An Epistemic Characterization of RSCE*

Xiao Luo[†], Ben Wang

Department of Economics, National University of Singapore, Singapore 117570

This version: April 2013

Abstract

Dekel et al. (J Econ Theory 89 (1999) 165-185) offered a solution concept of "rationalizable self-confirming equilibrium (RSCE)" as the steady state where rational individuals observe the played actions and use the information about opponents' payoffs in forming the beliefs about opponents' behavior off the equilibrium path. In this paper we investigate epistemic conditions for RSCE from a decision-theoretic point of view. Within a standard semantic framework, we formulate and show, by using the notion of "conditional probability system (CPS)," that RSCE is the logical consequence of common knowledge of rationality and mutual knowledge of the actions along the path of play. In this paper, we also apply this epistemic approach to other related solution concepts such that self-confirming equilibrium (SCE) and sequential rationalizable self-confirming equilibrium (SRSCE).

Keywords: RSCE; CPS; rationality; mutual knowledge; common knowledge; SCE; SRSCE

*Financial support from National University of Singapore is gratefully acknowledged. Acknowledgement to be added.

[†]Corresponding author. Fax:+65 6775 2646. *E-mail:* ecslx@nus.edu.sg (X. Luo), wangben@nus.edu.sg (B. Wang).

1 Introduction

In extensive games, Fudenberg and Levine (1993) presented a solution concept of "self-confirming equilibrium (SCE)" which arises as a steady state where players correctly predict the moves their opponents actually make, but may have misconceptions about what their opponents would do at information sets that are never reached when the equilibrium is played. That is, the notion of SCE is designed to model situations where players have no *a priori* information about opponents' play or payoffs and, when each time the game is played, they observe only the actions actually played by their opponents along the equilibrium path; cf. also Fudenberg and Kreps (1995) and Fudenberg and Levine (2006, 2009). A particular and noteworthy feature of SCE is that beliefs about off-path play are completely arbitrary so that players may hold false and inconsistent belief about off-path play; in particular, the notion of SCE allows players to use a "noncredible" threats in beliefs about off-path play (see Dekel et al. (1999, Fig. 2.1)). If, however, players can use information about opponents' payoffs and think strategically, players should be able to deduce and make use of information about opponents' payoff functions and, thus, can alleviate inconsistency in players' beliefs about off-path play. To fulfil this purpose, Dekel et al. (1999, 2001) provided a solution concept of "rationalizable self-confirming equilibrium (RSCE)" which refines SCE by requiring a player's rationality at the player's information sets that are not precluded by his own strategy. Dekel et al. (1999) showed that RSCE is robust to payoff uncertainty in the sense of Fudenberg et al. (1988). Dekel et al. (1999, Sec. 4) also defined a stronger concept of "sequentially rationalizable self-confirming equilibrium (SRSCE)" by requiring a player's rationality at all of the player's information sets, so that the sequential rationalizability notion implies backward induction in finite games of perfect information with generic payoffs; SRSCE is related to Greenberg et al.'s (2009) notion of "mutually acceptable course of action (MACA)."

The purpose of this paper is to offer a simple epistemic characterization for RSCE. This line of study can help to deepen our understanding of RSCE and other related solution concepts from an epistemic perspective. In doing so, a technical difficulty encountered in dynamic extensive-form game models is, when facing with strategic uncertainty, how to model a player's beliefs about opponents' play in every contingency, including information sets that the player thinks will not actually arise. Inspired by Selten's (1975) idea of "trembles," Dekel et al. (2002) defined the "extensive-form convex hull" of a set of behavior strategies to model a player's beliefs about the play of an opponent's strategic behavior in extensive games; cf. also Greenberg et al. (2009, pp.95-98) for related discussions. In this paper, we use the notion of "conditional probability system (CPS)" introduced by Myerson (1986) to represent players' beliefs and provide an epistemic characterization for the solution concept of RSCE. More specifically, each player is assumed to hold an "independent" CPS over on the product of action spaces in the agent-normal form of an extensive game, which is based on the information along the path of play.

Within a standard semantic framework or Aumann's model of knowledge, we formulate and show that RSCE is the logical consequence of mutual knowledge of actions and rationality along the path of play and common knowledge of rationality off the path of play (see Theorem 3.1 and Corollary 3.1). This result provides a unifying epistemic approach to other related solution concepts such as SCE and SRSCE; we demonstrate, in this paper, how various epistemic characterizations for related solution concepts can be derived by varying the restrictions of rationality (see Corollaries 3.2 and 3.3).

The rest of this paper is organized as follows. Section 2 contains some preliminary notation and definitions. Section 3 presents a simple epistemic characterization for RSCE and discusses its epistemic relations to other related solution concepts such as SCE and SRSCE. Section 4 offers concluding remarks.

2 Notation and Definitions

Since the formal description of an extensive game is by now standard (see, for instance, Kreps and Wilson (1982) and Kuhn (1954)), only the necessary notation is given below. Consider a (finite) extensive-form game with perfect recall:

$$T \equiv (N, V, H, \{A^h\}_{h\in H}, \{u_i\}_{i\in N}),$$

where $N = \{1, 2, \ldots, n\}$ is the (finite) set of players, V is the (finite) set of nodes (or vertices), H is the set of information sets (which is a partition of nonterminal nodes), A^h is the (finite) set of pure actions available at information set h, and u_i is player i's payoff function defined on terminal nodes. A mixed action at information set h is a probability measure on A^h. Denote the set of mixed actions at h by $\triangle A^h$. Denote the collection of player i's information sets by H^i. Denote by $A \equiv \times_{h \in H} A^h$ the set of actions.

A *behavior strategy* of player i is a function, π_i, that assigns some randomization $\pi_i(h) \in \triangle A^h$ to every $h \in H^i$. Let $\mathbf{\Pi}_i$ be the set of player i's behavior strategies. Denote the set of behavior strategy profiles by $\mathbf{\Pi}$, i.e. $\mathbf{\Pi} = \times_{j \in N} \mathbf{\Pi}_j$. For $\pi \in \mathbf{\Pi}$, we denote by $u_i(\pi)$ player i's (expected) payoff if strategy profile π is adopted from the root of the game. For $\pi \in \mathbf{\Pi}$, we denote by $\pi(h)$ the mixed action of π at h, and denote by $\pi(-h)$ the profile of mixed actions of π at all information sets other than h. Given $\pi \in \mathbf{\Pi}$, let H_π be the set of information sets reached with positive probability under π. Denote by $H^i_\pi = H_\pi \cap H^i$ the set of player i's information sets reached by π and $H^i_{\pi_i} = \cup_{\pi_{-i} \in \mathbf{\Pi}_{-i}} H^i_{(\pi_i, \pi_{-i})}$ the set of player i's information sets that are reachable under π_i.

