha}, \end{aligned} \tag{14}$$

where the second equality is obtained by substituting in (13).

The expected length a seller is in the market is

$$\delta \big/ \left[e^{-\delta\nu} + \left(1 - e^{-\kappa_*(1-F(r_*))}\right)\left(1 - e^{-\delta\nu}\right)\right]$$

which increases as κ_* decreases, r_* increases, and ν decreases.

4.1 Change in Inflow Rate ι

First, we see how an increase in the inflow rate, affects the equilibrium. $\mathbf{E}=1$ and $\mathbf{F}=0$, so (8) and (9) become

$$\begin{aligned}\underset{-}{\mathbf{A}}\frac{dr_*}{d\iota}+\underset{+}{\mathbf{B}}\frac{d\kappa_*}{d\iota} &= 1, \\ \underset{+}{\mathbf{C}}\frac{dr_*}{d\iota}+\underset{+}{\mathbf{D}}\frac{d\kappa_*}{d\iota} &= 0.\end{aligned}$$

By (10)-(14),

$$\frac{dr_*}{d\iota}<0, \frac{d\kappa_*}{d\iota}>0, \frac{d\rho_*}{d\iota}>0, \frac{d\left(\Delta U_*\right)}{d\iota}<0, \frac{d\pi_*}{d\iota}>0$$

An increase in the buyer inflow rate is an increase in the market's per-period relative demand. The steady state buyer-seller ratio, or demand per unit of supply, increases as a result of the increase in the buyer inflow rate. Although the equilibrium reserve type decreases, the equilibrium reserve price increases. Buyers' total discounted continuation payoff decreases, and the seller's optimal per period revenue increases. None of the results is striking as the relative demand increases, perhaps except for the fact that participation rate increases unambiguously, although the sellers adjust to have higher reserve price.

4.2 Change in Discount Rate β

Differentiate with respect to β, $\mathbf{E}=0$, and

$$\mathbf{F}=-\delta\frac{\Delta}{1-\Delta}U_*<0$$

Therefore, by (10) and (11), with strict inequalities,

$$\frac{dr_*}{d\beta}<0, \frac{d\kappa_*}{d\beta}<0$$

Straightforwardly by (12) and (13),

$$\frac{d\rho_*}{d\beta}>0, \frac{d\left(\Delta U_*\right)}{d\beta}<0$$

As the buyers become more impatient about the future periods, their discounted continuation payoff decreases, sellers raise both reserve price and reserve type to restrict participation. The steady state buyer-seller ratio decreases because the participation rate is increased - more buyers exit the market by obtaining the good despite of higher reserve price and transaction prices.

The per period revenue is affected in two ways. The expected number of buyers per seller decreases in the steady state, resulting in a lower expected revenue from bidding competition; on the other hand, however, buyer's discounted continuation payoff is discounted more, granting more power to the seller. The second effect dominates, so the revenue increases as the buyers become more impatient,

$$[*]: \quad \frac{d\pi_*}{d\beta} > 0.$$

4.3 Change in Buyer Exit Rate μ

In (8) and (9), when the variable is μ,

$$\begin{aligned}
\mathbf{E} &= -\delta e^{-\delta\mu}\left[\kappa_* - 1 + e^{-\kappa_*(1-F(r_*))}\right] < 0 \\
\mathbf{F} &= -\delta\frac{\Delta}{1-\Delta}U_* < 0
\end{aligned}$$

By Cramer's Rule,

$$\frac{d\kappa_*}{d\mu} < 0,$$

but the sign of $dr_*/d\mu$ is not directly determined from ,

$$\frac{dr_*}{d\mu} = \frac{\overset{-}{\mathbf{E}}\overset{+}{\mathbf{D}} - \overset{+}{\mathbf{B}}\overset{-}{\mathbf{F}}}{\underset{-}{\mathbf{A}}\underset{+}{\mathbf{D}} - \underset{+}{\mathbf{B}}\underset{+}{\mathbf{C}}}$$

and it has the same sign as $\mathbf{BF} - \mathbf{ED}$,

$$\begin{aligned}
& \frac{\delta\Delta}{1-\Delta}\left[-\frac{1-e^{-\delta\mu}+e^{-\delta\mu}e^{-\kappa_*(1-F(r_*))}\left(1-F\left(r_*\right)\right)}{1-\Delta}u_* - e^{-\delta\mu}\left(\kappa_* - 1 + e^{-\kappa_*(1-F(r_*))}\right)\frac{\partial u_*}{\partial\kappa_*}\right] \\
= \; & \frac{\delta\Delta}{1-\Delta}\int_{r_*}^{1}\left(1-F\left(v\right)\right)e^{-\kappa_*(1-F(v))} \\
& \left[-\frac{1-e^{-\delta\mu}+e^{-\delta\mu}e^{-\kappa_*(1-F(r_*))}\left(1-F\left(r_*\right)\right)}{1-e^{-\delta\mu}e^{-\delta\beta}} + \left(1-F\left(v\right)\right)e^{-\delta\mu}\left(\kappa_* - 1 + e^{-\kappa_*(1-F(r_*))}\right)\right]dv
\end{aligned}$$

which has an ambiguous sign. There are two effects that are in conflict with each other,

$$\left[\mathrm{MR}'\left(r_*\right) - \Delta\frac{\partial U_*}{\partial r_*}\right]\frac{dr_*}{d\mu} = \underset{(-)}{\left[\Delta\frac{\partial U_*}{\partial\kappa_*}\right]}\underset{(-)}{\frac{d\kappa_*}{d\mu}} + \underset{(-)}{\frac{\partial\left(\Delta U_*\right)}{\partial\mu}}.$$