Write $\pi^k_i \rightsquigarrow \pi_i$ for the "trembling" sequence $\left\{\pi^k_i\right\}^\infty_{k=1}$ of strictly positive behavior strategies in $\mathbf{\Pi}_i$ that converges to π_i.

2.1 RSCE: A Definition

Dekel et al. (1999) proposed a solution concept of "rationalizable self-confirming equilibrium (RSCE)" for extensive games where players learn the path of the play and incorporate the information of opponents' payoffs into the original notion of SCE. Following Dekel et al. (1999), an *assessment* η_i for player i is a function that assigns a probability measure over the nodes at each of his own information sets. A *belief* of player i is a pair (η_i, π^i_{-i}) where η_i is player i's assessment and $\pi^i_{-i} = (\pi^i_j)_{j \neq i}$ represents player i's conjecture about opponents' strategies. A *version* of player i is a strategy-belief pair $v_i = (\pi_i, (\eta_i, \pi^i_{-i}))$. Given a version $v_i = (\pi_i, (\eta_i, \pi^i_{-i}))$, $\pi_i(h)$ is a *best response with respect to* $(\pi_i, (\eta_i, \pi^i_{-i}))$ *at* $h \in H^i$ if

$$u_i\left(\pi_i, \pi^i_{-i} | h, \eta_i(h)\right) \geq u_i\left(a^h, \left(\pi_i, \pi^i_{-i}\right)(-h) | h, \eta_i(h)\right) \ \forall a^h \in A^h$$

where $u_i(\pi | h, \eta_i(h))$ represents player i's conditional expected payoff given that information set h is reached, that player i's assessment is given by $\eta_i(h)$, and that the strategies are π.

A version $v_i = (\pi_i, (\eta_i, \pi^i_{-i}))$ is *consistent* (Kreps and Wilson (1982)) if $\eta_{i,k} \rightarrow \eta_i$ where $\eta_{i,k}$ is obtained using Bayes rule from a trembling sequence $\pi^i_{-i,k} \rightsquigarrow \pi^i_{-i}$. A *belief model* $V = (V_1, V_2, ..., V_n)$ where V_i is the set of consistent versions for player i.

A strategy π_i of player i is in the *extensive-form convex hull of a subset* $\Pi_i \subseteq \Pi_i$ (Dekel et al. (2002)), denote by $co^e(\Pi_i)$, if there is an integer m, strategies $\{\pi_{i,t}\}_{t=1,...,m}$ in Π_i, sequences of strictly positive behavior strategies $\pi_{i,t,k} \rightsquigarrow \pi_{i,t}$, and a sequence $\alpha_k \to \alpha$ of probability distributions on $[1, ..., m]$, such that the behavior strategies $\pi_{i,k}$, which is outcome-equivalent to convex combination $\sum_{t=1}^{m} \alpha_{t,k}\, \pi_{i,t,k}$, converges to π_i (in this situation we denote by $\pi_{i,k} \rightsquigarrow \pi_i \in co^e(\Pi_i)$).

Dekel et al. (1999 & 2002) defined SCE, RSCE and SRSCE as strategy profiles. Since only the path of play is essential in these notions, we give the following alternative definition in terms of paths of play.

Definition 2.1 (Dekel et al. 1999 & 2002). Let $\widehat{\pi}$ be a path of play. Given a belief model $V = (V_1, V_2, ..., V_n)$, for every player $i \in N$ and every $\left(\pi_i, \left(\eta_i, \pi^i_{-i}\right)\right) \in V_i$, we consider the following conditions for V:

(1) $\forall h \in H^i_{\pi_i}$, $\pi_i(h)$ is a best response with respect to (η_i, π^i_{-i}).

(1') $\forall h \in H^i_{(\pi_i, \pi^i_{-i})}$, $\pi_i(h)$ is a best response with respect to (η_i, π^i_{-i}).

(1") $\forall h \in H^i$, $\pi_i(h)$ is a best response with respect to (η_i, π^i_{-i}).

(2) the path of play resulting from (π_i, π^i_{-i}) is $\widehat{\pi}$.

(3) $\forall j \neq i$, $\pi^i_j \in co^e(\Pi_j)$ where $\Pi_j = \{\pi_j : (\pi_j, (\eta_j, \pi^j_{-j})) \in V_j$ for some belief $(\eta_j, \pi^j_{-j})\}$.

The path $\widehat{\pi}$ is a *rationalizable self-confirming equilibrium (RSCE)* if there is a belief model V satisfying (1), (2) and (3), $\widehat{\pi}$ is a *self-confirming equilibrium (SCE)* if there is a belief model V satisfying (1'), (2) and (3), and $\widehat{\pi}$ is a *sequential rationalizable self-confirming equilibrium (SRSCE)* if there is a belief model V satisfying (1"), (2) and (3).

Dekel et al. (1999, p.173) showed, through the example of Selten's Horse, that arbitrage and heterogeneous false beliefs about off-path play can lead to non-Nash outcomes: SCE, RSCE and SRSCE all can arrive at a steady state that cannot arise in Nash equilibrium. The following example illustrates the differences in SCE, RSCE and SRSCE.

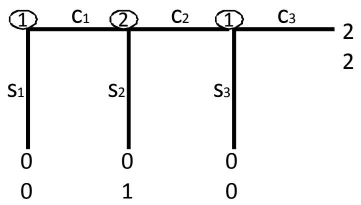

Fig. 1

In this game depicted in Fig. 1, it is easy to verify that the path outcomes of SCE, RSCE and SRSCE are as follows:

SCE: s_1; c_1s_2; $c_1c_2c_3$

RSCE: s_1; $c_1c_2c_3$

SRSCE: $c_1c_2c_3$

While the path s_1 can arise in RSCE by using a "rationalizable" belief that player 2 will play s_2 with probability 1, the path c_1s_2 cannot arise in RSCE (since player 1's second decision node is not precluded by his strategy and, thus, the rationality at this decision node requires player 1's choice to be c_3). The unique backward induction outcome: $c_1c_2c_3$ coincides with the unique path outcome of SRSCE.

2.2 CPS in Extensive Games

In this paper, we consider the "conditional probability system (CPS)" on the space, $A = \times_{h\in H}A^h$, of action profiles in the agent-normal form of T. Accordingly, a CPS on A can be viewed as a conditional-probability function which is defined at every information set, including those are not reached, a probability distribution on agents' actions. Formally, a CPS $\mu|$ on A is a function that specifies, for every nonempty subset $B \subseteq A$, a conditional probability distribution $\mu|_B$ given B and satisfies the property:

$$\mu|_B(D) = \mu|_C(D)\mu|_B(C) \text{ for } D \subseteq C \subseteq B \subseteq A.$$

See, e.g., Myerson (1991, Sec. 1.6).

Denote by

$$A\,(h) \equiv \{a \in A : a \text{ reaches } h\}$$

the set of action profiles by each of which h can be reached. For $i \in N$ and $h \in H^i$, $a^h \in A^h$ is a *best response with respect to a CPS $\mu|$ on A* if

$$\sum_{a^{-h} \in A^{-h}} \mu|^{-h}_{A(h)}(a^{-h}) u_i(a^h, a^{-h}) \geq \sum_{a^{-h} \in A^{-h}} \mu|^{-h}_{A(h)}(a^{-h}) u_i(b^h, a^{-h}) \;\; \forall b^h \in A^h$$

where $\mu|^{-h}_{A(h)}$ is the margin of $\mu|_{A(h)}$ on A^{-h},[1] which specifies the agent h's belief about opponents' choices given that information set h is reached.

By Myerson's (1986) Theorem 1, a CPS on a (finite) state space can be expressed by a convergent sequence of "full-support" probability distributions over the state space. A CPS $\mu|$ on A is *associated with a probability distribution* p (on A), denoted by $\mu|_{[p_k \rightsquigarrow p]}$, if there exists a sequence of probability distributions $p_k \to p$ such that:

(i) For $k = 1, 2, ...$ and every $a \in A$, $p_k(a) > 0$;

(ii) For any B, $C \subseteq A$ with $B \neq \emptyset$, $\mu|_B(C) = \lim_{k \to \infty} \frac{p_k(B \cap C)}{p_k(B)}$.

For the purpose of this paper, we say "a CPS $\mu|$ on A is independent" if $\mu| = \mu|_{[p_k \rightsquigarrow p]}$ where p_k are product measures on the (product) space A; cf., e.g., McLennan (1989) for more discussions.

The following lemma is an immediate implication of Myerson's (1986) Theorem 1, which states a relationship between "sequential rationality" and "conditionally preference ordering by CPS."

Lemma 1. *Let $\pi_{j,k} \rightsquigarrow \pi_j \; \forall j \in N$. For all $h \in H^i$, $\pi_i(h)$ is a best response with respect to a consistent version $(\pi_i, (\eta_i, \pi_{-i}))$ with $\pi_{j,k} \rightsquigarrow \pi_j \; \forall j \neq i$ if, and only if, $\pi_i(h)$ is preferred to a^h with respect to $\mu|_{[\pi_k \rightsquigarrow \pi]}$ for all $a^h \in A^h$.*

For any subset $\Pi \subseteq \mathbf{\Pi}$, let

$$co^e(\Pi) = \times_{j \in N} co^e(\Pi_j),$$

[1] The margin of $\mu|_{A(h)}$ on A^{-h} is defined as probability measure on A^{-h} such that

$$\forall a^{-h} \in A^{-h}, \; \mu|^{-h}_{A(h)}(a^{-h}) \equiv \sum_{a^h \in A^h} \mu|_{A(h)}(a^h, a^{-h}).$$

where $\Pi_j = \{\pi_j : (\pi_j, \pi_{-j}) \in \Pi\}$. Written $\pi_k \rightsquigarrow \pi \in co^e(\Pi)$ for "$\pi_{j,k} \rightsquigarrow \pi_j \in co^e(\Pi_j)$ $\forall j \in N$." Define

$$ICPS^e(\Pi) \equiv \left\{\mu| \ : \ \mu| = \mu|_{[\pi_k \rightsquigarrow \pi]} \text{ for some } \pi_k \rightsquigarrow \pi \text{ in } co^e(\Pi)\right\}.$$

That is, $ICPS^e(\Pi)$ is the set of all independent CPS on A that can be generated by $\pi \in co^e(\Pi)$.

3 Epistemic Characterization of RSCE

Following Aumann (1976, 1987, 1995, and 1999), we provide, within the standard partition model, epistemic conditions for RSCE by common knowledge of "rationality" and mutual knowledge of the equilibrium path. A model of knowledge for game T is given by

$$\mathcal{M}(T) =< \Omega, \{P_i\}_{i\in N}, \{\boldsymbol{\pi}_i\}_{i\in N}, \{\boldsymbol{\mu}_i|\}_{i\in N} > ,$$

where

$$\begin{array}{l} \Omega \text{ is the set of states} \\ P_i(\omega) \text{ is player } i\text{'s information partition at } \omega \\ \boldsymbol{\pi}_i(\omega) \text{ is player } i\text{'s behavior strategy at } \omega \\ \boldsymbol{\mu}_i|(\omega) \text{ is player } i\text{'s conditional belief systems at } \omega \end{array} .$$

We refer to a subset $E \subseteq \Omega$ as an *event*. For an event $E \subseteq \Omega$, we take the following standard definitions.

- $K_iE \equiv \{\omega \in \Omega |\ P_i(\omega) \subseteq E\}$ is the event that i *knows* E.
- $KE \equiv \cap_{i\in N} K_iE$ is the event that E is *mutually known*.
- $CKE \equiv KE \cap KKE \cap KKKE \cap \cdots$ is the event that E *is commonly known*.