The increase in buyer exit rate decreases survival rate - both the buyer's own survival rate as well as others'. The first effect is an increase in the buyer continuation payoff, resulted from decrease in the equilibrium buyer-seller ratio by the increase in other buyers' exit rate. When the buyers exit more often, holding inflow constant, there will be fewer buyers surviving in the equilibrium. As a result, competition among buyers is reduced, so the buyer's continuation value increases. This effect increases the optimal reserve type (and decreases the optimal reserve price - the two always move in opposite direction by (12)). The second effect is an decrease in the normalized discount factor caused by an increase in own buyer exit rate. Because the buyer survives with less probability, her discounted continuation value decreases, conflicting with the first effect her continuation value increases, however realized only when she survives to future periods. However, which effect is stronger depends on the composite discount factor and the inflow rate.

4.4 Change in the Length of a Period δ

When differentiating (8) and (9) with respect to δ,

$$\begin{aligned}
\mathbf{E} &= -\mu e^{-\delta\mu}\left[\kappa_* - 1 + e^{-\kappa_*(1-F(r_*))}\right] < 0 \\
\mathbf{F} &= -(\mu+\beta)\frac{\Delta}{1-\Delta}U_* < 0
\end{aligned}$$

By Cramer's Rule,

$$\frac{d\kappa_*}{d\delta} < 0,$$

and the sign of $\frac{dr_*}{d\delta}$, similar to the effects of change in μ, depends on the sign of $\mathbf{BF} - \mathbf{ED}$,

$$\begin{aligned}
& \frac{\Delta}{1-\Delta}\left[-(\mu+\beta)\frac{1-e^{-\delta\mu}+e^{-\delta\mu}e^{-\kappa_*(1-F(r_*))}(1-F(r_*))}{1-\Delta}u_* - \mu e^{-\delta\mu}\left(\kappa_* - 1 + e^{-\kappa_*(1-F(r_*))}\right)\frac{\partial u_*}{\partial \kappa_*}\right] \\
= \; & \frac{\Delta}{1-\Delta}\int_{r_*}^{1}(1-F(v))\,e^{-\kappa_*(1-F(v))} \\
& \left[-(\mu+\beta)\frac{1-e^{-\delta\mu}+e^{-\delta\mu}e^{-\kappa_*(1-F(r_*))}(1-F(r_*))}{1-e^{-\delta\mu}e^{-\delta\beta}} + \mu(1-F(v))\,e^{-\delta\mu}\left(\kappa_* - 1 + e^{-\kappa_*(1-F(r_*))}\right)\right]dv
\end{aligned}$$

5 Comparative Statics II

Now we do not assume $\nu = \infty$, and investigate the welfare effects of such changes.

$$
\begin{aligned}
\left(1 - e^{-\delta(\nu+\beta)} e^{-\kappa_*(1-F(r_*))}\right) \iota &= e^{-\delta\mu}\left(1 - e^{-\kappa_*(1-F(r_*))}\right) + \left(1 - e^{-\delta\mu}\right)\kappa_* \\
\mathrm{MR}\left(r_*\right) &= \frac{e^{-\delta(\mu+\beta)}}{1 - e^{-\delta(\mu+\beta)}} \int_{r_*}^{1} \left(1 - F\left(v\right)\right) e^{-\kappa_*(1-F(v))} dv + \\
&\quad \frac{e^{-\delta(\nu+\beta)}}{1 - e^{-\delta(\nu+\beta)} e^{-\kappa_*(1-F(r_*))}} \int_{r_*}^{1} \left(\mathrm{MR}\left(v\right) - e^{-\delta(\mu+\beta)} U_*\right) de^{-\kappa_*(1-F(v))}
\end{aligned}
$$

6 Discussions

Although some effects are straightforward, some effects are rather special results of the general equilibrium approach. Many important welfare effects are ambiguous with respect to changes in the market environment, so empirical estimations are important to determine these welfare effects.

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Appendix

A Notations

Primitive Parameters

t: time

δ: length of a period

β: discount rate

μ: instantaneous buyer exit rate

ν: instantaneous seller exit rate

ι: relative buyer inflow rate - measure of new buyers per unit of time (per time unit measure of new sellers is normalized to 1)

n: number of buyers a seller is matched with

$p_n(\kappa)$: the probability a seller is matched with n buyers when the number of buyers a seller is matched with is Poisson distributed with parameter κ

F: buyer value distribution

f: buyer value density

Period t and Equilibrium $_*$

B_t, B_*: measure of buyers per period

S_t, S_*: measure of sellers per period

κ_t, κ_*: buyer-seller ratio

r_t, r_*: reserve type

ρ_t, ρ_*: reserve price

U_*: buyer continuation payoff, buyer's discounted sum of expected utilities

Π_*: seller continuation payoff, seller's discounted sum of expected revenues