For $E \subseteq \Omega$, we denote by

$$\boldsymbol{\pi}(E) \equiv \{\boldsymbol{\pi}(\omega) : \ \omega \in E\}.$$

Throughout this paper, we assume $\boldsymbol{\pi}_i(\cdot)$ is measurable w.r.t. information partition P_i – i.e. $\boldsymbol{\pi}_i(\omega) = \boldsymbol{\pi}_i(\omega') \ \forall \omega' \in P_i(\omega)$.

Agent $h \in H^i$ is *rational at* ω if $\boldsymbol{\pi}_i(\omega)(h)$ is a best response with respect to $\boldsymbol{\mu}_i|(\omega) \in ICPS^e(\boldsymbol{\pi}(P_i(\omega))$. For every $i \in N$ and every $h \in H^i$, denote by

$$\mathring{R}^h \equiv \left\{\omega : \text{ agent } h \text{ is robust-rational at } \omega \text{ if } h \in H^i_{\boldsymbol{\pi}_i(\omega)}\right\},$$

i.e., $\mathring{R}^h$ represents the event that agent h is robust-rational whenever information set h is not excluded by his strategy choice. Let

$$\mathring{R}^i \equiv \cap_{h \in H^i} \mathring{R}^h \text{ and } \mathring{R} \equiv \cap_{i \in N} \mathring{R}^i.$$

For any given path of play $\hat{\pi}$, let

$$\mathring{R}^{\hat{\pi}} \equiv \cap_{h \in H_{\hat{\pi}}} \mathring{R}^h \text{ and } \mathring{R}^{-\hat{\pi}} \equiv \cap_{h \notin H_{\hat{\pi}}} \mathring{R}^h,$$

where $H_{\hat{\pi}}$ is the information sets That is, $\mathring{R}^{\hat{\pi}}$ is the event that players are robust-rational at the information sets along the path $\hat{\pi}$ and $\mathring{R}^{-\hat{\pi}}$ is the event that players are rational at the off-path information sets.

The *path of play* under π can be viewed as the restriction of π to reachable information sets:

$$\widehat{\pi} = \times_{h \in H_\pi} \pi(h).$$

Denote by $\hat{\boldsymbol{\pi}}$ the restriction of $\boldsymbol{\pi}$ to $H_{\hat{\pi}}$, i.e., $\hat{\boldsymbol{\pi}}(\omega)= \boldsymbol{\pi}|_{H_{\hat{\pi}}}(\omega)$ $\forall \omega \in \Omega$. Let

$$[\hat{\pi}] \equiv \{\omega : \hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}\},$$

i.e., $[\hat{\pi}]$ is the event that the path of play is $\hat{\pi}$.

We are now in a position to present the central result of this paper which provides a simple epistemic characterization for the notion of RSCE. Theorem 3.1 states that mutual knowledge of a path of play, robust-rationality along the information sets prescribed by the path, and common knowledge of robust-rationality at off-path information sets, imply an RSCE. Conversely, any RSCE can be attained by the aforementioned epistemic assumptions.

Theorem 3.1 *(a) Let* $\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$. *Then,* $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ *is an RSCE. (b) Let* $\hat{\pi}$ *be an RSCE. Then, there is a knowledge model* $\mathcal{M}(T)$ *such that* $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ *for all* $\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$.

Proof. (a) For any $i \in N$, define

$$\Pi_i = \{\boldsymbol{\pi}_i(\omega) : \omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}\},$$

and let $\Pi \equiv \times_{i \in N} \Pi_i$.

Clearly, if $h \in H_{\hat{\pi}}$, $\pi(h) = \hat{\pi}(h)$ for all $\pi \in \Pi$. That is, for all $\pi \in \Pi$, π has the same distribution over outcomes as induced by $\hat{\pi}$.

(i) For any $i \in N$, $\pi_i \in \Pi_i$, there exists $\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$ such that $\boldsymbol{\pi}_i(\omega) = \pi_i$. Since $\omega \in \mathring{R}^{\hat{\pi}} \cap CK\mathring{R}^{-\hat{\pi}}$, $\omega \in \mathring{R}$. Therefore, $\forall i \in N$ there is $\boldsymbol{\mu}_i|(\omega) \in ICPS^e(\boldsymbol{\pi}(P^i(\omega))$ such that $\forall h \in H_{\boldsymbol{\pi}_i(\omega)}, \boldsymbol{\pi}_i(\omega)(h)$ is a best response with respect to $\boldsymbol{\mu}_i|_{A(h)}(\omega)$.

(ii) Let $\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$. Since $\omega \in K_i[\hat{\pi}]$, for all $\omega' \in P_i(\omega)$, $\boldsymbol{\pi}(\omega')(h) = \hat{\pi}(h)$ for all $h \in H_{\hat{\pi}}$. That is, for all $\omega' \in P_i(\omega)$, $\boldsymbol{\pi}(\omega')(h) = \boldsymbol{\pi}(\omega)(h)$ for all $h \in H^{\hat{\pi}}$.

If $h \notin H_{\hat{\pi}}$, then $\forall \omega' \in P_i(\omega)$,

$$\begin{aligned} \boldsymbol{\pi}(\omega')(h) &\in \{\boldsymbol{\pi}(\omega'')(h) : \omega'' \in CK\mathring{R}^{-\hat{\pi}}\} \text{ (since } P_i(\omega) \subseteq CK\mathring{R}^{-\hat{\pi}}) \\ &= \{\boldsymbol{\pi}(\omega'')(h) : \omega'' \in CK\mathring{R}^{-\hat{\pi}} \cap (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}})\}. \end{aligned}$$

Therefore, $\boldsymbol{\pi}(P_i(\omega)) \subseteq \Pi$. Since $\boldsymbol{\pi}_i(\omega) = \boldsymbol{\pi}_i(\omega')$ $\forall \omega' \in P_i(\omega)$, $\boldsymbol{\pi}(P_i(\omega)) \subseteq \{\pi_i\} \times \Pi_{-i}$ for all $\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$.

By (i) and (ii), it follows that for every $i \in N$ and $\pi_i \in \Pi_i$, there is a $\mu_i| \in ICPS^e(\Pi)$ such that for every $h \in H_{\pi_i}$, $\pi_i(h)$ is a best response with respect to $\mu_i|_{A(h)}$. Thus, there exists $\pi_k \rightsquigarrow \pi \in co^e(\Pi)$ such that $\mu_i| = \mu|_{[\pi_k \rightsquigarrow \pi]}$, and $\forall h \in H_{\hat{\pi}}$, $\pi(h) = \hat{\pi}(h)$. By Lemma 1, $\forall i \in N$ and $\pi_i \in \Pi_i$, there exists (η_i, π^i_{-i}), which is consistent with $\pi_k \rightsquigarrow \pi$, such that $\forall h \in H_{\pi_i}$, $\pi_i(h)$ is best response with respect to (η_i, π^i_{-i}). As $\forall j \neq i$, $\pi_{j,k} \rightsquigarrow \pi_j \in co^e(\Pi_j)$ and $\pi_j = \pi^i_j$, $\pi^i_j \in co^e(\Pi_j)$. That is, $\forall h \in H_{\hat{\pi}}$ $\left(\pi_i, \pi^i_{-i}\right)(h) = \hat{\pi}(h)$.

For all $\forall i \in N$, let

$$V_i \equiv \left\{ (\boldsymbol{\pi}_i(\omega), (\eta_i, \pi^i_{-i})) \;:\; \begin{array}{c} (\eta_i, \pi^i_{-i}) \text{ is consistent with } \pi_k \rightsquigarrow \pi \\ \text{where } \mu|_{[\pi_k \rightsquigarrow \pi]} = \boldsymbol{\mu}_i|(\omega) \\ \text{and } \omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}} \end{array} \right\},$$

and $V \equiv (V_1, V_2, ..., V_n)$. Then, for all $i \in N$ and $\left(\pi_i, \left(\eta_i, \pi^i_{-i}\right)\right) \in V_i$, we have

(1) $\forall h \in H_{\pi_i}$, $\pi_i(h)$ is a best response with respect to (η_i, π^i_{-i}).

(2) $\left(\pi_i, \pi^i_{-i}\right)$ has the distribution over outcomes induced by $\hat{\pi}$.

(3) $\forall j \neq i$, there exists $\pi^i_j \in co^e(\Pi_j)$ where $\Pi_j = \{\pi'_j : (\pi'_j, (\eta_i, \pi^j_{-j})) \in V_j$ for some

belief $(\eta_i, \pi^j_{-j})\}$.

That is, $\forall\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ and $\hat{\pi}$ is an RSCE.

(b)Let $\hat{\pi}$ be an RSCE that is supported by $V = (V_1, V_2, ..., V_n)$.

We proceed to show a stronger result that there is $\mathcal{M}(T)$ such that $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in CK([\hat{\pi}] \cap \mathring{R}) \neq \emptyset$. For all $i \in N$, for every $(\pi_i, (\eta_i, \pi^i_{-i})) \in V_i$,

(1) $\forall h \in H_{\pi_i}$, $\pi_i(h)$ is a best response with respect to (η_i, π^i_{-i}).

(2) (π_i, π^i_{-i}) has the distribution over outcomes induced by $\hat{\pi}$.

(3) $\forall j \neq i$, there exists $\pi^i_j \in co^e(\Pi_j)$ where $\Pi_j = \{\pi'_j : (\pi'_j, (\eta_i, \pi^j_{-j})) \in V_j$ for some belief $(\eta_i, \pi^j_{-j})\}$.

Let $\mu_i|(\pi_i) = \mu|_{[\pi_k \rightsquigarrow \pi]}$ such that $\pi_{i,k} \rightsquigarrow \pi_i \in co^e(\{\pi_i\})$, and $\pi^i_{j,k} \rightsquigarrow \pi^i_j \in co^e(\Pi_j)$ $\forall j \neq i$. Clearly, $\mu_i|(\pi_i) \in ICPS^e(\{\pi_i\} \times \Pi_{-i})$. Define a knowledge model for game T:

$$\mathcal{M}(T) =< \Omega, \{P_i\}_{i\in N}, \{\boldsymbol{\pi}_i\}_{i\in N}, \{\boldsymbol{\mu}_i|\}_{i\in N} > ,$$

such that $\Omega = \left\{(\pi_j, \mu_j|(\pi_j))_{j\in N} : \ \pi_j \in \Pi_j, \forall j \in N\right\}$ and for all $i \in N$ and $\omega = (\pi_j, \mu_j|(\pi_j))_{j\in N}$ in Ω,

$$\boldsymbol{\pi}_i(\omega) = \pi_i,\ \boldsymbol{\mu}_i|(\omega) = \mu_i|(\pi_i) \text{ and}$$
$$P_i(\omega) = \{\omega' \in \Omega : \ \boldsymbol{\pi}_i(\omega') = \pi_i \text{ and } \boldsymbol{\mu}_i(\omega') = \mu_i(\pi_i)\}.$$

Now, consider any arbitrary $\omega = (\pi_j, \mu_j|(\pi_j))_{j\in N} \in \Omega$. By Lemma 1, it follows that for all $i \in N$ and $h \in H_{\pi_i}$, $\boldsymbol{\pi}_i(\omega)(h)$ is a best response with respect to $\boldsymbol{\mu}_i|_{A(h)}(\omega)$. Since $\mu_i|(\pi_i) \in ICPS^e(\{\pi_i\} \times \Pi_{-i})$, $\boldsymbol{\mu}_i|(\omega) \in ICPS^e(\boldsymbol{\pi}(P_i(\omega)))$ $\forall i \in N$. Therefore, $\omega \in \mathring{R}$. But, since $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$, $\omega \in [\hat{\pi}]$. Therefore, $\Omega = \mathring{R} \cap [\hat{\pi}]$ and, hence, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in CK([\hat{\pi}] \cap \mathring{R}) = \Omega$. ■

An immediate corollary of Theorem 3.1 gives a more readily expressible and readable form of epistemic assumptions of RSCE: The notion of RSCE can be viewed as the logical consequence of common knowledge of robust-rationality plus mutual knowledge of a path of play.

Corollary 3.1 *(a) Let $\omega \in K[\hat{\pi}] \cap CK\mathring{R}$. Then, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ is an RSCE. (b) Let $\hat{\pi}$ be an RSCE. Then, there is a knowledge model $\mathcal{M}(T)$ such that $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in K[\hat{\pi}] \cap CK\mathring{R}$.*

This theorem says that mutual knowledge of the on-path actions, robust-rationality along on-path information sets, and common knowledge of robust-rationality at off-path information sets lead to "rationalizable self-confirming equilibrium (RSCE)." The "robust-rationality" is defined only at reachable information sets, rather than at all information sets. In particular, this "rationality" at off-path information sets does require that each player be optimal at all these information sets, but it requires only that each player be optimal at the information sets that are not precluded by the player's strategy at the state. Dekel et al. (1999) showed that the notion of RSCE is not robust to the presence of a small amount of payoff uncertainty in the sense of Fudenberg et al. (1988). The epistemic assumption of "common knowledge of robust-rationality at off-path information sets" can be justified by using the prior payoff information (cf. Dekel et al. (1999)).

In Fudenberg and Levine (1993) and Fudenberg and Kreps (1995), players are assumed to have no a prior information about each others' payoffs, and only observe the actions chosen by their opponents. In such an environment, Fudenberg and Kreps (1995) proposed the solution concept of "self-confirming equilibrium (SCE)" in which players' behavior is required to be optimal only at the observed information sets and players' behavior at off the equilibrium path information sets imposes no requirement of rationality. Without imposing any "rationality" restriction on the off-path behavior, we obtain an epistemic characterization for SCE as a corollary of Theorem 3.1.

Corollary 3.2 *(a) Let $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$. Then, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ is an SCE. (b) Let $\hat{\pi}$ be an SCE. Then, there is a knowledge model $\mathcal{M}(T)$ such that $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$.*

Proof. (a) For any $i \in N$, define

$$\Pi_i = \{\boldsymbol{\pi}_i(\omega) : \omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}},$$

and let $\Pi \equiv \times_{i \in N} \Pi_i$.

Clearly, if $h \in H_{\hat{\pi}}$, $\pi(h) = \hat{\pi}(h)$ for all $\pi \in \Pi$. That is, for all $\pi \in \Pi$, π has the same distribution over outcomes as induced by $\hat{\pi}$.

(i) For any $i \in N$, $\pi_i \in \Pi_i$, there exists $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$ such that $\boldsymbol{\pi}_i(\omega) = \pi_i$. Since $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$, $\omega \in \mathring{R}^{\hat{\pi}}$. Therefore, $\forall i \in N$ there is $\boldsymbol{\mu}_i|(\omega) \in ICPS^e(\boldsymbol{\pi}(P^i(\omega))$ such that $\forall h \in H_{\hat{\pi}} \cap H^i$, $\boldsymbol{\pi}_i(\omega)(h)$ is a best response with respect to $\boldsymbol{\mu}_i|_{A(h)}(\omega)$.

(ii) Since $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$, $\omega \in K_i[\hat{\pi}] \subseteq [\hat{\pi}]$. Then, for all $\omega' \in P_i(\omega)$, $\boldsymbol{\pi}(\omega')(h) = \hat{\pi}(h)$ for all $h \in H_{\hat{\pi}}$. That is, for all $\omega' \in P_i(\omega)$, $\boldsymbol{\pi}(\omega')(h) = \boldsymbol{\pi}(\omega)(h)$ for all $h \in H_{\hat{\pi}}$. Therefore,

$\boldsymbol{\pi}(P_i(\omega)) \subseteq \Pi$. Since $\boldsymbol{\pi}_i(\omega) = \boldsymbol{\pi}_i(\omega')\ \forall\omega' \in P_i(\omega)$, $\boldsymbol{\pi}(P_i(\omega)) \subseteq \{\pi_i\} \times \Pi_{-i}$ for all $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$.

By (i) and (ii), it follows that for every $i \in N$ and $\pi_i \in \Pi_i$, there is a $\mu_i \in ICPS^e(\Pi)$ such that for all $h \in H_{\hat{\pi}} \cap H^i$, $\pi_i(h)$ is best response with respect to μ_i. Thus, there exists $\pi_k \rightsquigarrow \pi \in co^e(\Pi)$ such that $\mu_i| = \mu|_{[\pi_k \rightsquigarrow \pi]}$, and $\forall h \in H_{\hat{\pi}}$, $\pi(h) = \hat{\pi}(h)$. By Lemma 1, $\forall i \in N$ and $\pi_i \in \Pi_i$, there exists (η_i, π^i_{-i}), which is consistent with $\pi_k \rightsquigarrow \pi$, such that $\forall h \in H_{(\pi_i, \pi^i_{-i})} \cap H^i$, $\pi_i(h)$ is best response with respect to (η_i, π^i_{-i}). As $\forall j \neq i$, $\pi_{j,k} \rightsquigarrow \pi_j \in co^e(\Pi_j)$ and $\pi_j = \pi^i_j$, $\pi^i_j \in co^e(\Pi_j)$. That is, $\forall h \in H_{\hat{\pi}}\ \left(\pi_i, \pi^i_{-i}\right)(h) = \hat{\pi}(h)$.

For all $i \in N$, let

$$V_i \equiv \left\{ \left(\boldsymbol{\pi}_i(\omega), (\eta_i, \pi^i_{-i})\right) \ : \ \begin{array}{c} (\eta_i, \pi^i_{-i}) \text{ is consistent with } \pi_k \rightsquigarrow \pi \\ \text{where } \mu|_{[\pi_k \rightsquigarrow \pi]} = \boldsymbol{\mu}_i|(\omega) \\ \text{and } \omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}} \end{array} \right\},$$

and $V \equiv (V_1, V_2, ..., V_n)$. Then, for all $i \in N$ and $\left(\pi_i, \left(\eta_i, \pi^i_{-i}\right)\right) \in V_i$, we have

(1') $\forall h \in H_{(\pi_i, \pi^i_{-i})} \cap H^i$, $\pi_i(h)$ is a best response with respect to (η_i, π^i_{-i}).

(2) $\left(\pi_i, \pi^i_{-i}\right)$ has the distribution over outcomes induced by $\hat{\pi}$.

(3) $\forall j \neq i$, there exists $\pi^i_j \in co^e(\Pi_j)$ where $\Pi_j = \{\pi'_j : (\pi'_j, (\eta_i, \pi^j_{-j})) \in V_j$ for some belief $(\eta_i, \pi^j_{-j})\}$.

That is, $\forall \omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ and $\hat{\pi}$ is an SCE.

(b) Let $\hat{\pi}$ be an SCE that is supported by $V = (V_1, V_2, ..., V_n)$.

We proceed to show a stronger result that there is $\mathcal{M}(T)$ such that $\boldsymbol{\chi}(\omega) = \hat{\pi}$ for all $\omega \in K([\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \neq \emptyset$. For all $i \in N$, for every $(\pi_i, (\eta_i, \pi^i_{-i})) \in V_i$,

(1') $\forall h \in H_{(\pi_i, \pi^i_{-i})} \cap H^i$, $\pi_i(h)$ is a best response with respect to (η_i, π^i_{-i}).

(2) $\left(\pi_i, \pi^i_{-i}\right)$ has the distribution over outcomes induced by $\hat{\pi}$.

(3) $\forall j \neq i$, there exists $\pi^i_j \in co^e(\Pi_j)$ where $\Pi_j = \{\pi'_j : (\pi'_j, (\eta_i, \pi^j_{-j})) \in V_j$ for some belief $(\eta_i, \pi^j_{-j})\}$.

Let $\mu_i|(\pi_i) = \mu|_{[\pi_k \rightsquigarrow \pi]}$ such that $\pi_{i,k} \rightsquigarrow \pi_i \in co^e(\{\pi_i\})$, and $\pi^i_{j,k} \rightsquigarrow \pi^i_j \in co^e(\Pi_j)$ $\forall j \neq i$. Clearly, $\mu_i|(\pi_i) \in ICPS^e(\{\pi_i\} \times \Pi_{-i})$. Define a knowledge model for game T:

$$\mathcal{M}(T) = <\Omega, \{P_i\}_{i \in N}, \{\boldsymbol{\pi}_i\}_{i \in N}, \{\boldsymbol{\mu}_i|\}_{i \in N}>,$$

such that $\Omega = \left\{\left(\pi_j, \mu_j|(\pi_j)\right)_{j \in N} : \ \pi_j \in \Pi_j, \forall j \in N\right\}$ and for all $i \in N$ and $\omega = \left(\pi_j, \mu_j|(\pi_j)\right)_{j \in N}$

in Ω,

$$\boldsymbol{\pi}_i(\omega) = \pi_i,\ \boldsymbol{\mu}_i(\omega) = \mu_i(\pi_i) \text{ and}$$
$$P_i(\omega) = \{\omega' \in \Omega : \boldsymbol{\pi}_i(\omega') = \pi_i \text{ and } \boldsymbol{\mu}_i(\omega') = \mu_i(\pi_i)\}.$$

Since $\left(\pi_i, \pi^i_{-i}\right)$ has the distribution over outcomes induced by $\hat{\pi}$ and perfect recall, $\forall h \in H_{\hat{\pi}}$, $\left(\pi_i, \pi^i_{-i}\right)(h) = \hat{\pi}(h)$. Now, consider any arbitrary $\omega = \left(\pi_j, \mu_j|(\pi_j)\right)_{j\in N} \in \Omega$. By lemma 1, it follows that for all $i \in N$ and $h \in H_{\hat{\pi}} \cap H^i$, $\boldsymbol{\pi}_i(\omega)(h)$ is a best response with respect to $\boldsymbol{\mu}_i|_{A(h)}(\omega)$. Since $\mu_i|(\pi_i) \in ICPS^e(\{\pi_i\} \times \Pi_{-i})$, $\boldsymbol{\mu}_i|(\omega) \in ICPS^e(\boldsymbol{\pi}(P_i(\omega)))$ $\forall i \in N$. Therefore, $\omega \in \mathring{R}^{\hat{\pi}}$. But, since $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$, $\omega \in [\hat{\pi}]$. Therefore, $\Omega = \mathring{R}^{\hat{\pi}} \cap [\hat{\pi}]$ and, hence, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in K([\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) = \Omega$. ■

As pointed out, Dekel et al. (1999) defined RSCE by using robust-rationality. If "rationality" is defined as the conventional (sequential) rationality in the sense of Kreps and Wilson (1982) – i.e., it requires to be sequentially rational at every information set, including those unreachable information sets, we can obtain a stronger version of "sequentially rationalizable self-confirming equilibrium (SRSCE)"; see Dekel et al. (1999, Sec. 4). Denoted by

$$R^h \equiv \{\omega : \text{ agent } h \text{ is rational at } \omega\}.$$

Denoted by $R^i \equiv \cap_{h\in H^i} R^h$ the event that player i is (sequential) rational at every his own information sets and let $R \equiv \cap_{i\in N} R^i$.

For any given path of play $\hat{\pi}$, let

$$R^{\hat{\pi}} \equiv \cap_{h\in H_{\hat{\pi}}} R^h \text{ and } R^{-\hat{\pi}} \equiv \cap_{h\notin H_{\hat{\pi}}} R^h.$$

$R^{\hat{\pi}}$ is the event that players are (sequential) rational along on-path information sets specified by strategy profile $\hat{\pi}$, and $R^{-\hat{\pi}}$ is the event that players are (sequential) rational at off-path information sets.

Corollary 3.3 *(a) Let $\omega \in (K[\hat{\pi}] \cap R^{\hat{\pi}}) \cap CKR^{-\hat{\pi}}$. Then, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ is an SRSCE. (b) Let $\hat{\pi}$ be an SRSCE. Then, there is a knowledge model $\mathcal{M}(T)$ such that $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in (K[\hat{\pi}] \cap R^{\hat{\pi}}) \cap CKR^{-\hat{\pi}}$.*

Proof. Corollary 3.3 follows immediately from Theorem 3.2 since $R^h \subseteq \mathring{R}^h$. ■

4 Concluding Remarks

In extensive-form games, Fudenberg and Levine (1993) and Fudenberg and Kreps (1995) presented a solution concept of "self-confirming equilibrium (SCE)" which arise as a steady states where players have no prior information about opponents' payoff functions or strategies, and each player observes only the actions played by opponents at each round of the game. Dekel et al. (1999) offered a solution concept of "rationalizable self-confirming equilibrium (RSCE)," where each player observes only the actions played by opponents at each round of the game and behaves rationally at all of his information sets that are not precluded by his own strategy, as a refinement of SCE. In this paper, we have carried out the epistemic program in game theory to explore epistemic conditions for RSCE.

We have established a simple epistemic characterization of RSCE. More specifically, by using the notion of "conditional probability system (CPS)" introduced by Myerson (1986), we have defined "rationality" as conditional maximization through CPS beliefs and, within a standard semantic framework, we have formulated and shown that RSCE is the logical consequence of common knowledge of "robust-rationality" and mutual knowledge of actions along the path. This paper therefore provides an epistemic counterpart of RSCE in terms of what players know and believe about "rationality," actions, information, and knowledge in complex social environments with emerging a commonly observed path.

This paper provides a unifying epistemic approach to other related game-theoretic solution concepts such as SCE, and "sequential rationalizable self-confirming equilibrium (SRSCE)." In this paper, we have shown how epistemic characterizations for various related solution concepts can be obtained, in a direct and simple way, by varying the requirements of "rationality," as well as assuming different epistemic conditions to players in the game. For instance, SCE can be formally represented as the result of mutual knowledge of actions along the path and rationality along the path; it coincides with the motivation of SCE where each player's strategy is a best response to his beliefs about the play of his opponents, and each player's beliefs are correct along the equilibrium path of play. The study of this paper is useful to deepen our understanding of RSCE and related solution concepts in the literature.

We would like to point out that, in this paper, we define "rationality" as conditional expected maximization by "independent" CPS beliefs. This formalism is used to capture the conventional notion of sequential rationality in Kreps and Wilson (1982). Greenberg et al. (2009) presented a unified game-theoretic solution concept of "mutually acceptable course of action (MACA)" suitable for situations where "perfectly" and "cautiously" rational individuals with different beliefs and views of the world agree to a shared course of action. When the underlying course of action is taken as the form of "path of play," MACA delivers a strong perfect-version of SRSCE which can rule out weakly dominated strategies. Luo and Wang (2013) provided expressible epistemic characterization for MACA by using "lexicographic probability system (Blume et al. (1991a,b))."

References

[1] G.B. Asheim and A. Perea, Sequential and quasi-perfect rationalizability in extensive games, Games Econ Behav 53 (2005), 15-42

[2] R.J. Aumann, Subjectivity and correlation in randomized strategies, J Math Econ 1 (1974), 67-96.

[3] R.J. Aumann, Agreeing to disagree, Ann Statist 4 (1976), 1236-1239

[4] R.J. Aumann, Correlated equilibrium as an expression of Bayesian rationality, Econometrica 55 (1987), 1-18.

[5] R.J. Aumann, Backward induction and common knowledge of rationality, Games Econ Behav 8 (1995), 6-19.

[6] R.J. Aumann, Interactive epistemology I: Knowledge, Int J Game Theory 28 (1999), 263-300.

[7] R.J. Aumann and A. Brandenburger, Epistemic conditions for Nash equilibrium, Econometrica 63 (1995), 1161-1180.

[8] B.D. Bernheim, Rationalizable strategic behavior, Econometrica 52 (1984), 1007-1028.

[9] L. Blume., A. Brandenburger, and E. Dekel, Lexicographic probabilities and choice under uncertainty, Econometrica 59 (1991a), 61-79.

[10] L. Blume., A. Brandenburger, and E. Dekel, Lexicographic probabilities and equilibrium refinements, Econometrica 59 (1991b), 81-98.

[11] A. Brandenburger., A. Friedenberg, and H.J. Keisler, Admissibility in games, Econometrica 76 (2008), 307-352.

[12] E. Dekel and D. Fudenberg, Rational behavior with payoff uncertainty, J Econ Theory 52 (1990), 243-267.

[13] E. Dekel, D. Fudenberg, and D.K. Levine, Payoff information and self-confirming equilibrium, J Econ Theory 89 (1999), 165-185.

[14] E. Dekel, D. Fudenberg, and D.K. Levine, Subjective uncertainty over behavior strategies: A correction, J Econ Theory 104 (2002), 473-478.

[15] D. Fudenberg and D.M. Kreps, Learning in extensive games. I. Self-confirming equilibrium, Games Econ Behav 8 (1995), 20-55.

[16] D. Fudenberg and D.M. Kreps, and D.K. Levine, On the robustness of equilibrium refinements, J Econ Theory 44 (1988), 354 380.

[17] D. Fudenberg and D.K. Levine, Self-confirming equilibrium, Econometrica 61 (1993), 523-546.

[18] D. Fudenberg and D.K. Levine, Superstition and rational learning, Amer Econ Rev 96 (2006), 630-651.

[19] D. Fudenberg and D.K. Levine, Self-confirming equilibrium and the Lucas critique, J Econ Theory 144 (2009) , 2354-2371.

[20] J. Greenberg., S. Gupta., and X. Luo, Mutually acceptable courses of action, Econ Theory 40 (2009), 91-112.

[21] J.Y. Halpern, Lexicographic probability, conditional probability, and nonstandard probability, Games Econ Behav 68 (2010), 155-179.

[22] D.M. Kreps and R. Wilson, Sequential equilibria, Econometrica 50 (1982), 863-894.

[23] H. Kuhn., Extensive games and the problem of information. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games, vol. II, pp. 193-216. New York, NJ: Princeton Univ. Press (1953).

[24] X. Luo and B. Wang, An epistemic approach to MACA, Working paper, National University of Singapore (2013).

[25] A. McLennan, Consistent conditional systems in noncooperative game theory, Inter J Game Theory 18 (1989), 141-174.

[26] R. Myerson, Multistage games with communication, Econometrica 54 (1986), 323-358.

[27] R. Myerson, Game Theory: Analysis of Conflict. Harvard Univ. Press, MA, 1991.

[28] D. Pearce, Rationalizable strategic behavior and the problem of perfection, Econometrica 52 (1984), 1029-1050.

[29] R. Selten, Reexamination of the perfectness concept for equilibrium points in extensive games, Int J Game Theory 4 (1975), 25-55.

[30] T. Tan, and S.R.C. Werlang, The Bayesian foundations of solution concepts of games, J Econ Theory 45 (1988), 370-391.